ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 6, 2018

ON EXISTENCE AND UNIQUENESS OF L_1-SOLUTIONS FOR QUADRATIC INTEGRAL EQUATIONS VIA A KRASNOSELSKII-TYPE FIXED POINT THEOREM

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ABSTRACT. Using a Krasnoselskii-type fixed point theorem due to Burton **[7]**, we discuss the existence of integrable solutions of general quadratic-Urysohn integral equations on a bounded interval (a, b). Uniqueness of the solution is also studied. An example to illustrate our theory is also included.

1. Introduction. Quadratic integral equations play an important role in the theory of radioactive transfer, kinetic theory of gases, neutron transport theory, and in traffic theory **[2, 6, 8]**. In this paper, we study the equation

$$x(t) = g(t, T_1x(t)) + T_2x(t) \cdot \int_a^b u(t, s, x(s))\, ds, \quad t \in I = (a, b), \tag{1.1}$$

where T_1 and T_2 are two operators.

Equation (1.1) is very general and includes the following equations as special cases:

(1) $T_1x = x$, $T_2x = 1$, $u(t, s, x) = K(t, s)f(s, x)$, and the integral equation is of Hammerstein type **[15]**;
(2) $g(t, T_1x) = h(t)$, and $T_2x = x$ is the functional-integral equation **[14]**;
(3) $g(t, T_1x) = h(t)$, $u(t, s, x) = K(t, s)f(s, x)$, and the quadratic integral equation was discussed in Orlicz spaces **[9, 10]**;

2010 AMS *Mathematics subject classification.* Primary 45G10, 47H30, 47N20.
Keywords and phrases. Quadratic-Urysohn integral equations, integrable solutions, Krasnoselskii's fixed point theorem, uniqueness of the solution.
Received by the editors on October 27, 2017, and in revised form on January 4, 2018.
DOI:10.1216/RMJ-2018-48-6-1743

(4) for continuous solutions with $T_1x = x$, $T_2x = f(t,x)$ and

$$u(t,s,x) = \frac{u_1(t,s,x)}{\Gamma(\alpha)\cdot(t-s)^{1-\alpha}}, \quad \text{see } [\mathbf{5}];$$

(5) $T_1x = x$, $T_2x = \lambda$ is the functional Urysohn integral equation (for continuous solutions see [**3**]);
(6) $T_1x = x$, $T_2x = x$ is the quadratic (functional) Urysohn integral equation (see, for example, [**4**]);
(7) $T_1x = \int_0^1 u(t,s,x(s))\,ds$, $T_2x(t) = 0$ and was discussed in [**12**].

In this paper, we discuss the existence of integrable solutions of (1.1). We will distinguish between two different cases: when an operator takes its values in the Lebesgue spaces $L_p(I)$ or in a space of essentially bounded functions $L_\infty(I)$. Uniqueness of the solution is also discussed in each case.

2. Notation and auxiliary facts. Let $\mathbb{R}$ be the field of real numbers, I an interval (a,b) and $L_1(I)$ the space of Lebesgue integrable functions (equivalence classes of functions) on a measurable subset I of $\mathbb{R}$, with the standard norm

$$\|x\|_{L_1(I)} = \int_I |x(t)|\,dt.$$

Recall that, by L_p, $1 \le p < \infty$, we will denote the space of (equivalences classes of) functions x, satisfying

$$\int_I |x(t)|^p < \infty.$$

By $\|\cdot\|_p$, we will denote the norm in $L_p(I)$. In addition, $L_\infty(I)$ denotes the Banach space of essentially bounded measurable functions together with the essential supremum norm (denoted by $\|.\|_{L_\infty}$). We will write L_1, L_p and L_∞ instead of $L_1(I)$, $L_p(I)$ and $L_\infty(I)$, respectively. Denote by $B(x,r)$ the closed ball with the center at x and with radius r and B_r for the ball $B(\theta,r)$ centered at zero element θ.

Now, let $I \subset \mathbb{R}$ be a given interval.

Definition 2.1. Assume that a function $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, i.e., it is measurable in t for any

$x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every measurable function x on I, we shall define the operator

$$F_f(x)(t) = f(t,\ x(t)), \quad t \in I.$$

The operator F_f, defined in such a way, is called the *superposition* (*Nemytskii*) *operator* generated by the function f [**1**].

Theorem 2.2 ([**1**]). *Suppose that f satisfies Carathéodory conditions. The superposition operator F_f maps the space L_p into L_q, $p,q \geq 1$, if and only if*:

$$|f(t,x)| \leq a(t) + b|x|^{p/q}, \tag{2.1}$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L_q$ and $b \geq 0$. Moreover, this operator is continuous.

For Nemytiskii operators, we have the following theorem.

Theorem 2.3 ([**1,** Theorem 3.17]). *The superposition operator F_f maps L_p into L_∞ if and only if*

$$|f(t,x)| \leq a(t), \quad x \in \mathbb{R}$$

for some $a \in L_\infty$, i.e., f is independent of x.

We now recall some basic facts concerning the Urysohn operators $Ux(t) = \int_a^b u(t,s,x(s))\,ds$.

Theorem 2.4 ([**17,** Theorem 10.1.10]). *Let $u : I \times I \times \mathbb{R}$ satisfy the Carathéodory condition, i.e., it is measurable in (t,s) for any $x \in \mathbb{R}$ and continuous in x for almost all $(t,s) \in I \times I$. Assume that the operator U maps L_p into $L_q (q < \infty)$ and, for each $h > 0$, the function*

$$R_h(t,s) = \max_{|x| \leq h} |u(t,s,x)|$$

is integrable with respect to s for almost every $t \in I$. If, moreover, for each $h > 0$ and $D \subset I$, we have

$$\lim_{\mathrm{meas} D \to 0} \sup_{|x| \leq h} \left\| \int_D u(t,s,x)\,ds \right\|_{L_q} = 0$$

and, for any nonnegative $z \in L_q$,

$$\lim_{\text{meas} D \to 0} \sup_{|x| \le z} \left\| \int_D u(t,s,x)\,ds \right\|_{L_q} = 0,$$

then U *is a continuous operator. The first two conditions are satisfied when* $\int_I R_h(t,s)\,ds \in L_q$.

Theorem 2.5. **[17]** *Let* $u : I \times I \times \mathbb{R} \to \mathbb{R}$ *satisfy the Carathéodory condition, i.e., it is measurable in* (t,s) *for any* $x \in \mathbb{R}$ *and continuous in* x *for almost all* (t,s). *Assume that*

$$|u(t,s,x)| \le k(t,s),$$

where the nonnegative function k *is measurable in* (t,s) *such that the linear integral operator* K_0 *with the kernel* $k(t,s)$ *maps* L_1 *into* L_∞. *Then, the operator* U *maps* L_1 *into* L_∞. *Moreover, if, for arbitrary* $h > 0$, $x_i \in \mathbb{R}$, $i = 1,2$,

$$\lim_{\delta \to 0} \left\| \int_I \max_{\substack{|x_i| \le h \\ |x_1 - x_2| \le \delta}} |u(t,s,x_1) - u(t,s,x_2)|\,ds \right\|_{L_\infty} = 0,$$

then U *is a continuous operator.*

We note that some particular conditions ensuring the continuity of the operator U may be found in **[16, 17]**.

Next, we state the compactness criteria due to Kolmogorov, see **[11]**.

Theorem 2.6 (**[13]**). *Let* $\Omega \subseteq L_p[0,1]$, $1 \le p < \infty$. *If*

(i) Ω *is bounded in* $L_p[0,1]$,
(ii) $v_h \to v$ (*converges in* $L_p[0,1]$) *as* $h \to 0$ *uniformly with respect to* $v \in \Omega$, *then* Ω *is relatively compact in* $L_p[0,1]$; *here,*

$$v_h(t) = \frac{1}{h} \int_t^{t+h} v(s)\,ds.$$

Finally, we state a Krasnoselskii-type fixed point theorem due to Burton **[7]**.

Theorem 2.7. *Let S be a nonempty, closed, convex and bounded subset of the Banach space E, and let $A : E \to E$ and $B : S \to E$ be two operators such that*:

(a) *A is contraction*;
(b) *B is completely continuous*;
(c) *$x = Ax + By \Rightarrow x \in S$ for all $y \in S$.*

Then, the equation $Ax + Bx = x$ has a solution in S.

3. Main result.

Rewrite (1.1) as

$$x = Ax + Bx,$$

where

$$(Ax)(t) = F_g(T_1x)(t), \qquad (Bx)(t) = (T_2x)(t) \cdot (Ux)(t),$$

$$F_g(T_1x) = g(t, T_1x), \quad \text{and} \quad (Ux)(t) = \int_a^b u(t, s, x(s))\, ds.$$

3.1. The case when the operator U has values in L_∞. We consider (1.1) and the following assumptions.

(i) $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. The operators $T_i : L_1 \to L_1$, $i = 1, 2$, are continuous. There are positive functions $a_i \in L_1$, $i = 1, 2, 3$, such that

$$|g(t, 0)| \leq a_3(t), \qquad |T_1(0)| \leq a_1(t)$$

and

$$|T_2x(t)| \leq a_2(t) + b_2|x(t)|, \quad b_2 \geq 0$$

for almost every $t \in I$ and $x \in L_1$. Also, there are constants $b_j \geq 0$, $j = 1, 3$, such that, for almost every $t \in I$:

$$|g(t, x) - g(t, y)| \leq b_3|x - y|, \quad x, y \in \mathbb{R}$$

and

$$|T_1(x(t)) - T_1(y(t))| \leq b_1|x(t) - y(t)|, \quad x, y \in L_1.$$

(ii) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. Moreover, for arbitrary fixed $s \in I$ and $x \in \mathbb{R}$, the function $t \to u(t,s,x)$ is integrable.

(iii) Assume that $|u(t,s,x)| \le k(t,s)$, for all $t,s \ge 0$ and $x \in \mathbb{R}$, where the function k is measurable in (t,s). Assume that the linear integral operator K_0 with the kernel $k(t,s)$ maps L_1 into L_∞. Moreover, assume that, for arbitrary $h > 0$, $x_i \in \mathbb{R}$, $i = 1,2$,

$$\lim_{\delta \to 0} \left\| \int_I \max_{\substack{|x_i| \le h \\ |x_1 - x_2| \le \delta}} |u(t,s,x_1) - u(t,s,x_2)|\, ds \right\|_{L_\infty} = 0,$$

and let

$$\lim_{h \to 0} \left\| \frac{1}{h} \int_t^{t+h} |k(\theta,s) - k(t,s)|\, d\theta \right\|_{L_1} = 0. \tag{3.1}$$

(iv) $b_1 b_3 + b_2 \|K_0\|_{L_\infty} < 1$.

(v) Let

$$r = \frac{\|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + \|K_0\|_{L_\infty} \cdot \|a_2\|_{L_1}}{1 - (b_1 b_3 + b_2\|K_0\|_{L_\infty})},$$

and assume that

$$\left\| \frac{1}{h} \int_t^{t+h} |T_2 x(\theta) - T_2 x(t)|\, d\theta \right\|_{L_1} \longrightarrow 0, \tag{3.2}$$

as $h \to 0$ uniformly with respect to $x \in B_r$, where B_r is the closed ball with center 0 and radius r, i.e., $B_r = \{x \in L_1 : \|x\|_{L_1} \le r\}$.

Theorem 3.1. *Suppose that assumptions* (i)–(v) *hold. Then,* (1.1) *has at least one integrable solution $x \in L_1$ on I.*

Proof. The proof will be given in five steps.

Step 1. The operator $A : L_1 \to L_1$ is a contraction.

Step 2. The operator B maps the ball B_r into L_1 and is continuous.

Step 3. $B(B_r(I))$ is relatively compact in L_1 using Theorem 2.6.

Step 4. We prove that Theorem 2.7 (c) holds.

Step 5. We apply Theorem 2.7.

Step 1. Let $x \in L_1$. From assumption (i), we have, for almost every $t \in I$, that

$$\begin{aligned} ||T_1(x(t))| - |T_1(0)|| &\leq |T_1(x(t)) - T_1(0)| \leq b_1|x(t)| \\ \Longrightarrow |T_1(x(t))| &\leq |T_1(0)| + b_1|x(t)| \leq a_1(t) + b_1|x(t)|, \end{aligned}$$

and

$$\begin{aligned} |g(t, T_1x(t))| &\leq a_3(t) + b_3|T_1(x(t))| \\ &\leq a_3(t) + b_3[a_1(t) + b_1|x(t)|] \\ &\leq [a_3(t) + b_3a_1(t)] + b_1b_3|x(t)|. \end{aligned}$$

Thus, A maps L_1 into itself. In addition,

$$\begin{aligned} \int_I |(Ax)(t) - (Ay)(t)|\, dt &= \int_a^b |g(t, T_1x(t)) - g(t, T_1y(t))|\, dt \\ &\leq \int_a^b b_3|T_1x(t) - T_1y(t)|\, dt \\ &\leq b_3 \int_a^b b_1|x(t) - y(t)|\, dt \\ &\leq b_1b_3 \int_a^b |x(t) - y(t)|\, dt, \end{aligned}$$

which implies that

$$\|Ax - Ay\|_{L_1} \leq b_1b_3\|x - y\|_{L_1}. \tag{3.3}$$

From assumption (iv), we deduce that A is a contraction.

Step 2. From assumptions (ii) and (iii), we deduce that the operator U maps L_1 into L_∞ continuously. From assumption (i), the operator T_2 maps L_1 into itself continuously, which implies that B transforms the ball $B_r(L_1)$ into L_1 and is continuous.

Step 3. Now, for $x \in B_r(I)$, we have

$$\begin{aligned} \|B(x)\|_{L_1} &= \int_I |(Bx)(t)|\, dt \leq \int_I |T_2x(t) \cdot U(x)(t)|\, dt \\ &\leq \int_I |T_2x(t)| \cdot \int_I |u(t, s, x(s))|\, ds\, dt \\ &\leq \int_I [a_2(t) + b_2|x(t)|]\, dt \cdot \int_I k(t, s)\ ds \\ &= \|K_0\|_{L_\infty}[\|a_2\|_{L_1} + b_2\|x\|_{L_1}]; \end{aligned}$$

thus, $B(B_r(I))$ is bounded in L_1, i.e., Theorem 2.6 (i) is satisfied. Letting $x \in B_r(I)$, then

$$\begin{aligned}
|(Bx)_h(t) - (Bx)(t)| &= \left| \frac{1}{h} \int_t^{t+h} (Bx)(\theta)\, d\theta - (Bx)(t) \right| \\
&\le \frac{1}{h} \int_t^{t+h} |(Bx)(\theta) - (Bx)(t)|\, d\theta \\
&= \frac{1}{h} \int_t^{t+h} \bigg| T_2x(\theta) \cdot \int_a^b u(\theta, s, x(s))\, ds \\
&\qquad\qquad - T_2x(t) \cdot \int_a^b u(t, s, x(s))\, ds \bigg|\, d\theta \\
&\le \frac{1}{h} \int_t^{t+h} \bigg| T_2x(\theta) \cdot \int_a^b u(\theta, s, x(s))\, ds - T_2x(t) \\
&\qquad\qquad \cdot \int_a^b u(\theta, s, x(s))\, ds \bigg|\, d\theta \\
&\quad + \frac{1}{h} \int_t^{t+h} \bigg| T_2x(t) \cdot \int_a^b u(\theta, s, x(s))\, ds \\
&\qquad\qquad - T_2x(t) \cdot \int_a^b u(t, s, x(s))\, ds \bigg|\, d\theta \\
&\le \frac{1}{h} \int_t^{t+h} |T_2x(\theta) - T_2x(t)| \\
&\quad \cdot \int_a^b |u(\theta, s, x(s))|\, ds \,|d\theta + \frac{1}{h} \int_t^{t+h} |T_2x(t)| \\
&\quad \cdot \int_a^b |u(\theta, s, x(s)) - u(t, s, x(s))|\, ds\, d\theta \\
&\le \frac{1}{h} \int_t^{t+h} |T_2x(\theta) - T_2x(t)| \cdot \int_a^b k(\theta, s)\, ds\, d\theta \\
&\quad + \int_a^b [a_2(t) + b_2|x(t)|] \\
&\quad \cdot \left(\frac{1}{h} \int_t^{t+h} |k(\theta, s) - k(t, s)|\, d\theta \right) ds,
\end{aligned}$$

which implies that

$$
\begin{aligned}
\|(Bx)_h - (Bx)\|_{L_1} &= \int_a^b |(Bx)_h(t) - (Bx)(t)|\, dt \\
&\le \|K_0\|_{L_\infty} \int_a^b \left(\frac{1}{h} \int_t^{t+h} |T_2x(\theta) - T_2x(t)|\, d\theta \right) dt \\
&\quad + \int_a^b \int_a^b [a_2(t) + b_2|x(t)|] \\
&\quad \cdot \left(\frac{1}{h} \int_t^{t+h} |k(\theta,s) - k(t,s)|\, d\theta \right) ds\, dt \\
&\le \|K_0\|_{L_\infty} \left\| \frac{1}{h} \int_t^{t+h} |T_2x(\theta) - T_2x(t)|\, d\theta \right\|_{L_1} \\
&\quad + [\|a_2\|_{L_1} + b_2 \cdot r] \cdot \left\| \frac{1}{h} \int_t^{t+h} |k(\theta,s) - k(t,s)|\, d\theta \right\|_{L_1}.
\end{aligned}
$$

From (3.1) and (3.2), we deduce that $(Bx)_h \to (Bx)$, (converges in L_1) as $h \to 0$ uniformly with respect to $x \in B_r(I)$. Now, Theorem 2.6 guarantees that $B(B_r(I))$ is relatively compact in the space L_1.

Step 4. Fix $x \in L_1$, and assume that the equality $x = Ax + By$ holds for some $y \in B_r$. Then,

$$
\begin{aligned}
\int_I |x(t)|\, dt &\le \int_I |Ax(t) + By(t)|\, dt \\
&\le \int_I \left(|g(t, T_1x(t))| + |T_2y(t)| \cdot \left| \int_I u(t,s,y(s))\, ds \right| \right) dt \\
&\le \int_I \left(a_3(t) + b_3|T_1x(t)| + [a_2(t) + b_2|y(t)|] \int_I k(t,s)\, ds \right) dt \\
&\le \|a_3\|_{L_1} + b_3 \int_I |T_1x(t)|\, dt \\
&\quad + \int_I \left([a_2(t) + b_2|y(t)|] \cdot \int_I k(t,s)\, ds \right) dt \\
&\le \|a_3\|_{L_1} + b_3 \int_a^b [a_1(t) + b_1|x(t)|]\, dt \\
&\quad + \left(\int_I a_2(t) dt + b_2 \int_a^b |y(t)|\, dt \right) \|K_0\|_{L_\infty} \\
&\le \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + b_1b_3\|x\|_{L_1} + [\|a_2\|_{L_1} + b_2\|y\|_{L_1}]
\end{aligned}
$$

$$\begin{aligned}
&\cdot \|K_0\|_{L_\infty} \\
&\le \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + b_1 b_3\|x\|_{L_1} + \|K_0\|_{L_\infty} \\
&\quad \cdot \left[\|a_2\|_{L_1} + \ b_2 \cdot r\right].
\end{aligned}$$

The above inequality yields

$$(1 - b_1 b_3)\|x\|_{L_1} \le \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + \|K_0\|_{L_\infty} \cdot \left[\|a_2\|_{L_1} + b_2 \cdot r\right].$$

Since $(1 - b_1 b_3) > 0$, this implies

$$\|x\|_{L_1} \le \frac{\|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + \|K_0\|_{L_\infty} \cdot [\|a_2\|_{L_1} + b_2 \cdot r]}{(1 - b_1 b_3)}.$$

Now, recall that

$$\frac{\|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + \|K_0\|_{L_\infty} \cdot [\|a_2\|_{L_1} + b_2 \cdot r]}{(1 - b_1 b_3)} = r.$$

Hence, $\|x\|_{L_1} \le r$, i.e., $x \in B_r$. Thus, Theorem 2.7 (c) is satisfied.

Step 5. From the above steps, we can apply Theorem 2.7. Thus, (1.1) has at least one integrable solution $x \in L_1$ in $B_r(I)$. $\square$

3.1.1. Uniqueness of the solution. Consider the next two assumptions.

(vi) $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. The operators $T_i : L_1 \to L_1$, $i = 1, 2$, are continuous. There are positive functions $a_i \in L_1$, $i = 1, 2, 3$, such that

$$|g(t, 0)| \le \ a_3(t), \qquad |T_j(0)| \le a_j(t), \quad j = 1, 2,$$

for almost every $t \in I$. In addition, there are constants $b_i \ge 0$, $i = 1, 2, 3$, such that, for almost every $t \in I$:

$$|g(t, x) - g(t, y)| \le b_3|x - y|, \quad x, y \in \mathbb{R},$$

and

$$|T_j(x(t)) - T_j(y(t))| \le b_j|x(t) - y(t)|, \quad j = 1, 2, \ x, y \in \ L_1.$$

(vii) There exists a positive constant M (which may depend upon r) such that

$$|u(t, s, x(s)) - u(t, s, y(s))| \le M|x(s) - y(s)|,$$

for almost every $t \in I$, almost every $s \in I$ and $x, y \in B_r$ (where r is given in Theorem 3.1).

Theorem 3.2. *Suppose that assumptions* (ii)–(vii) *hold. If*

$$M \leq \frac{[1-(b_1b_3+b_2\|K_0\|_{L_\infty})]^2}{(1-b_1b_3)\|a_2\|_{L_1}+b_2\|a_3\|_{L_1}+b_2b_3\|a_1\|_{L_1}},$$

then (1.1) *has a unique, integrable solution $x \in L_1$ in $B_r(I)$.*

Proof. Let x and y be any two solutions of (1.1) in $B_r(I)$. Then, for almost every $t \in I$,

$$\begin{aligned}
|x(t)-y(t)| &\leq |g(t,T_1x(t))-g(t,T_1y(t))| \\
&\quad + \left|T_2x(t)\cdot\int_I u(t,s,x(s))\,ds - T_2y(t)\cdot\int_I u(t,s,y(s))\,ds\right| \\
&\leq b_3|T_1x(t)-T_1y(t)| \\
&\quad + \left|T_2x(t)\cdot\int_I u(t,s,x(s))\,ds - T_2x(t)\cdot\int_I u(t,s,y(s))\,ds\right| \\
&\quad + \left|T_2x(t)\cdot\int_I u(t,s,y(s))\,ds - T_2y(t)\cdot\int_I u(t,s,y(s))\,ds\right| \\
&\leq b_1b_3|x(t)-y(t)| \\
&\quad + |T_2x(t)|\cdot\int_I |u(t,s,x(s))-u(t,s,y(s))|\,ds \\
&\quad + |T_2x(t)-T_2y(t)|\int_I |u(t,s,y(s))|\,ds \\
&\leq b_1b_3|x(t)-y(t)| + [a_2(t)+b_2|x(t)|]\int_I M|x(s)-y(s)|\,ds \\
&\quad + b_2|x(t)-y(t)|\int_I k(t,s)\,ds;
\end{aligned}$$

thus,

$$\begin{aligned}
\|x-y\|_{L_1} = \int_I |x(t)-y(t)|\,dt &\leq b_1b_3\int_I |x(t)-y(t)|\,dt \\
&\quad + \int_I [a_2(t)+b_2|x(t)|]\int_I M|x(s)-y(s)|\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ b_2 \int_I |x(t) - y(t)| \int_I k(t,s)\,ds\,dt \\
&= b_1 b_3 \|x-y\|_{L_1} + M[\|a_2\|_{L_1} + b_2\|x\|_{L_1}] \cdot \|x-y\|_{L_1} \\
&\quad + b_2\|K_0\|_{L_\infty}\|x-y\|_{L_1} \\
&\le b_1 b_3 \|x-y\|_{L_1} + M[\|a_2\|_{L_1} + b_2 \cdot r] \cdot \|x-y\|_{L_1} \\
&\quad + b_2\|K_0\|_{L_\infty}\|x-y\|_{L_1}.
\end{aligned}
$$

The above inequality yields

$$[1 - (b_1 b_3 + b_2\|K_0\|_{L_\infty} + M[\|a_2\|_{L_1} + b_2 \cdot r])] \cdot \|x-y\|_{L_1} \le 0,$$

which implies that

$$\|x-y\|_{L_1} = 0, \qquad \Longrightarrow x = y.$$

This completes the proof. □

3.2. The existence of solutions when the operator U has values in L_p. In this section, we use Theorem 2.4 (with $1/p + 1/q = 1$).

We consider (1.1) with the following assumptions:

(i)′ $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. The operator $T_1 : L_1 \to L_1$ is continuous, and the operator $T_2 : L_1 \to L_p$ is continuous. There are positive functions $a_1, a_3 \in L_1$ and $a_2 \in L_p$ such that

$$|g(t,0)| \le \; a_3(t), \qquad |T_1(0)| \le a_1(t)$$

and

$$|T_2 x(t)| \le a_2(t) + b_2|x(t)|^{1/p}, \quad b_2 \ge 0,$$

for all $t \in I$ and $x \in L_1$. In addition, there are constants $b_j \ge 0$, $j = 1, 3$, such that, for almost every $t \in I$:

$$|g(t,x) - g(t,y)| \le b_3|x-y|, \quad x, y \in \mathbb{R},$$

and

$$|T_1(x(t))\; T_1(y(t))| \le b_1|x(t) - y(t)|, \quad x, y \in L_1.$$

(ii)′ $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. Suppose that, for any nonnegative $z \in L_q$ and for $D \subset I$,

$$\lim_{\operatorname{meas} D \to 0} \sup_{|x| \le z} \left\| \int_D u(t,s,x(s))\,ds \right\|_{L_q} = 0,$$

and that

$$|u(t,s,x)| \le k(t,s)(a_4(s) + b_4|x|^{1/p}) \quad \text{for all } t,s \ge 0 \text{ and } x \in \mathbb{R},$$

where the function k is measurable in (t,s), a_4 is a positive function in L_p and $b_4 \ge 0$. Assume that the linear integral operator K_0 with the kernel $k(t,s)$ maps L_p into L_q. Moreover, assume that

$$\lim_{h\to 0} \left\| \frac{1}{h}\int_t^{t+h} \| \, k(\theta,\cdot) - k(t,\cdot)\|_{L_q} \, d\theta \right\|_{L_q} = 0. \tag{3.4}$$

(iii)$'$ Assume that r' is a positive solution of the equation

$$\begin{aligned} &\|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + \|a_2\|_{L_p}\|a_4\|_{L_p}\|K_0\| + (b_1b_3 - 1)\cdot w \\ &\qquad + w^{1/p}\cdot \|K_0\|(b_2\|a_4\|_{L_p} + b_4\|a_2\|_{L_p}) + b_2b_4\|K_0\|\cdot w^{2/p} = 0, \end{aligned}$$

where $\|K_0\| = \| \|k(t,\cdot)\|_{L_q}\|_{L_q}$, and assume that

$$\left\| \frac{1}{h}\int_t^{t+h} |T_2x(\theta) - T_2x(t)| \, d\theta \right\|_{L_p} \longrightarrow 0, \tag{3.5}$$

as $h \to 0$ uniformly with respect to $x \in B_{r'}$.

Theorem 3.3. *Suppose that assumptions* (i)$'$–(iii)$'$ *hold. If* $b_1b_3 < 1$, *then* (1.1) *has at least one integrable solution* $x \in L_1$ *in* $B_{r'}(I)$.

Proof.

Step 1 is the same as in Theorem 3.1.

Step 2. From assumption (ii)$'$, we deduce that the operator U maps L_1 into L_q continuously. From assumption (i)$'$, the operator T_2 maps L_1 into L_p continuously, which implies that B transforms the ball $B_{r'}(L_1)$ into L_1 and is continuous.

Step 3. Now, for $x \in B_{r'}(I)$, we have

$$\begin{aligned} \|B(x)\|_{L_1} &= \|T_2x\cdot U(x)\|_{L_1} \\ &\le \|T_2x\|_{L_p}\cdot \left\| \int_a^b u(t,s,x(s))\,ds \right\|_{L_q} \\ &\le \|a_2 + b_2x^{1/p}\|_{L_p}\cdot \left\| \int_a^b k(t,s)(a_4(s) + b_4|x(s)|^{1/p})\,ds \right\|_{L_q} \end{aligned}$$

$$\leq (\|a_2\|_{L_p} + b_2\|x^{1/p}\|_{L_p}) \cdot \|\|k(t,\cdot)\|_{L_q}\|a_4 + b_4 x^{1/p}\|_{L_p}\|_{L_q}$$
$$\leq (\|a_2\|_{L_p} + b_2\|x\|_{L_1}^{1/p}) \cdot \|K_0\|(\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p});$$

thus, $B(B_{r'}(I))$ is bounded in L_1, i.e., Theorem 2.6 (i) is satisfied. Let $x \in B_{r'}(I)$. Then,

$$\begin{aligned}
&|(Bx)_h(t) - (Bx)(t)| \\
&\quad \leq \frac{1}{h}\int_t^{t+h} |T_2 x(\theta) - T_2 x(t)| \cdot \int_a^b k(\theta,s)(a_4(s) + b_4|x(s)|^{1/p})\,ds\,d\theta \\
&\quad + \frac{1}{h}\int_t^{t+h} \bigg([a_2(t) + b_2|x(t)|^{1/p}] \\
&\qquad\qquad \cdot \int_a^b |u(\theta,s,x(s)) - u(t,s,x(s))|\,ds\bigg)\,d\theta \\
&\quad \leq \frac{1}{h}\int_t^{t+h} |T_2 x(\theta) - T_2 x(t)| \cdot \|k(\theta,\cdot)\|_{L_q} \\
&\quad \cdot \|a_4 + b_4 x^{1/p}\|_{L_p}\,d\theta + [a_2(t) + b_2|x(t)|^{1/p}]\frac{1}{h} \\
&\quad \cdot \int_t^{t+h} \bigg(\int_a^b |k(\theta,s) - k(t,s)|(a_4(s) \\
&\qquad\qquad + b_4|x(s)|^{1/p})\,ds\bigg)\,d\theta \\
&\leq \frac{1}{h}\int_t^{t+h} |T_2 x(\theta) - T_2 x(t)| \cdot \|k(\theta,\cdot)\|_{L_q} \\
&\quad \cdot (\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p})\,d\theta + [a_2(t) + b_2|x(t)|^{1/p}](\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p}) \\
&\quad \cdot \frac{1}{h}\int_t^{t+h} \|k(\theta,\cdot) - k(t,\cdot)\|_{L_q}\,d\theta,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|(Bx)_h - (Bx)\|_{L_1} \leq &\bigg\|\frac{1}{h}\int_t^{t+h} |T_2 x(\theta) - T_2 x(t)| \cdot \|k(\theta,\cdot)\|_{L_q} \\
&\cdot (\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p})\,d\theta\bigg\|_{L_1} \\
&+ \bigg\|[a_2 + b_2|x|^{1/p}](\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p})
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \cdot \frac{1}{h}\int_t^{t+h} \|k(\theta,\cdot)-k(t,\cdot)\|_{L_q}\,d\theta\Big\|_{L_1}\\
&\le \Big\|\frac{1}{h}\int_t^{t+h} |T_2x(\theta)-T_2x(t)|\,d\theta\Big\|_{L_p}\\
&\quad \cdot \Big\|\|k(\theta,\cdot)\|_{L_q}(\|a_4\|_{L_p}+b_4\|x\|_{L_1}^{1/p})\Big\|_{L_q}\\
&\quad + \Big\|a_2+b_2|x|^{1/p}\Big\|_{L_p}\\
&\quad \cdot \Big\|(\|a_4\|_{L_p}+b_4\|x\|_{L_1}^{1/p})\frac{1}{h}\\
&\qquad\qquad \cdot \int_t^{t+h} \|k(\theta,\cdot)-k(t,\cdot)\|_{L_q}\,d\theta\Big\|_{L_q}\\
&\le \|K_0\|(\|a_4\|_{L_p}+b_4\|x\|_{L_1}^{1/p})\\
&\quad \cdot \Big\|\frac{1}{h}\int_t^{t+h} |T_2x(\theta)-T_2x(t)|\,d\theta\Big\|_{L_p}\\
&\quad + [\|a_2\|_{L_p}+b_2\|x\|_{L_1}^{1/p}](\|a_4\|_{L_p}+b_4\|x\|_{L_1}^{1/p})\\
&\quad \cdot \Big\|\frac{1}{h}\int_t^{t+h} \|k(\theta,\cdot)-k(t,\cdot)\|_{L_q}\,d\theta\Big\|_{L_q}.
\end{aligned}
$$

From (3.4) and (3.5), we deduce that $(Bx)_h \to (Bx)$, (converges in L_1) as $h \to 0$ uniformly with respect to $x \in B_{r'}(I)$. Now, Theorem 2.6 guarantees that $B(B_{r'}(I))$ is relatively compact in the space L_1.

Step 4. Fix $x \in L_1$, and assume that the equality $x = Ax + By$ holds for some $y \in B_{r'}$. Then

$$
\begin{aligned}
\|x\|_{L_1} &\le \|Ax+By\|_{L_1}\\
&\le \|Ax\|_{L_1} + \|T_2y\|_{L_p}\|Uy\|_{L_q}\\
&= \|g(t,T_1x)\|_{L_1} + \|T_2y\|_{L_p}\Big\|\int_I u(t,s,y(s))\,ds\Big\|_{L_q}\\
&\le \|a_3+b_3T_1x\|_{L_1} + \|a_2+b_2|y|^{1/p}\|_{L_p}\\
&\quad \cdot \Big\|\int_I k(t,s)(a_4(s)+b_4|y(s)|^{1/p})\,ds\Big\|_{L_q}\\
&\le \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + b_1b_3\|x\|_{L_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\|a_2\|_{L_p} + b_2\|y\|_{L_1}^{1/p})\|\|k(t,\cdot)\|_{L_q}(\|a_4\|_{L_p} + b_4\|y\|_{L_1}^{1/p})\|_{L_q} \\
\leq\ & \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} + b_1 b_3\|x\|_{L_1} \\
&+ \|K_0\|(\|a_2\|_{L_p} + b_2 \cdot r'^{1/p})(\|a_4\|_{L_p} + b_4 \cdot r'^{1/p}).
\end{aligned}
$$

The above inequality yields

$$
\begin{aligned}
(1 - b_1 b_3)\|x\|_{L_1} \leq\ & \|a_3\|_{L_1} + b_3\|a_1\|_{L_1} \\
&+ \|K_0\|(\|a_2\|_{L_p} + b_2 \cdot r'^{1/p})(\|a_4\|_{L_p} + b_4 \cdot r'^{1/p}).
\end{aligned}
$$

Since $(1 - b_1 b_3) > 0$, this implies that

$$
\begin{aligned}
\|x\|_{L_1} \leq\ & \frac{\|a_3\|_{L_1} + b_3\|a_1\|_{L_1}}{(1 - b_1 b_3)} \\
&+ \frac{\|K_0\|(\|a_2\|_{L_p} + b_2 \cdot r'^{1/p})(\|a_4\|_{L_p} + b_4 \cdot r'^{1/p})}{(1 - b_1 b_3)}.
\end{aligned}
$$

Now, recall that

$$
\frac{\|a_3\|_{L_1}+b_3\|a_1\|_{L_1}+\|K_0\|(\|a_2\|_{L_p}+b_2 \cdot r'^{1/p})(\|a_4\|_{L_p}+b_4 \cdot r'^{1/p})}{(1 - b_1 b_3)} = r'.
$$

Hence, $\|x\|_{L_1} \leq r'$, i.e., $x \in B_{r'}$. Thus, Theorem 2.7 (c) is satisfied.

Step 5. From the above steps, we can apply Theorem 2.7. Thus, (1.1) has at least one integrable solution $x \in L_1$ in $B_{r'}(I)$. $\square$

3.2.1. Uniqueness of the solution. Consider the following two assumptions

(iv)$'$ $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. The operator $T_1 : L_1 \to L_1$ is continuous, and the operator $T_2 : L_1 \to L_p$ is continuous. There are positive functions $a_1, a_3 \in L_1$ and $a_2 \in L_p$ such that

$$
|g(t,0)| \leq a_3(t), \qquad |T_j(0)| \leq a_j(t), \quad j = 1,2,
$$

for almost every $t \in I$. In addition, there are positive constants $b_i \geq 0$, $i = 1,2,3$, such that, for almost every $t \in I$:

$$
\begin{aligned}
|g(t,x) - g(t,y)| &\leq b_3|x - y|, & x,y \in \mathbb{R}, \\
|T_1(x(t)) - T_1(y(t))| &\leq b_1|x(t) - y(t)|, & x,y \in L_1,
\end{aligned}
$$

and

$$|T_2(x(t)) - T_2(y(t))| \le b_2|x(t) - y(t)|^{1/p}, \quad x, y \in L_1.$$

(v)′ Assume that

$$|u(t, s, x(s)) - u(t, s, y(s))| \le k(t, s)|x(s) - y(s)|^{1/p},$$

for almost every $t \in I$, almost every $s \in I$ and $x, y \in B_{r'}$ (where r' is defined in assumption (iii)′); here, k is a measurable function.

Theorem 3.4. *Suppose that assumptions* (ii)′–(v)′ *hold. If*

$$(b_1 b_3 + \|K_0\|(2r')^{1/p-1}[\|a_2\|_{L_p} + b_2\|a_4\|_{L_p} + b_2(1 + b_4) \cdot r'^{1/p}]) \le 1,$$

then (1.1) *has a unique integrable solution* $x \in L_1$ *in* $B_{r'}(I)$.

Proof. Let x and y be any two solutions of (1.1) in $B_{r'}(I)$. Then, for almost every $t \in I$,

$$\begin{aligned}
|x(t) - y(t)| \le{} & b_1 b_3|x(t) - y(t)| \\
& + |T_2 x(t)| \cdot \int_I |u(t, s, x(s)) - u(t, s, y(s))|\, ds \\
& + |T_2 x(t) - T_2 y(t)| \int_I |u(t, s, y(s))|\, ds \\
\le{} & b_1 b_3|x(t) - y(t)| + [a_2(t) + b_2|x(t)|^{1/p}] \\
& \cdot \int_I k(t, s)|x(s) - y(s)|^{1/p}\, ds \\
& + b_2|x(t) - y(t)|^{1/p} \int_I k(t, s)(a_4(s) + b_4|x(s)|^{1/p})\, ds;
\end{aligned}$$

thus,

$$\begin{aligned}
\|x - y\|_{L_1} \le{} & b_1 b_3\|x - y\|_{L_1} \\
& + \|a_2 + b_2|x|^{1/p}\|_{L_p} \left\| \int_I k(t, s)|x(s) - y(s)|^{1/p}\, ds \right\|_{L_q} \\
& + b_2\|(x - y)^{1/p}\|_{L_p} \left\| \int_I k(t, s)(a_4(s) + b_4|x(s)|^{1/p})\, ds \right\|_{L_q} \\
\le{} & b_1 b_3\|x - y\|_{L_1} + (\|a_2\|_{L_p} + b_2\|x\|_{L_1}^{1/p})
\end{aligned}$$

$$
\begin{aligned}
&\quad \cdot \| \| k(t,\cdot)\|_{L_q} \|(x-y)^{1/p}\|_{L_p}\|_{L_q} \\
&\quad + b_2\|x-y\|_{L_1}^{1/p} \| \| k(t,\cdot)\|_{L_q} \|a_4 + b_4|x|^{1/p}\|_{L_p}\|_{L_q} \\
&\le b_1 b_3\|x-y\|_{L_1} + (\|a_2\|_{L_p} + b_2\|x\|_{L_1}^{1/p})\|K_0\| \cdot \|x-y\|_{L_1}^{1/p} \\
&\quad + b_2\|x-y\|_{L_1}^{1/p}\|K_0\|(\|a_4\|_{L_p} + b_4\|x\|_{L_1}^{1/p}) \\
&\le b_1 b_3\|x-y\|_{L_1} + \|K_0\|(2r')^{1/p-1}(\|a_2\|_{L_p} + b_2 \cdot r'^{1/p}) \\
&\quad \cdot \|x-y\|_{L_1} + b_2\|K_0\|(2r')^{1/p-1}(\|a_4\|_{L_p} + b_4 \cdot r'^{1/p}) \\
&\quad \cdot \|x-y\|_{L_1}.
\end{aligned}
$$

The above inequality yields

$$
\begin{aligned}
[1-(b_1b_3 + \|K_0\|(2r')^{1/p-1}[\|a_2\|_{L_p} + b_2\|a_4\|_{L_p} + b_2(1+b_4)\cdot r'^{1/p}])] \\
\cdot \|x-y\|_{L_1} \le 0,
\end{aligned}
$$

which implies that

$$
\|x-y\|_{L_1} = 0, \quad \Longrightarrow x = y.
$$

This completes the proof. $\square$

4. Examples.

Example 4.1. For $t \in (0,1)$, consider the following equation

$$
x(t) = \frac{1}{1+t^2} + \frac{1}{4}\left[t^2 + \frac{1}{3}x(t)\right] + \int_0^1 \frac{t\cos(ts)}{1+(x(s))^2}\, ds. \tag{4.1}
$$

Note that (4.1) is a particular case of (1.1), where

$$
g(t,x) = \frac{1}{1+t^2} + \frac{1}{4}x, \qquad T_1x(t) = t^2 + \frac{1}{3}x(t), \qquad T_2x(t) = 1
$$

and

$$
u(t,s,x) = \frac{t\cos(ts)}{1+x^2}.
$$

Also, note that

$$
|u(t,s,x)| \le 1 = k(t,s).
$$

Now,

$$
\int_0^1 k(t,s)\, ds = 1;
$$

thus, $\|K_0\|_{L_\infty} = 1$. Moreover, given arbitrary $h > 0$ such that $|x_1| \le h$, $|x_2| \le h$ and $|x_2 - x_1| \le \delta$, we have

$$|u(t,s,x_1) - u(t,s,x_2)| \le |t\cos(ts)| \left| \frac{x_1^2 - x_2^2}{(1+x_1^2)(1+x_2^2)} \right| \le 2h\delta\, t,$$

so assumption (iii) holds.

Note that

- g and T_2 satisfy assumption (i) with $a_1(t) = t^2$, $a_2(t) = 1$, $a_3(t) = 1/(1+t^2)$, and with constants $b_1 = 1/3$, $b_2 = 0$, $b_3 = 1/4$;
- $b_1 b_3 + b_2 \|K_0\|_{L_\infty} = (1/3)\cdot(1/4) + 0 \cdot 1 = 1/12 < 1$;
- Assumption (v) is satisfied with $r = (1/11)(3\pi + 13)$.

Thus, all of the assumptions of Theorem 3.1 are satisfied so that the integral equation (4.1) has at least one integrable solution in $(0,1)$.

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