ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 4, 2018

THE TRACIAL ROKHLIN PROPERTY FOR ACTIONS OF AMENABLE GROUPS ON C^*-ALGEBRAS

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ABSTRACT. In this paper, we present a definition of the tracial Rokhlin property for (cocyclic) actions of countable discrete amenable groups on simple C^*-algebras, which generalize Matui and Sato's definition. We show that generic examples, like Bernoulli shift on the tensor product of copies of the Jiang-Su algebra, has the weak tracial Rokhlin property, while it is shown in [**8**] that such an action does not have finite Rokhlin dimension. We further show that forming a crossed product from actions with the tracial Rokhlin property preserves the class of C^*-algebras with real rank 0, stable rank 1 and has strict comparison for projections, generalizing the structural results in [**23**]. We use the same idea of the proof with significant simplification. In another joint paper with Chris Phillips and Joav Orovitz, we shall show that pureness and $\mathcal{Z}$-stability could be preserved by crossed product of actions with the weak tracial Rokhlin property. The combination of these results yields an application to the classification program, which is discussed in the aforementioned paper. These results indicate that we have the correct definition of tracial Rokhlin property for actions of general countable discrete amenable groups.

1. Preliminaries and notation. The tracial Rokhlin property for finite group actions on simple C^*-algebras was introduced in [**24**] for studying the structure of the crossed product. It is much more flexible than the Rokhlin property, but still produces good structural theorems, [**6, 24**]. It should be viewed as the C^*-version of outness that has the closest relationship with outness of actions on von Neumann algebras, while the latter has been well developed [**4, 9, 20**]. The tracial Rokhlin property for actions of $\mathbb{Z}$ has been studied by many authors [**13, 14, 16, 22, 23**]. Matui and Sato gave a definition of the tracial Rokhlin property for actions of discrete amenable groups [**18, 19**]. They studied both the structure of the crossed product

2010 AMS *Mathematics subject classification.* Primary 46L55.

Keywords and phrases. Tracial Rokhlin property, amenable group, C^*-algebra.

Received by the editors on July 17, 2015, and in revised form on February 2, 2016.

DOI:10.1216/RMJ-2018-48-4-1307

and classification of actions. However, their definition is (at least formally) stricter than the standard definition for finite group actions or $\mathbb{Z}$ actions, and their results work only for a special class of amenable groups.

Let A be a C^*-algebra in the following. For $a, b \in A$, we denote by $[a, b]$ the commutator $ab - ba$. For $\varepsilon > 0$, we write $a =_\varepsilon b$ for $\|a - b\| < \varepsilon$. For $B \subset A$, we write $a \in_\varepsilon B$ if there is some $b \in B$ such that $a =_\varepsilon b$. If h is a real function, then h_+ is the function defined by $h_+(t) = \mathrm{Max}\{0, h(t)\}$. If $a \in A$ is self-adjoint, then $a_+ = \iota_+(a)$, where ι is the identity function. The set of tracial states on A is denoted by $\mathrm{T}(A)$. For $a \in A$ and $\tau \in \mathrm{T}(A)$, we define

$$\|a\|_{2,\tau} = \|\tau(a^*a)^{1/2}\|, \qquad \|a\|_2 = \sup_{\tau \in \mathrm{T}(A)} \|a\|_{2,\tau}.$$

If $\mathrm{T}(A)$ is non-empty, then $\|\cdot\|_2$ is a semi-norm. For $\tau \in T(A)$, we let π_τ, H_τ denote the GNS representation of A associated with τ. The dimension function d_τ associated with τ is given by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}),$$

for a positive element $a \in A$. The term $\mathrm{V}(A)$ denotes the Murray-von Neumann semigroup and $\mathrm{W}(A)$ denotes the Cuntz semigroup. (See [**3,** Section 2] for an introduction to the Cuntz semigroup). The space of states on $\mathrm{W}(A)$ is denoted by $\mathrm{DF}(A)$, where DF stands for dimension functions. For any $\tau \in \mathrm{T}(A)$, d_τ give rise to lower semicontinuous dimension functions on A. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Define

$$\begin{aligned} c_\infty(A) &= \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\}, \\ A^\infty &= \ell^\infty(\mathbb{N}, A)/c_\infty(A); \\ c_\omega(A) &= \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|a_n\| = 0\}, \\ A^\omega &= \ell^\infty(\mathbb{N}, A)/c_\omega(A). \end{aligned}$$

Identify A with the subalgebra of A^∞ (A^ω) consisting of constant sequences. Let

$$A_\infty = A^\infty \cap A', \qquad A_\omega = A^\omega \cap A',$$

and call them the central sequence algebras of A. For a sequence $x =$

$(x_i)_{i\in\mathbb{N}}$, define $\|x\|_{2,\omega} = \lim_{n\to\omega} \|x_n\|_2$ for a seminorm in A^ω. Let

$$J_A = \{x \in A^\omega \mid \|x\|_{2,\omega} = 0\}. \tag{1.1}$$

Then J_A is a well defined, two-sided closed ideal in A^ω. The cardinality of a set F is written as $|F|$.

Definition 1.1. Let G be a countable discrete group.

(1) For a finite subset $K \in G$ and $\varepsilon > 0$, we say that a finite subset $T \subset G$ is (K, ε)-invariant if

$$\left| T \cap \bigcap_{g\in F} gT \right| \geq (1-\varepsilon)|T|.$$

(2) Group G is amenable if, for any finite subsets $K \in G$ and $\varepsilon > 0$, there exists a (K, ε)-invariant finite subset $T \in G$.

Let G be any discrete group. We write $\mathrm{Act}_G(A)$ to be the set of all actions $\alpha\colon G \to \mathrm{Aut}(A)$.

When α is an automorphism or an action of A, we can consider its natural extensions on A^ω and A_ω. We shall denote it by the same symbol α. For $\alpha \in \mathrm{Aut}(A)$, we let

$$\mathrm{T}^\alpha(A) = \{\tau \in \mathrm{T}(A) \mid \tau \circ \alpha = \tau\}. \tag{1.2}$$

Definition 1.2. Let A be a unital C^*-algebra, and let G be a discrete group.

(1) A pair (α, u) of a map $\alpha\colon G \to \mathrm{Aut}(A)$ and a map $u\colon G \times G \to U(A)$ is called a *cocyclic action* of G on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in G$. We always assume that $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in G$. Note that α gives rise to a genuine action of G on A_ω.

(2) A cocyclic action (α, u) is said to be *outer* if α_g is outer for every $g \in G$ except for the identity element.

(3) Two cocyclic actions $(\alpha, u)\colon G \curvearrowright A$ and $(\beta, v)\colon G \curvearrowright B$ are said to be *cocyclic conjugate* if there exist a family of unitaries $(w_g)_{g\in G}$ in B and an isomorphism $\theta\colon A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \operatorname{Ad} w_g \circ \beta_g \tag{1.3}$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^* \tag{1.4}$$

for every $g, h \in G$.

Definition 1.3. Let $(\alpha, u)\colon G \curvearrowright A$ be a cocyclic action of a discrete group G on a unital C^*-algebra A. The (full) twisted crossed product $A \rtimes_{\alpha,u} G$ is the universal C^*-algebra generated by A and a family of unitaries $(\lambda_g^\alpha)_{g\in G}$ satisfying

$$\lambda_g^\alpha \lambda_h^\alpha = u(g,h)\lambda_{gh}^\alpha \quad \text{and} \quad \lambda_g^\alpha a (\lambda_g^\alpha)^* = \alpha_g(a) \tag{1.5}$$

for all $g, h \in G$ and $a \in A$.

If two cocyclic actions $(\alpha, u)\colon G \curvearrowright A$ and $(\beta, v)\colon G \curvearrowright B$ are cocyclic conjugate, then $A \rtimes_{\alpha,u} G$ and $B \rtimes_{\beta,v} G$ are canonically isomorphic.

We introduce the following comparison for the convenience of studying the tracial Rokhlin property.

Definition 1.4. Let $f \in (A^\omega)_+$ and a be an element of A_+. We say f is pointwisely Cuntz subequivalent to a and write $f \precsim_{\text{p.w.}} a$ if f has a representative $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$ such that each f_n is positive and $f_n \precsim a$ in A for all $n \in \mathbb{N}$.

2. Equivalent definitions of the tracial Rokhlin property.

Throughout this paper, we let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be some fixed free ultrafilter. We shall also assume that the groups acting on C^*-algebras are countable, discrete and amenable.

Definition 2.1. Let A be a simple unital C^*-algebra. Let $(\alpha, u) : G \curvearrowright A$ be a cocyclic action. We say that α has the tracial Rokhlin property if, for any finite subset K of G, any $\varepsilon > 0$, and any $z \in A_+ \setminus \{0\}$, there exist (K, ε)-invariant finite subsets $T_1, T_2, \ldots, T_n$ and projections $\{e_i \mid 1 \leq i \leq n\} \subset A_\omega$ such that

(1) $\alpha_g(e_i)\alpha_h(e_j) = 0$ for any $g \in T_i, h \in T_j$ such that $g \neq h$ or $i \neq j$.
(2) With

$$e = \sum_{\substack{g \in T_i \\ 1 \leq i \leq n}} \alpha_g(e_i),$$

$1 - e \precsim_{\text{p.w.}} z$. (See Definition 1.4.)

If the e_i are weakened to a positive contraction, then we say that α has the weak tracial Rokhlin property.

Alternatively, the (weak) tracial Rokhlin property can be defined in terms of the original C^*-algebra A with approximate relations.

Proposition 2.2. *Let A be a simple separable unital C^*-algebra. Let $(\alpha, u)\colon G \curvearrowright A$ be a cocyclic action. Then, (α, u) has the weak tracial Rokhlin property if, and only if, for any finite subset K of G, any $\varepsilon_0 > 0$, and any non-zero positive element $z \in A$, there is a (K, ε_0)-invariant subset $T_1, \ldots, T_n$ of G such that, for any finite subset F of A, any $\varepsilon_1 > 0$, there exist mutually orthogonal positive contractions $\{e_{g,i}\}_{g \in T_i, 1 \leq i \leq n}$ with the following properties:*

(1) *$\|[e_{g,i}, f]\| < \varepsilon_1$ for any $g \in T_i$ and any $f \in F$.*
(2) *$\|\alpha_{hg^{-1}}(e_{g,i}) - \lambda_{hg^{-1},g} e_{h,i} \lambda^*_{hg^{-1},g}\| < \varepsilon_1$ for any g and h in T_i.*
(3) *With*

$$e = \sum_{\substack{g \in T_i \\ 1 \leq i \leq n}} e_{g,i},$$

we have $1 - e \precsim z$.

Furthermore, if α has the tracial Rokhlin property, then the positive contractions e_g may always be chosen to be non-zero projections.

Remark 2.3. Proposition 2.2 shows that the definition of the (weak) tracial Rokhlin property is independent of the choice of the free ultrafilter. In addition, A_∞ can be used instead of A_ω in the definition. For most of the results and proofs of this paper, it does not matter which one is used. However, one advantage of using a free ultrafilter instead of the sequence algebra is that, if (M,τ) is a tracial von Neumann algebra, then $M^\omega = \ell^\infty(M)/c_{\omega,\tau}(M)$ is again a von Neumann algebra, where

$$c_{\omega,\tau}(M) = \{x \in M \mid \lim_{n\to\omega} \tau(xx^*)^{1/2} = 0\}.$$

The analogue algebra $M^\infty = \ell^\infty(M)/c_{0,\tau}(M)$ is not a von Neumann algebra. We will make use of this fact in the proof of Proposition 3.7.

When the C^*-algebra has strict comparison, the Cuntz comparison is equivalent to comparison by traces, which yields:

Proposition 2.4. *Let A be a simple unital C^*-algebra with strict comparison. Let $(\alpha,u)\colon G \curvearrowright A$ be a cocyclic action. Then, (α,u) has the weak tracial Rokhlin property if, and only if, for any finite subset K of G, any $\varepsilon_0,\varepsilon_1 > 0$, there exist (K,ε_0)-invariant subsets $T_1,\dots,T_n$ and positive contractions $e_1,\dots,e_n \in A_\omega$, such that*

(1) *$\alpha_g(e_i)\alpha_h(e_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.*
(2) *Let*

$$e = \sum_{\substack{g\in T_i\\ 1\le i\le n}} (\alpha_g(e_i)).$$

There is a representative $(e^{(n)})_{n\in\mathbb{N}}$ of e such that

$$\lim_{n\to\omega} \max_{\tau\in \mathrm{T}(A)} d_\tau(1-e^{(n)}) < \varepsilon_1. \tag{2.1}$$

In the case of the tracial Rokhlin property, positive contraction may be replaced by non-zero projection in the above statement.

Proposition 2.4 leads to the next definition:

Definition 2.5. Let $(\alpha,u)\colon G \curvearrowright A$ be a cocyclic action. Let $S \subset \mathrm{T}(A)$. We say that (α,u) has the (weak) tracial Rokhlin property with respect to S if, for any finite subset K of G, any $\varepsilon_0,\varepsilon_1 > 0$, there exist (K,ε_0)-invariant subsets $T_1,\dots,T_n$ and projections (positive contractions) $e_1,\dots,e_n \in A_\omega$, such that

(1) $\alpha_g(e_i)\alpha_h(e_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.

(2) Let

$$e = \sum_{\substack{g\in T_i\\ 1\le i\le n}} (\alpha_g(e_i)).$$

There is a representative $(e^{(n)})_{n\in\mathbb{N}}$ of e such that

$$\lim_{n\to\omega} \max_{\tau\in S} d_\tau(1-e^{(n)}) < \varepsilon_1. \tag{2.2}$$

In the following, we shall show that, if A has tracial rank zero, then the weak tracial Rokhlin property actually implies the tracial Rokhlin property. The case $G = \mathbb{Z}$ was proven by Phillips and Osaka [**22,** Theorem 2.14] and [**25,** Proposition 1.3]). We need the next lemma before proving it.

Lemma 2.6. *Let A be a C^*-algebra, and let B be a finite-dimensional subalgebra. Let $\{e_{ij}^l\}$ be the standard matrix units of B. Then, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, whenever a projection $p \in A$ satisfies $\|[p, e_{ij}^l]\| < \delta$ for all i, j, l, there is a projection q in the relative commutant $A \cap B'$ such that $\|p - q\| < \varepsilon$.*

Proof. Fix some $\varepsilon > 0$. Choose δ_0 according to $\varepsilon/2$ as in [**15,** Lemma 2.5.10]. Choose δ_1 according to δ_0 as in [**15,** Theorem 2.5.9]. (It is easy to see that this lemma generalizes to finite-dimensional C^*-algebras.) Set $\delta = \delta_1/2$. Let $p \in A$ be a projection satisfying $\|[p, e_{ij}^l]\| < \delta$. Identify $C = pAp \oplus (1-p)A(1-p)$ as a subalgebra of A. Let

$$a_{ij}^l = pe_{i,j}^l p + (1-p)e_{i,j}^l(1-p) \in C. \tag{2.3}$$

Then, $\|a_{i,j}^l - e_{i,j}^l\| < \delta_1$. Hence, by [**15,** Theorem 2.5.9], there are matrix units $\{f_{i,j}^l\} \subset C$ such that $\|f_{i,j}^l - e_{i,j}^l\| < \delta_0$. By [**15,** Lemma 2.5.10], there is a unitary $u \in A$ such that $uf_{i,j}^l u^* = e_{i,j}^l$ and $\|u - 1\| < \varepsilon/2$. Now, let $q = upu^*$. Then $\|q - p\| < \varepsilon$. We shall show that q commutes with B by showing that q commutes with each $e_{i,j}^l$. Since $\{f_{i,j}^l\} \subset C$, we have $f_{i,j}^l = (1-p)f_{i,j}^l(1-p) + pf_{i,j}^l p$ for any i, j, l. Hence,

$$qe_{i,j}^l = upf_{i,j}^l u^* = upf_{i,j}^l pu^* = uf_{i,j}^l pu^* = e_{i,j}^l q. \tag{2.4}$$

□

Theorem 2.7. *Let $(\alpha, u)\colon G \curvearrowright A$ be a cocyclic action with the weak tracial Rokhlin property. If A is a simple C^*-algebra tracial rank 0, then (α, u) indeed has the tracial Rokhlin property.*

Proof. If A has tracial rank 0, then A is tracially approximately divisible [**17,** Theorem 5.4]. If we define tracial $\mathcal{Z}$-absorption [**7,** Definition 2.1] using finite-dimensional C^*-algebras whose simple component has arbitrarily large size instead of matrix algebras, tracial approximate divisibility will imply tracial $\mathcal{Z}$-absorption. This implies that [**7,** Theorem 3.3] will still hold, using essentially the same proof. Hence, A has strict comparison. (It turns out that, in the simple case, the aforementioned definition of tracial $\mathcal{Z}$-absorbing coincides with [**7,** Definition 2.1], although we do not need it.) Let K be a finite subset of G, and let $\varepsilon_0 > 0$ be given. By Proposition 2.4, there exist (K, ε_0)-invariant subsets $T_1, \ldots, T_n$ and positive contractions $e_1, \ldots, e_n \in A_\omega$, such that

(1) $\alpha_g(e_i)\alpha_h(e_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.
(2) Let
$$e = \sum_{\substack{g\in T_i \\ 1\le i\le n}} (\alpha_g(e_i)).$$
There is a representative $(e^{(n)})_{n\in\mathbb{N}}$ of e such that
$$\lim_{n\to\omega} \max_{\tau\in \mathrm{T}(A)} d_\tau(1 - e^{(n)}) < \varepsilon_1/3. \tag{2.5}$$

The rest of the proof amounts to perturbation of the e_is to projections with the desired properties. Let F be a finite subset of A. Let $\eta > 0$ be arbitrary. Set $M = |T_1| + \cdots + |T_n|$. Choose δ such that
$$\delta = \varepsilon_1/3(M + 1). \tag{2.6}$$

Since A has tracial rank 0, there is a finite-dimensional subalgebra $B \subset A$ with $1_B = p$, such that

(1) $\|[p, a]\| < \eta$ for any $a \in F$.
(2) $pap \in_\eta B$.
(3) $\tau(1 - p) < \delta$ for any $\tau \in \mathrm{T}(A)$.

Consider $C = A^\omega \cap B' \supset A_\omega$. The sequence algebra of a real rank 0 C^*-algebra is again real rank 0. As a consequence of Lemma 2.6 we obtain $C = (A \cap B')^\omega$. In particular, the C^*-algebras C and $\{\overline{pe_iCe_ip}\}_{1\le i\le n}$

have real rank 0. Note here that we regard $p \in A$ as constant sequence in A^ω, which commutes with each e_i. Choose a projection $q_i \in \overline{pe_iCe_ip}$ such that $\|q_ie_iq_i - pe_ip\| < \delta$. We first claim that $\alpha_g(q_i)\alpha_h(q_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$. In order to prove this, since $q_i \in \overline{pe_iCe_ip}$ for any $\gamma > 0$, we can find $d_i \in C$ such that $\|q_i - pe_id_ie_ip\| < \gamma$. Hence,

$$\begin{aligned}\|\alpha_g(q_i)\alpha_h(q_j)\| &\leq \|\alpha_g(q_i - pe_id_ie_ip)\alpha_h(q_j)\| \\ &\quad + \|\alpha_g(pe_id_ie_ip)\alpha_h(q_j - pe_jd_je_jp)\| \\ &\quad + \|\alpha_g(pe_id_ie_ip)\alpha_h(pe_jd_je_jp)\| \\ &\leq \gamma + (1+\gamma)\gamma + 0 = (2+\gamma)\gamma.\end{aligned}$$

Since γ is arbitrary, we have $\alpha_g(q_i)\alpha_h(q_j) = 0$. Let

$$q = \sum_{\substack{g \in T_i \\ 1 \leq i \leq n}} \alpha_g(q_i).$$

Let $(q^{(m)})_{m\in\mathbb{N}}$ be a representative of q such that each $q^{(m)}$ is a projection. We can estimate:

$$\begin{aligned}&\lim_{m\to\omega} \max_{\tau \in \mathrm{T}(A)} \{\tau(1 - q^{(m)})\} \\ &\leq \lim_{m\to\omega} \max_{\tau \in \mathrm{T}(A)} \{\tau(1 - q^{(m)}e^{(m)}q^{(m)})\} \\ &\leq \lim_{m\to\omega} \max_{\tau \in \mathrm{T}(A)} \{\tau(1 - pe^{(m)}p) + \|pe^{(m)}p - q^{(m)}e^{(m)}q^{(m)}\|\} \\ &\leq \lim_{m\to\omega} \max_{\tau \in \mathrm{T}(A)} \{d_\tau(1 - e^{(m)}) + \tau(1-p) + \|pep - qeq\|\} \\ &\leq \frac{\varepsilon_1}{3} + \delta + M\delta < \varepsilon_1.\end{aligned}$$

Finally, for any $a \in F$, find $b \in B$ such that $\|pap - b\| < \eta$. Then,

$$\begin{aligned}\|q_ia - aq_i\| &= \|pq_ipa - apq_ip\| \\ &\leq \|pq_ipa - pq_ipap\| \\ &\quad + \|pq_ipap - papq_ip\| + \|papq_ip - apq_ip\| \\ &\leq \|pq_ib - bq_ip\| + 2\eta + \eta + \eta = 4\eta.\end{aligned}$$

Now we choose an increasing sequence of finite subsets $\{F_k\}$ whose union is dense in A. Letting $\eta = 1/k$, we can obtain a sequence of projections $\{q_{i,k}\}_{1\leq i\leq n, k\in\mathbb{N}}$ in A^ω satisfying:

(1) $\|q_{i,k}a - aq_{i,k}\| \leq 4/k$ for any $a \in F_k$.
(2) $\alpha_g(q_{i,k})\alpha_h(q_{j,k}) = 0$ for any $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.
(3) Let

$$q_k = \sum_{\substack{g\in T_i\\ 1\leq i\leq n}} \alpha_g(q_{i,k}).$$

There is a representative $(q_k^{(m)})_{m\in\mathbb{N}}$ of q_k, such that

$$\lim_{m\to\omega} \max_{\tau\in \mathrm{T}(A)} \tau(1 - q_k^{(m)}) < \varepsilon_1. \tag{2.7}$$

We can then use Cantor's diagonal argument to select projections p_i in A_ω satisfying the conditions in Proposition 2.4; therefore, α has the tracial Rokhlin property. □

3. Examples of actions with the weak tracial Rokhlin property.

Definition 3.1. Let G be a discrete group, let A and B be unital C^*-algebras and let $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ be actions of G on A and B. We say that B admits an approximate equivariant central unital homomorphism from A if there is a sequence of unital completely positive maps $\phi_i\colon A \to B$ such that, for any $a, a_1 \in A$ and $b \in B$, we have

(1) $\lim_{i\to\infty} \phi_i(a)\phi_i(a_1) - \phi_i(aa_1) = 0$.
(2) $\lim_{i\to\infty} \phi_i(a)b - b\phi_i(a) = 0$.
(3) $\lim_{i\to\infty} \phi_i(\alpha_g(a)) - \beta_g(\phi_i(a)) = 0$.

It is immediate from the definition that an approximate equivariant central unital homomorphism from A to B induces an equivariant unital homomorphism from A to the central sequence algebra B_ω.

Theorem 3.2. *Let $\alpha \in \mathrm{Act}_G(A)$, where G is amenable. Let X be a compact metrizable space with a Borel probability measure μ. Let $\beta\colon G \curvearrowright (X, \mu)$ be a free and measure-preserving action which is also a topological action (i.e., it acts on X by homeomorphisms). It induces an action on $C(X)$. Let τ be a tracial state on A. Suppose that there* *are approximate equivariant central unital homomorphisms*

$$\iota_i\colon C(X) \longrightarrow A$$

with μ_i the measure induced by $\tau \circ \iota_i$. If μ is the ω-limit of $(\mu_i)_{i\in\mathbb{N}}$, then α has the weak tracial Rokhlin property with respect to τ. Furthermore, if X is totally disconnected, then α has the tracial Rokhlin property.

Proof. Let $K \in G$ be a finite subset and let $\varepsilon_0, \varepsilon_1 > 0$. Since $\beta\colon G \curvearrowright X$ is a free and measure preserving action, by **[21,** page 59, Theorem 5, remark after proof], there exist (K, ε_0)-invariant subsets $T_1, T_2, \ldots, T_n$ and measurable subsets $B_1, \ldots, B_n$ such that:

(1) gB_i and hB_j are disjoint for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.

(2)
$$\mu\left(X \setminus \bigcup_{\substack{g\in T_i\\ 1\le i\le n}} gB_i\right) < \varepsilon_1.$$

Any finite measure on a compact metrizable space is regular. Without loss of generality, we may assume that each B_i is compact.

Now, we shall construct open sets $U_i \supset B_i$ such that $g\overline{U_i}$ and $h\overline{U_j}$ are disjoint for $g \in T_i$ and $h \in T_j$ with $g \neq h$ or $i \neq j$. Since X is a normal topological space, for any i and any $g \in T_i$, we can inductively find an open set $V_{g,i} \supset gB_i$ such that $\overline{V_{g,i}}$ and $\overline{V_{h,j}}$ are disjoint for any $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$. For each i, define

$$U_i = \bigcap_{g\in T_i} g^{-1}V_{g,i}. \tag{3.1}$$

It is easy to see from our construction that U_i satisfies the requirement previously mentioned. Furthermore, if X is totally disconnected and compact, we may choose the $V_{g,i}$s to be clopen sets. It is clear from the construction that the U_is are clopen sets as well.

Repeating the above argument and replacing B_i by $\overline{U_i}$, we can obtain open sets $W_i \supset \overline{U_i}$ such that $g\overline{W_i}$ and $h\overline{W_j}$ are disjoint for any $g \in T_i, h \in T_j$ such that $(g,i) \neq (h,j)$. Now, by Urysohn's lemma, we can find the continuous function

$$f_i \colon X \longrightarrow [0,1],$$

which is 1 on $\overline{U_i}$ and 0 outside W_i. If X is totally disconnected, we let f_i be the characteristic function on the clopen set W_i. Let $e_i = (\iota_k(f_i))_{k\in\mathbb{N}}$ for $1 \leq i \leq n$. Now, we can see that $\{e_i\}_{1\leq i\leq n}$ are positive contractions (or projections, if X is totally disconnected) in A_ω such that

(1) $\alpha_g(e_i)$ and $\alpha_h(e_j)$ are disjoint for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.

(2) With

$$f = \sum_{\substack{g\in T_i \\ 1\leq i\leq n}} \alpha_g(f_i) \quad \text{and} \quad e^{(k)} = \iota_k(f),$$

we see that $(e^{(k)})_{k\in\mathbb{N}}$ is a representative of

$$e = \sum_{\substack{g\in T_i \\ 1\leq i\leq n}} e_i$$

such that

$$\begin{aligned}\lim_{k\to\omega} d_\tau(1-e^{(k)}) = \lim_{k\to\omega} d_{\mu_i}(1-f) &= \lim_{k\to\omega} \mu_i(\{x\in X \mid 1-f(x)\neq 0\}) \\ &\leq \lim_{k\to\omega} \mu_i\Bigg(X \setminus \bigcup_{\substack{g\in T_i \\ 1\leq i\leq n}} gU_i\Bigg) \\ &\leq \mu\Bigg(X \setminus \bigcup_{\substack{g\in T_i \\ 1\leq i\leq n}} gU_i\Bigg) < \varepsilon_1.\end{aligned}$$

The second equality follows from [**1,** Proposition I.2.1]. By Proposition 2.4, the action α has the weak tracial Rokhlin property with respect to τ and has the tracial Rokhlin property with respect to τ if X is further assumed to be totally disconnected. $\square$

Let A be a separable unital C^*-algebra and G a countable discrete group. Let $\otimes_G A$ be the minimal tensor product of countably many copies of A indexed by the elements of G. The left multiplication of G on itself induces an action on $\otimes_G A$ (permuting the indices), which we shall call the *Bernoulli shift* on $\otimes_G A$. We can generalize [**25,** Corollary] to actions of amenable groups.

Proposition 3.3. *Let A be a unital C^*-algebra. Let $\tau \in \mathrm{T}(A)$ be such that, with π_τ the associated* GNS *representation, the von Neumann algebra $\pi_\tau(A)''$ has no minimal projections. When G is infinite and amenable the Bernoulli shift on $\otimes_G A$ has the weak tracial Rokhlin property with respect to τ.*

Proof. By [**25,** Proposition 2.8], there is some $a \in A$ with $0 \leq a \leq 1$ such that the spectral measure μ_0 on $[0,1]$, defined by

$$\int_0^1 f \, d\mu_0 = \tau(f(a)) \quad \text{for } f \in C([0,1])$$

satisfies $\mu_0(\{t\}) = 0$ for all $t \in [0,1]$. Using functional calculus, there is a unital embedding

$$\iota \colon C([0,1]) \longrightarrow A$$

defined by $f \to f(a)$. Let

$$X = \prod_G [0,1]$$

be the product of countably many copies of $[0,1]$ indexed by elements of G. There is a natural isomorphism between $C(X)$ and $\otimes_G C([0,1])$. We use

$$f_1^{(g_1)} \otimes f_2^{(g_2)} \otimes \cdots \otimes f_n^{(g_n)}$$

to indicate the elementary tensor in $\otimes_G A$ which is f_i in the g_ith tensor factor for $1 \leq i \leq n$ and is $1 = 1_A$ in all other places. Let α be the Bernoulli shift on $\otimes_G C([0,1])$, determined by

$$\alpha_g(f_1^{(g_1)} \otimes f_2^{(g_2)} \otimes \cdots \otimes f_n^{(g_n)}) = f_1^{(gg_1)} \otimes f_2^{(gg_2)} \otimes \cdots \otimes f_n^{(gg_n)}. \tag{3.2}$$

Using the natural isomorphism between $C(X)$ and $\otimes_G C([0,1])$ and the duality between actions on $C(X)$ and actions on X, we obtain an induced action β on X, defined by

$$\beta_g(x_1^{(g_1)}, x_2^{(g_2)}, \ldots, x_n^{(g_n)}, \ldots) = (x_1^{(gg_1)}, x_2^{(gg_2)}, \ldots, x_n^{(gg_n)}, \ldots). \tag{3.3}$$

Let μ be the product measure on X induced by μ_0. It is easy to see that β is measure preserving. Now, we check that it is also free. Given

$g \in G \setminus \{1\}$, let

$$S = \{x \in X \mid \beta_g(x) = x\}.$$

We have

$$S = \{(x_h)_{h\in G} \mid x_h = x_k \text{ for all } h, k \in G\}.$$

Using Fubini's theorem along with the assumption that the single point set in $[0,1]$ has measure 0, we get $\mu(S) = 0$. Next, list the elements in G by $h_1, h_2, \dots$. Let $\phi_k\colon C(X) \to \otimes_G A$ be the *right index shift* by h_k determined by

$$\begin{aligned} f_1^{(g_1)} \otimes f_2^{(g_2)} \otimes \cdots \otimes f_n^{(g_n)} &\longrightarrow f_1(a)^{(g_1 h_k)} \\ &\otimes f_2(a)^{(g_2 h_k)} \otimes \cdots \otimes f_n(a)^{(g_n h_k)}. \end{aligned} \tag{3.4}$$

We can see that $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of equivariant unital homomorphisms. We now check that it is approximately central. Let $f \in C(X)$ and $b \in \otimes_G(A)$. Without loss of generality, we may assume that f, b are elementary tensors:

$$\begin{aligned} f &= f_1^{(g_1)} \otimes f_2^{(g_2)} \otimes \cdots \otimes f_n^{(g_n)}, \\ b &= b_1^{(h_1)} \otimes b_2^{(h_2)} \otimes \cdots \otimes b_n^{(h_n)}. \end{aligned} \tag{3.5}$$

There are only finitely many $g \in G$ such that $g_i g = h_j$ for some $1 \le i \le j \le n$; hence, $\lim_{k\to\infty} \phi_k(a)b - b\phi_k(a) = 0$. By Theorem 3.2, the action α has the weak tracial Rokhlin property with respect to τ. $\square$

In particular, for the Jiang-Su algebra $\mathcal{Z}$, there is a central embedding of $C([0,1])$ such that the unique trace τ on $\mathcal{Z}$ induces the Lebesgue measure on $[0,1]$. Hence, we have:

Corollary 3.4. *If G is countable, discrete and amenable, then the Bernoulli shift on $\otimes_G \mathcal{Z} \cong \mathcal{Z}$ has the weak tracial Rokhlin property.*

A cocyclic action $(\alpha, u)\colon G \curvearrowright A$ is called *strongly outer*, if and only if, for any $g \neq 1$ and any $\tau \in \mathrm{T}^{\alpha_g}(A)$, the weak extension of α_g on $\pi_\tau(A)''$ is not weakly inner.

Proposition 3.5. *Let G be a countable discrete amenable group, let A be a unital simple infinite dimensional C^*-algebra, let $(\alpha, u)\colon G \curvearrowright A$ be an action with the weak tracial Rokhlin property. Suppose that the* *tracial state space* $\mathrm{T}(A)$ *has finitely many extreme points. Then* α *is strongly outer.*

Proof. Let $1 \neq g \in G$ be given, and let τ be an α_g-invariant trace. Let $E\colon A \rtimes_{\alpha_g} \mathbb{Z} \to A$ be the conditional expectation determined by $E(a_n\lambda_g^n) = a_0$, where $a_n \in A$ and λ_g is the canonical unitary in $A \rtimes_{\alpha_g} \mathbb{Z}$ implementing the action. We will show that, for any trace $\Phi \in \mathrm{T}(A \rtimes_{\alpha_g} \mathbb{Z})$, we have $\Phi(a\lambda_g) = 0$. If this is done, then the proof of [**13,** Lemma 4.4] shows that α_g is not weakly inner.

For any $\tau \in \mathrm{T}(A)$ and $x = (x_n)_{n\in\mathbb{N}}$, let $\tau_\omega(x) = \lim_{n\to\omega} \tau(x_n)$, which is a trace on A_ω. Let $\varepsilon > 0$ be arbitrary. Since α has the weak tracial Rokhlin property, by Proposition 2.4, we can find a $(\{g\}, \varepsilon)$-invariant subsets $T_1, \ldots, T_n$ of G, and positive contractions $e_1, \ldots, e_n$ in A_∞, such that:

(1) $\alpha_g(e_i)\alpha_h(e_j) = 0$ for $h \in T_i$ and $k \in T_j$ such that $h \neq k$ or $i \neq j$.
(2) Let

$$e = \sum_{\substack{h\in T_i\\ 1\leq i\leq n}} (\alpha_h(e_i)).$$

There is a representative $(e^{(n)})_{n\in\mathbb{N}}$ of e such that

$$\lim_{n\to\omega} \max_{\tau\in\mathrm{T}(A)} d_\tau(1 - e^{(n)}) < \varepsilon. \tag{3.6}$$

Now let $\tau_1, \tau_2, \ldots, \tau_k$ be the extreme tracial states of A. Identify $a\lambda_g$ with the constant sequence in $(A \rtimes_{\alpha_g} \mathbb{Z})^\infty$, without loss of generality, assume $\|a\| = 1$. We have

(3.7a)
$$|\Phi_\omega(a\lambda_g)| \leq \left|\Phi_\omega\Big(\sum_{h\in T_i,i} \alpha_h(e_i)a\lambda_g\Big)\right| + \left|\Phi_\omega\Big(\Big(1 - \sum_{h\in T_i,i} \alpha_h(e_i)\Big)a\lambda_g\Big)\right|$$

$$\leq \left|\Phi_\omega\Big(\sum_{h\in T_i\cap gT_i,i} \alpha_h(e_i)a\lambda_g\Big)\right| + \left|\Phi_\omega\Big(\sum_{h\in T_i\setminus gT_i,i} \alpha_h(e_i)a\lambda_g\Big)\right| \tag{3.7b}$$

$$+ \Phi_\omega\Big(1 - \sum_{h\in T_i,i} \alpha_h(e_i)\Big)\|a\lambda_g\| \tag{3.7c}$$

$$\leq \left|\Phi_\omega\Big(\sum_{h\in T_i\cap gT_i,i} \alpha_h(e_i^{1/2})a\lambda_g\alpha_{g^{-1}h}(e_i^{1/2})\Big)\right| \tag{3.7d}$$

$$+\sum_{h\in T_i\setminus gT_i,i}\Phi_\omega(\alpha_h(e_i)))\|a\lambda_g\|+\varepsilon \tag{3.7e}$$

$$\leq 0+\left|\sum_{1\leq j\leq k}\sum_{h\in T_i\setminus gT_i,i}\tau_{j,\omega}(\alpha_h(e_i))\right|+\varepsilon \tag{3.7f}$$

$$\leq \sum_{1\leq j\leq k}\sum_{h\in T_i,i}\frac{|T_i\setminus gT_i|}{|T_i|}\tau_{j,\omega}((\alpha_h(e_i))+\varepsilon \tag{3.7g}$$

$$\leq \varepsilon\sum_{1\leq j\leq k}\tau_{j,\omega}\bigg(\sum_{h\in T_i,i}\alpha_h(e_i)\bigg)+\varepsilon\leq (k+1)\varepsilon. \tag{3.7h}$$

The estimation in (3.7g) used the fact that

$$\sum_{1\leq j\leq k}\tau_j((\alpha_h(a))$$

is independent of h, since $\tau\to\tau\circ\alpha_h$ permutes the set of extreme tracial states. Since ε is arbitrary, this shows that $\Phi_\omega(a\lambda_g)=0$, and therefore, $\Phi(a\lambda_g)=0$. □

Corollary 3.6. *Let $\alpha\in \mathrm{Act}_G(A)$ be an action with the weak tracial Rokhlin property. Then, the canonical embedding*

$$A\longrightarrow A\rtimes_\alpha G$$

induces a bijection between $\mathrm{T}^\alpha(A)$ and $\mathrm{T}(A\rtimes_\alpha G)$.

Proof. Let r be the map from $\mathrm{T}(A\rtimes_\alpha G)$ to $\mathrm{T}^\alpha(A)$ induced by the canonical embedding $A\to A\rtimes_\alpha G$. Let s be the map from $\mathrm{T}^\alpha(A)$ to $\mathrm{T}(A\rtimes_\alpha G)$, defined by

$$s(\tau)\Big(\sum a_g\lambda_g\Big)=\tau(a_1)\quad\text{for all }\tau\in\mathrm{T}^\alpha(A). \tag{3.8}$$

It is easy to check that $r\circ s$ is the identity map. In order to prove that $s\circ r$ is the identity map, it suffices to show that, for any trace Φ in $\mathrm{T}(A\rtimes_\alpha G)$, $g\neq 1$, we have $\Phi(a\lambda_g)=0$. We repeat the same argument as in Proposition 3.5 except for the last three inequalities (3.7f), (3.7g) and (3.7h). Note that Φ is now a trace on $\mathrm{T}(A\rtimes_\alpha G)$, not merely a trace on $\mathrm{T}(A\rtimes_{\alpha_g}\mathbb{Z})$; we dropped the assumption that A has finitely many extremal tracial states. Let $\tau=r(\Phi)\in\mathrm{T}^\alpha(A)$, and, adopting the same notation as in Proposition 3.5 yields the following estimation:

$$\begin{aligned}\sum_{h\in T_i\setminus gT_i,i}\Phi_\omega(\alpha_h(e_i))) &= \sum_{h\in T_i\setminus gT_i,i}\tau_\omega(\alpha_h(e_i))\\ &= \sum_{h\in T_i,i}\frac{|T_i\setminus gT_i|}{|T_i|}\tau_\omega(\alpha_h(e_i))+\varepsilon\\ &\leq \varepsilon\tau_\omega\left(\sum_{h\in T_i,i}\alpha_h(e_i)\right)\leq\varepsilon.\end{aligned}$$

Hence, $\Phi_\omega(a\lambda_g)<2\varepsilon$. Since ε is arbitrary, we have $\Phi(a\lambda_g)=0$. □

We can now reestablish a Rokhlin-type lemma for outer actions on the hyperfinite II_1 factor R (as was discussed in **[20,** Chapter 6] for actions of more general von Neumann algebras. The formulation is slightly different; our projections are exactly permuted by the action but do not sum up exactly to 1). Let $p_\omega\colon R^\omega\to R^\omega/J_R$, where J_R is the trace-kernel defined in Section 1.

Proposition 3.7. *Let G be a countable, discrete and amenable group. Let R be the hyperfinite II_1 factor. Let $\alpha\colon G\curvearrowright R$ be any outer action. Then, for any finite set $K\Subset G$ and $\varepsilon,\varepsilon_1>0$, there exist (K,ε)-invariant sets $T_1,\dots,T_n$ in G and projections $p_1,\dots,p_n\in R^\omega/J_R\cap p_\omega(R)'$ such that*

(1) *$\alpha_g(p_i)\alpha_h(p_j)=0$ for $g\in T_i$ and $h\in T_j$ such that $g\neq h$ or $i\neq j$.*
(2)

$$\tau_\omega\left(1-\sum_{\substack{g\in T_i\\1\leq i\leq n}}\alpha_g(p_i)\right)<\varepsilon_1.$$

Proof. Any two outer actions on the hyperfinite II_1 are cocyclic conjugate **[20,** Theorem 1.4]; hence, we need only check one of them. Let $\mathcal{Z}$ be the Jiang-Su algebra. Let α be the Bernoulli shift on $\otimes_G\mathcal{Z}\cong\mathcal{Z}$. Let τ be the unique tracial state on $\mathcal{Z}$. Then $\pi_\tau(\mathcal{Z})''$ is the hyperfinite II_1 factor R. The induced action on R, still denoted α, is outer by Corollary 3.4 and Proposition 3.5. Let $K\Subset G$ be any finite set. Let $\varepsilon,\varepsilon_1>0$. Since α has the weak tracial Rokhlin property, there exist (K,ε)-invariant sets $T_1,\dots,T_n$ in G and positive contractions $e_1,\dots,e_n\in\mathcal{Z}_\omega$ such that

(1) $\alpha_g(e_i)\alpha_h(e_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$.
(2) Let

$$e = \sum_{\substack{g \in T_i \\ 1 \leq i \leq n}} (\alpha_g(e_i)).$$

There is a representative $(e^{(n)})_{n\in\mathbb{N}}$ of e such that

$$\lim_{n\to\omega}\{d_\tau(1 - e^{(n)})\} < \varepsilon_1. \tag{3.9}$$

We can lift $\{\alpha_g(e_i)\}_{g\in T_i,i}$ to mutually orthogonal positive contractions $\{e_{g,i} = (e^{(n)}_{g,i})_{g\in T_i,i}\}$ using semiprojectivity of direct sums of $C_0((0,1])$. Let

$$\widetilde{e}^{(n)} = \sum_{g\in T_i,i} (e^{(n)}_{g,i}),$$

and set $\delta_n = \|\widetilde{e}^{(n)} - e^{(n)}\|$. Then, $\lim_{n\to\omega}\delta_n = 0$. Let

$$h_{\delta_n}(x) = 1 - \frac{1}{1-\delta_n}(1 - x - \delta_n)_+ \quad \text{for all } x \in [0,1]. \tag{3.10}$$

Note that $h_{\delta_n} \in C_0((0,1])$ tends to the identity function as $n \to \omega$. Let $f^{(n)}_{g,i} = h_{\delta_n}(e^{(n)}_{g,i})$, and set $f_{g,i} = (f^{(n)}_{g,i})_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$. Then, $f_{g,i}$ is a representative of $\alpha_g(e_i)$. Let

$$f^{(n)} = \sum_{g\in T_i,i} (f^{(n)}_{g,i}).$$

We have

$$1 - f^{(n)} = 1 - h_{\delta_n}(\widetilde{e}^{(n)}) \approx (1 - \widetilde{e}^{(n)} - \delta_n)_+ \precsim 1 - e^{(n)}. \tag{3.11}$$

The algebra R^ω/J_R is again a von Neumann algebra. For each i, let p_i be the support projection of $p_\omega(e_i) \in R^\omega/J_R$. Since multiplication is strongly continuous on bounded sets and $\mathcal{Z}$ is strongly dense in R, we have $p_i \in R^\omega/J_R \cap p_\omega(R)'$ and $\alpha_g(p_i)\alpha_h(p_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $g \neq h$ or $i \neq j$. Let $\widetilde{p}^{(n)}_{g,i}$ be the support projection of $f^{(n)}_{g,i}$, and set $\widetilde{p}_{g,i} = (\widetilde{p}^{(n)}_{g,i})_{n\in\mathbb{N}}$. Since p_ω is strongly continuous, we see that $\widetilde{p}_{g,i}$ is a lift of $\alpha_g(p_i)$. Let

$$p^{(n)} = \sum_{g\in T_i,i} (p^{(n)}_{g,i}).$$

Since $f^{(n)}p^{(n)} = f^{(n)}$, an easy calculation shows that $(1 - f^{(n)})(1 - p^{(n)}) = 1 - p^{(n)}$. If we let $q^{(n)}$ be the support projection of $1 - f^{(n)}$, then $1 - p^{(n)} \leq q^{(n)}$. Using the fact that $d_\tau(1 - f^{(n)}) = \tau(q^{(n)})$, we have

$$\lim_{n\to\omega} \tau(1 - p^{(n)}) \leq \lim_{n\to\omega} d_\tau(1 - f^{(n)}) < \varepsilon_1. \tag{3.12}$$

□

Theorem 3.8. *Let A be a unital, simple, separable, infinite dimensional C^*-algebra with finitely many extremal tracial states. Suppose that A is either nuclear or has tracial rank 0. Let G be a countable discrete amenable group. For a cocyclic action (α, u) of G on A, it is strongly outer if and only if it has the weak tracial Rokhlin property.*

Proof. If A is either nuclear or has tracial rank 0, then every trace τ is uniformly amenable, see [**2,** Definition 3.2.1, Theorem 4.2.1, Proposition 4.5]. The von Neumann algebra $\phi_\tau(A)''$ is hyperfinite by [**2,** Theorem 3.2.2]. Since A is unital, simple and infinite-dimensional, $\phi_\tau(A)''$ is the hyperfinite II_1 factor. Now we see that the proof of [**19,** Theorem 3.7] may be generalized to actions of discrete amenable groups. The only change needed is to replace property (Q) by the property in Proposition 3.7 and accordingly modify the estimations. □

Another type of example comes from product-type actions. We begin with the next definition.

Definition 3.9. Let

$$A = \bigotimes_{i=1}^{\infty} \mathrm{B}(H_i),$$

where H_i is a finite-dimensional Hilbert space for each i. An action $\alpha \in \mathrm{Act}_G(A)$ is called a *product-type action* if and only if, for each i, there exists a unitary representation $\pi_i\colon G \to \mathrm{B}(H_i)$, which induces an inner action $\alpha_i\colon g \mapsto \mathrm{Ad}(\pi_i(g))$, such that

$$\alpha = \bigotimes_{i=1}^{\infty} \alpha_i.$$

Definition 3.10. Let $\alpha \in \mathrm{Act}_G(A)$ be a product-type action on a UHF-algebra A. A *telescope* of the action is a choice of an infinite sequence of positive integers $1 = n_1 < n_2 < \cdots$ and a reexpression of the action such that

$$A = \bigotimes_{i=1}^{\infty} B(T_i)$$

where

$$T_i = \bigotimes_{j=n_i}^{n_{i+1}-1} H_j,$$

and the action on $B(T_i)$ is

$$\bigotimes_{j=n_i}^{n_{i+1}-1} \alpha_j.$$

Theorem 3.11. *Let $\alpha \in \mathrm{Act}_G(A)$ be a product-type action where G is countable, discrete and amenable. Let H_i, π_i and α_i be defined as in Definition* 3.9*. Let d_i be the dimension of H_i and χ_i the character of π_i. We will use the same notation if we perform a telescope to the action. Define*

$$\chi \colon G \longmapsto \mathbb{C}$$

to be the characteristic function on 1_G. Then, the action α has the tracial Rokhlin property if and only if there exists a telescope such that, for any $n \in \mathbb{N}$, the infinite product

$$\prod_{n \le i < \infty} \frac{1}{d_i} \chi_i = \chi. \tag{3.13}$$

Proof. Any UHF algebra has tracial rank 0 and is monotracial. By Theorem 3.8, that α has the tracial Rokhlin property is equivalent to that α is strongly outer. In this case, α has the tracial Rokhlin property if and only if $\alpha|_H$ has tracial Rokhlin property for any cyclic subgroup $H \subset G$. Let $\chi_{H,i}$ be the restriction of χ_i to the subgroup H, which is exactly the character of the restricted action $\pi_i|_H$. We observe that

$$\prod_{n \le i < \infty} \frac{1}{d_i} \chi_i = \chi$$

if and only if

$$\prod_{n\leq i<\infty} \frac{1}{d_i}\chi_{H,i} = \chi \quad \text{for all cyclic subgroups } H \subset G. \tag{3.14}$$

Hence, the theorem is proven if we can show that it is true for any cyclic group G. If G is finite, then it is proven in [**29**]. If G is infinite, Let x be a generator, and let U_i be the unitary in $B(H_i)$ such that $\pi_i(x) = \operatorname{Ad} U_i$. Let $S_{k,l}$ be a sequence consisting of eigenvalues of $\otimes_{i=k}^{l} U_i$, repeated as often as multiplicity is indicated. Kishimoto has shown that, in the case of an infinite cyclic group acting on UHF algebra, the tracial Rokhlin property coincides with the Rokhlin property [**12,** Theorem 1.3]. He also showed [**12,** Lemma 5.2] that the product-type action α has the Rokhlin property if and only $\{S_{k,l}\}_{l=k}^{\infty}$ is uniformly distributed for any $k \in \mathbb{N}$. Now fix some $k \in \mathbb{N}$. For any sequence $S = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\mathbb{T}$, we let μ_S be the measure on $\mathbb{T}$ such that $\mu_S = (1/n)\sum_i \delta_{\lambda_i}$, where δ_{λ_i} is the Dirac measure concentrated at the point $\lambda_i \in \mathbb{T}$. By definition, $\{S_{k,l}\}_{l=k}^{\infty}$ is uniformly distributed if and only if

$$\lim_{l\to\infty} \mu_{S_{k,l}}(f) = \int_{\mathbb{T}} f\, d\mu \quad \text{for all } f \in C(\mathbb{T}), \tag{3.15}$$

where μ is the normalized Haar measure. Now it is not difficult to see that

$$\prod_{k\leq i<l} \frac{1}{d_i}\chi_i(n) = \mu(S_{k,l})(z^n) \quad \text{for all } n \in \mathbb{Z}, \tag{3.16}$$

where $z^n \in C(\mathbb{T})$ stands for the function $z \to z^n$. Hence,

$$\prod_{k\leq i<\infty} \frac{1}{d_i}\chi_i = \chi \tag{3.17}$$

is equivalent to

$$\lim_{l\to\infty} \mu_{S_{k,l}}(z^n) = \delta(n,0) = \int_{\mathbb{T}} z^n\, d\mu \quad \text{for all } n \in \mathbb{Z}, \tag{3.18}$$

and therefore, further equivalent to $\{S_{k,l}\}_{l=k}^{\infty}$ being uniformly distributed, since any continuous function in $C(\mathbb{T})$ can be uniformly approximated by finite linear combinations of the functions z^n. $\square$

Another example is derived from actions on non-commutative tori. Let θ be a nondegenerate anti-symmetric bicharacter on $\mathbb{Z}^d$. We identify it with its matrix under the canonical basis of $\mathbb{Z}^d$. Then, the associated non-commutative tori A_θ is a simple, unital $\mathbf{A}\mathbb{T}$ algebra with a unique trace. A_θ is generated by unitaries $\{U_x \mid x \in \mathbb{Z}^d\}$, subject to the relation

$$U_y U_x = \exp(\pi i \langle x, \theta y\rangle) U_{x+y} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{3.19}$$

For any $T \in \mathrm{M}_d(\mathbb{Z})$, the map

$$U_x \longrightarrow U_{Tx}$$

gives rise to an endomorphism α_T of A_θ if and only if $(T^t\theta T - \theta)/2 \in \mathrm{M}_d(\mathbb{Z})$ (This relation is automatically satisfied for $d = 2$.) It is an automorphism if and only if T is invertible. Let

$$G_\theta = \{T \in GL_n(\mathbb{Z}) \mid \tfrac{1}{2}(T^t\theta T - \theta) \in \mathrm{M}_d(\mathbb{Z})\}. \tag{3.20}$$

Proposition 3.12. *Let θ be a non-degenerate anti-symmetric bicharacter on $\mathbb{Z}^d$. Let G be any amenable subgroup of G_θ. Then, the action $\alpha \in \mathrm{Act}_G(A_\theta)$, defined by $T \to \alpha_T$, is strongly outer, and hence has the tracial Rokhlin property.*

Proof. Let τ be the unique trace state on A_θ. By [**5,** Lemma 5.10], for each $T \in G \setminus \{e\}$, the automorphism α_T is not weakly inner. Hence, α is strongly outer. □

If we can find one example of actions with (weak) tracial Rokhlin property, we can actually find many by forming inner tensors. More specifically, we have the following:

Proposition 3.13. *Let $\alpha \in \mathrm{Act}_G(A)$ be an action with the weak tracial Rokhlin property, and let $\beta \in \mathrm{Act}_G(B)$ be arbitrary, where A, B are both simple and unital. Then, the inner tensor of these two actions $\gamma = \alpha \otimes \beta \in \mathrm{Act}_G(A \otimes_{\min} B)$ has the weak tracial Rokhlin property. If α has the tracial Rokhlin property, then γ has the tracial Rokhlin property.*

Proof. Let $K \subset G$ be any finite subset and $\varepsilon > 0$ arbitrary. Since α has the weak tracial Rokhlin property, we can find (K, ε)-invariant subsets $T_1, \ldots, T_n$ of G with the property stated in the definition of weak tracial Rokhlin property. Let $x \in A \otimes_{\min} B$ be a non-zero positive element.

We first show that there is a non-zero positive element $d \in A$ such that $d \otimes 1 \precsim x$. By Kirchberg's slice lemma ([**10,** Lemma 2.7] or [**28,** Lemma 4.1.9]), there are non-zero positive elements $a \in A_+$ and $b \in B_+$ and some $z \in A \otimes_{\min} B$ such that $zz^* = a \otimes b$ and $z^*z \in \mathrm{Her}(x)$. This, in particular, shows that $a \otimes b \precsim x$. Since B is simple and unital, we can find elements $s_1, s_2, \ldots, s_n$ in B such that

$$\sum_i s_i b s_i^* = 1.$$

By [**11,** Proposition 4.10], we can find a non-zero positive contraction $d \in A$ such that $d^{\oplus n} \precsim a$. Hence,

$$\begin{aligned} d \otimes 1 &= \sum_i (1 \otimes s_i)(d \otimes b)(1 \otimes s_i)^* \\ &\precsim (d \otimes b)^{\oplus n} \sim d^{\oplus n} \otimes b \precsim a \otimes b \precsim x. \end{aligned} \tag{3.21}$$

Since α has the weak tracial Rokhlin property, there exist positive contractions $f_i \in A_\omega$ such that:

(1) $\alpha_g(f_i)\alpha_h(f_j) = 0$ for any $g \in T_i, h \in T_j$ such that $(g,i) \neq (h,j)$.
(2) With

$$e = \sum_{\substack{g \in T_i \\ 1 \le i \le n}} \alpha_g(f_i),$$

$1 - e \precsim_{\text{p.w.}} d$.

Now consider the positive contractions $f_i \otimes 1$. It is clear that $f_i \otimes 1 \in (A \otimes_{\min} B)_\omega$, and:

(1) $\gamma_g(f_i \otimes 1)\gamma_h(f_j \otimes 1) = (\alpha_g(f_i)\alpha_h(f_j)) \otimes 1 = 0$ for any $g \in T_i, h \in T_j$ such that $(g,i) \neq (h,j)$.
(2) With

$$\widetilde{e} = \sum_{\substack{g_i \in T_i \\ 1 \le i \le n}} \gamma_g(f_i \otimes 1),$$

we have

$$1 - \widetilde{e} \precsim_{\text{p.w.}} d \otimes 1 \precsim x. \tag{3.22}$$

Hence, $\gamma = \alpha \otimes \beta$ has the weak tracial Rokhlin property. If α has the tracial Rokhlin property, then we can require f_i to have non-zero projections; then, $f_i \otimes 1$ also contain projections, and the above proof shows that γ has the tracial Rokhlin property. □

Remark 3.14. Let G be any countable, discrete amenable group. It admits at least one action on $\mathcal{Z}$ with the weak tracial Rokhlin property (Corollary 3.4). We then obtain many actions with the weak tracial Rokhlin property on any $\mathcal{Z}$-stable C^*-algebra A, by Proposition 3.13. Following the same argument as in **[25]**, we can actually show that the set of actions with the weak tracial Rokhlin property is G_δ-dense in $\mathrm{Act}_G(A)$, where $\mathrm{Act}_G(A)$ is endowed with the topology of pointwise convergence. In particular, by Theorem 2.7, if A is simple with tracial rank 0, then actions with the tracial Rokhlin property form a G_δ-dense subset of $\mathrm{Act}_G(A)$.

In the next two sections, we generalize the results in **[23]**, from which we adapt our ideas.

4. The Murray–von Neumann semigroup.

For a C^*-algebra A, we let $\mathrm{V}(A)$ be the Murray-von Neumann semigroup of A. We say that $\mathrm{V}(A)$ has *strict comparison* if, for any $p, q \in V(A)$, we have that $\tau(p) < \tau(q)$ for any $\tau \in \mathrm{T}(A)$ implies $p \precsim q$. Note that such a C^*-algebra is said to satisfy Blackadar's second fundamental comparability question, which states in different literature that the order of projections is determined by traces. We say that $\mathrm{V}(A)$ is *almost divisible* if, for any $p \in \mathrm{V}(A)$ and any $n \in \mathbb{N}$, there is some $q \in \mathrm{V}(A)$ such that $nq \leq p \leq (n+1)q$. Note that, if A is simple infinite-dimensional with real rank 0, then $\mathrm{V}(A)$ is almost divisible, by **[23,** Lemma 2.3].

Lemma 4.1. *Let A be a unital simple separable C^*-algebra with property* (SP). *Suppose that* $\mathrm{V}(A)$ *has strict comparison and is almost divisible. Let $(\alpha, u)\colon G \curvearrowright A$ be a cocycle action with the tracial Rokhlin property. Then, for every finite subset $F \subset A \rtimes_{\alpha,u} G$, every $\varepsilon > 0$, and every nonzero $z \in (A \rtimes_{(\alpha,u)} G)_+$, there exist some finite subset K of G and (K, ε)-invariant subsets $T_1, T_2, \ldots, T_n$ of G, projections $f_1, \ldots, f_n \in A$ and an embedding*

$$\phi\colon \bigoplus_i M_{|T_i|} \otimes f_i A f_i \longrightarrow A \rtimes_{\alpha,u} G,$$

whose image shall be called D, such that

(1) *there is a* $g_i \in T_i$ *for each* i *such that* $\phi(e^{(i)}_{g_i,g_i} \otimes a) = \alpha_{g_i}(a)$ *for any* $a \in f_iAf_i$.
(2) $\phi(e_{g,g} \otimes f_i) \in A$ *for any* $g \in T_i$ *and* $1 \leq i \leq n$.
(3) $\|\phi(e^{(i)}_{g,h} \otimes a) - \lambda_g a \lambda_h^*\| \leq \varepsilon\|a\|$ *for any* $g, h \in T_i$ *and* $a \in f_iAf_i$.
(4) *Let*

$$\widetilde{T}_i = \bigcap_{g\in K} gT_i \cap T_i.$$

Let

$$p = \sum_{\substack{g\in\widetilde{T}_i \\ 1\leq i\leq n}} \phi(e^{(i)}_{g,g}).$$

We have

$$pb \subset_\varepsilon D \quad \text{and} \quad bp \subset_\varepsilon D \quad \text{for any } b \in F. \tag{4.1}$$

(5) *With* p *defined as in* (4), $1 - p \precsim z$.

Proof. We first choose two nonzero orthogonal positive elements $z_0, z_1 \in A_+$ such that $z_0 \oplus z_1 \precsim z$ according to [**7,** Lemma 5.1]. Since A has property (SP), we may assume that z_0 and z_1 are projections. Let $\eta = \mathrm{Min}_{\tau\in T(A)} \tau(z_0) > 0$. Let $\varepsilon_0 = \mathrm{Min}\{(\eta/2), \varepsilon\}$. Without loss of generality, assume that there is a symmetric finite set $K \subset G$ such that elements of F are all of the form

$$\sum_{g\in K} a_g\lambda_g,$$

where a_g are elements of A and λ_g are the canonical unitaries implementing the action. By Definition 2.1, we can find (K, ε_0)-invariant subsets $T_1, T_2, \ldots, T_n$ of G and central projections $q_i \in A_\infty$, such that

(1) $\alpha_g(q_i)\alpha_h(q_j) = 0$ for $g \in T_i$ and $h \in T_j$ such that $(g, i) \neq (h, j)$.
(2) $1 - \sum_{g\in T} \alpha_g(q) \precsim_{\text{p.w.}} z_1$.

For $1 \leq i \leq n$, let $\{e^{(i)}_{g,h}\}$ be the standard matrix units of $M_{|T_i|}$. By the universal property of finite dimensional C^*-algebras, there is an embedding

$$\psi\colon \bigoplus_{1\leq i\leq n} M_{|T_i|} \longrightarrow (A \rtimes_{\alpha,u} G)^\infty \tag{4.2}$$

such that $\psi(e^{(i)}_{g,h}) = \lambda_g q_i \lambda_h^*$. Using semiprojectivity of M, we can lift ψ to a sequence of embeddings

$$\psi_k\colon \bigoplus_i M_{|T_i|} \longrightarrow (A \rtimes_{\alpha,u} G).$$

We may further assume that $\psi_k(e^{(i)}_{g,g}) \in A$ for $g \in T_i$ by the standard perturbation argument, see [**15,** Lemma 2.5.7].

Now, fix some $g_i \in T_i$ for each i. Let $q_{i,k} = \lambda_{g_i}^* \psi_k(e^{(i)}_{g_i,g_i})\lambda_{g_i} \in A$. We see that $(q_{i,k})_{k\in\mathbb{N}}$ is a representative of q_i. Let

$$\begin{aligned} F_0 = &\left\{\alpha_g(a_h) \mid \sum_{h\in K} a_h\lambda_h \in F, g \in \cup_i T_i\right\} \\ &\cup \{\alpha_k(u_{g,h}) \mid g,h,k \in \cup_i (T_i \cup T_i^{-1})\}. \end{aligned} \tag{4.3}$$

Let $L = \mathrm{Max}\{\|a\| \mid a \in F_0\}$. Define

$$\delta = \mathrm{Min}\left\{\frac{1}{2}, \frac{\varepsilon}{|K|(\sum_i |T_i|)(L+5)}, \frac{\varepsilon}{2}\right\} \tag{4.4}$$

We can find a large enough k such that:

(1′) letting $e^{(i)}_g = \psi_k(e^{(i)}_{g,g}) \in A$, we have $\|[e_g, a]\| < \delta$ for any $g \in T$ and any $a \in F_0$.

(2′) Letting $f_i = q_{i,k}$, we have $\|\psi_k(e^{(i)}_{g,h}) - \lambda_g f_i \lambda_h^*\| < \delta$ for any $g \in T$.

(3′) With

$$e = \sum_{\substack{g\in T_i\\ 1\le i\le k}} e^{(i)}_g,$$

we have $1 - e \precsim z_1$.

The last condition comes from the fact that if two projections are close enough, then they are unitarily equivalent.

We now define an embedding

$$\phi\colon \bigoplus_i M_{|T_i|} \otimes f_i A f_i \longrightarrow A \rtimes_{\alpha,u} G$$

by

$$\phi(e^{(i)}_{g,h} \otimes a) = \psi_k(e^{(i)}_{g,g_i})\alpha_{g_i}(a)\psi_k(e^{(i)}_{g_i,h}), \tag{4.5}$$

and extend linearly. Let $D = \phi(\oplus_i M_{|T_i|} \otimes f_i A f_i)$ be the image of ϕ. Let $\widetilde{T}_i = \cap_{g\in K} gT_i \bigcap T_i$ and

$$p = \phi\left(\sum_{\substack{g\in\widetilde{T}_i \\ 1\le i\le n}} (e^{(i)}_{g,g} \otimes f_i)\right) = \sum_{\substack{g\in\widetilde{T}_i \\ 1\le i\le n}} e^{(i)}_g. \tag{4.6}$$

We now verify the conditions required in this lemma. Conditions (1) and (2) follow from the definition.

For condition (3), we have the following estimation:

$$\begin{aligned}\phi(e^{(i)}_{g,h} \otimes a) &=_{2\delta\|a\|} \lambda_g f_i \lambda^*_{g_i} \alpha_{g_i}(a) \lambda_{g_i} f \lambda^*_h \\ &= \lambda_g f_i a f_i \lambda^*_h = \lambda_g a \lambda^*_h.\end{aligned}$$

Hence, $\|\phi(e^{(i)}_{g,h} \otimes a) - \lambda_g a \lambda^*_h\| \le 2\delta\|a\| \le \varepsilon\|a\|$. In addition, for condition (3), we write

$$1 - p = \left(1 - \sum_{g\in T_i, i} e^{(i)}_g\right) + \sum_{g\in T_i\setminus\widetilde{T}_i, i} e^{(i)}_g. \tag{4.7}$$

For $g \in T_i$, we have $\|e^{(i)}_g - \alpha_g(f_i)\| < \delta < 1$, which implies that the two projections are unitarily equivalent in A. Hence, for any α-invariant trace τ and any $g, h \in T_i$, we have $\tau(e^{(i)}_g) = \tau(\alpha_g(f_i)) = \tau(f_i) = \tau(e^{(i)}_h)$. Therefore,

$$\tau\left(\sum_{g\in T_i\setminus\widetilde{T}_i, i} e^{(i)}_g\right) = \varepsilon_0 \tau\left(\sum_{g\in T_i, i} e^{(i)}_g\right) \le \varepsilon_0 < \tau(z_0). \tag{4.8}$$

For condition (4), let

$$b = \sum_{h\in K} b_h \lambda_h \in F.$$

We have:

$$pb = \sum_{\substack{g\in\widetilde{T}_i\\1\le i\le n\\h\in K}} e_g^{(i)} b_h\lambda_h =_{\delta|K|(\sum_i|\widetilde{T}_i|)L} \sum_{\substack{g\in\widetilde{T}_i\\1\le i\le n\\h\in K}} \lambda_g f_i \lambda_g^* b_h \lambda_h$$

$$= \sum_{\substack{g\in\widetilde{T}_i\\1\le i\le n\\h\in K}} \lambda_g f_i \lambda_{g^{-1}} u(g,g^{-1}) b_h \lambda_{g^{-1}}^* u(g^{-1},h) u(g^{-1}h, hg^{-1})\lambda_{h^{-1}g}^*$$

$$=_{\delta_1} \sum_{\substack{g\in\widetilde{T}_i\\1\le i\le n\\h\in K}} \lambda_g f_i \alpha_{g^{-1}}(u(g,g^{-1})b_h) u(g^{-1},h) u(g^{-1}h, hg^{-1}) f_i \lambda_{h^{-1}g}^*$$

$$=_{\delta_2} \phi\bigg(\sum_{g,i,h} e_{g,h^{-1}g} \otimes f_i \alpha_{g^{-1}}(u(g,g^{-1})b_h) u(g^{-1},h) u(g^{-1}h, hg^{-1}) f_i\bigg),$$

where

$$\delta_1 = 4\delta|K|\bigg(\sum_i |\widetilde{T}_i|\bigg) \quad \text{and} \quad \delta_2 = 2\delta|K|\bigg(\sum_i |\widetilde{T}_i|\bigg)L.$$

This yields $pb \subset_\varepsilon D$. The proof that $bp \subset_\varepsilon D$ is similar.

By [**23,** Proposition 2.4] (although stated for real rank 0 C^*- algebra, all that is needed is for $\mathrm{V}(A)$ to be almost divisible, and the same proof works for cocyclic actions), we have

$$\sum_{g\in T_i\setminus\widetilde{T}_i, i} e_g^{(i)} \precsim z_0 \quad \text{in } A \rtimes_\alpha G.$$

Hence,

$$1 - p = \bigg(1 - \sum_{g\in T_i, i} e_g^{(i)}\bigg) + \sum_{g\in T_i\setminus\widetilde{T}_i, i} e_g^{(i)} \precsim z_1 \oplus z_0 \precsim z. \tag{4.9}$$ □

In the next theorem, we assume that A has sufficiently many projections. Any real rank 0 C^*-algebra will satisfy the requirement stated in the theorem.

Theorem 4.2. *Let A be a unital, simple, separable C^*-algebra with the property that, for any positive element $x \in M_\infty(A)$ and any $\varepsilon > 0$, there exists a projection $p \in M_\infty(A)$ such that $(x-\varepsilon)_+ \precsim p \precsim x$.*

Suppose that $\mathrm{V}(A)$ *has strict comparison and is almost divisible. Let* $(\alpha, u)\colon G \curvearrowright A$ *be an action with the tracial Rokhlin property. Then,* $\mathrm{V}(A \rtimes_{\alpha,u} G)$ *has strict comparison.*

Proof. Let r and s be projections such that $\tau(p) < \tau(q)$ for any $\tau \in \mathrm{T}(A)$. Let

$$\varepsilon = \min_{\tau \in \mathrm{T}(A)} \{\tau(r) - \tau(s)\} > 0. \tag{4.10}$$

Let $\delta = \varepsilon/3$. By Lemma 4.1, there exist a finite subset K of G and (K, δ)-invariant subsets $T_1, T_2, \ldots, T_n$ of G, projections $f_1, \ldots, f_n \in A$, and an embedding

$$\phi\colon \bigoplus_i M_{|T_i|} \otimes f_i A f_i \longrightarrow A \rtimes_\alpha G,$$

whose image is called D such that

(1) there is a $g_i \in T_i$ for each i such that $\phi(e^{(i)}_{g_i,g_i} \otimes a) = \alpha_{g_i}(a)$ for any $a \in f_i A f_i$.
(2) There is a projection $p \in D$ such that

$$pr, ps \subset_\delta D \quad \text{and} \quad rp, sp \subset_\delta D. \tag{4.11}$$

(3) Using the same p as in (2) $\tau(1-p) < \varepsilon/3$ for any $\tau \in \mathrm{T}(A)$.

Let $x \in D$ be a positive element such that $\|x - prp\| < \delta$. Since D is isomorphic to the finite direct sum of matrix algebras over A, there is a projection $\widetilde{r} \in D$ such that $(x - 2\delta)_+ \precsim \widetilde{r} \precsim (x - \delta)_+$ in D. We estimate that $\widetilde{r} \precsim (x - \delta)_+ \precsim prp \precsim r$. Let $r_0 = \widetilde{r} \oplus (1 - p)$. By **[26,** Lemma 1.8], we have

$$\begin{aligned} r \approx (r - \delta)_+ &\precsim (prp - \delta)_+ \oplus (1 - p) \\ &\precsim (x - 2\delta)_+ \oplus (1 - p) \precsim r_0. \end{aligned} \tag{4.12}$$

For any $\tau \in \mathrm{T}(A \rtimes_{\alpha,u} G)$, we have $\tau(r) > \tau(r_0) - \varepsilon/3$. Similarly, there is a projection $s_0 \in D$ such that $s_0 \precsim s$ and $\tau(s_0) > \tau(s) - \varepsilon/3$ for any $\tau \in \mathrm{T}(A \rtimes_{\alpha,u} G)$. Let

$$d = \bigoplus_i d_i, \qquad e = \bigoplus_i e_i$$

be projections in $\oplus_i M_{|T_i|} \otimes f_i A f_i$ such that $\widetilde{r} = \phi(d)$ and $s_0 = \phi(e)$. Each $M_{|T_i|} \otimes f_i A f_i$ is simple; hence, $d_i \precsim \mathrm{diag}\{e^{(i)}_{g_i,g_i} \otimes f_i, \ldots e^{(i)}_{g_i,g_i} \otimes f_i\}$

in $M_{k_i}(M_{|T_i|} \otimes f_i A f_i)$ for some $k_i \in \mathbb{N}$. Let

(4.13)
$$f = \mathrm{diag}\{e^{(1)}_{g_1,g_1} \otimes f_1, \dots e^{(1)}_{g_1,g_1} \otimes f_1, \dots, e^{(n)}_{g_n,g_n} \otimes f_n, \dots e^{(n)}_{g_n,g_n} \otimes f_n\}$$

Let
$$k = \sum_i k_i |T_i|.$$

Define
$$\iota\colon \bigoplus_i M_{|T_i|} \otimes f_i A f_i \longrightarrow M_k(A)$$

by

$$(a_1, a_2, \dots, a_n) \longmapsto \mathrm{diag}\{a_1, a_2, \dots, a_n, 0, 0, \dots\}. \tag{4.14}$$

Then, we have $\iota(d) \precsim f$ in $fM_k(A)f$. Let $\widetilde{d} \in fM_k(A)f$ be a projection such that $\iota(d) \approx \widetilde{d}$. Since $\phi(e^{(i)}_{g_i,g_i} \otimes a) = \alpha_{g_i}(a)$ for any $a \in f_i A f_i$, we can see that, for any element $a \in \oplus_i M_{|T_i|} \otimes f_i A f_i$, we have $\phi(a) \approx \iota(a)$ in $M_\infty(A \rtimes_{\alpha,u} G)$. Let $r_1 = \widetilde{d} \oplus (1-p) \in M_{k+1}(A)$. We have

$$r_1 \approx \iota(d) \oplus (1-p) \approx \phi(d) \oplus (1-p) = r_0. \tag{4.15}$$

Similarly, there is a projection $s_1 \in M_l(A)$ such that $s_1 \approx s_0$. Let τ be any α-invariant trace on A, which derives from a trace ω on $A \rtimes_{\alpha,u} G$ by Corollary 3.6. We can compute

$$\omega(s_1) - \omega(r_1) = \omega(s_0) - \omega(r_0) > \omega(s) - \varepsilon/3 - (\omega(r) + \varepsilon/3) > 0. \tag{4.16}$$

By [**23,** Proposition 2.4], we have $r_1 \precsim s_1$. Hence,

$$r \precsim r_0 \approx r_1 \precsim s_1 \approx s_0 \precsim s. \tag{4.17}$$ □

5. Real and stable rank of the crossed product. The next lemma states that any single self-adjoint element of the crossed product may be *tracially* approximated by subalgebras with real rank 0. It is weaker than the tracial approximation formulated in [**6,** Definition 2.2]; however, it is good enough to deduce that the crossed product has real rank 0, at least when we know the crossed product has strict comparison for projections.

Lemma 5.1. *Let A be a simple infinite-dimensional C^*-algebra with real rank 0 which has strict comparison for projections. Let (α, u):*

$G \curvearrowright A$ be a cocyclic action with the tracial Rokhlin property, where G is a countable discrete amenable group. Then, for any self-adjoint element $a \in A \rtimes_{\alpha,u} G$, any $\varepsilon > 0$ and any nonzero positive element $z \in A \rtimes_{\alpha,u} G$, there is a C^*-subalgebra D of $A \rtimes_{\alpha,u} G$ with real rank 0 and a projection $p \in D$ such that:

(1) $\|pa - ap\| < \varepsilon$.
(2) $pap \in_{\varepsilon} D$.
(3) $1 - p \precsim z$.

Proof. Let a, ε and z be given as in this lemma. Without loss of generality, assume that $\|a\| \leq 1$. Choose two nonzero orthogonal projections z_0 and z_1 in A such that $z_0 + z_1 \precsim z$. Let $\eta = \mathrm{Min}\{\tau(z_0) \mid \tau \in \mathrm{T}(A)\}$. In addition, let

$$\delta = \mathrm{Min}\left\{\frac{\varepsilon}{4}, \frac{1-\eta}{2}, \frac{\eta\varepsilon}{4 + (3+\eta)\varepsilon}\right\}. \tag{5.1}$$

By Lemma 4.1, there exist projections $f_i \in A$, finite subsets $\widetilde{T}_i \subset T_i \subset G$ with $|\widetilde{T}_i|/|T_i| > 1 - \delta$, an embedding

$$\phi\colon \bigoplus_i M_{|T_i|} \otimes f_i A f_i \longrightarrow A \rtimes_{\alpha} G,$$

whose image is called D, and a projection $q \in D$ such that the following hold.

(1) Letting $e_g^{(i)} = \phi(e_{g,g}^{(i)} \otimes f_i)$, for $g \in T_i$, we have $e_g^{(i)} \in A$.
(2) $q = \sum_{g \in \widetilde{T}_i, i} e_g^{(i)}$.
(3) There exist d_1 and d_2 in D such that $\|qa - d_1\| < \delta$ and $\|aq - d_2\| < \delta$.
(4) $1 - q \precsim z_1$.

We can write

$$\begin{aligned} a &= qa + (1-q)aq + (1-q)a(1-q) \\ &=_{2\delta} d_1 + (1-q)\,d_2 + (1-q)a(1-q). \end{aligned} \tag{5.2}$$

Let $d = d_1 + (1-q)\,d_2$ and $\overline{d} = (d + d^*)/2$. Then $\overline{d}$ is a self-adjoint element in D such that $\|a - (\overline{d} + (1-q)a(1-q))\| < 2\delta$. Let $c = \oplus_i c_i$ be a self-adjoint element in $\oplus_i M_{|T_i|} \otimes f_i A f_i$ such that $\overline{d} = \phi(c)$. Let N

be an integer such that $2/\varepsilon \le N \le 2/\varepsilon + 1$. By our choice of δ, we have

$$(2N+1)|T_i \setminus \widetilde{T}_i| \le (4/\varepsilon + 3)\frac{\delta}{1-\delta}|\widetilde{T}_i| \le \eta|\widetilde{T}_i|. \tag{5.3}$$

Choose a subset $S_i \subset \widetilde{T}_i$ such that $|S_i| = (2N+1)|T_i \setminus \widetilde{T}_i|$. Let

$$r_i = \sum_{g \in S_i} e_{g,g}^{(i)} \otimes f_i,$$
$$q_i = \sum_{g \in \widetilde{T}_i} e_{g,g}^{(i)} \otimes f_i$$

and

$$e_i = \sum_{g \in T_i} e_{g,g}^{(i)} \otimes f_i.$$

In addition, let $e = \phi(\oplus_i e_i)$. Note that $q = \phi(\oplus_i q_i)$. By [**23,** Lemma 4.4], there is a projection s_i in $M_{|\widetilde{T}_i|} \otimes f_i A f_i$ such that

$$e_i - q_i \le s_i \precsim r_i, \quad \|s_i c_i - c_i s_i\| < \frac{1}{N} \le \frac{\varepsilon}{2}. \tag{5.4}$$

Let $s = \phi(\oplus_i s_i) \ge e - q$. We have $\|s\phi(c) - \phi(c)s\| < \varepsilon/2$. Let $p = e - s \le q$. For condition (1), we have:

$$\begin{aligned}\|pa - ap\| &= \|p(a - ((1-q)a(1-q)) - (a - (1-q)a(1-q))p\| \\ &\le 2\delta + \|p\overline{d} - \overline{d}p\| = 2\delta + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

Since $p \le q$, we have $pap = pqaqp \in {}_{\varepsilon}D$, which proves condition (2).

Finally, for any $\tau \in T(A \rtimes_\alpha G)$, we have

$$\tau(s) = \tau(\phi(\oplus_i s_i)) \le \eta\tau(\phi(\oplus_i q_i)) \le \tau(z_0). \tag{5.5}$$

By [**23,** Proposition 2.4], this shows that $s \precsim z_0$. Hence,

$$1 - p = (1-e) + s \le (1-q) \oplus s \precsim z_1 \oplus z_0 \precsim z. \tag{5.6}$$

□

Proposition 5.2. *Let A be a unital, simple C^*-algebra with strict comparison for projections. Suppose that, for any self-adjoint element $a \in A$, any $\varepsilon > 0$ and any nonzero positive element $z \in A$, there is a unital C^*-subalgebra D of A with real rank 0 and $1_D = p$ such that:*

(1) $\|pa - ap\| < \varepsilon$,
(2) $pap \in_\varepsilon D$,
(3) $1 - p \precsim z$.

Then, A has real rank 0.

Proof. Let a be a self-adjoint element in A and $\varepsilon > 0$ be given. Without loss of generality, assume that $\|a\| = 1$. Assume that a is not invertible; otherwise, there is nothing to prove. Let $\varepsilon_0 = \varepsilon/(26)$. Let

$$g\colon [-1, 1] \longrightarrow [0, 1]$$

be a continuous function such that

$$\text{supp}\, g = [-\varepsilon_0, \varepsilon_0] \quad \text{and} \quad g(0) = 1. \tag{5.7}$$

Let

$$\varepsilon_1 = \text{Min}\{\varepsilon_0, \tfrac{1}{4}\,\text{Min}\{\tau(g(a)) \mid \tau \in \mathrm{T}(A)\}\} > 0. \tag{5.8}$$

Choose $\delta > 0$ such that, whenever a, b are normal elements with norm ≤ 1 and $\|a - b\| < \delta$, then $\|g(a) - g(b)\| \leq \varepsilon_1$, according to [**15,** Lemma 2.5.11]. We further require that $\delta \leq \varepsilon_1$. Since A has strict comparison, we can find a C^*-subalgebra D of A with real rank 0 and a projection $p \in D$ such that:

(1) $\|pa - ap\| < \delta/2$.
(2) There is some self-adjoint element $d \in D$ such that $\|pap - d\| < \delta$.
(3) $\tau(1 - p) < \delta/2$ for any $\tau \in \mathrm{T}(A)$.

Replacing d by pdp, we may assume that $d \in pDp$. We may also assume that $\|d\| \leq 1$. Since pDp has real rank 0 for a corner of real rank 0 C^*-algebra, there is a projection $r \in g(d)Dg(d)$ such that $\|rg(d)r - g(d)\| < \delta$. In the following, we shall show that $1 - p \precsim r \leq p$ and, for any projection $s \leq r$, we have $\|sa\| < \varepsilon$, $\|as\| < \varepsilon$. The choice of δ shows that

$$g(a) =_{\varepsilon_1} g(pap + (1 - p)a(1 - p)) = g(pap) + g((1 - p)a(1 - p)) \tag{5.9}$$

and $g(pap) =_{\varepsilon_1} g(d)$. Hence, for any $\tau \in \mathrm{T}(A)$, we can compute:

$$\begin{aligned}
\tau(r) &\geq \tau(rg(d)r) \geq \tau(g(d)) - \delta \geq \tau(g(pap)) - \varepsilon_1 - \varepsilon_1 \\
&\geq \tau(g(a)) - \tau(g((1 - p)a(1 - p))) - 3\varepsilon_1 \\
&\geq \tau(g(a)) - \tau(1 - p) - 3\varepsilon_1 > \tau(1 - p).
\end{aligned}$$

Since A has strict comparison, this shows that $1-p \precsim r$.

Next, since $r \in g(d)Dg(d)$, we have

$$\|rd\| = \lim_{n\to\infty} \|rg(d)^{1/n}d\| \le \varepsilon_0. \tag{5.10}$$

Hence, for any projection $s \le r$, $\|sd\| = \|srd\| \le \varepsilon_0$. Similarly, $\|ds\| \le \varepsilon_0$. Now, combining the facts that $\|pa-pa\| < \delta/2 < \varepsilon_0$ and $\|pap-d\| < \varepsilon_0$, we obtain

$$\begin{aligned} \|sa\| &= \|s(pap) + spa(1-p) + s(1-p)a\| \\ &\le (\|sd\| + \varepsilon_0) + \varepsilon_0 \le \varepsilon, \end{aligned} \tag{5.11}$$

and similarly, $\|as\| < \varepsilon$. Now, since $1-p \precsim r$, let v be a partial isometry such that $vv^* = 1-p$ and $v^*v = s \le r \le p$. Using the decomposition $1 = (p-s) \oplus s \oplus (1-p)$, we may write a in matrix form:

$$a = \begin{pmatrix} (p-s)a(p-s) & (p-s)as & (p-s)a(1-p) \\ sa(p-s) & sas & sa(1-p) \\ (1-p)a(p-s) & (1-p)as & (1-p)a(1-p) \end{pmatrix}.$$

Further, $(p-s)a(p-s) =_{\varepsilon_0} (p-s)\,d(p-s) \in (p-s)D(p-s)$. Since $(p-s)D(p-s)$ has real rank 0, there is an invertible self-adjoint element $d_1 \in (p-s)D(p-s)$ such that $\|(p-s)\,d(p-s) - d_1\| < \varepsilon_0$. Hence,

$$\begin{aligned} a &=_{23\varepsilon_0} \begin{pmatrix} (p-s)\,d(p-s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1-p)a(1-p) \end{pmatrix} \\ &=_{2\varepsilon_0} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & \varepsilon_0 v^* \\ 0 & \varepsilon_0 v & (1-p)a(1-p) \end{pmatrix}. \end{aligned}$$

The last matrix corresponds to an invertible self-adjoint element a_0 in A. By our choice of ε_0, we have $\|a-a_0\| < \varepsilon$. □

Combining Theorem 4.2, Lemma 5.1 and Proposition 5.2, we get the following.

Theorem 5.3. *Let A be a simple unital C^*-algebra with real rank 0 and containing strict comparison for projections. Let $(\alpha, u)\colon G \curvearrowright A$ be a cocyclic action with the tracial Rokhlin property. Then, $A \rtimes_\alpha G$ has real rank 0.*

Now, let us turn to the case of stable rank 1. We first see that [**23,** Lemma 5.2] may be generalized to actions of general amenable groups since its proof depends only upon [**23,** Lemma 2.5, Lemma 2.6] and some other lemmas unrelated to the crossed product. We could use Lemma 4.1 to replace the first one, and [**23,** Lemma 2.6] could be generalized to actions of amenable groups with the same proof. Hence, we have the following.

Lemma 5.4. *Let A be a simple C^*-algebra with real rank* 0 *and strict comparison for projections. Let $(\alpha, u)\colon G \curvearrowright A$ be a cocyclic action with the tracial Rokhlin property. Then, for any nonzero projections $p_1, \ldots, p_n \in A \rtimes_{\alpha,u} G$ and arbitrary elements $a_1, \ldots, a_m \in A \rtimes_{\alpha,u} G$, any $\varepsilon > 0$, there exist a unital subalgebra $A_0 \subset A \rtimes_{\alpha,u} G$, stably isomorphic to A, a projection $p \in A_0$ and subprojections $r_1, \ldots, r_n$ of p such that*:

1. *$pa \in_\varepsilon A_0$, $ap \in_\varepsilon A_0$.*
2. *$p_k r_k =_\varepsilon r_k$ for any k.*
3. *$1 - p \precsim r_k$ for any k.*

Proposition 5.5. *Let A be a unital, simple stably finite C^*-algebra with property* (SP). *If, for any $x \in A$, any $\varepsilon > 0$ and any projection $p_1, \ldots, p_n$, there is a unital simple subalgebra D with stable rank* 1 *and property* (SP), *a projection $p \in D$ and subprojections $r_1, \ldots, r_n$ of p such that*:

1. *$pxp \in_\varepsilon D$,*
2. *$r_k p_k =_\varepsilon r_k$,*
3. *$1 - p \precsim r_k$,*

then, A has stable rank 1.

Proof. Let x be an arbitrary element of A, and let $\varepsilon > 0$ be given. Without loss of generality, assume that $\|x\| = 1$. Since A is stably finite, every one-sided invertible element is two-sided invertible; hence, by [**27,** Theorem 3.3 (a)], we may assume that x is a two-sided zero divisor. Since A has property (SP), we can find nonzero projections e and f such that $ex = xf = 0$. Let $\varepsilon_0 = \varepsilon/11$. We can then find a unital simple subalgebra D with stable rank 1 and property (SP), a projection $p \in D$ and subprojections e_0 and f_0 of p such that

$$e_0 e =_{\varepsilon_0} e_0, \qquad f_0 f =_{\varepsilon_0} f_0. \tag{5.12}$$

Consider $x_0 = (1 - e_0)x(1 - f_0)$. Then,

$$x_0 =_{2\varepsilon_0} (1 - e_0 e)x(1 - f f_0) = x. \tag{5.13}$$

Since D is a simple C^*-algebra with property (SP), there is a nonzero projection $r \leq e_0$ and $r \precsim f_0$. Since D has stable rank 1, there exists some unitary u such that $uru^* \leq f_0$. Hence, $r(x_0 u) = (x_0 u)r = 0$.

Next, we shall approximate $x_1 = x_0 u$ by an invertible element. To this end, we find a unital subalgebra D_1 of A with stable rank 1, a projection $p \in D_1$ and subprojection r_1 of p, and an element $d \in D_1$ such that

$$px_1 p =_{\varepsilon_0} d, \qquad r_1 r =_{\varepsilon_0} r_1 \quad \text{and} \quad 1 - p \precsim r_1. \tag{5.14}$$

Choose a partial isometry v such that $vv^* = 1-p$ and $v^*v = s \leq r_1 \leq p$. According to the decomposition $1 = (1-p) \oplus (p-s) \oplus s$, we can write x_1 in matrix form:

$$\begin{pmatrix} (1-p)x_1(1-p) & (1-p)x_1(p-s) & (1-p)x_1 s \\ (p-s)x_1(1-p) & (p-s)x_1(p-s) & (p-s)x_1 s \\ sx_1(1-p) & sx_1(p-s) & sx_1 s \end{pmatrix}.$$

Since $(p-s)D_1(p-s)$ has stable rank 1, there is an invertible element $d_1 \in (p-s)D_1(p-s)$ such that

$$d_1 =_{\varepsilon_0} (p-s)\, d(p-s) =_{\varepsilon_0} (p-s)x_1(p-s). \tag{5.15}$$

We also have $sx_1 = sr_1 x_1 =_{\varepsilon_0} sr_1 r x_1 = 0$, and similarly, $x_1 s =_{\varepsilon_0} 0$. Therefore,

$$x_1 =_{7\varepsilon_0} \begin{pmatrix} a & b & 0 \\ c & d_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =_{2\varepsilon_0} \begin{pmatrix} a & b & \varepsilon_0 \\ c & d_1 & 0 \\ \varepsilon_0 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

We call the last matrix x_2, which is invertible. Then

$$\|x - x_2 u^*\| \leq \|x - x_0\| + \|(x_0 u - x_2)u^*\| < 11\varepsilon_0 < \varepsilon. \tag{5.17}$$

Hence, A has stable rank 1. □

Combining Lemma 5.4 and Proposition 5.5, we obtain:

Theorem 5.6. *Let A be a simple unital C^*-algebra with real rank 0, stable rank 1 and with strict comparison for projections. Let $(\alpha, u)\colon G \curvearrowright$*

A be a cocyclic action with the tracial Rokhlin property. Then, $A \rtimes_{\alpha,u} G$ has stable rank 1.

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