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GLOBAL EXISTENCE AND DECAY RATE OF STRONG SOLUTION TO INCOMPRESSIBLE OLDROYD TYPE MODEL EQUATIONS

BAOQUAN YUAN AND YUN LIU

ABSTRACT. This paper investigates the global existence and the decay rate in time of a solution to the Cauchy problem for an incompressible Oldroyd model with a deformation tensor damping term. There are three major results. The first is the global existence of the solution for small initial data. Second, we derive the sharp time decay of the solution in L^2-norm. Finally, the sharp time decay of the solution of higher order Sobolev norms is obtained.

1. Introduction. In this paper, we consider the incompressible Oldroyd model with a deformation tensor damping term

$$\begin{cases} \partial_t u - \mu\Delta u + u\cdot\nabla u + \nabla p = \nabla\cdot(FF^T), \\ \partial_t F + \nu F + u\cdot\nabla F = \nabla u F, \\ \operatorname{div} u = 0 \end{cases} \tag{1.1}$$

for any $t > 0$, $x \in \mathbb{R}^3$, where $u = u(t,x)$ is the velocity of the flow, $\mu > 0$ the kinematic viscosity, $\nu > 0$ a constant, p the scalar pressure and F the deformation tensor of the fluid. We define $(\nabla\cdot F)_i = \partial_{x_j} F_{ij}$ for the matrix F. When $\nu = 0$, equation (1.1) reduces to the classic Oldroyd model which exhibits an incompressible non-Newtonian fluid. Many hydrodynamic behaviors of complex fluids can be regarded as a consequence of the interaction between fluid motions and internal elastic properties. Physical background on this model may be found in **[1, 4, 10]**.

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Supposing $\mathrm{div} F^T(0,x) = 0$, it can be proven that $\mathrm{div} F^T(t,x) = 0$ almost everywhere for any time $t > 0$. In fact, from the second equation in (1.1), we have

$$\partial_t(\nabla \cdot F^T) + \nu \nabla \cdot F^T + u \cdot \nabla(\nabla \cdot F^T) = 0. \tag{1.2}$$

Multiplying equation (1.2) by $\nabla \cdot F^T$, integrating over $\mathbb{R}^3$ and then using the divergence-free condition of u yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \cdot F^T\|_{L^2}^2 + 2\nu\|\nabla \cdot F^T\|_{L^2}^2 = 0,$$

which implies $\|\nabla \cdot F^T\|_{L^2} = 0$ for any time $t > 0$. Therefore, $\nabla \cdot (FF^T) = (F_{\cdot i} \cdot \nabla)F_{\cdot i}$, and system (1.1) can be written in an equivalent form

$$\begin{cases} \partial_t u - \mu\Delta u + u \cdot \nabla u + \nabla p = F_{\cdot i} \cdot \nabla F_{\cdot i}, \\ \partial_t F_{\cdot j} + \nu F_{\cdot j} + u \cdot \nabla F_{\cdot j} = F_{\cdot j} \cdot \nabla u, \qquad j = 1, \dots, n, \\ \mathrm{div}\, u = 0, \quad \mathrm{div}\, F^T = 0. \end{cases} \tag{1.3}$$

If $\nu = 0$, (1.3) is the classical incompressible Oldroyd model equation. For this model equation, existence of the local or global solution is a concern. Lin, Liu and Zhang [**14**] proved the local existence of smooth solutions and the global existence of classical solutions with small initial data in both the entire space and the periodic domain, if the initial data is sufficiently close to the equilibrium state for the global existence case. Later, Lei, Liu and Zhou [**12**] established a similar existence result for both local and global smooth solutions to the Cauchy problem of incompressible Oldroyd model equations, provided that the initial data is sufficiently close to the equilibrium state.

Theorem A. *For the divergence-free smooth initial data $(u_0, F_0) \in H^2(\mathbb{R}^n)$ for $n = 2$ or 3, there exists a positive time $T = T(\|u_0\|_{H^2}, \|F_0\|_{H^2})$ such that system* (1.3) *with $\nu = 0$ and $\mu > 0$ possesses a unique smooth solution on $[0,T]$ with*

$$u \in L^\infty([0,T]; H^2(\mathbb{R}^n)) \cap L^2([0,T]; H^3(\mathbb{R}^n)),$$
$$F \in L^\infty([0,T]; H^2(\mathbb{R}^n)).$$

Moreover, if T^* is the maximal time of existence, then

$$\int_0^{T^*} \|\nabla u\|_{H^2}^2 \,\mathrm{d}s = +\infty.$$

In a bounded domain, Lin and Zhang **[15]** showed the local well-posedness of the initial-boundary value problem of the Oldroyd model with Dirichlet condition and the global well-posedness of the initial-boundary value problem when the initial data is sufficiently close to the equilibrium state. Qian **[19]** obtained the local existence of the solution with initial data in the critical Besov space and discovered that, if the initial data is sufficiently close to the equilibrium state in the critical Besov, the solution is global in time. For more results on the topic of the Oldroyd model, the reader is referred to **[2, 5, 6, 11, 13, 16, 18, 22, 23, 25]**.

Recently **[24]**, we established a local well-posedness result in $H^s(\mathbb{R}^3)$ for $s > 3/2$ for the classical incompressible Oldroyd model equations by virtue of a new commutator estimate proven by Fefferman, et al. **[3]**, that is:

Theorem B. *Assume $u_0, F_0 \in H^s(\mathbb{R}^3)$ with $s > 3/2$. Then, there exists a time $T = T(\|u_0\|_{H^s}, \|F_0\|_{H^s}) > 0$ such that equations* (1.3) *with $\nu = 0$ and $\mu > 0$ have a unique strong solution (u, F) with $u, F \in C([0,T]; H^s(\mathbb{R}^3))$.*

This paper is dedicated to the study of the Cauchy problem for system (1.3) with the initial condition

$$(u,F)(0,x) = (u_0(x), F_0(x)) \in H^m(\mathbb{R}^3) \quad \text{for } m \geq 3. \tag{1.4}$$

The purpose of this paper is to obtain the global existence of a small initial datum and the decay rate of the smooth solution for model (1.3). For system (1.3) with $\nu = 0$, the local in-time existence and uniqueness of solution in H^s for $s > 3/2$ is derived. However, the global existence of the small initial data solution is an open problem. If we have a deformation tensor term F in the second equation of system (1.3), the local existence of a strong solution in H^m for $m \geq 3$ still holds, formulated in the following theorem.

Theorem C. *Assume $u_0, F_0 \in H^s(\mathbb{R}^3)$ with $s > 1 + 3/2$. Then, there exists a time $T = T(\|u_0\|_{H^s}, \|F_0\|_{H^s}) > 0$ such that equations* (1.3) *with $\nu > 0$ and $\mu > 0$ have a unique strong solution (u, F) with $u, F \in C([0,T]; H^s(\mathbb{R}^3))$. Moreover, the local solution (u, F) satisfies the following estimate*

$$\begin{aligned} &\|u(\cdot,t)\|_{H^s}^2 + \|F(\cdot,t)\|_{H^s}^2 + \int_0^t \|F(\cdot,\tau)\|_{H^s}^2 + \|\nabla u(\cdot,\tau)\|_{H^s}^2 \,\mathrm{d}\tau \\ &\qquad \leq C_1(\|u_0\|_{H^s}^2 + \|F_0\|_{H^s}^2) \quad \textit{for any } t \in [0,T]. \end{aligned} \tag{1.5}$$

Remark 1.1. In Theorem C, if we only require $s > 3/2$, the local existence of the strong solution also holds. In order to have the a priori estimate (1.5), the condition $s > 1 + 3/2$ is required.

Towards this end, we state our main results as follows:

Theorem 1.2. *Let $m \geq 3$ be an integer and $\nu > 0, \mu > 0$. Assume that $(u_0, F_0) \in H^m(\mathbb{R}^3)$ and the initial data satisfies*

$$\|u_0\|_{H^m} + \|F_0\|_{H^m} \leq \delta_0$$

for a small constant $\delta_0 > 0$. Then, there exists a unique, globally smooth solution (u, F) to the Cauchy problem (1.3) *and* (1.4) *satisfying*

$$\|(u, F)(\cdot, t)\|_{H^m}^2 + \int_0^t \|\nabla u(\cdot, \tau)\|_{H^m}^2 + \|F(\cdot, \tau)\|_{H^m}^2 \,\mathrm{d}\tau \leq C_1 \|u_0, F_0\|_{H^m}^2$$

for all $t > 0$.

Theorem 1.3. *Under the assumptions of Theorem* 1.2*, if, in addition, $(u_0, F_0) \in L^1(\mathbb{R}^3) \cap H^m(\mathbb{R}^3)$ for $m \geq 3$, then the smooth solution (u, F) has the following optimal decay rate*:

$$\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 \leq C(t+1)^{-3/2}.$$

The decay rate of the higher order derivative of the solution also holds.

Theorem 1.4. *Under the assumptions of Theorem* 1.3*, for any integer $j \geq 0$, there exists a T_0 such that the small global in-time solution satisfies*

$$\|\nabla^j u(t)\|_{L^2}^2 + \|\nabla^j F(t)\|_{L^2}^2 \leq C(t+1)^{-3/2-j}$$

for all $t > T_0$, where C is a constant which depends upon j and the initial data.

This paper is organized as follows. In Section 2, we briefly recall some lemmas which will be used in our proof. In Section 3, we prove global existence of the smooth solution by the local existence result and the a priori estimate. Section 4 is devoted to the proof of Theorem 1.3 by the classical Fourier splitting method first used by Schonbek in **[20]**. In Section 5, an induction argument will be applied to obtain the optimal decay estimate of higher order derivative of the solution in the L^2 norm.

Throughout this paper, C denotes a generic positive constant which may be different in each occurrence. Since the specific values of the constants $\mu > 0$ and $\nu > 0$ are not important for our arguments, in the following sections, we take $\mu = \nu = 1$.

2. Preliminaries.

In this section, we present some lemmata which will be used in the proof.

In the following sections, we will apply the following commutator estimate and the product estimate of two functions; for details, readers may refer to Kato and Ponce [**8**] and Kenig, Ponce and Vega [**9**] or Majda and Bertozzi [**17**].

Lemma 2.1. *Let $1 < p < \infty$ and $0 < s$. Then, there exists an abstract constant C such that*

$$\|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}) \tag{2.1}$$

for $f \in \dot{W}^{1,p_1} \cap \dot{W}^{s,p_3}$ and $g \in \dot{W}^{s-1,p_2} \cap L^{p_4}$;

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}) \tag{2.2}$$

for $f \in L^{p_1} \cap \dot{W}^{s,p_3}$, $g \in \dot{W}^{s,p_2} \cap L^{p_4}$ and $1 < p_2, p_3 < \infty$, such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$ and $\Lambda = (-\Delta)^{1/2}$.

We shall use the following L^2 estimate of the Fourier transform of the initial datum in a ball, which can be proved by the Hausdorff-Young theorem. Readers may also refer to [**7,** Proposition 3.3], [**21**].

Lemma 2.2. *Let $u_0 \in L^p(\mathbb{R}^3)$, $1 \le p < 2$. Then:*

$$\int_{S(t)} |\mathcal{F}u_0(\xi)|^2 \mathrm{d}\xi \le C(t+1)^{-3(2/p-1)/2}, \tag{2.3}$$

where $S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le g(t)\}$ *is a ball with*

$$g(t) = \left(\frac{\gamma}{t+1}\right)^{1/2}.$$

Here, $\gamma > 0$ *is a constant which will be determined later and* C *is a constant which depends upon* γ *and the* L^p *norm of* u_0.

Proof. Let $\mathcal{F}f$ denote the Fourier transform of a function f. For $1 \le p < 2$, by the Hausdorff-Young inequality, $\mathcal{F}$ is a bounded map from $L^p \to L^q$ and

$$\|\mathcal{F}u_0\|_{L^q} \le C\|u_0\|_{L^p}, \quad 1/p + 1/q = 1. \tag{2.4}$$

Hence, the Hölder inequality yields

$$\int_{S(t)} |\mathcal{F}u_0|^2 \mathrm{d}\xi \le \left(\int_{S(t)} |\mathcal{F}u_0|^q \mathrm{d}\xi\right)^{2/q} \left(\int_{S(t)} \mathrm{d}\xi\right)^{1-2/q}. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\int_{S(t)} |\mathcal{F}u_0|^2 \mathrm{d}\xi \le C\left(\int_{S(t)} \mathrm{d}\xi\right)^{1-2/q},$$

which implies estimate (2.3), and this completes the proof of Lemma 2.2. □

3. Proof of global existence. In order to prove the global existence of a smooth solution, we first prove the following a priori estimate.

Lemma 3.1. *For an integer* $m \ge 3$, *if there exists a small number* $\delta > 0$ *such that*

$$\sup_{0 \le t \le T} \|u(\cdot,t)\|_{H^m} + \|F(\cdot,t)\|_{H^m} \le \delta, \tag{3.1}$$

then, for any $t \in [0,T]$, *there exists a constant* $C_1 > 1$ *such that*

$$\begin{aligned} &\|u(\cdot,t)\|_{H^m}^2 + \|F(\cdot,t)\|_{H^m}^2 + \int_0^t \|F(\cdot,\tau)\|_{H^m}^2 + \|\nabla u(\cdot,\tau)\|_{H^m}^2 \,\mathrm{d}\tau \\ &\quad \le \|(u_0, F_0)\|_{H^m}^2. \end{aligned}$$

Proof. We divide the a priori estimate into three steps.

Step 1. *L^2-norms of u, F.* Taking the L^2 inner product of the equations (1.3) with u and F, then summing, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2+\|F\|_{L^2}^2)+\|F\|_{L^2}^2+\|\nabla u\|_{L^2}^2=0, \tag{3.2}$$

where we have used $(u\cdot\nabla u,u)=0$, $(u\cdot\nabla F\cdot j,F\cdot j)=0$, $(\nabla p,u)=0$ and $(F_{\cdot i}\cdot\nabla F_{\cdot i},u)+(F_{\cdot j}\cdot\nabla u,F_{\cdot j})=0$ by the divergence free conditions of u and $F_{\cdot j}$.

Step 2. *L^2-norms of $\nabla^m u$, $\nabla^m F$.* Applying the operator ∇^m to both sides of (1.3) and taking the L^2 inner product of the resulting equations with $\nabla^m u$ and $\nabla^m F_{\cdot j}$, respectively, adding and then integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^m u\|_{L^2}^2+\|\nabla^m F\|_{L^2}^2)+\|\nabla^m F\|_{L^2}^2+\|\nabla^{m+1}u\|_{L^2}^2\\&\quad\le-\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla u)\cdot\nabla^m u\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^m(F_{\cdot i}\cdot\nabla F_{\cdot i})\cdot\nabla^m u\,\mathrm{d}x\\&\qquad-\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla F_{\cdot j})\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^m(F_{\cdot j}\cdot\nabla u)\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x\\&\quad\triangleq\sum_{i=1}^4 I_i.\end{aligned} \tag{3.3}$$

In what follows, we estimate each term on the right-hand side of the above equation separately. For the term I_1, we obtain

$$\begin{aligned}I_1&=-\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla u)\cdot\nabla^m u\,\mathrm{d}x\\&=-\sum_{0\le l\le m}C_m^l\int_{\mathbb{R}^3}(\nabla^l u\cdot\nabla^{m-l}\nabla u)\cdot\nabla^m u\,\mathrm{d}x\\&\le\sum_{0\le l\le m}C_m^l\|\nabla^l u\nabla^{m-l}\nabla u\|_{L^{6/5}}\|\nabla^m u\|_{L^6}\,\mathrm{d}x.\end{aligned}$$

For $0\le l\le[m/2]$, applying the Gagliardo-Nirenberg inequality leads to

$$\begin{aligned}&\|\nabla^l u\nabla^{m-l}\nabla u\|_{L^{6/5}}\le C\|\nabla^l u\|_{L^3}\|\nabla^{m-l+1}u\|_{L^2}\\&\quad\le C\|\Lambda^\alpha u\|_{L^2}^{1-l/m}\|\nabla^{m+1}u\|_{L^2}^{l/m}\|\nabla u\|_{L^2}^{l/m}\|\nabla^{m+1}u\|_{L^2}^{1-l/m}\le C\delta\|\nabla^{m+1}u\|_{L^2},\end{aligned}$$

where α satisfies

$$\frac{l}{3}-\frac{1}{3}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\times\left(1-\frac{l}{m}\right)+\left(\frac{m+1}{3}-\frac{1}{2}\right)\times\frac{l}{m}$$

with

$$\alpha=\frac{m-2l}{2(m-l)}\in\left[0,\frac{1}{2}\right].$$

However, for $[m/2]+1\le l\le m$, we have

$$\begin{aligned}
&\|\nabla^l u\nabla^{m-l}\nabla u\|_{L^{6/5}}\\
&\quad\le C\|\nabla^l u\|_{L^2}\|\nabla^{m-l+1}u\|_{L^3}\\
&\quad\le C\|u\|_{L^2}^{1-l/(m+1)}\|\nabla^{m+1}u\|_{L^2}^{l/(m+1)}\|\Lambda^\alpha u\|_{L^2}^{l/(m+1)}\|\nabla^{m+1}u\|_{L^2}^{1-l/(m+1)}\\
&\quad\le C\delta\|\nabla^{m+1}u\|_{L^2},
\end{aligned}$$

where α satisfies

$$\frac{m-l+1}{3}-\frac{1}{3}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\times\left(\frac{l}{m+1}\right)+\left(\frac{m+1}{3}-\frac{1}{2}\right)\times\left(1-\frac{l}{m+1}\right)$$

with

$$\alpha=\frac{m+1}{2l}\in\left(\frac{1}{2},1\right].$$

In both cases, we obtain

$$I_1\le C\delta\|\nabla^{m+1}u\|_{L^2}^2.$$

For the term I_2, an application of estimate (2.2) and integration by parts directly yields

$$\begin{aligned}
I_2&=\int_{\mathbb{R}^3}\nabla^m(F_{\cdot i}\cdot\nabla F_{\cdot i})\cdot\nabla^m u\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}\nabla^{m-1}(F_{\cdot i}\cdot\nabla F_{\cdot i})\cdot\nabla^{m+1}u\,\mathrm{d}x\\
&\le\|F_{\cdot i}\|_{L^\infty}\|\nabla^{m-1}\nabla F_{\cdot i}\|_{L^2}\|\nabla^{m+1}u\|_{L^2}\\
&\quad+\|\nabla^{m-1}F_{\cdot i}\|_{L^6}\|\nabla F_{\cdot i}\|_{L^3}\|\nabla^{m+1}u\|_{L^2}\\
&\le C\delta(\|\nabla^{m+1}u\|_{L^2}^2+\|\nabla^m F_{\cdot i}\|_{L^2}^2).
\end{aligned}$$

For the term I_3, we obtain

$$\begin{aligned}
I_3 &= -\int_{\mathbb{R}^3} \nabla^m(u\cdot\nabla F_{\cdot j})\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^3} \nabla^m(u\cdot\nabla F_{\cdot j})\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x + \int_{\mathbb{R}^3}(u\cdot\nabla)\nabla^m F_{\cdot j}\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}([\nabla^m, u\cdot\nabla]F_{\cdot j})\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x\\
&\le (\|\nabla u\|_{L^\infty}\|\nabla^m F_{\cdot j}\|_{L^2} + \|\nabla^m u\|_{L^6}\|\nabla F_{\cdot j}\|_{L^3})\|\nabla^m F_{\cdot j}\|_{L^2}\\
&\le C\delta(\|\nabla^{m+1}u\|_{L^2}^2 + \|\nabla^m F_{\cdot j}\|_{L^2}^2),
\end{aligned}$$

where use has been made of the fact

$$\int_{\mathbb{R}^3}(u\cdot\nabla)\nabla^m F_{\cdot j}\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x = 0$$

and the commutator estimate (2.1).

For the last term, by means of the estimate (2.2), it yields

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3}\nabla^m(F_{\cdot j}\cdot\nabla u)\cdot\nabla^m F_{\cdot j}\,\mathrm{d}x\\
&\le C(\|F_{\cdot j}\|_{L^\infty}\|\nabla^{m+1}u\|_{L^2} + \|\nabla^m F_{\cdot j}\|_{L^2}\|\nabla u\|_{L^\infty})\|\nabla^m F_{\cdot j}\|_{L^2}\\
&\le C\delta(\|\nabla^{m+1}u\|_{L^2}^2 + \|\nabla^m F_{\cdot j}\|_{L^2}^2).
\end{aligned}$$

Substituting the estimates I_1–I_4 into (3.3), the key estimate is obtained by choosing δ small enough.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^m u\|_{L^2}^2 + \|\nabla^m F\|_{L^2}^2) + \|\nabla^m F\|_{L^2}^2 + \|\nabla^{m+1}u\|_{L^2}^2 \le 0. \tag{3.4}$$

Step 3. *Conclusion.* Summing up (3.2) and (3.4), we thereby obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^m}^2 + \|F\|_{H^m}^2) + \|F\|_{H^m}^2 + \|\nabla u\|_{H^m}^2 \le 0.$$

Integrating the above inequality directly in time leads to

$$\begin{aligned}
&\|u(\cdot,t)\|_{H^m}^2 + \|F(\cdot,t)\|_{H^m}^2\\
&\qquad + \int_0^t \|F(\cdot,\tau)\|_{H^m}^2 + \|\nabla u(\cdot,\tau)\|_{H^m}^2\mathrm{d}\tau \le \|(u_0,F_0)\|_{H^m}^2.
\end{aligned}$$

We thus finish the proof of Lemma 3.1. ∎

Combining the local existence Theorem 1.2 and the a priori estimate Lemma 3.1, we will complete the proof of the global existence of the smooth solution by a continuous extension argument.

Proof of Theorem 1.2. Assume

$$E_0 := \|u_0\|_{H^m} + \|F_0\|_{H^m} < \delta/\sqrt{C_1}, \tag{3.5}$$

where δ is defined in Lemma 3.1. By choosing $\delta_0 = \delta/\sqrt{C_1}$, we can prove that there exists a global-in-time solution to system (1.3). Since the initial data satisfies $E_0 < \delta/\sqrt{C_1}$, then, according to Theorem C, there exists a positive constant $T_1 > 0$ such that the smooth solution of (1.3) and (1.4) exists on $[0, T_1]$, and the following holds:

$$\|u(\cdot,t)\|_{H^m}^2 + \|F(\cdot,t)\|_{H^m}^2 + \int_0^t \|F(\cdot,\tau)\|_{H^m}^2 + \|\nabla u(\cdot,\tau)\|_{H^m}^2 \mathrm{d}\tau \leq C_1 E_0^2$$

for $t \in [0, T_1]$, which implies

$$E_1 := \sup_{0\leq t\leq T_1} \|(u,F)(\cdot,t)\|_{H^m} \leq \sqrt{C_1} E_0 < \delta.$$

For

$$E_k := \sup_{0\leq t\leq kT_1} \|(u,F)(\cdot,t)\|_{H^m} \leq \sqrt{C_1} E_0 < \delta, \tag{3.6}$$

Lemma 3.1 and (3.5) yield $E_k \leq E_0 < \delta/\sqrt{C_1}$.

Considering $(u, F)(x, kT_1)$ as the initial data, Theorem C admits

$$\begin{aligned}\|u(\cdot,t)\|_{H^m}^2 + \|F(\cdot,t)\|_{H^m}^2 + \int_{kT_1}^t \|F(\cdot,\tau)\|_{H^m}^2 + \|u(\cdot,\tau)\|_{H^{m+1}}^2 \mathrm{d}\tau \\ \leq C_1(\|u(\cdot,kT_1)\|_{H^m}^2 + \|F(\cdot,kT_1)\|_{H^m}^2),\end{aligned}$$

for $t \in [kT_1, (k+1)T_1]$. Taking into account (3.6), we obtain

$$\sup_{kT_1\leq t\leq (k+1)T_1} \|(u,F)(\cdot,t)\|_{H^m} \leq \sqrt{C_1} E_k < \delta,$$

and thus,

$$E_{k+1} := \sup_{0\leq t\leq (k+1)T_1} \|(u,F)(\cdot,t)\|_{H^m} \leq E_0 < \delta/\sqrt{C_1}.$$

By the bootstrap argument, the proof of Theorem 1.2 is complete. □

4. Proof of Theorem 1.3. In this section, we prove the decay rate of the smooth solution to equations (1.3) in the L^2 space. For ease of presentation, we denote the Fourier transform of f by $\mathcal{F}f$ or $\widehat{f}$ in the subsequences.

In Section 3, we have already obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|F\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|F\|_{L^2}^2 = 0. \tag{4.1}$$

Applying Plancherel's theorem to (4.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2)\,\mathrm{d}\xi = -\int_{\mathbb{R}^3}(|\xi|^2|\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2)\,\mathrm{d}\xi.$$

By decomposing the frequency domain into two time-dependent subsets, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}(|\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2)\,\mathrm{d}\xi \\
&\quad\le -\int\limits_{|\xi|\ge g(t)} g(t)^2|\widehat{u}(\xi)|^2\mathrm{d}\xi - \int\limits_{|\xi|\le g(t)} |\xi|^2|\widehat{u}(\xi)|^2\mathrm{d}\xi - \int\limits_{\mathbb{R}^3}|\widehat{F}(\xi)|^2\mathrm{d}\xi \\
&\quad= -\int\limits_{\mathbb{R}^3} g(t)^2|\widehat{u}(\xi)|^2\mathrm{d}\xi + \int\limits_{|\xi|\le g(t)} g(t)^2|\widehat{u}(\xi)|^2\mathrm{d}\xi - \int\limits_{\mathbb{R}^3}|\widehat{F}(\xi)|^2\mathrm{d}\xi,
\end{aligned}$$

where $g(t)$ is defined in Lemma 2.2 and γ is a constant to be determined later. There exists a time $T_0 > 0$ such that, when $t > T_0$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2\mathrm{d}\xi + \frac{\gamma}{1+t}\int\limits_{\mathbb{R}^3}|\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2\mathrm{d}\xi \\
&\qquad\le \frac{\gamma}{1+t}\int\limits_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 + |\widehat{F}(\xi)|^2\mathrm{d}\xi.
\end{aligned} \tag{4.2}$$

Multiplying (4.2) by the integrating factor $(t+1)^\gamma$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}((t+1)^\gamma(\|u(t)\|_{L^2}^2+\|F(t)\|_{L^2}^2)) \leq \gamma(t+1)^{\gamma-1}\int\limits_{|\xi|\leq g(t)}(|\widehat{u}(\xi)|^2+|\widehat{F}(\xi)|^2)\,\mathrm{d}\xi. \tag{4.3}$$

In order to finish the proof, we prove the estimates of $|\widehat{u}(\xi)|$ and $|\widehat{F}(\xi)|$ as follows.

Lemma 4.1. *Let (u, F) be a smooth solution to the Cauchy problem (1.3) with the small initial data $(u_0, F_0)\in L^1\cap H^m$, $m\geq 3$. Then, there exist*

$$|\widehat{u}(\xi,t)|\leq C\left(|\widehat{u_0}(\xi)|+\frac{1}{|\xi|}\right) \tag{4.4}$$

and

$$|\widehat{F}(\xi,t)|\leq C(|\widehat{F_0}(\xi)|+|\xi|). \tag{4.5}$$

Proof. Taking the Fourier transform of equations (1.3), we have

$$\widehat{u_t}(\xi,t)+|\xi|^2\widehat{u}(\xi,t)=H(\xi,t), \tag{4.6}$$

where $H(\xi,t)=-\widehat{u\cdot\nabla u}(\xi,t)-\widehat{\nabla p}(\xi,t)+\widehat{F\cdot\nabla F_{\cdot i}}(\xi,t)$ and

$$\widehat{F}_t(\xi,t)+\widehat{F}(\xi,t)=G(\xi,t), \tag{4.7}$$

where $G(\xi,t)=-\widehat{u\cdot\nabla F}(\xi,t)+\widehat{F\cdot\nabla u}(\xi,t)$. Multiplying (4.6) and (4.7) by the integrating factors $\mathrm{e}^{|\xi|^2t}$ and e^t, respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{|\xi|^2t}\widehat{u}(\xi,t))\leq \mathrm{e}^{|\xi|^2t}H(\xi,t) \tag{4.8}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{t}\widehat{F}(\xi,t))\leq \mathrm{e}^{t}G(\xi,t). \tag{4.9}$$

Integrating (4.8) and (4.9) in time from 0 to t yields

$$\widehat{u}(\xi,t)\leq \mathrm{e}^{-|\xi|^2t}\widehat{u_0}(\xi,t)+\int_0^t \mathrm{e}^{-|\xi|^2(t-\tau)}H(\xi,\tau)\,\mathrm{d}\tau \tag{4.10}$$

and

$$\widehat{F}(\xi,t) \le \mathrm{e}^{-t}\widehat{F_0}(\xi,t) + \int_0^t \mathrm{e}^{-(t-\tau)} G(\xi,\tau)\,\mathrm{d}\tau.$$

Now, we derive the estimates for $H(\xi,t)$ and $G(\xi,t)$. Taking the divergence operator from the first equation of (1.3) and using the divergence free condition of u and F, we have

$$\triangle p = -\nabla\cdot\operatorname{div}(u\otimes u) + \nabla\cdot\operatorname{div}(F\otimes F).$$

Since the Fourier transform is a bounded map from L^1 to L^∞, this leads to

$$\begin{aligned} |\widehat{\nabla p}(\xi,t)| &\le |\xi||\widehat{p}(\xi,t)| \le |\xi|(\|u(t)u(t)\|_{L^1} + \|F(t)F(t)\|_{L^1}) \\ &\le C|\xi|(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2). \end{aligned} \tag{4.11}$$

Similarly, for the convective terms, we also have

$$|\widehat{u\cdot\nabla u}(\xi,t)| \le C|\xi|\|u(t)\|_{L^2}^2 \tag{4.12}$$

and

$$|\widehat{F\cdot\nabla F}(\xi,t)| \le C|\xi|\|F(t)\|_{L^2}^2, \tag{4.13}$$

as well as the following estimates

$$|\widehat{u\cdot\nabla F}(\xi,t)| \le C|\xi|(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2) \tag{4.14}$$

and

$$|\widehat{F\cdot\nabla u}(\xi,t)| \le C|\xi|(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2). \tag{4.15}$$

Combining the estimates (4.11)–(4.13) together, we get

$$|H(\xi,t)| \le C|\xi|(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2). \tag{4.16}$$

Combining the estimates (4.14)–(4.15), we obtain

$$|G(\xi,t)| \le C|\xi|(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2).$$

Inserting $|H(\xi,t)|$ into (4.10) and using the boundedness of L^2 norms of the solution, we deduce

$$|\widehat{u}(\xi,t)| \le |\widehat{u_0}(\xi)| + \frac{C}{|\xi|}(\|u_0\|_{L^2}^2 + \|F_0\|_{L^2}^2)(1-\mathrm{e}^{-|\xi|^2 t}) \le C\left(|\widehat{u_0}(\xi)| + \frac{1}{|\xi|}\right).$$

Using a similar argument, we have

$$|\widehat{F}(\xi,t)| \le |\widehat{F_0}(\xi)| + C|\xi|(\|u_0\|_{L^2}^2 + \|F_0\|_{L^2}^2)(1-\mathrm{e}^{-t}) \le C(|\widehat{F_0}(\xi)| + |\xi|).$$

We thus derive the estimates of $|\widehat{u}(\xi)|$ and $|\widehat{F}(\xi)|$. $\square$

Placing (4.4) and (4.5) into the right-hand side of (4.3) and applying Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}((t+1)^\gamma(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2)) \\
&\quad \le C(t+1)^{\gamma-1} \int_{|\xi|\le g(t)} (|\widehat{u_0}(\xi)|^2 + |\widehat{F_0}(\xi)|^2)\,\mathrm{d}\xi \\
&\qquad + C(t+1)^{\gamma-1} \int_{|\xi|\le g(t)} \frac{1}{|\xi|^2}\,\mathrm{d}\xi + C(t+1)^{\gamma-1} \int_{|\xi|\le g(t)} |\xi|^2\,\mathrm{d}\xi \\
&\quad \le C(t+1)^{\gamma-1-3/2} + C(t+1)^{\gamma-1-1/2} + C(t+1)^{\gamma-1-5/2}.
\end{aligned}$$

Integrating the above inequality in time from 0 to t leads to

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 \\
&\quad \le C((t+1)^{-\gamma} + (t+1)^{-3/2} + C(t+1)^{-1/2} + C(t+1)^{-5/2}).
\end{aligned}$$

By choosing $\gamma > 1/2$, we obtain

$$\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 \le C(t+1)^{-1/2}. \tag{4.17}$$

Again inserting the above estimate (4.17) of $\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2$ into estimate (4.16), it follows that

$$\begin{aligned}
\int_0^t \mathrm{e}^{-|\xi|^2(t-\tau)}|H(\xi,\tau)|\,\mathrm{d}\tau &\le C|\xi| \int_0^t (\tau+1)^{-1/2}\mathrm{d}\tau \\
&\le C|\xi|((t+1)^{1/2} - 1)) \\
&\le C(t+1)^{-1/2}((t+1)^{1/2}) \le C,
\end{aligned} \tag{4.18}$$

if $|\xi|$ is in the ball $S(t)$ defined in Lemma 2.2. Putting (4.18) into (4.10), we get $|\widehat{u}(\xi,t)| \le C(|\widehat{u_0}(\xi)| + 1)$. Arguing similarly, we obtain $|\widehat{F}(\xi,t)| \le C(|\widehat{F_0}(\xi)|+1)$. Inserting these estimates of $\widehat{u}(\xi,t)$ and $\widehat{F}(\xi,t)$ into (4.3) and, by Lemma 2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}((t+1)^\gamma(\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2)) \le \gamma(t+1)^{\gamma-1}(t+1)^{-3/2}.$$

Integrating the above estimate in time and choosing $\gamma > 3/2$ leads to

$$\|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 \leq C(t+1)^{-3/2},$$

which completes the proof of Theorem 1.3. $\square$

5. Proof of Theorem 1.4. This section is devoted to showing the higher order derivative's optimal decay estimate of a smooth solution to equations (1.3) in the L^2 norm.

Proof. As is standard, we denote $S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \leq f(t)\}$, with $f(t) = (\gamma/(t+1))^{1/2}$, where γ is a constant to be determined later. For the order $m+1$ derivative term, again by the Fourier-splitting method, it is deduced as

$$\begin{aligned} \|\Lambda^{m+1}u\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^2 |\mathcal{F}\Lambda^m u(\xi,t)|^2 \mathrm{d}\xi \\ &\geq \int_{|\xi|\geq f(t)} |\xi|^2 |\mathcal{F}\Lambda^m u(\xi,t)|^2 \mathrm{d}\xi \\ &\geq f^2(t)\|\Lambda^m u\|_{L^2}^2 - f^2(t)\int_{S(t)} |\mathcal{F}\Lambda^m u(\xi,t)|^2 \mathrm{d}\xi \\ &\geq f^2(t)\|\Lambda^m u\|_{L^2}^2 - f^4(t)\int_{\mathbb{R}^3} |\mathcal{F}\Lambda^{m-1} u(\xi,t)|^2 \mathrm{d}\xi, \end{aligned} \tag{5.1}$$

where $m \geq 1$ is an integer.

Inserting estimate (5.1) into (3.4), it follows that, for $t > T_0$ with some $T_0 > 0$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m F\|_{L^2}^2) + \frac{\gamma}{t+1}\|\Lambda^m F\|_{L^2}^2 + \frac{\gamma}{t+1}(t)\|\Lambda^m u\|_{L^2}^2 \\ &\qquad \leq \left(\frac{\gamma}{t+1}\right)^2 (\|\Lambda^{m-1}u\|_{L^2}^2 + \|\Lambda^{m-1}F\|_{L^2}^2). \end{aligned} \tag{5.2}$$

If $m = 1$, multiplying both sides of inequality (5.2) by $(t+1)^\gamma$ yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}((t+1)^\gamma(\|\Lambda u\|_{L^2}^2 + \|\Lambda F\|_{L^2}^2)) &\leq (t+1)^{\gamma-2}(\|u\|_{L^2}^2 + \|F\|_{L^2}^2) \\ &\leq C(t+1)^{\gamma-2-(3/2)}. \end{aligned} \tag{5.3}$$

Integrating inequality (5.3) from T_0 to t, we have

$$\begin{aligned}(t+1)^{\gamma}(\|\Lambda u\|_{L^2}^2+\|\Lambda F\|_{L^2}^2)\\ \leq (T_0+1)^{\gamma}(\|\Lambda u(T_0)\|_{L^2}^2+\|\Lambda F(T_0)\|_{L^2}^2)+C(t+1)^{\gamma-1-3/2}.\end{aligned}$$

Therefore, we can obtain, by choosing $\gamma > 5/2$,

$$\|\Lambda u\|_{L^2}^2+\|\Lambda F\|_{L^2}^2 \leq C(t+1)^{-3/2-1}. \tag{5.4}$$

In order to finish the proof of Theorem 1.4, we use the argument of induction by m. Assume that

$$\|\Lambda^{m-1} u\|_{L^2}^2+\|\Lambda^{m-1} F\|_{L^2}^2 \leq C_{m-1}(t+1)^{-3/2-m-1}. \tag{5.5}$$

After inserting (5.5) into (5.2), and multiplying $(t+1)^{\gamma}$ on both sides of the resulting inequality, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}((t+1)^{\gamma}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m F\|_{L^2}^2)) \leq \gamma^2 C_{m-1}(t+1)^{\gamma-3/2-m-1-2}.$$

Integrating the above inequality in time from T_0 to t, we get

$$\begin{aligned}(t+1)^{\gamma}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m F\|_{L^2}^2)\\ \leq (T_0+1)^{\gamma}(\|\Lambda^m u(T_0)\|_{L^2}^2+\|\Lambda^m F(T_0)\|_{L^2}^2)\\ +\gamma^2 C_{m-1}(t+1)^{\gamma-3/2-m-1-1}.\end{aligned}$$

Similarly, by choosing $\gamma > 3/2+m$, we obtain

$$\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m F\|_{L^2}^2 \leq C_m(t+1)^{-3/2-m}.$$

We thus have completed the proof of Theorem 1.4. □

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Henan Polytechnic University, School of Mathematics and Information Science, Henan, 454000, China
Email address: **bqyuan@hpu.edu.cn**

Henan Polytechnic University, School of Mathematics and Information Science, Henan, 454000, China
Email address: **liuyunlexie@163.com**