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THE UPPER VIETORIS TOPOLOGY ON THE SPACE OF INVERSE-CLOSED SUBSETS OF A SPECTRAL SPACE AND APPLICATIONS

CARMELO A. FINOCCHIARO, MARCO FONTANA AND DARIO SPIRITO

ABSTRACT. Given an arbitrary spectral space X, we consider the set $\boldsymbol{\mathcal{X}}(X)$ of all nonempty subsets of X that are closed with respect to the inverse topology. We introduce a Zariski-like topology on $\boldsymbol{\mathcal{X}}(X)$ and, after observing that it coincides the upper Vietoris topology, we prove that $\boldsymbol{\mathcal{X}}(X)$ is itself a spectral space, that this construction is functorial, and that $\boldsymbol{\mathcal{X}}(X)$ provides an extension of X in a more "complete" spectral space. Among the applications, we show that, starting from an integral domain D, $\boldsymbol{\mathcal{X}}(\mathrm{Spec}(D))$ is homeomorphic to the (spectral) space of all the stable semistar operations of finite type on D.

1. Introduction. The first study of the set of prime ideals from a topological point of view is due to Stone [**42, 43**], who developed the theory in the context of distributive lattices and Boolean algebras. Later, Hochster [**28**] defined a *spectral space* as a topological space that is homeomorphic to the prime spectrum of a (commutative) ring endowed with the Zariski topology and proceeded to show that this class of topological spaces can be characterized in a purely topological way. More precisely, he proved that a topological space X is spectral if and only if it is T_0, quasi-compact, admits a basis of quasi-compact open subspaces that is closed under finite intersections, and is sober

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(i.e., every irreducible closed subset of X has a (unique) generic point). Spectral spaces can also be viewed through the lens of ordered topological spaces (via the concept of the *Priestley space*) **[6, 37, 38]**, of bitopological spaces (through *pairwise Stone spaces*) **[4]**, or through domain theory (using the notion of the *stably compact space*) **[30]**.

The first example of a spectral space which occurs naturally in commutative algebra, but is not defined as a spectrum, is the Riemann-Zariski space $\mathrm{Zar}(K \mid D)$ of all valuation domains with quotient field K and containing D; this was proven by explicitly providing a Bézout domain whose prime spectrum is naturally homeomorphic to $\mathrm{Zar}(K \mid D)$ (see **[7, 11, 27]**). Recently, several other spaces, which naturally occur in multiplicative ideal theory, have been shown to be spectral: for example, this occurs for the spaces $\mathrm{Overr}(D)$ and $\mathrm{Overr}_{\mathrm{ic}}(D)$ consisting, respectively, of the overrings and of the integrally closed overrings of an integral domain D. This result was later extended to the space $\mathrm{SStar}_{\mathrm{f}}(D)$ of all semistar operations of finite type on D, providing an appropriate and natural topological extension of the spectral space $\mathrm{Overr}(D)$ (and, in particular, of both $\mathrm{Spec}(D)$ and $\mathrm{Zar}(K \mid D)$) **[15]**. Unlike the proof of the spectrality of $\mathrm{Zar}(K \mid D)$, these spaces were shown to be spectral using a criterion based on ultrafilters **[10]**, which is well suited to this kind of space; however, this criterion is not constructive, that is, it does not explicitly provide a ring whose spectrum is homeomorphic to the given spectral space.

If X is a topological space, we denote by X^{d} the set X endowed with the *co-compact topology*, i.e., the topology on X having, as a base of open sets, the complements of the subsets of X that are both quasi-compact and obtained as an intersection of open sets **[19,** Definition O-5.10]. In the context of spectral spaces, the co-compact topology of X is called the *inverse topology* of X and plays a crucial role in Hochster's study of spectral spaces; it owes its name to the fact that the order canonically associated to the inverse topology coincides with the reverse order of the one induced by the spectral topology. Subsets of a spectral space that are closed in the inverse topology are strictly related to the study of representations of integrally closed domains as intersections of collections of valuation domains (see also **[33, 34, 35]**), and they represent a method of classifying several distinguished classes of semistar operations of finite type. It was shown in **[11, 15]** that complete, or eab, semistar operations (respectively, stable semistar operations; definitions recalled later) correspond to the subsets of $\mathrm{Zar}(D)$ (respectively, $\mathrm{Spec}(D)$) that are closed in the inverse topology. Moreover, these two spaces are spectral extensions of the spaces $\mathrm{Zar}(D)$ and $\mathrm{Spec}(D)$, also see [**14**].

The aim of this paper is to study, for an arbitrary spectral space X, the space $\boldsymbol{\mathcal{X}}(X)$ of all nonempty subsets of X that are closed with respect to the inverse topology; in particular, this study is carried out using the same ultrafilter-theoretic approach of [**10, 11**], using techniques closer to commutative algebra than to general topology, in an attempt to bridge the gap between the two communities. After endowing $\boldsymbol{\mathcal{X}}(X)$ with a natural topology, we show that it is a spectral space and a spectral extension of the original space X. It is worth noting that this construction, which arises in the topological context associated to commutative ring theory, is a special case of the construction of the Smyth powerdomain of a general topological space X, endowed with the upper Vietoris topology [**31, 44**] (the definitions are recalled later), usually studied from the point of view of domain theory (see [**30,** Section 5], [**41**]). In Section 5, we see that the two spaces of distinguished semistar operations recalled above are examples of the space $\boldsymbol{\mathcal{X}}(X)$, when applied to the spectral spaces $X = \mathrm{Zar}(D)$ and $X = \mathrm{Spec}(D)$. We also show that the extension $X \hookrightarrow \boldsymbol{\mathcal{X}}(X)$ represents, in a certain sense, a spectral "completion" of the original space X, matching the possibility of extending the spectral space $\mathrm{Overr}(D)$ inside the more "complete" spectral space of the semistar operations of finite type $\mathrm{SStar}_{\mathrm{f}}(D)$. The "completeness" mentioned above is related to a universal-like property satisfied by $\boldsymbol{\mathcal{X}}(X)$. Broadly speaking, $\boldsymbol{\mathcal{X}}(X)$ is the completion of X with respect to the existence of the supremum for families of quasi-compact subspaces.

2. Preliminaries. It is well known that the prime spectrum of a commutative ring endowed with the Zariski topology is always T_0 and quasi-compact, but almost never Hausdorff (it is Hausdorff only in the zero-dimensional case). Thus, many authors have considered a finer topology on the prime spectrum of a ring, known as the *constructible topology* (see [**3,** Chapter 3, Exercises 27, 28, 30], [**5**], [**21,** pages 337–339]) or as the *patch topology* [**28**].

Following [**38, 39**], it is possible to introduce the constructible topology by a Kuratowski closure operator: if X is a spectral space, for each subset Y of X, we set:

$$\mathrm{Cl}^{\mathrm{cons}}(Y) := \bigcap \{U \cup (X \setminus V) \mid U \text{ and } V \text{ open and quasi-compact in } X, \\ U \cup (X \setminus V) \supseteq Y\}.$$

We denote by X^{cons} the set X, equipped with the constructible topology. For Noetherian topological spaces, this definition of constructible topology coincides with the classical one given in [**5**]. It is well known that the constructible topology is a refinement of the given topology, and it is always Hausdorff.

Given a topology on a set X, we can define a preorder $\leq_X$ on X by setting $x \leq_X y$ if $y \in \mathrm{Cl}(\{x\})$, where $\mathrm{Cl}(Y)$ denotes the closure of a subset Y of X. This order is the opposite of the specialization order generally used in topology; however, it is the one more commonly used in commutative algebra and algebraic geometry, since, on the spectrum of a ring, it coincides with the set-theoretic containment (for example, this is the order used in [**28**]). The set

$$Y^{\mathrm{gen}} := \downarrow Y := \{x \in X \mid x \leq y, \text{ for some } y \in Y\}$$

is called *closure under generizations* of Y. Similarly, using the opposite order, the set

$$Y^{\mathrm{sp}} := \uparrow Y := \{x \in X \mid y \leq x, \text{ for some } y \in Y\}$$

is called *closure under specializations* of Y. We say that Y is *closed under generalizations* or a *down set* (respectively, *closed under specializations* or an *upper set*) if $Y = Y^{\mathrm{gen}}$ (respectively, $Y = Y^{\mathrm{sp}}$). It is straightforward that, for two elements x and y in a spectral space X, we have:

$$x \leq y \iff \{x\}^{\mathrm{gen}} \subseteq \{y\}^{\mathrm{gen}} \iff \{x\}^{\mathrm{sp}} \supseteq \{y\}^{\mathrm{sp}}.$$

Given a spectral space X, Hochster [**28,** Proposition 8] introduced a new topology on X, called here the *inverse topology*, by defining a Kuratowski closure operator for each subset Y of X, as follows:

$$\mathrm{Cl}^{\mathrm{inv}}(Y) := \bigcap \{U \mid U \text{ open and quasi-compact in } X, \, U \supseteq Y\}.$$

If we denote by X^{inv} the set X equipped with the inverse topology, Hochster proved that X^{inv} is still a spectral space, and the partial order on X induced by the inverse topology is the opposite order of that induced by the given topology on X. In particular, the closure under generizations $\{x\}^{\mathrm{gen}}$ of a singleton is closed in the inverse topology of

X, since $\{x\}^{\mathrm{gen}} = \bigcap\{U \mid U \subseteq X$ quasi-compact and open, $x \in U\}$ [**28,** Proposition 8]. On the other hand, it is trivial, by the definition, that the closure under specializations of a singleton $\{x\}^{\mathrm{sp}}$ is closed in the given topology of X, since $\{x\}^{\mathrm{sp}} = \mathrm{Cl}(\{x\})$.

Recall that it is well known that $\mathrm{Cl}^{\mathrm{inv}}(Y) = (\mathrm{Cl}^{\mathrm{cons}}(Y))^{\mathrm{gen}}$ (see, for instance, [**16,** Lemma 1.1] applied to the inverse topology or, explicitly, [**11,** Remark 2.2]; a more general situation is considered in [**30,** subsection 2.2]). It follows that each closed set in the inverse topology (for short, *inverse-closed*) is closed under generizations and, from [**11,** Proposition 2.6], that a quasi-compact subspace Y of X closed for generizations is inverse-closed.

We mention here the existence of several different point of views that might shed further light on the theory of spectral spaces.

One perspective is through the language of *ordered topological spaces.* Let X be a topological space and $\leq$ an order on X: then, the pair $(X, \leq)$ is a *Nachbin space* if X is quasi-compact and the set $\{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$. A *Priestley space* is a Nachbin space $(X, \leq)$ such that, for every $x, y \in X$ with $x \nleq y$, there exists a clopen subset Γ of X that is closed under specialization (with respect to $\leq$) such that $x \in \Gamma$ and $y \notin \Gamma$. It is well known that there is an isomorphism between the category of Priestly spaces (and continuous monotone maps) and the category of spectral spaces (and spectral maps): if X is a spectral space, and $\leq$ is the order induced by the topology, then $(X^{\mathrm{cons}}, \leq)$ is a Priestley space, while, if $(X, \leq)$ is a Priestley space, then the topology on X generated by the open subsets of X that are closed under generizations (with respect to $\leq$) is a spectral space. In this context, reversing the order defining a Priestley space amounts to passing from a spectral topolgy to its inverse topology, while the case where $\leq$ is the indiscrete order (i.e., $x \leq y$ if and only if $x = y$) corresponds to the case where the spectral space X is Hausdorff, i.e., when the topology on X is equal to its own constructible topology. For a deeper insight on this topic, see, for instance, [**4, 6, 37, 38**] and [**19,** Chapter VI].

Another point of view is offered by domain theory. A topological space X is said to be *stably compact* (see, for instance, [**30**]) if it satisfies the following properties:

(1) X is T_0 and quasi-compact.

(2) X is *locally quasi-compact* (that is, for any open set U of X and any $x \in U$, there are a quasi-compact subspace K of X and an open set $V \subseteq X$ such that $x \in V \subseteq K \subseteq U$).

(3) X is *coherent* (that is, any finite intersection of quasi-compact saturated subsets of X is quasi-compact).

(4) X is sober.

Note that stably compact spaces can also be defined as the retracts of the spectral spaces [**40,** Lemma 3.13(b)]; further connections are outlined in the following well-known results.

Recall that a subset of a topological space is called a saturated subset if it is an intersection of a family of open sets.

Lemma 2.1. *Let X be a topological space having a basis for the open sets given by the quasi-compact open subspaces.*

(i) *If $K \subseteq U \subseteq X$, K is quasi-compact and U is open in X, then there exists a quasi-compact open subspace Ω of X such that $K \subseteq \Omega \subseteq U$.*

(ii) *If X is spectral, then a subset of X is closed, with respect to the inverse topology, if and only if it is saturated and quasi-compact.*

Under this terminology, a spectral space is exactly a stably compact space such that the quasi-compact open subspaces are a basis:

Lemma 2.2. *Let X be a topological space. Then, the following conditions are equivalent.*

(i) *X is a spectral space.*

(ii) *X is a stably compact space with a basis for the open sets given by the quasi-compact open subspaces.*

Note that the notion of stably compact space is strictly more general than that of spectral space. For instance, it is easy to see that the subspace $[0, 1]$ of the real line is stably compact but not a spectral space, for lack of quasi-compact open subspaces.

Finally, we observe that the isomorphism between the category of Priestley spaces and spectral spaces (recalled above) naturally extends to an isomorphism between the categories of Nachbin spaces (and continuous monotone maps) and of stably compact spaces (and proper maps). See [**19,** Chapter VI].

3. The space of inverse-closed subsets of a spectral space. Let X be a spectral space. The main object of study of this paper is the space

$$\boldsymbol{\mathcal{X}}(X) := \{Y \subseteq X \mid Y \neq \emptyset,\, Y = \mathrm{Cl}^{\mathrm{inv}}(Y)\},$$

that is, $\boldsymbol{\mathcal{X}}(X)$ is the set of all nonempty subsets of X that are closed in the inverse topology. From the point of view of ordered topological spaces, if $(X, \leq)$ is a Priestley space, then $\boldsymbol{\mathcal{X}}(X)$ is the space of nonempty closed downsets of X.

If X is understood from the context, we shall simply write $\boldsymbol{\mathcal{X}}$ instead of $\boldsymbol{\mathcal{X}}(X)$. If $X = \mathrm{Spec}(R)$ for some ring R, we write, for short, $\boldsymbol{\mathcal{X}}(R)$ instead of $\boldsymbol{\mathcal{X}}(\mathrm{Spec}(R))$.

We define a *Zariski topology* on $\boldsymbol{\mathcal{X}}(X)$ by taking, as subbasis of open sets, the sets of the form

$$\boldsymbol{\mathcal{U}}(\Omega) := \{Y \in \boldsymbol{\mathcal{X}} \mid Y \subseteq \Omega\},$$

where Ω varies among the quasi-compact open subspaces of X. Note that the previous subbasis is, in fact, a basis, since $\boldsymbol{\mathcal{U}}(\Omega) \cap \boldsymbol{\mathcal{U}}(\Omega') = \boldsymbol{\mathcal{U}}(\Omega \cap \Omega')$, and $\Omega \cap \Omega'$ is a quasi-compact open subspace of X for any pair Ω, Ω' of quasi-compact open subspaces of X. Moreover, $\emptyset \neq \Omega \in \boldsymbol{\mathcal{U}}(\Omega)$, since a quasi-compact open subset Ω of X is closed in the inverse topology of X. Note also that, when $X = \mathrm{Spec}(R)$ for some ring R, a generic basic open set of the Zariski topology on $\boldsymbol{\mathcal{X}}(R)$ is of the form

$$\boldsymbol{\mathcal{U}}(J) := \boldsymbol{\mathcal{U}}(\mathtt{D}(J)) = \{Y \in \boldsymbol{\mathcal{X}}(R) \mid Y \subseteq \mathtt{D}(J)\},$$

where J is any finitely generated ideal of R.

The construction $\boldsymbol{\mathcal{X}}(X)$ can also be understood in terms of the traditional domain-theoretic definition of the Smyth powerdomain in the setting of topological spaces. More precisely, let X be a topological space. Following, for example, [**30,** Definition 5.2], the *Smyth powerdomain of* X is the collection $\boldsymbol{\mathcal{Q}}(X)$ of all nonempty quasi-compact saturated subsets of X, equipped with the *upper Vietoris topology*, that is, the topology on $\boldsymbol{\mathcal{Q}}(X)$ whose basic open sets are sets of the form

$$U^+ := \{Q \in \boldsymbol{\mathcal{Q}}(X) \mid Q \subseteq U\},$$

for any open set U of X.

In view of Lemma 2.1 (i), if X is a spectral space, then $\boldsymbol{\mathcal{Q}}(X) = \boldsymbol{\mathcal{X}}(X)$, as sets. Now, we show that this equality holds at a topological level.

Proposition 3.1. *Let X be a spectral space. Then, the space $\boldsymbol{\mathcal{X}}(X)$, endowed with the Zariski topology, coincides with the space $\boldsymbol{\mathcal{Q}}(X)$, endowed with the upper Vietoris topology.*

Proof. Clearly, it is sufficient to show that, if U is an open subset of X, then U^+ is open, with respect to the Zariski topology on $\boldsymbol{\mathcal{X}}(X)$. Take a set $Q \in U^+$. Since Q is, in particular, quasi-compact, Lemma 2.1 (i) implies the existence of a quasi-compact open subspace Ω of X such that $Q \subseteq \Omega \subseteq U$. It follows immediately that $U \in \boldsymbol{\mathcal{U}}(\Omega) = \Omega^+ \subseteq U^+$. The proof is now complete. $\square$

On the other hand, from the theory of stably compact spaces, the following property holds.

Theorem 3.2 ([**30,** Theorem 5.9])**.** *Let X be a stably compact space. Then, the Smyth powerdomain $\boldsymbol{\mathcal{Q}}(X)$ of X, equipped with the upper Vietoris topology, is stably compact.*

From Lemma 2.2, the fact that $\boldsymbol{\mathcal{X}}(X)$ is a spectral space can be seen in the frame of the theory of stably compact spaces. We begin with the following, easy lemma, the proof of which is left to the reader.

Lemma 3.3. *Let X be any spectral space. Then, $\boldsymbol{\mathcal{X}}(X)$, endowed with the Zariski topology, is a T_0-space.*

Theorem 3.4. *Let X be a spectral space.*

(i) *The space $\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{X}}(X)$, endowed with the Zariski topology, i.e., with the upper Vietoris topology, is a spectral space.*

(ii) *Let $Y_1, Y_2 \in \boldsymbol{\mathcal{X}}$. Then, $Y_1 \subseteq Y_2$ if and only if $Y_1 \leq_{\boldsymbol{\mathcal{X}}} Y_2$.*

(iii) *The canonical map* $\varphi : X \to \boldsymbol{\mathcal{X}}$, *defined by* $\varphi(x) := \{x\}^{\text{gen}}$, *for each* $x \in X$, *is a spectral embedding* (*and, in particular, an order-preserving embedding between ordered sets, with the ordering induced by the Zariski topologies*).

(iv) $\boldsymbol{\mathcal{X}}$ *has a unique maximal point, i.e.,* X.

Proof.

(i) Let $\boldsymbol{\mathcal{U}}(\Omega)$ be a member of the canonical basis of $\boldsymbol{\mathcal{X}}(X)$, where $\Omega \neq \emptyset$ is a quasi-compact open subspace of X. If $\mathcal{A}$ is an open cover of $\boldsymbol{\mathcal{U}}(\Omega)$, then there is a set $A \in \mathcal{A}$ such that $\Omega \in A$. Hence, there is a nonempty quasi-compact open set V of X such that $\Omega \in \boldsymbol{\mathcal{U}}(V) \subseteq A$. Now, if $U \in \boldsymbol{\mathcal{U}}(\Omega)$, then $U \subseteq \Omega \subseteq V$, and thus, $U \in A$; it follows that the singleton $\{A\}$ is an open subcover of $\boldsymbol{\mathcal{U}}(\Omega)$. Therefore, $\boldsymbol{\mathcal{U}}(\Omega)$ is quasi-compact.

From Proposition 3.1 and Theorem 3.2, $\boldsymbol{\mathcal{X}}(X)$ is stably compact; by Lemma 2.2, and the previous reasoning, it follows that $\boldsymbol{\mathcal{X}}(X)$ is a spectral space.

Statements (ii), (iii) and (iv) are straightforward. $\square$

Remark 3.5. As was done in the first version of the present paper, it is also possible to prove the spectrality of $\boldsymbol{\mathcal{X}}(X)$ by using, instead of [**30,** Theorem 5.9], ultrafilter-theoretic techniques developed by ring theorists for studying spectral spaces; we sketch how to do it. From [**10,** Corollary 3.3], it suffices to show that, if $\mathscr{U}$ is an ultrafilter on $\boldsymbol{\mathcal{X}}$, then the set

$$\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{T}}}(\mathscr{U}) := \{Y \in \boldsymbol{\mathcal{X}} \mid (\text{for each } \boldsymbol{\mathcal{U}}(\Omega),\ Y \in \boldsymbol{\mathcal{U}}(\Omega) \Longleftrightarrow \boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U})\}$$

is nonempty. Set

$$\mathscr{F}(\mathscr{U}) := \{\Omega \mid \Omega \subseteq X \text{ quasi-compact open and } \boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U}\}.$$

Then, $\mathscr{F}(\mathscr{U})$ does not contain the empty set and has the finite intersection property; therefore,

$$Y_0 := \bigcap \{\Omega \mid \Omega \in \mathscr{F}(\mathscr{U})\}$$

is a nonempty inverse-closed subset of X, i.e., $Y_0 \in \boldsymbol{\mathcal{X}}(X)$.

Furthermore, if $Y_0 \in \boldsymbol{\mathcal{U}}(\Omega_0)$ and $\Omega_0 \notin \mathscr{U}$, then, since $\mathscr{U}$ is closed by finite intersection,

$$\mathscr{C} := \{\Omega \cap (X \setminus \Omega_0) \mid \Omega \text{ quasi-compact open in } X \text{ and } \boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U}\}$$

is a collection of sets having the finite intersection property, and each element of $\mathscr{C}$ is closed in the constructible topology. Therefore, its intersection is nonempty, and any point in the intersection belongs to $Y_0 \setminus \Omega_0$, a contradiction. Thus, $\Omega_0 \in \mathscr{U}$. Conversely, if $\Omega_0 \in \mathscr{U}$, then

$$\Omega_0 \supseteq \bigcap\{\Omega \mid \Omega \subseteq X \text{ quasi-compact open and } \boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U}\} = Y_0,$$

i.e., $Y_0 \in \boldsymbol{\mathcal{U}}(\Omega_0)$. Hence, $Y_0 \in \boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{T}}}(\mathscr{U})$, and $\boldsymbol{\mathcal{X}}(X)$ is a spectral space.

Remark 3.6.

(a) Let X be a spectral space and, as above, let X^{inv} denote the set X, endowed with the inverse topology. Then, keeping in mind Hochster's duality (i.e., sketchy, $(X^{\text{inv}})^{\text{inv}} = X$), the set $\boldsymbol{\mathcal{X}}'(X) := \boldsymbol{\mathcal{X}}(X^{\text{inv}})$ consists of all of the nonempty closed sets of X, with respect to the given spectral topology. Keeping in mind that the quasi-compact open subspaces of X^{inv} are precisely the complements of the quasi-compact open subspaces of X, it follows immediately, by definition, that the Zariski topology of $\boldsymbol{\mathcal{X}}'(X)$ has as a basis of open sets the collection of sets of the type:

$$\boldsymbol{\mathcal{U}}'(\Omega) := \boldsymbol{\mathcal{U}}(X \setminus \Omega) = \{C \in \boldsymbol{\mathcal{X}}'(X) \mid C \cap \Omega = \emptyset\},$$

for Ω varying among the quasi-compact open subspaces of X^{inv}. Dually, the canonical map

$$\varphi' : X^{\text{inv}} \longrightarrow \boldsymbol{\mathcal{X}}'(X),$$

defined by $x \mapsto \{x\}^{\text{sp}}$, is a spectral topological embedding. Now, let X be the prime spectrum of a ring R, and let $\text{Rd}(R)$ be the set of all proper radical ideals of R, endowed with the so called *hull-kernel topology*, that is, the topology whose subbasic open sets are those of the form

$$\mathtt{D}(x_1, x_2, \ldots, x_n) := \{H \in \text{Rd}(R) \mid (x_1, x_2, \ldots, x_n)R \nsubseteq H\}.$$

In **[12]**, it is proven that $\text{Rd}(R)$ is a spectral space that naturally extends the space $\text{Spec}(R)$, endowed with the Zariski topology. Moreover, it is proven that there is a canonical homeomorphism

$$\lambda : \boldsymbol{\mathcal{X}}'(R) := \boldsymbol{\mathcal{X}}'(\text{Spec}(R)) \longrightarrow \text{Rd}(R)^{\text{inv}},$$

mapping a nonempty closed set $C \subseteq \operatorname{Spec}(R)$ to the radical ideal $\lambda(C) := \bigcap\{P \mid P \in C\}$.

(b) Recall that, for any topological space X, the co-compact topology on X is the topology having as a base for the open sets the complements of quasi-compact saturated subsets of X [**19,** Definition O-5.10]. The topological space X endowed with this topology, denoted X^{d}, is called the *de Groot dual* of X. It is known that, if X is a stably compact space, X^{d} is also stably compact and $(X^{\mathrm{d}})^{\mathrm{d}} = X$ [**30,** Proposition 3.6]. For a spectral space X, X^{inv} coincides with the de Groot dual X^{d} (Lemma 2.1 (ii)).

We set forth in the following remark some observations concerning Theorem 3.4.

Remark 3.7. The notation of Theorem 3.4 is preserved.

(a) The subspace $\varphi(X)$ is dense in $\boldsymbol{\mathcal{X}}(X)$. In fact, let $\boldsymbol{\mathcal{U}}$ be a nonempty open subset of $\boldsymbol{\mathcal{X}}(X)$, and take an element $C \in \boldsymbol{\mathcal{U}}$ and a quasi-compact open subspace Ω of X such that $C \in \boldsymbol{\mathcal{U}}(\Omega) \subseteq \boldsymbol{\mathcal{U}}$. If $c \in C$, then $\{c\}^{\mathrm{gen}} \subseteq C \subseteq \Omega$, and thus, $\{c\}^{\mathrm{gen}} \in \boldsymbol{\mathcal{U}}(\Omega) \subseteq \boldsymbol{\mathcal{U}}$. This proves that $\varphi(X) \cap \Omega \neq \emptyset$.

(b) Following [**21,** Definition (2.6.3)], recall that a subset X_0 of a topological space X is said to be very dense in X if, for any open sets $U, V \subseteq X$, the equality $U \cap X_0 = V \cap X_0$ implies $U = V$.

The subspace $\varphi(X)$ is not very dense in $\boldsymbol{\mathcal{X}}(X)$. Indeed, let V_1 and V_2 be two discrete rank-one valuation domains having the same quotient field. Then, the prime spectrum X of the ring $D := V_1 \cap V_2$ consists exactly of (0) and of the two maximal ideals M_1 and M_2 which are the (incomparable) contractions in D of the maximal ideals of V_1 and V_2. Then, in the present situation,

$$\begin{aligned}\boldsymbol{\mathcal{X}}(X) &= \{\{(0)\}, \{(0), M_1\}, \{(0), M_2\}, X\};\\ \varphi(X) &= \{\{(0)\}, \{(0), M_1\}, \{(0), M_2\}\}.\end{aligned}$$

Since $\{X\}$ is closed in $\boldsymbol{\mathcal{X}}(X)$, it follows that $\varphi(X)$ is open in $\boldsymbol{\mathcal{X}}(X)$. From this fact, we immediately deduce that $\varphi(X)$ is dense but not very dense in $\boldsymbol{\mathcal{X}}(X)$.

(c) Let X be a spectral space, and let

$$\widehat{\boldsymbol{\mathcal{X}}}(X) := \widehat{\boldsymbol{\mathcal{X}}} := \{Y \subseteq X \mid Y = \mathrm{Cl}^{\mathrm{inv}}(Y)\} = \boldsymbol{\mathcal{X}}(X) \cup \{\emptyset\}.$$

Note that the techniques used in the proof of Theorem 3.4 (i) also allows us to show that $\widehat{\boldsymbol{\mathcal{X}}}$ (endowed with an obvious extension of the topology of $\boldsymbol{\mathcal{X}}$) is a spectral space. Since $\boldsymbol{\mathcal{U}}(\emptyset) = \{\emptyset\}$ is open in $\widehat{\boldsymbol{\mathcal{X}}}$, then $\boldsymbol{\mathcal{X}}$ is a closed (spectral) subspace of $\widehat{\boldsymbol{\mathcal{X}}}$.

Before stating the next result, we observe that $X \in \varphi(X)$ if and only if X has a unique closed point (in the given spectral topology).

Proposition 3.8. *Let X be a spectral space, and let $\varphi : X \to \boldsymbol{\mathcal{X}}(X)$ be the topological embedding defined in Theorem* 3.4 (iii). *Then, $\varphi(X) = \boldsymbol{\mathcal{X}}(X)$ if and only if $(X, \leq)$ is linearly ordered.*

Proof. Set, as usual, $\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{X}}(X)$. In order to avoid the trivial case, we can assume that X is not a singleton. First, suppose that $(X, \leq)$ is linearly ordered, and let $Y \in \boldsymbol{\mathcal{X}}$. Consider the collection $\mathscr{C} := \{\mathrm{Cl}(\{y\}) \cap Y \mid y \in Y\}$ of closed sets of Y (with respect to the subspace topology induced by the given topology of X). Since $(X, \leq)$ is linearly ordered, $\mathscr{C}$ has the finite intersection property. On the other hand, Y is a quasi-compact subspace of X, since, in particular, it is closed in the constructible topology of X. Thus, it is quasi-compact in the constructible topology, and, a fortiori, in the given topology of X. Therefore, there is a point $y_0 \in \cap\{C \mid C \in \mathscr{C}\}$. Now, it is easy to infer that $Y = \{y_0\}^{\mathrm{gen}}$.

Conversely, assume that $\varphi(X) = \boldsymbol{\mathcal{X}}$, and take two points $x, y \in X$. Clearly, the set $Z := \{x, y\}^{\mathrm{gen}} = \{x\}^{\mathrm{gen}} \cup \{y\}^{\mathrm{gen}}$ is nonempty and closed with respect to the inverse topology on X, i.e., $Z \in \boldsymbol{\mathcal{X}}$. By assumption, there is a point $z \in X$ such that $\varphi(z) = \{z\}^{\mathrm{gen}} = \{x\}^{\mathrm{gen}} \cup \{y\}^{\mathrm{gen}}$. The inclusion $\supseteq$ implies $x, y \leq z$. On the other hand, the inclusion $\subseteq$ implies that $z \leq x$ or $z \leq y$. From these facts, it easily follows that $(X, \leq)$ is linearly ordered. $\square$

Next, we compare the dimensions of X and $\boldsymbol{\mathcal{X}}(X)$ with the cardinality $|X|$ of the spectral space X.

Proposition 3.9. *Let X be a spectral space. Then,* $\dim(\boldsymbol{\mathcal{X}}(X)) = |X| - 1 \geq \dim(X)$. *Moreover, in the finite-dimensional case,* $\dim(\boldsymbol{\mathcal{X}}(X)) = \dim(X)$ *if and only if X is linearly ordered.*

Proof. Suppose first that X is finite. If $Y_0 <_{\boldsymbol{\mathcal{X}}(X)} Y_1 <_{\boldsymbol{\mathcal{X}}(X)} \cdots <_{\boldsymbol{\mathcal{X}}(X)} Y_n$ is a chain of points in $\boldsymbol{\mathcal{X}}(X)$, then $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ is a chain of nonempty subsets of X. In particular, $|Y_{k-1}| < |Y_k|$ for all k, $1 \leq k \leq n$. Therefore, $n+1 \leq |X|$ and $\dim(\boldsymbol{\mathcal{X}}(X)) \leq |X|-1$.

On the other hand, we can write X as a sequence $x_1, x_2, \ldots, x_t$ (where $t := |X|$) such that x_i is not bigger than x_j for every $i < j$ (simply, take x_1 as a minimal element of X and x_i as a minimal element of $X \setminus \{x_1, \ldots, x_{i-1}\}$ for $i \geq 2$). In particular, each $X_i := \{x_1, x_2, \ldots, x_i\}$ is inverse-closed in X, so that $X_1 <_{\boldsymbol{\mathcal{X}}(X)} X_2 <_{\boldsymbol{\mathcal{X}}(X)} \cdots <_{\boldsymbol{\mathcal{X}}(X)} X_t$ is a chain of points in $\boldsymbol{\mathcal{X}}(X)$ of length $t-1$. Therefore, $\dim(\boldsymbol{\mathcal{X}}(X)) \geq |X|-1$ and, by the previous paragraph, we conclude that $\dim(\boldsymbol{\mathcal{X}}(X)) = |X|-1$.

Suppose now that X is infinite. Take a positive integer t, and let X' be a subset of X of cardinality t. As before, we can enumerate the elements $x_1, x_2, \ldots, x_t$ of X' in such a way that x_i is not bigger than x_j for every $i < j$. Then, for each $i \in \{1, 2, \ldots, t\}$, the set $C_i := \{x_1, x_2, \ldots, x_i\}^{\text{gen}}$ is closed in the inverse topology of X, i.e., $C_i \in \boldsymbol{\mathcal{X}}(X)$. Clearly, $C_i \subsetneq C_{i+1}$ for each $i = 1, 2, \ldots, t-1$, since $x_{i+1} \in C_{i+1} \setminus C_i$. This proves that, for any positive integer t, there is a chain of length $t-1$ in $\boldsymbol{\mathcal{X}}(X)$. Thus, $\dim(\boldsymbol{\mathcal{X}}(X)) = \infty$.

If X is finite, $\dim(X) = |X|-1$ if, and only if, in X, there is a chain of the type $x_0 < x_1 < \cdots x_{|X|-1}$. This means that all elements of X are in such a chain, i.e., X is linearly ordered. $\square$

Remark 3.10. While the inequality $|X|-1 \geq \dim(X)$ is sharp by Proposition 3.9, the more incomparable elements the set X contains, the more $\dim(X)$ is small with respect to $|X|$. For example, if X is homeomorphic to the prime spectrum of the direct product of $n+1$ fields, $n \geq 1$, then $\dim(X) = 0$ while $|X|-1 = n$.

If $\dim(X)$ is not finite, then, clearly, $\dim(\boldsymbol{\mathcal{X}}(X)) = \dim(X)$; however, we can easily choose X not to be linearly ordered.

4. Functorial properties. A map $\psi : X_1 \to X_2$ of spectral spaces is called *spectral* if $\psi^{-1}(\Omega)$ is a quasi-compact open subset of X_1 for every quasi-compact open subset Ω of X_2.

Proposition 4.1. *Let $\psi : X_1 \to X_2$ be a spectral map of spectral spaces, and denote by $\varphi_1 : X_1 \to \boldsymbol{\mathcal{X}}(X_1)$ and $\varphi_2 : X_2 \to \boldsymbol{\mathcal{X}}(X_2)$ the topological embeddings defined in Theorem* 3.4 (iii). *Then, there is a spectral map $\boldsymbol{\mathcal{X}}(\psi) : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$ such that $\boldsymbol{\mathcal{X}}(\psi) \circ \varphi_1 = \varphi_2 \circ \psi$.*

Proof. First, note that each $C \in \boldsymbol{\mathcal{X}}(X_1)$ is quasi-compact in X_1 and, thus, $\psi(C)$ is quasi-compact in X_2. Thus, $\mathrm{Cl}^{\mathrm{inv}}(\psi(C)) = \psi(C)^{\mathrm{gen}} = \cup\{\{x_2\}^{\mathrm{gen}} \mid x_2 \in \psi(C)\} = \sup\{\{x_2\}^{\mathrm{gen}} \mid x_2 \in \psi(C)\}$ [**11,** Remark 2.2, Proposition 2.6]. For every $C \in \boldsymbol{\mathcal{X}}(X_1)$, define $\boldsymbol{\mathcal{X}}(\psi)(C) := \psi(C)^{\mathrm{gen}}$. In particular, we have that $\boldsymbol{\mathcal{X}}(\psi)(\{x\}^{\mathrm{gen}}) = \{\psi(x)\}^{\mathrm{gen}}$, for each $x \in X_1$.

Let Ω be a quasi-compact open subset of X_2. We claim that

$$(\boldsymbol{\mathcal{X}}(\psi))^{-1}(\boldsymbol{\mathcal{U}}(\Omega)) = \boldsymbol{\mathcal{U}}(\psi^{-1}(\Omega)),$$

which is quasi-compact open in $\boldsymbol{\mathcal{X}}(X_2)$, since ψ is spectral (and, thus, $\psi^{-1}(\Omega)$ is quasi-compact open in X_1). As a matter of fact, let $C \in (\boldsymbol{\mathcal{X}}(\psi))^{-1}(\boldsymbol{\mathcal{U}}(\Omega))$, i.e., $\boldsymbol{\mathcal{X}}(\psi)(C) \subseteq \Omega$; therefore, $\psi^{-1}(\boldsymbol{\mathcal{X}}(\psi)(C)) \subseteq \psi^{-1}(\Omega)$, and thus, clearly, $C \subseteq \psi^{-1}(\boldsymbol{\mathcal{X}}(\psi)(C))$. Conversely, let $C \subseteq \psi^{-1}(\Omega)$. Then, $\boldsymbol{\mathcal{X}}(\psi)(C) \leq \boldsymbol{\mathcal{X}}(\psi)(\psi^{-1}(\Omega))$. Moreover, we have that

$$\boldsymbol{\mathcal{X}}(\psi)(\psi^{-1}(\Omega)) = (\psi(\psi^{-1}(\Omega)))^{\mathrm{gen}} \subseteq \Omega^{\mathrm{gen}} = \Omega.$$

Therefore, $\boldsymbol{\mathcal{X}}(\psi)(C) \in \boldsymbol{\mathcal{U}}(\Omega)$. We conclude that $\boldsymbol{\mathcal{X}}(\psi)$ is a spectral map. $\square$

It is well known that, for compact Hausdorff spaces, and hence for Stone spaces, the upper Vietoris construction is functorial. Similarly, we now show that the assignment $\boldsymbol{\mathcal{X}}$ defined by the pair $(X \mapsto \boldsymbol{\mathcal{X}}(X), \psi \mapsto \boldsymbol{\mathcal{X}}(\psi))$ can be interpreted as a functor from the category of spectral spaces into itself.

Proposition 4.2. *We preserve the notation of Proposition* 4.1.

(i) *If* $X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} X_3$ *is a chain of spectral maps, then the spectral map* $\boldsymbol{\mathcal{X}}(\psi_2 \circ \psi_1) : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_3)$, *induced by* $\psi_2 \circ \psi_1$, *is equal to the composition* $\boldsymbol{\mathcal{X}}(\psi_2) \circ \boldsymbol{\mathcal{X}}(\psi_1)$. *It follows that the assignment* $(X \mapsto \boldsymbol{\mathcal{X}}(X), \psi \mapsto \boldsymbol{\mathcal{X}}(\psi))$ *defines a functor from the category of spectral spaces into itself.*

(ii) *Let* $\Psi : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$ *be a spectral map. Assume that there exists a spectral map* $\psi : X_1 \to X_2$ *such that* $\Psi \circ \varphi_1 = \varphi_2 \circ \psi$. *Then,* $\boldsymbol{\mathcal{X}}(\psi) \leq \Psi$, *i.e.,* $\boldsymbol{\mathcal{X}}(\psi)(C) \subseteq \Psi(C)$ *for each* $C \in \boldsymbol{\mathcal{X}}(X_1)$.

Proof.

(i) The proof is straightforward.

(ii) Let $C \in \boldsymbol{\mathcal{X}}(X_1)$. For every $c \in C$, we have

$$C \supseteq \varphi_1(c) = \{c\}^{\text{gen}}$$

(i.e., $C \geq \varphi_1(c)$ with respect to the order of $\boldsymbol{\mathcal{X}}(X_1)$ induced by the Zariski topology). Since Ψ is continuous, it is order-preserving, and thus, $\Psi(C) \geq \Psi(\varphi_1(c)) = \varphi_2(\psi(c)) = \{\psi(c)\}^{\text{gen}}$. Hence, $\psi(c) \in \Psi(C)$, and therefore, $\psi(C) \subseteq \Psi(C)$. Since $\Psi(C)$ is closed in the inverse topology on X_2, then $\text{Cl}^{\text{inv}}(\psi(C)) \subseteq \Psi(C)$. On the other hand, by definition, $\boldsymbol{\mathcal{X}}(\psi)(C) = \psi(C)^{\text{gen}} = \text{Cl}^{\text{inv}}(\psi(C)) \leq \Psi(C)$; hence, $\boldsymbol{\mathcal{X}}(\psi) \leq \Psi$. $\square$

Remark 4.3. The previous result is very similar to the statement concerning the functoriality of the Smyth powerdomain construction $\boldsymbol{\mathcal{Q}}(X)$, proven in [**19,** page 371, Proposition IV.8.19], when X is a *directed-complete partial order* (that is, a partially ordered set where each directed subset has a supremum) endowed with the topology generated by the upper sets (called the *Scott topology*). However, despite the similarity of the construction, the Scott topology does not coincide with the given spectral topology, but, in general, it is stronger than the inverse topology [**26,** Proposition 2.9]. Nevertheless, by order-theoretic reasons, the functoriality of the Smyth powerdomain construction $\boldsymbol{\mathcal{Q}}(X)$ given in [**19**] is closer to functoriality of the construction $\boldsymbol{\mathcal{X}}'(X) := \boldsymbol{\mathcal{X}}(X^{\text{inv}})$ [**12**], recalled briefly in Remark 3.6 (a).

The next example shows that it is possible to have $\Psi \neq \boldsymbol{\mathcal{X}}(\psi)$, i.e., it is possible to have more than one "extension" of $\psi : X_1 \to X_2$ between the spaces $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$. On the other hand, we will show in Proposition 4.5 that this situation does not occur when ψ is a homeomorphism.

Example 4.4. Let $X_1 = \{a_1, a_2, b\}$ and $X_2 := \{c_1, c_2\}$. Suppose that a_1 and a_2 are incomparable but both are smaller than b, and also suppose that $c_1 < c_2$. It is straightforward that the order structures of X_1 and X_2 are compatible with the order of suitable spectral topologies on X_1 and X_2. When X_1 and X_2 are equipped with these spectral topologies, it is easy to see that $\boldsymbol{\mathcal{X}}(X_1) = \{\{a_1\}, \{a_2\}, \{a_1, a_2\}, \{b, a_1, a_2\}\}$, while $\boldsymbol{\mathcal{X}}(X_2) = \{\{c_1\}, \{c_1, c_2\}\}$. Let

$$\psi : X_1 \longrightarrow X_2$$

be the spectral map defined by $\psi(a_1) := \psi(a_2) := c_1$ and $\psi(b) := c_2$. Let

$$\Psi : \boldsymbol{\mathcal{X}}(X_1) \longrightarrow \boldsymbol{\mathcal{X}}(X_2)$$

be the map defined by $\Psi(\{a_1\}) := \Psi(\{a_2\}) := \{c_1\}$ and $\Psi(\{b, a_1, a_2\}) := \Psi(\{a_1, a_2\}) := \{c_1, c_2\}$. Clearly, Ψ is a spectral map of spectral spaces, since

$$\Psi^{-1}(\boldsymbol{\mathcal{U}}(\{c_1\})) = \{\{a_1\}, \{a_2\}\} = \boldsymbol{\mathcal{U}}(\{a_1\}) \cup \boldsymbol{\mathcal{U}}(\{a_2\}),$$

and $\Psi^{-1}(\boldsymbol{\mathcal{U}}(\{c_1, c_2\})) = \Psi^{-1}(X_2) = X_1$. Moreover, it is obvious that Ψ "extends" ψ. However, the "natural extension" $\boldsymbol{\mathcal{X}}(\psi)$ of ψ, defined in Proposition 4.1, is such that $\boldsymbol{\mathcal{X}}(\psi)(\{a_1, a_2\}) = \{c_1\}$, and thus, $\Psi \neq \boldsymbol{\mathcal{X}}(\psi)$. This situation is illustrated in Figure 1.

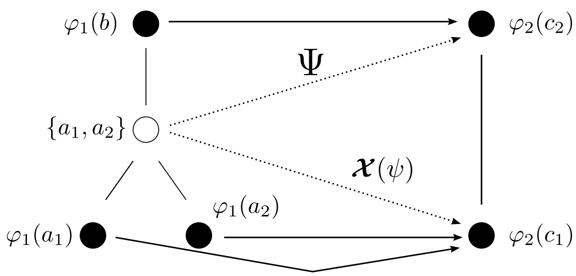

FIGURE 1. Illustration of Example 4.4. Black circles represent elements of $\varphi_1(X_1)$ and $\varphi_2(X_2)$.

Proposition 4.5. *Let X_1 and X_2 be spectral spaces, and let $\varphi_1 : X_1 \to \boldsymbol{\mathcal{X}}(X_1)$ and $\varphi_2 : X_2 \to \boldsymbol{\mathcal{X}}(X_2)$ be the canonical embeddings* (*as in Theorem* 3.4 (iii)).

(i) *If $\psi : X_1 \to X_2$ is a topological embedding* (*respectively, a homeomorphism*), *then $\boldsymbol{\mathcal{X}}(\psi) : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$* (*as defined in Proposition* 4.1) *is a topological embedding* (*respectively, homeomorphism*).

(ii) *If $\Psi : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$ is a homeomorphism, then there exists a unique homeomorphism $\psi : X_1 \to X_2$ such that $\Psi = \boldsymbol{\mathcal{X}}(\psi)$* (*and thus, $\Psi \circ \varphi_1 = \varphi_2 \circ \psi$*).

(iii) *In particular, X_1 and X_2 are homeomorphic if and only if $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$ are homeomorphic.*

Proof.

(i) By Proposition 4.1, $\boldsymbol{\mathcal{X}}(\psi) \circ \varphi_1 = \varphi_2 \circ \psi$. Since φ_1 and φ_2 are topological embeddings, if ψ is also an embedding, so is $\varphi_2 \circ \psi$. Thus, so is $\boldsymbol{\mathcal{X}}(\psi) \circ \varphi_1$; hence, $\boldsymbol{\mathcal{X}}(\psi)$ is also an embedding. If ψ is a homemorphism, and $C \in \boldsymbol{\mathcal{X}}(X_2)$, then $C = \boldsymbol{\mathcal{X}}(\psi)(\psi^{-1}(C))$ so that $\boldsymbol{\mathcal{X}}(\psi)$ is surjective and, thus, a homeomorphism.

(ii) We begin with showing the following.

Claim 4.5.1. *Let X be a spectral space, and let $\varphi : X \to \boldsymbol{\mathcal{X}}(X)$ be the canonical embedding. Then, $\varphi(X)$ is precisely the set of all irreducible closed subsets of X, endowed with the inverse topology.*

As a matter of fact, it is well known that the space X^{inv}, i.e., the set X endowed with the inverse topology, is itself a spectral space [**28,** Proposition 8], and thus, any irreducible closed subspace C of X^{inv} has a unique generic point, say x, that is, $C = \mathrm{Cl}^{\mathrm{inv}}(\{x\}) = \{x\}^{\mathrm{gen}} = \varphi(x)$. On the other hand, it is trivial that $\varphi(X)$ is contained in the set of all irreducible closed subsets of X^{inv}.

Claim 4.5.2. *Assume that $\Psi : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$ is a homeomorphism. Let C be an irreducible and closed subspace of X_1^{inv}. Then, $\Psi(C)$ is an irreducible* (*and closed*) *subset of X_2^{inv}.*

Let $D', D'' \in \boldsymbol{\mathcal{X}}(X_2)$ be such that $D' \cup D'' = \Psi(C)$. Since Ψ is a homeomorphism and also is an isomorphism of ordered sets, we see that $C = \Psi^{-1}(D') \cup \Psi^{-1}(D'')$. Since C is irreducible, we have either $C = \Psi^{-1}(D')$ or $C = \Psi^{-1}(D'')$, and thus, either $\Psi(C) = D'$ or $\Psi(C) = D''$.

Now, fix a point $x \in X_1$. By Claim 4.5.2, the set $\Psi(\{x\}^{\mathrm{gen}})$ is irreducible in X_2; thus, by Claim 4.5.1, there is a unique point $x_\Psi \in X_2$ such that $\{x_\Psi\}^{\mathrm{gen}} = \Psi(\{x\}^{\mathrm{gen}})$. Thus, Ψ naturally induces a map $\psi : X_1 \longrightarrow X_2$ by setting $\psi(x) := x_\Psi$ for any $x \in X$. Clearly, $\varphi_2 \circ \psi = \Psi \circ \varphi_1$. Next, we want to prove that $\psi : X_1 \to X_2$ is a homeomorphism.

Claim 4.5.3. *Assume that $\Psi : \boldsymbol{\mathcal{X}}(X_1) \to \boldsymbol{\mathcal{X}}(X_2)$ is a homeomorphism. Let Ω be a quasi-compact open subspace of X_1, in particular, $\Omega \in \boldsymbol{\mathcal{X}}(X_1)$. Then, $\Psi(\Omega)$ is a quasi-compact open subspace of X_2.*

Note that the quasi-compact open subspace $\boldsymbol{\mathcal{U}}(\Omega)$ of $\boldsymbol{\mathcal{X}}(X_1)$ coincides with $\{\Omega\}^{\text{gen}}$ (where the generizations are taken in $\boldsymbol{\mathcal{X}}(X_1)$). Since Ψ is a homeomorphism, then $\Psi(\boldsymbol{\mathcal{U}}(\Omega)) = \Psi(\{\Omega\}^{\text{gen}}) = \{\Psi(\Omega)\}^{\text{gen}}$ is a quasi-compact open set of $\boldsymbol{\mathcal{X}}(X_2)$ which is irreducible as an inverse-closed subspace of $\boldsymbol{\mathcal{X}}(X_2)$. In order to show that $\Psi(\Omega)$ is a quasi-compact open subspace of X_2, we observe that

$$\begin{aligned}\Psi(\{\Omega\}^{\text{gen}}) = \Psi(\boldsymbol{\mathcal{U}}(\Omega)) &= \bigcup\{\boldsymbol{\mathcal{U}}(V_i) \mid 1 \leq i \leq n\} \\ &= \bigcup\{\{V_i\}^{\text{gen}} \mid 1 \leq i \leq n\},\end{aligned}$$

for a finite family of quasi-compact open subspaces $\{V_i \mid 1 \leq i \leq n\}$ of X_2. Therefore, $\Psi(\{\Omega\}^{\text{gen}}) = \{V_{\tilde{i}}\}^{\text{gen}}$ for some $\tilde{i}$ and so $\Psi(\Omega) = V_{\tilde{i}}$.

In order to prove that $\psi : X_1 \to X_2$ is a homeomorphism, we start by showing that ψ is continuous. Let $V \subseteq X_2$ be a quasi-compact open. We claim that $\psi^{-1}(V) = \Psi^{-1}(V)$, where $\Psi^{-1}(V) \in \boldsymbol{\mathcal{X}}(X_1)$ is a quasi-compact open subspace of X_1 since Ψ is a homeomorphism. Moreover,

$$\boldsymbol{\mathcal{U}}(\Psi^{-1}(V)) = \{\Psi^{-1}(V)\}^{\text{gen}} = \Psi^{-1}(\{(V)\}^{\text{gen}}) = \Psi^{-1}(\boldsymbol{\mathcal{U}}(V)).$$

Now, take a point $x \in X_1$. Then,

$$\begin{aligned}\psi(x) \in V \iff \{x_\psi\}^{\text{gen}} \subseteq V \iff \Psi(\{x\}^{\text{gen}}) \in \boldsymbol{\mathcal{U}}(V) \\ \iff \{x\}^{\text{gen}} \in \Psi^{-1}(\boldsymbol{\mathcal{U}}(V)) \iff x \in \Psi^{-1}(V),\end{aligned}$$

i.e., $\psi^{-1}(V) = \Psi^{-1}(V)$.

Now, we show that $\psi : X_1 \to X_2$ is open. Let Ω be a quasi-compact open subspace of X_1. From Claim 4.5.3, $\Psi(\Omega)$ is a quasi-compact open subspace of X_2 and, obviously, $\Omega = \Psi^{-1}(\Psi(\Omega))$. Moreover, by the previous observation, $\Psi^{-1}(\Psi(\Omega)) = \psi^{-1}(\Psi(\Omega))$, and, since ψ is bijective, $\psi(\Omega) = \Psi(\Omega)$.

Finally, we show that $\boldsymbol{\mathcal{X}}(\psi) = \Psi$. Take a set $C \in \boldsymbol{\mathcal{X}}(X_1)$. Since ψ is a homeomorphism, it is also a homeomorphism between X_1^{inv} and X_2^{inv}, and, in particular, it is a closed map (with respect to the inverse topologies). Therefore, it suffices to prove that $\boldsymbol{\mathcal{X}}(\psi)(C) = \psi(C)^{\text{gen}} = \psi(C)$ coincides with $\Psi(C)$. Let $\{C_i \mid i \in I\}$ be the collection of the irreducible (and closed) components of C in X_1^{inv}. From Claim 4.5.1, for any $i \in I$, let $x_i \in X_1$ be the unique generic point of C_i in X_1^{inv}. Keeping in mind that both ψ and Ψ are also isomorphisms of partially ordered sets (orderings induced by the topologies), we have

$$\begin{aligned}\Psi(C) &= \Psi(\sup\{C_i \mid i \in I\}) = \sup\{\Psi(C_i) \mid i \in I\}\\ &= \sup\{\Psi(\{x_i\}^{\mathrm{gen}}) \mid i \in I\} = \sup\{\{\psi(x_i)\}^{\mathrm{gen}} \mid i \in I\}\\ &= \bigcup\{\{\psi(x_i)\}^{\mathrm{gen}} \mid\ i \in I\} = \psi\Big(\bigcup\{\{x_i\}^{\mathrm{gen}} \mid i \in I\}\Big)\\ &= \psi\Big(\bigcup\{C_i \mid i \in I\}\Big) = \psi(C).\end{aligned}$$

The proof of (ii) is now complete. Part (iii) is an immediate consequence of statements (i) and (ii). $\square$

It is not difficult to see that $\varphi(= \varphi_X) : X \to \boldsymbol{\mathcal{X}}(X)$ does not provide a unique way for embedding a spectral space X into a larger, "natural" spectral space. However, φ satisfies a universal-like property.

We begin with a lemma.

Lemma 4.6. *Let Z be a spectral space, and let Y be a closed set in the constructible topology of Z; in particular, Y is a spectral space. Assume that the map*

$$\Sigma_{Y,Z} \colon \boldsymbol{\mathcal{X}}(Y) \longrightarrow Z, \qquad C \longmapsto \sup_Z(C)$$

for each $C \in \boldsymbol{\mathcal{X}}(Y)$ is well defined. Then, the following statements hold.

- (i) *If each point of Z has a local basis consisting of sets of the form $\{\omega\}^{\mathrm{gen}}$ for suitable elements $\omega \in Z$, then $\Sigma_{Y,Z}$ is continuous, spectral and open onto its image.*
- (ii) *If $Y = Z$, then the converse holds.*

Proof.

(i) For the sake of simplicity, set $\Sigma := \Sigma_{Y,Z}$. Let $x \in Z$ and V_x be a basic open set of Z containing x; then, we claim that $\Sigma^{-1}(V_x) = \boldsymbol{\mathcal{U}}(V_x \cap Y)$ and that $\Sigma(\boldsymbol{\mathcal{U}}(V_x \cap Y)) = V_x \cap \Sigma(\boldsymbol{\mathcal{X}}(Y))$. (Note that, since Y is closed, with respect to the constructible topology, $V_x \cap Y$ is open in Y and quasi-compact and thus determines a basic open set of $\boldsymbol{\mathcal{X}}(Y)$.) Indeed, take a point $K \in \Sigma^{-1}(V_x)$. Then, $k := \sup_Z(K) \in V_x$, and thus, $K \subseteq \{k\}^{\mathrm{gen}} \subseteq V_x$. Since, clearly, $K \subseteq Y$, we have $K \in \boldsymbol{\mathcal{U}}(V_x \cap Y)$. Conversely, take a point $K \in \boldsymbol{\mathcal{U}}(V_x \cap Y)$, in particular, $K \subseteq V_x$, and thus, we have $k := \sup_Z(K) \leq x$, or, equivalently, $k \in V_x$. Hence, $K \in \Sigma^{-1}(V_x)$. This reasoning also shows the second equality.

The hypotheses on Z now imply that Σ is continuous, spectral and open onto its image.

(ii) Let now $\Sigma := \Sigma_{Z,Z}$. Take a point $z \in Z$ and an open neighborhood U of Z. Since $z = \Sigma(\{z\}^{\text{gen}})$, and Σ is continuous, there is a quasi-compact open subspace Ω of Z such that $\{z\}^{\text{gen}} \in \boldsymbol{\mathcal{U}}(\Omega)$, i.e., $z \in \Omega$, and $\Sigma(\boldsymbol{\mathcal{U}}(\Omega)) \subseteq U$. Since $\Omega \in \boldsymbol{\mathcal{U}}(\Omega)$, the last statement implies that $\omega := \sup_Z(\Omega) \in U$. It follows that $z \in \Omega \subseteq \{\omega\}^{\text{gen}} \subseteq U$. $\square$

Remark 4.7. Let Z be a spectral space, and let $\varphi_Z : Z \to \boldsymbol{\mathcal{X}}(Z)$ be the spectral embedding introduced in Theorem 3.4 (iii). Under the assumptions and the equivalent conditions of Lemma 4.6, the map Σ_Z ($= \Sigma_{Z,Z}$) gives rise to a topological retraction since $\Sigma_Z \circ \varphi_Z$ is the identity map on Z.

We say that a map $f : X \to Y$ of spectral spaces is sup-*preserving* if, whenever F is a finite subset of X and there exists an $\sup_X(F)$, then there exist $\sup_Y(f(F))$ and $f(\sup_X(F)) = \sup_Y(f(F))$.

Theorem 4.8. *Let X be a spectral space, and let φ $(= \varphi_X) : X \to \boldsymbol{\mathcal{X}}(X)$ be the canonical spectral embedding* (*Theorem* 3.4 (iii)). *Let Z be a spectral space, and let $\lambda : X \to Z$ be a spectral map. Suppose that the map Σ $(= \Sigma_{\lambda(X),Z}) : \boldsymbol{\mathcal{X}}(\lambda(X)) \to Z$, introduced in Lemma* 4.6, *is* (*well defined and*) *spectral.*

(i) *There is a* sup-*preserving spectral map $\boldsymbol{\lambda}^\sharp : \boldsymbol{\mathcal{X}}(X) \to Z$, defined by setting $\boldsymbol{\lambda}^\sharp(C) := \sup_Z(\lambda(C)^{\text{gen}})$, for each $C \in \boldsymbol{\mathcal{X}}(X)$, such that $\boldsymbol{\lambda}^\sharp \circ \varphi = \lambda$.*

(ii) *If $\boldsymbol{\Lambda} : \boldsymbol{\mathcal{X}}(X) \to Z$ is a spectral map such that $\boldsymbol{\Lambda} \circ \varphi = \lambda$, then $\boldsymbol{\lambda}^\sharp(K) \leq \boldsymbol{\Lambda}(K)$ for every $K \in \boldsymbol{\mathcal{X}}(X)$* (*where $\leq$ is the order induced on Z by the topology*).

(iii) *If, moreover, $\boldsymbol{\Lambda}$ is sup-preserving, then $\boldsymbol{\Lambda} = \boldsymbol{\lambda}^\sharp$.*

Proof.

(i) Since λ is a spectral map, it is also continuous when X and Z are both endowed with the constructible topology. In particular, since the constructible topology is both quasi-compact and Hausdorff, λ is a closed map when considered in the constructible topology, and thus, $\lambda(X)$ is a closed set in the constructible topology of Z; therefore, $\lambda(X)$ is a spectral space (so that $\boldsymbol{\mathcal{X}}(\lambda(X))$ is well defined) and the inclusion $j : \lambda(X) \hookrightarrow Z$ is a spectral map. In particular, it is possible to define the map Σ $(= \Sigma_{\lambda(X),Z})$.

Let $\boldsymbol{\lambda}^\sharp : \boldsymbol{\mathcal{X}}(X) \to Z$ be the map defined above.

Keeping in mind [**35,** Propositions 2.1, 2.2] and the fact that any point of a quasi-compact T_0 space is less than or equal to a maximal point of the space, we easily infer that $\boldsymbol{\lambda}^\sharp = \Sigma \circ \boldsymbol{\mathcal{X}}(\lambda)$, and thus, by assumption, $\boldsymbol{\lambda}^\sharp$ is spectral. Moreover, both Σ and $\boldsymbol{\mathcal{X}}(\lambda)$ are sup-preserving (which is easily verified), and thus, $\boldsymbol{\lambda}^\sharp$ is sup-preserving; by definition, it follows that $\boldsymbol{\lambda}^\sharp \circ \varphi = \lambda$.

(ii) Suppose now that $\boldsymbol{\Lambda} : \boldsymbol{\mathcal{X}}(X) \to Z$ is such that $\boldsymbol{\Lambda} \circ \varphi = \lambda$, and fix $K \in \boldsymbol{\mathcal{X}}(X)$.

For each $x \in K$, we have $\{x\}^{\text{gen}} \subseteq K$, and since, in particular, $\boldsymbol{\Lambda}$ is continuous, it follows that

$$\lambda(x) = \boldsymbol{\Lambda}(\varphi(x)) = \boldsymbol{\Lambda}(\{x\}^{\text{gen}}) \leq \boldsymbol{\Lambda}(K).$$

By definition, $\boldsymbol{\lambda}^\sharp(K)$ is equal to the supremum in Z of the set $\lambda(K)^{\text{gen}}$; moreover, it is equal to the supremum of $\lambda(K)$ since, if $y \in \lambda(K)^{\text{gen}}$, then $y \leq \lambda(x)$ for some $x \in K$. By the previous calculation, $\lambda(x) \leq \boldsymbol{\Lambda}(K)$ for every $x \in K$; therefore, $\boldsymbol{\lambda}^\sharp(K) \leq \boldsymbol{\Lambda}(K)$, as claimed.

(iii) Suppose now that the spectral map $\boldsymbol{\Lambda}$ is sup-preserving, and, as above, let $K \in \boldsymbol{\mathcal{X}}(X)$. Take any open neighborhood V of $z := \boldsymbol{\lambda}^\sharp(K)$ in Z. Then, by definition as well as by (ii), in order to prove that $\boldsymbol{\lambda}^\sharp(K) = \boldsymbol{\Lambda}(K)$, it suffices to show that $\boldsymbol{\Lambda}(K) \in V$. Since Σ is continuous, there exist an element $v \in V$ and a quasi-compact open subspace W of Z such that $z \in W \subseteq \{v\}^{\text{gen}} \subseteq V$, in view of Lemma 4.6. For any $x \in K$, we have

$$\boldsymbol{\Lambda}(\{x\}^{\text{gen}}) = \boldsymbol{\Lambda}(\varphi(x)) = \lambda(x) \leq \sup{}_Z(\lambda(K)^{\text{gen}}) = z \in W.$$

Since W is (in particular) closed under generalizations, it follows that $\boldsymbol{\Lambda}(\{x\}^{\text{gen}}) \in W$. Since $\boldsymbol{\Lambda}$ is continuous, there is a quasi-compact open subspace A_x of Z such that $\{x\}^{\text{gen}} \in \boldsymbol{\mathcal{U}}(A_x)$, i.e., $x \in A_x$, and $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{U}}(A_x)) \subseteq W$. Thus, $\cup_{x\in K} A_x \supseteq K$ and, since K is (in particular) quasi-compact, there are finitely many elements $x_1, x_2, \ldots, x_n \in K$ such that $K \subseteq \cup_{i=1}^n A_{x_i}$. Note that $\cup_{i=1}^n A_{x_i} \in \boldsymbol{\mathcal{X}}(Z)$ since any $A_{x_i} \in \boldsymbol{\mathcal{X}}(Z)$ is open and quasi-compact. Keeping in mind that $\mathbf{\Lambda}$ is continuous (and thus, an order-preserving map), we have

$$\begin{aligned}\mathbf{\Lambda}(K) \leq \mathbf{\Lambda}\left(\bigcup_{i=1}^n A_{x_i}\right) &= \mathbf{\Lambda}(\sup{}_{\boldsymbol{\mathcal{X}}(Z)}(\{A_{x_i} \mid 1 \leq i \leq n\})) \\ &= \sup{}_Z(\{\mathbf{\Lambda}(A_{x_i}) \mid 1 \leq i \leq n\}).\end{aligned}$$

Since $\mathbf{\Lambda}(\boldsymbol{\mathcal{U}}(A_{x_i})) \in W \subseteq \{v\}^{\text{gen}}$, for $1 \leq i \leq n$, it follows that $\sup_Z(\{\mathbf{\Lambda}(A_{x_i}) \mid 1 \leq i \leq n\}) \in \{v\}^{\text{gen}} \subseteq V$, and a fortiori, $\mathbf{\Lambda}(K) \in V$. The proof is now complete. $\square$

Remark 4.9. Theorem 4.11 (iii) provides a slight generalization of [**30,** Proposition 5.6]. Indeed, under the equivalence between the construction $\boldsymbol{\mathcal{X}}(X)$ (with the Zariski topology) and the Smyth powerdomain $\boldsymbol{\mathcal{Q}}(X)$ (with the upper Vietoris topology) established in Proposition 3.1, a sup-preserving map becomes a homomorphism of semilattices, and the map Σ coincides with the map $\wedge$ considered in [**30**]. The difference between Theorem 4.8 and [**30,** Proposition 5.6] is that we do not require the map Σ to exist on the whole of $\boldsymbol{\mathcal{X}}(Z)$, but only on $\boldsymbol{\mathcal{X}}(\lambda(X))$.

Proposition 4.10. *Preserve the notation and hypotheses of Theorem* 4.8*, and suppose that the map* Σ $(= \Sigma_{\lambda(X),Z}) : \boldsymbol{\mathcal{X}}(\lambda(X)) \to Z$ *is injective. Then, the following hold.*

(i) $\boldsymbol{\lambda}^\sharp$ *is a spectral embedding.*

(ii) *If, furthermore,* $z = \sup_Z\{\lambda(x) \mid x \in \lambda^{-1}(\{z\}^{\text{gen}})\}$ *for every* $z \in Z$*, and* $\mathbf{\Lambda} : \boldsymbol{\mathcal{X}}(X) \to Z$ *is a spectral embedding such that* $\mathbf{\Lambda} \circ \varphi = \lambda$*, then* $\mathbf{\Lambda} = \boldsymbol{\lambda}^\sharp$.

Proof.

(i) The proof of Lemma 4.6 shows that Σ is a spectral embedding whenever it is injective. Since φ is also a spectral embedding, so is $\Sigma \circ \varphi$, i.e., $\boldsymbol{\lambda}^\sharp$.

(ii) In the present situation, we claim that $\mathbf{\Lambda}$ is sup-preserving. Let $C_1, C_2 \in \boldsymbol{\mathcal{X}}(X)$, and consider $\mathbf{\Lambda}(C_1 \cup C_2)$ (note that the order on $\boldsymbol{\mathcal{X}}(X)$ is the set-theoretic inclusion; thus, the union is exactly their supremum). Clearly, $\mathbf{\Lambda}(C_1 \cup C_2)$ is bigger than both $\mathbf{\Lambda}(C_1)$ and $\mathbf{\Lambda}(C_2)$, and thus, also of their supremum.

Let x be such that $\lambda(x) \leq \mathbf{\Lambda}(C_1 \cup C_2)$, or equivalently, such that $x \in \lambda^{-1}(\mathbf{\Lambda}(C_1 \cup C_2))$. Since $\lambda(x) = \mathbf{\Lambda}(\{x\}^{\text{gen}})$, the previous inequality can be rewritten as $\mathbf{\Lambda}(\{x\}^{\text{gen}}) \leq \mathbf{\Lambda}(C_1 \cup C_2)$. On the other hand, $\mathbf{\Lambda}$ is an embedding, i.e., it is a homeomorphism onto its image, and thus, $\{x\}^{\text{gen}} \leq C_1 \cup C_2$ in $\boldsymbol{\mathcal{X}}(X)$. Hence, $x \in C_1 \cup C_2$, which means $x \in C_1$ or $x \in C_2$. Therefore,

$$\mathbf{\Lambda}(\{x\}^{\text{gen}}) \leq \sup\{\mathbf{\Lambda}(C_1), \mathbf{\Lambda}(C_2)\}.$$

By hypothesis, we have

$$\mathbf{\Lambda}(C_1 \cup C_2) = \sup\{\mathbf{\Lambda}(\{x\}^{\text{gen}}) \mid x \in X \text{ such that } \lambda(x) \leq \mathbf{\Lambda}(C_1 \cup C_2)\}.$$

Therefore, by the previous inequality, we deduce that $\mathbf{\Lambda}(C_1 \cup C_2) \leq \sup\{\mathbf{\Lambda}(C_1),\ \mathbf{\Lambda}(C_2)\}$. As observed above, the opposite inequality also holds; thus, we have the equality, and so, $\mathbf{\Lambda}$ is sup-preserving.

By Theorem 4.8 (iii), we conclude that $\mathbf{\Lambda} = \boldsymbol{\lambda}^\sharp$. $\square$

Remark 4.11. In general, it is possible for a spectral map $\lambda : X \to Z$ to have more than one extension $\mathbf{\Lambda} : \boldsymbol{\mathcal{X}}(X) \to Z$, even under the hypothesis $z = \sup_Z\{\lambda(x) \mid x \in \lambda^{-1}(\{z\}^{\text{gen}})\}$ (the previous proposition merely guarantees the unicity of an extension $\mathbf{\Lambda}$, which is an *embedding*).

For example, suppose that $Z = \boldsymbol{\mathcal{X}}(X)$, and let $\lambda = \varphi$ be the canonical inclusion of X in $\boldsymbol{\mathcal{X}}(X)$. Clearly, if $z \in Z = \boldsymbol{\mathcal{X}}(X)$, then $A := \lambda^{-1}(\{z\}^{\text{gen}})$ is composed of the elements of X that belong to $\{z\}^{\text{gen}}$, and thus, the supremum of the set $\{\lambda(x) \mid x \in A\}$ is exactly z. Moreover, it is clear that the homeomorphism

$$\boldsymbol{\lambda}^\sharp : \boldsymbol{\mathcal{X}}(X) \longrightarrow Z = \boldsymbol{\mathcal{X}}(X),$$

whose existence is guaranteed by Theorem 4.8, is merely the identity $\mathrm{id}_{\boldsymbol{\mathcal{X}}(X)}$.

On the other hand, suppose that $X = \{a, b, c\}$ is composed of three elements and endowed with the discrete topology, that is, suppose that every subset of X is open. Then, X is a spectral space; denote by $\mathbf{\Lambda} : \boldsymbol{\mathcal{X}}(X) \to \boldsymbol{\mathcal{X}}(X)$ the function defined by

$$\mathbf{\Lambda}(C) := \begin{cases} C & \text{if } C \neq \{a, b\}, \\ X & \text{if } C = \{a, b\}. \end{cases}$$

Then, $\mathbf{\Lambda}$ is order-preserving (in the order induced by the Zariski topology), and since $\boldsymbol{\mathcal{X}}(X)$ is finite, this implies that $\mathbf{\Lambda}$ is continuous and spectral. Moreover, if C is in $\varphi(X)$, i.e., if C is a singleton, then $\mathbf{\Lambda}(C) = C$. Therefore, $\mathbf{\Lambda} \circ \varphi = \varphi = \mathrm{id}_{\boldsymbol{\mathcal{X}}(X)} \circ \varphi$.

5. Applications. In this section, we apply the topological results of the previous sections to various algebraic settings. In particular, in subsection 5.1, we show how the construction $\boldsymbol{\mathcal{X}}$ relates a spectral space associated to a family of modules with the space of all possible intersections of the family, and we prove that the space of all overrings of an integral domain D that are integrally closed is a spectral space and it is a topological quotient of the spectral space obtained using the construction $\boldsymbol{\mathcal{X}}$ from the Riemann-Zariski space $\mathrm{Zar}(D)$. In subsection 5.2, we use $\boldsymbol{\mathcal{X}}$ to represent some distinguished spaces of semistar operations and provide a different general proof of some results shown in [**13**].

5.1. Spaces of modules and overrings. Let R be a ring, let M be an R-module, and let $\mathrm{SMod}_R(M)$ be the set of R-submodules of M. The *Zariski topology* on $\mathrm{SMod}_R(M)$ is the topology having, as a subbasis of open sets, the sets in the form

$$\mathtt{B}_f := \{N \in \mathrm{SMod}_R(M) \mid f \in N\},$$

where f runs in M; equivalently, the sets in the form

$$\mathtt{B}_F := \{N \in \mathrm{SMod}_R(M) \mid F \subseteq N\},$$

where F runs among the finite subsets of M. Under this topology, $\mathrm{SMod}_R(M)$ is a spectral space [**12,** Proposition 2.1], and the order induced by the topology is exactly the inverse of the containment order. In particular, the supermum of a subset $\mathscr{X} \subseteq \mathrm{SMod}_R(M)$ is exactly the intersection of the elements of $\mathscr{X}$. Therefore, Lemma 4.6 translates immediately to the following.

Proposition 5.1. *Let $\mathscr{X} \subseteq \mathrm{SMod}_R(M)$ be a subset that is closed in the constructible topology, in particular, $\mathscr{X}$ is a spectral space. Then, the map*

$$\begin{aligned} \Sigma \colon \boldsymbol{\mathcal{X}}(\mathscr{X}) &\longrightarrow \mathrm{SMod}_R(M) \\ \Delta &\longmapsto \bigcap \{N \mid N \in \Delta\} \end{aligned}$$

is well defined, continuous, spectral and open onto its image.

Now, let D be an integral domain. An *overring* of D is an integral domain contained between D and its quotient field K; the collection of all overrings of D is denoted $\mathrm{Overr}(D)$. Under the Zariski topology, this space is closed in the constructible topology of $\mathrm{SMod}_D(K)$ (this essentially follows from [**36**]); in particular, $\mathrm{Overr}(D)$ is a spectral space, and a subbase for the open sets of $\mathrm{Overr}(D)$ is formed by the sets in the form

$$\mathtt{O}_F := \{B \in \mathrm{Overr}(D) \mid B \supseteq F\},$$

where F runs among the finite subsets of K.

A distinguished subset of $\mathrm{Overr}(D)$ is the *Riemann-Zariski space* of D, i.e., the space $\mathrm{Zar}(D)$ of all of the valuation overrings of D. Then, $\mathrm{Zar}(D)$ is a closed set in the constructible topology of $\mathrm{Overr}(D)$ (and thus, of $\mathrm{SMod}_D(K)$), and in particular, it is a spectral space.

Proposition 5.2 (ii) can also be directly proved using the same methods as those used to show that $\mathrm{Overr}(D)$ is a spectral space, see [**10,** Propositions 3.5, 3.6].

Proposition 5.2. *Let D be an integral domain, and let $\mathscr{X} := \mathrm{Overr}_{\mathrm{ic}}(D) \subseteq \mathrm{Overr}(D)$ be the space of overrings of D that are integrally closed.*

(i) *$\mathscr{X}$ is a topological quotient of $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$.*

(ii) *$\mathscr{X}$ is closed in the constructible topology of* $\mathrm{Overr}(D)$*, in particular, it is a spectral space.*

(iii) *If D is a Prüfer domain, then* $\mathrm{Overr}(D)$ *is homeomorphic to* $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$.

Proof.

(i) Consider the map

$$\begin{aligned} \lambda\colon \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D)) &\longrightarrow \mathrm{SMod}_D(K) \\ Y &\longmapsto \bigcap\{V \mid V \in Y\}. \end{aligned}$$

From Proposition 5.1, λ is well defined, continuous, spectral and open onto its image. Moreover, the image of λ is exactly $\mathscr{X}$: indeed, any intersection of valuation domains is an integrally closed ring, while, if $T \in \mathscr{X}$, then $T = \lambda(\mathrm{Zar}(T))$, and $\mathrm{Zar}(T)$ is an inverse-closed subset of $\mathrm{Overr}(D)$ (since it is quasi-compact and closed under generizations).

Therefore, $\mathscr{X}$ is a topological quotient of $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$ since the map $\lambda : \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D)) \to \mathscr{X}$ is open, continuous and surjective.

(ii) $\mathscr{X}$ is closed in the constructible topology of $\mathrm{SMod}_D(K)$ and of $\mathrm{Overr}(D)$ since it is the image of the spectral map λ.

(iii) Assume that D is a Prüfer domain. We claim that λ establishes a homeomorphism between $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$ and $\mathrm{Overr}(D)$. Indeed, since D is Prüfer, every overring of D is integrally closed [**20,** Theorem 26.2], and thus, the image of λ is exactly $\mathrm{Overr}(D)$.

Now, let $C_1, C_2 \in \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$ be such that $R := \cap\{V \mid V \in C_1\} = \cap\{V \mid V \in C_2\}$. Since R is itself a Prüfer domain (as an overring of a Prüfer domain), it is vacant, and thus, by [**11,** Corollary 4.16], C_1 and C_2 are dense subspaces of $\mathrm{Zar}(R)$ with respect to the inverse topology of $\mathrm{Zar}(R)$. Keeping in mind that $C_1, C_2 \in \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$, it immediately follows that $C_1 = C_2 = \mathrm{Zar}(R)$. This proves that λ is injective. Therefore, in the present situation, $\lambda : \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D)) \to \mathrm{Overr}(D)$ is bijective, continuous and open, and, thus, it is a homeomorphism. $\square$

5.2. Spaces of semistar operations. Let D be an integral domain, and let K be the quotient field of D. Let $\overline{\boldsymbol{F}}(D)$ be the set of D-submodules of K. A *semistar operation* on D is a map

$$\star : \overline{\boldsymbol{F}}(D) \longrightarrow \overline{\boldsymbol{F}}(D), \qquad I \longmapsto I^\star,$$

such that, for every $I, J \in \overline{\boldsymbol{F}}(D)$, we have

($\star_1$) $I \subseteq I^\star$;

($\star_2$) if $I \subseteq J$, then $I^\star \subseteq J^\star$;

($\star_3$) $(I^\star)^\star = I^\star$;

($\star_4$) $xI^\star = (xI)^\star$ for every $x \in K$.

For the basic properties of star, semistar and closure operations, the reader is referred to [**1, 2, 8, 9, 20, 22, 23, 32**].

In [**15**], a natural topology, called the *Zariski topology*, on the space $\mathrm{SStar}(D)$ of all the semistar operations on D, was defined by declaring as a subbasis of open sets the collection of all of the sets of the type

$$\mathtt{U}_F := \{\star \in \mathrm{SStar}(D) \mid 1 \in F^\star\},$$

where F runs among nonzero D-submodules of K.

A semistar operation $\star$ is *stable* if $(I \cap J)^\star = I^\star \cap J^\star$ for all nonzero D-submodules I and J of K, and is of *finite type* if

$$I^\star = \bigcup\{J^\star \mid J \subseteq I, J \text{ finitely generated over } D\}$$

for every nonzero D-submodule I of K. By [**15,** Corollary 4.4], a semistar operation $\star$ is simultaneously stable and of finite type if and only if there is a quasi-compact subset $Y \subseteq \mathrm{Spec}(D)$ such that $\star = \mathrm{s}_Y$, where s_Y is defined as

$$I^{\mathrm{s}_Y} := \bigcap\{ID_P \mid P \in Y\}.$$

We denote by $\widetilde{\mathrm{SStar}}(D)$ the set of all stable semistar operations of finite type. It is quite simple to show that the set $\widetilde{\mathrm{SStar}}(D)$, endowed with the subspace topology induced by that of $\mathrm{SStar}(D)$, has basic open sets of the type

$$\widetilde{\mathtt{U}}_J := \{\star \in \widetilde{\mathrm{SStar}}(D) \mid 1 \in J^\star\},$$

as J ranges among the nonzero finitely generated ideal of D (see [**13,** Proposition 4.1(1)]). Under this topology, $\widetilde{\mathrm{SStar}}(D)$ is a spectral space [**13,** Theorem 4.6] that can be thought of as a natural "extension" of $\mathrm{Spec}(D)$ since the canonical map $\mathrm{s}\colon \mathrm{Spec}(D) \to \widetilde{\mathrm{SStar}}(D)$, defined by $P \mapsto \mathrm{s}_{\{P\}}$, is a topological embedding. The construction $\boldsymbol{\mathcal{X}}$ introduces a new way to represent $\widetilde{\mathrm{SStar}}(D)$.

Proposition 5.3. *Let D be an integral domain.*

(i) *The map $\mathbf{s}^\sharp\colon \boldsymbol{\mathcal{X}}(D) \to \widetilde{\mathrm{SStar}}(D)$, defined by $Y \mapsto \mathrm{s}_Y$, and the map $\Delta\colon \widetilde{\mathrm{SStar}}(D) \to \boldsymbol{\mathcal{X}}(D)$, defined by $\star \mapsto \mathrm{QSpec}^\star(D) := \{P \in \mathrm{Spec}(D) \mid P^\star \cap D = P\}$, are homeomorphisms and are inverses of each other.*

(ii) *If $\varphi : \mathrm{Spec}(D) \to \boldsymbol{\mathcal{X}}(D)$ is defined by $P \mapsto \{P\}^{\mathrm{gen}}$ and s: $\mathrm{Spec}(D) \to \widetilde{\mathrm{SStar}}(D)$ is defined by $P \mapsto \mathrm{s}_{\{P\}}$, then $\mathbf{s}^\sharp \circ \varphi = \mathrm{s}$.*

Proof.

(i) The fact that $\mathbf{s}^\sharp$ and Δ are well defined and bijective follows from [**15,** Corollaries 4.4, 5.2, Proposition 5.1].

Let $\widetilde{\mathtt{U}}_J$ be a subbasic open set of $\widetilde{\mathrm{SStar}}(D)$, where J is a nonzero finitely generated ideal of D. Then, $\mathbf{s}^\sharp(Y) \in \widetilde{\mathtt{U}}_J$ if and only if

$1 \in J^{s_Y}$, that is, if and only if $Y \subseteq \mathtt{D}(J)$. Thus, by definition, ${\mathbf{s}^\sharp}^{-1}(\widetilde{\mathtt{U}}_J) = \boldsymbol{\mathcal{U}}(\mathtt{D}(J))$ is open.

Conversely, a subbasic open set of $\boldsymbol{\mathcal{X}}(D)$ has the form $\boldsymbol{\mathcal{U}}(\mathtt{D}(J))$ for some nonzero finitely generated ideal J. As above, $\mathbf{s}^\sharp(\boldsymbol{\mathcal{U}}(\mathtt{D}(J))) = \mathbf{s}^\sharp({\mathbf{s}^\sharp}^{-1}(\widetilde{\mathtt{U}}_J)) = \widetilde{\mathtt{U}}_J$, so that $\mathbf{s}^\sharp$ is open. Hence, $\mathbf{s}^\sharp$ is a homeomorphism. The bijective map $\Delta : \widetilde{\mathrm{SStar}}(D) \to \boldsymbol{\mathcal{X}}(D)$ is also a homeomorphism since it is the inverse map of $\mathbf{s}^\sharp$ which is, in particular, continuous and open.

(ii) This follows from the fact that $s_{\{P\}}$ and $s_{\{P\}^{\mathrm{gen}}}$ coincide for each prime ideal P of D. □

Corollary 5.4. *Let D be an integral domain. Then, $\widetilde{\mathrm{SStar}}(D)$ is a spectral space.*

Proof. Immediate from Propositions 3.4 and 5.3. □

Corollary 5.5. *Let D_1 and D_2 be two integral domains. Then, $\mathrm{Spec}(D_1)$ and $\mathrm{Spec}(D_2)$ are homeomorphic if and only if so are $\widetilde{\mathrm{SStar}}(D_1)$ and $\widetilde{\mathrm{SStar}}(D_2)$.*

Proof. From Proposition 5.3, $\widetilde{\mathrm{SStar}}(D_i) \simeq \boldsymbol{\mathcal{X}}(\mathrm{Spec}(D_i))$ for $i = 1, 2$. The claim now follows from Proposition 4.5 (iii). □

Using classical terminology which originated with Krull [**29**] (and later was adjusted by Gilmer [**20**]), a semistar operation $\star$ on D is called *endlich arithmetisch brauchbar* (eab) if, given finitely generated nonzero D-submodules F, G, H of K, then $(FG)^\star \subseteq (FH)^\star$ implies $G^\star \subseteq H^\star$. Note that any $Y \subseteq \mathrm{Zar}(D)$ induces a semistar operation $\wedge_Y$ on D, defined by $E^{\wedge_Y} := \bigcap\{EV \mid V \in Y\}$, for each $E \in \overline{\boldsymbol{F}}(D)$, and a semistar operation of type $\wedge_Y$ is eab [**18,** Proposition 7]. We denote by $\mathrm{SStar}_{f,\mathrm{eab}}(D)$ the set of all semistar operations that are at the same time eab and of finite type, endowed with the subspace Zariski topology induced by that of $\mathrm{SStar}(D)$.

Theorem 5.6. *Let D be an integral domain. Then, the map*

$$\epsilon : \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D)) \longrightarrow \mathrm{SStar}_{f,\mathrm{eab}}(D),$$

defined by $\epsilon(Y) := \wedge_Y$ for each $Y \in \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$, is a homeomorphism.

Proof. Let K be the quotient field of D, let V be a valuation overring of D and let $\mathfrak{m}_V$ be the maximal ideal of V. Then, the localization $V(T) := V[T]_{\mathfrak{m}_V[T]}$ of the polynomial ring $V[T]$ is a valuation domain of $K(T)$, called the *trivial extension* of V to $K(T)$ [**20,** Proposition 18.7]. For any nonempty subspace Y of $\mathrm{Zar}(D)$, consider the following subring of $K(T)$:

$$\mathrm{Kr}(Y) := \bigcap \{V(T) \mid V \in Y\}$$

[**24, 25**]. In particular, set $R := \mathrm{Kr}(\mathrm{Zar}(D))$. Then, R (like $\mathrm{Kr}(Y)$) is a Bézout domain with quotient field $K(T)$ [**17,** Theorems 3.11(3), 5.1], such that $\mathrm{Zar}(R)$ consists of the trivial extensions of the valuation domains in $\mathrm{Zar}(D)$ [**11,** Propositions 3.2 (2,5), 3.3, Corollary 3.6(2)]. In particular, $\mathrm{Zar}(R)$ is homeomorphic to $\mathrm{Zar}(D)$, and thus, by Proposition 4.5 (i), $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(R)) \simeq \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$. From Proposition 5.2, the map

$$\lambda : \boldsymbol{\mathcal{X}}(\mathrm{Zar}(R)) \longrightarrow \mathrm{Overr}(R),$$

defined by $\lambda(Z) := \cap\{V \mid V \in Z\}$, for each $Z \in \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$, is a homeomorphism. Therefore, every overring of R is in the form $\mathrm{Kr}(Y)$ for a unique closed set $Y \subseteq \mathrm{Zar}(D)$, with respect to the inverse topology, and thus the claim will follow if we prove that the map $\epsilon_0 \colon \mathrm{Overr}(R) \to \mathrm{SStar}_{f,\mathrm{eab}}(D)$, defined by setting $\epsilon_0(\mathrm{Kr}(Y)) := \wedge_Y$, for each $Y \in \boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$, is a homeomorphism.

By [**11,** Corollary 4.17], ϵ_0 is clearly well defined; it is also injective by [**17,** Remark 3.5(b)]. If, now, $\star \in \mathrm{SStar}_{f,\mathrm{eab}}(D)$, there must be a quasi-compact subspace Y of $\mathrm{Zar}(R)$ such that $\star = \wedge_Y$ [**11,** Theorem 4.13], and thus, $\star = \epsilon_0(\mathrm{Kr}(Y^{\mathrm{gen}}))$. Hence, ϵ_0 is bijective.

In order to show that ϵ_0 is continuous, take a nonzero finitely-generated fractional ideal $F = (f_0, f_1, \dots, f_n)D$ of D, and let $\mathtt{V}_F = \mathtt{U}_F \cap \mathrm{SStar}_{f,\mathrm{eab}}(D)$. By [**17,** Corollary 3.4(3), Theorem 3.11(2)], we have $F^{\wedge_Y} = f\,\mathrm{Kr}(Y) \cap K$, where f is the polynomial $f_0 + f_1T + \cdots + f_nT^n$; therefore,

$$\begin{aligned}\epsilon_0^{-1}(\mathtt{V}_F) &= \{A \in \mathrm{Overr}(R) \mid 1 \in FA \cap K\} \\ &= \{A \in \mathrm{Overr}(R) \mid f^{-1} \in \mathrm{Kr}(Y)\} = \mathtt{O}_{f^{-1}},\end{aligned}$$

which is, by definition, an open set of $\mathrm{Overr}(R)$.

Now, let $\mathtt{O}_G$ be a subbasic open set of $\mathrm{Overr}(R)$, where G is a nonzero finite subset of $K(T)$. Since R is a Bézout domain, $GR = \gamma R$ for some $\gamma \in K(T)$; therefore, $\mathtt{O}_G = \mathtt{O}_\gamma$. Let $\alpha, \beta \in K[T]$ be two nonzero polynomials such that $\gamma = \alpha/\beta$. Then,

$$\begin{aligned}\epsilon_0(\mathtt{O}_{\alpha/\beta}) &= \{\wedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid \alpha/\beta \in \mathrm{Kr}(Y)\}\\ &= \{\wedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid \alpha \in \beta\,\mathrm{Kr}(Y)\}.\end{aligned}$$

With the same reasoning as above, there are $b_0, b_1, \ldots, b_m,\ a_0, a_1, \ldots, a_k \in K$ such that

$$\beta\,\mathrm{Kr}(Y) = (b_0, b_1, \ldots, b_m)\,\mathrm{Kr}(Y)$$

and

$$\alpha\,\mathrm{Kr}(Y) = (a_0, a_1, \ldots, a_k)\,\mathrm{Kr}(Y);$$

therefore,

$$\begin{aligned}\epsilon_0(\mathtt{O}_{\alpha/\beta}) &= \Big\{ \bigwedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid a_0, a_1, \ldots, a_n \in \beta\,\mathrm{Kr}(Y) \Big\}\\ &= \Big\{ \bigwedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid a_0, a_1, \ldots, a_n \in (b_0, \ldots, b_m)\,\mathrm{Kr}(Y) \cap K \Big\}\\ &= \Big\{ \bigwedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid a_0, a_1, \ldots, a_n \in (b_0, b_1, \ldots, b_m)^{\wedge_Y} \Big\}\\ &= \bigcap_{i=0}^{n} \Big\{ \bigwedge_Y \in \mathrm{SStar}_{f,\mathrm{eab}}(R) \mid a_i \in (b_0, b_1, \ldots, b_m)^{\wedge_Y} \Big\}.\end{aligned}$$

The sets in the last line are each equal to $\mathtt{V}_{a_i^{-1}(b_0\ldots,b_m)}$, in particular, $\epsilon_0(\mathtt{O}_{\alpha/\beta})$ is an intersection of a finite number of open sets, and thus, is open. It follows that ϵ_0 is open, and thus, a homeomorphism, as claimed. $\square$

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University of Technology, Institute of Analysis and Number Theory, Graz, Steyrergasse 30/II, 8010 Graz, Austria
Email address: **finocchiaro@math.tugraz.at, carmelo@math.unipd.it**

Università degli Studi "Roma Tre," Dipartimento di Matematica e Fisica, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy
Email address: **fontana@mat.uniroma3.it**

Università degli Studi "Roma Tre," Dipartimento di Matematica e Fisica, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy
Email address: **spirito@mat.uniroma3.it**