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CHERN-DIRAC BUNDLES ON NON-KÄHLER HERMITIAN MANIFOLDS

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ABSTRACT. We introduce the notions of Chern-Dirac bundles and Chern-Dirac operators on Hermitian manifolds. They are analogues of classical Dirac bundles and Dirac operators, with the Levi-Civita connection replaced by the Chern connection. We then show that the tensor product of the canonical and the anticanonical spinor bundles, called the $\mathcal{V}$-spinor bundle, is a bigraded Chern-Dirac bundle with spaces of harmonic sections isomorphic to the full Dolbeault cohomology class. A similar construction establishes isomorphisms among other types of harmonic sections of the $\mathcal{V}$-spinor bundle and twisted cohomology.

1. Introduction. A *Dirac bundle* over a Riemannian manifold (M, g) is a real or complex vector bundle

$$\pi : E \longrightarrow M$$

endowed with a Riemannian or Hermitian metric h, a metric connection D and a Clifford multiplication

$$c : \mathcal{C}\ell\, M \longrightarrow \mathrm{End}(E),$$

i.e., a structure of the left $\mathcal{C}\ell\, M$-module with respect to which the multiplication by tangent vectors is fiber wise, skew-adjoint and covariantly constant. For every such bundle, there is a distinguished operator, called the *Dirac operator*, which plays a central role in many areas of differential geometry and theoretical physics (see, e.g., **[5, 8]** for an introduction to this topic). The most notable examples of Dirac bundles are spinor bundles on the so-called spin or spin$^{\mathbb{C}}$ manifolds.

One of the most important properties of Dirac operators is the fact that they are first order, elliptic and formally self-adjoint operators,

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whose squares have the same principal symbol of the rough Laplacian. On the basis of such properties, one may expect the existence of Hodge-type theorems for Dirac operators, relating the null spaces of these operators with appropriate cohomology groups of the manifold. This expectation is, however, contradicted by Hitchin's results in [**7**], where it was shown that the dimensions of the null spaces of Dirac operators cannot be expressed in purely topological terms. However, in the special case of Kähler geometry, there exist such strong interactions between Clifford multiplications and complex structures that give rise to some notable isomorphisms between the null space of the Dirac operators and certain cohomology groups of the manifold. More precisely, given a compact Kähler $2n$-manifold (M, g, J), the following facts hold.

(i) M admits a *canonical* $\mathrm{spin}^{\mathbb{C}}$ spinor bundle which is isomorphic to $\Lambda^{0,\cdot}(T^*M)$ and whose Dirac operator coincides with $\sqrt{2}(\overline{\partial}+\overline{\partial}^*)$. From this, one obtains that the space of harmonic spinors is isomorphic to the Dolbeault cohomology class $H^{0,\cdot}_{\overline{\partial}}(M)$ (see, e.g., [**5,** subsection 3.4], or [**8,** Appendix D]).

(ii) The complex Clifford bundle $\mathcal{C}\ell^{\mathbb{C}} M := \mathcal{C}\ell\, M \otimes_{\mathbb{R}} \mathbb{C}$ of M always carries a very rich algebraic structure, which has been systematically studied by Michelsohn [**10**]. There, the author determined a natural bigradation on $\mathcal{C}\ell^{\mathbb{C}} M$ and, using Dirac operators, constructed a natural elliptic cochain complex which defines the so-called *Clifford cohomology* of the Kähler manifold.

The aim of this paper is to determine analogues of (i) and (ii) in the more general setting of Hermitian geometry. This is indeed achieved by making use of the Chern connection, instead of the Levi-Civita connection. Following this idea, we first define the *Chern-Dirac bundles* on an Hermitian (possibly non-Kähler) manifold and the associated *Chern-Dirac operators*. They carry all the nice properties of the usual Dirac bundle and Dirac operators, respectively, and they are equal to them in the case where M is Kähler. Then, on any Hermitian manifold (M, g, J), we explicitly construct a distinguished Chern-Dirac bundle $\mathcal{V}M$, naturally isomorphic to $\mathcal{C}\ell^{\mathbb{C}} M$, called the *$\mathcal{V}$-spinor bundle.* Further, in the same spirit of [**10**], we show that $\mathcal{V}M$ is naturally bigraded and that the kernels of the Chern-Dirac operators on $\mathcal{V}$-spinors are naturally isomorphic to the De Rham and

Dolbeault cohomology classes of M. Finally, using these new tools, we obtain a spinorial characterization for the Bott-Chern, Aeppli and twisted cohomologies. We also determine explicit expressions for the squares of Chern-Dirac operators, which might be used to determine useful Bochner-type theorems on non-Kähler Hermitian manifolds with appropriate conditions on curvature and torsion. Although we proceed in a similar way, our construction is very different from that given in [**10**]. For the sake of clarity, we will often point out differences and similarities with Michelsohn's framework.

The paper is structured as follows. After the first two sections, where some basic properties of Hermitian manifolds, spin groups and spin$^{\mathbb{C}}$ structures are recalled, in Section 4, we define Chern-Dirac bundles and Chern-Dirac operators and prove their main properties. In Section 5, we introduce the Chern-Dirac bundle of $\mathcal{V}$-spinors and prove the main results of this paper. In Section 6, applications of $\mathcal{V}$-spinors in twisted cohomology are given.

2. Preliminaries and notation. In this section, we briefly summarize some basic notation and properties of spin$^{\mathbb{C}}$ structures and spinors over Hermitian manifolds. We refer to [**5,** subsection 3.4], for a more detailed treatment of these tools. However, we stress the fact that we are using the definition of Clifford algebra of [**8**], based on formula (2.5). The sign convention used in [**5**] is opposite to ours, and this causes differences in some formulas of this paper from those found in that book.

2.1. Hermitian manifolds and Chern connections. Let (M, g, J) be a $2n$-dimensional Hermitian manifold, with the fundamental form $\omega := g(J\cdot, \cdot)$. The J-holomorphic and J-antiholomorphic subbundles of $T^{\mathbb{C}}M$ are denoted by $T^{10}M$ and $T^{01}M$, respectively. Analogously, the corresponding dual subbundles of $T^{*\mathbb{C}}M$, determined by the J-action on covectors $(J\lambda)(\cdot) := -\lambda(J\cdot)$, are denoted by $T^{*10}M$ and $T^{*01}M$, respectively. The bundle of (p, q)-forms is indicated by $\Lambda^{p,q}(T^*M)$ and the space of its global sections by $\Omega^{p,q}(M)$. The decomposition $d = \partial + \overline{\partial}$ is the usual expression of the exterior differential d as sum of the classical ∂ and $\overline{\partial}$ operators.

We denote by

$$\pi : \mathrm{SO}_g(M) \longrightarrow M$$

the SO_{2n}-bundle of oriented g-orthonormal frames, and by

$$\mathrm{U}_{g,J}(M) \subset \mathrm{SO}_g(M)$$

the U_n-subbundle of (g,J)-unitary frames, that is, of the frames (e_j) satisfying the conditions $g(e_j, e_\ell) = \delta_{j\ell}$ and $Je_{2j-1} = e_{2j}$. Further, for each unitary frame $(e_j) \subset T_xM$, $x \in M$, we denote by

$$\left(\epsilon_s := \frac{e_{2s-1} - ie_{2s}}{\sqrt{2}},\ \overline{\epsilon}_s := \frac{e_{2s-1} + ie_{2s}}{\sqrt{2}}\right)_{1 \leq s \leq n} \tag{2.1}$$

the complex frame given by the normalized holomorphic and anti-holomorphic parts of the vectors e_s. We call it the *associated normalized complex frame.*

Note that, given a unitary frame (e_j) for T_xM, the Kähler form $\omega_x = \omega_{g,J}|_x$ is equal to

$$\omega_x = e^1 \wedge e^2 + \cdots + e^{2n-1} \wedge e^{2n} = i\big(\epsilon^1 \wedge \overline{\epsilon}^1 + \cdots + \epsilon^n \wedge \overline{\epsilon}^n\big).$$

The Levi-Civita connection of (M,g) is the torsion-free $\mathfrak{so}_{2n}$-valued 1-form ω^{LC} on $\mathrm{SO}_g(M)$, and its corresponding covariant derivative on vector fields of M is denoted by D^{LC}. Similarly, the Chern connection of (M,g,J) is the $\mathfrak{u}_n$-valued connection form $\omega^{\mathcal{C}}$ on $\mathrm{U}_{g,J}(M)$, whose associated covariant derivative $D^{\mathcal{C}}$ on vector fields of M possesses torsion satisfying $T(J\cdot,\cdot) = T(\cdot, J\cdot)$. The covariant derivatives of the Levi-Civita and Chern connections are related by

$$D^{\mathcal{C}}{}_XY = D^{LC}{}_XY + S(X,Y), \tag{2.2}$$

where S is the uniquely determined *contorsion tensor of the Chern connection.* It is well known that the contorsion and the torsion of the Chern connection are given by

$$\begin{aligned} S(X,Y,Z) &= -\frac{1}{2}\, d\omega(JX,Y,Z), \\ T(X,Y,Z) &= -\frac{1}{2}(d\omega(JX,Y,Z) + d\omega(X,JY,Z)), \end{aligned} \tag{2.3}$$

and they are related by

$$\begin{aligned} T(X,Y) &= S(X,Y) - S(Y,X), \\ 2S(X,Y,Z) &= T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y), \end{aligned} \tag{2.4}$$

where $S(X,Y,Z) := g(S(X,Y),Z)$ and $T(X,Y,Z) := g(T(X,Y),Z)$.

Finally, we recall that the *Lee form* of (M, g, J) is the 1-form

$$\vartheta(X) := \operatorname{Tr}(T(X, \cdot)) = \sum_j T(X, e_j, e_j),$$

where (e_j) is an arbitrary choice of a (local) unitary frame field on M. It can easily be verified that the fundamental form ω and the Lee form ϑ are related by

$$\vartheta = -J d^* \omega,$$

where d^* is the adjoint of d with respect to g.

2.2. Complex spin representations and $\mathrm{Spin}_n^{\mathbb{C}}$. We recall that the Clifford algebra $\mathcal{C}\ell_n$ is the real associative algebra with unit generated by n elements (e_j) satisfying

$$e_j \cdot e_k + e_k \cdot e_j = -2\delta_{jk}, \quad 1 \le j,\ k \le n. \tag{2.5}$$

As a vector space, $\mathcal{C}\ell_n$ can be identified with $\Lambda^{\cdot}(\mathbb{R}^n)$ in such a way that

$$v \cdot w = v \wedge w - v \lrcorner w \quad \text{for every } v \in \mathbb{R}^n,\ w \in \Lambda^{\cdot}(\mathbb{R}^n). \tag{2.6}$$

The spin group is the subset

$$\mathrm{Spin}_n := \{v_1 \cdot \ldots \cdot v_{2r} : v_j \in \mathbb{R}^n,\ \|v_j\| = 1\},$$

equipped with the multiplication of $\mathcal{C}\ell_n$. If $n \ge 3$, it is simply connected, and it is the universal covering of SO_n by means of the map

$$\tau_n : \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n,$$

$$\tau_n(v_1 \cdot \ldots \cdot v_{2r}) := \mathrm{refl}_{v_1} \circ \ldots \circ \mathrm{refl}_{v_{2r}},$$

where refl_v is the reflection of $\mathbb{R}^n$ with respect to $v^\perp$. We denote by $\mathcal{S}_n$ the space of complex n-spinors, by

$$\cdot : \mathcal{C}\ell_n \otimes \mathcal{S}_n \longrightarrow \mathcal{S}_n$$

the standard Clifford multiplication, and by

$$\kappa_n : \mathrm{Spin}_n \longrightarrow \mathrm{SU}(\mathcal{S}_n)$$

the *spin representation* of Spin_n, where we consider $\mathcal{S}_n$ endowed with a positive definite Hermitian scalar product which is invariant under Clifford multiplication by vectors $v \in \mathbb{R}^n \subset \mathcal{C}\ell_n$.

Now, let $\mathcal{C}\ell_n^{\mathbb{C}} = \mathcal{C}\ell_n \otimes_{\mathbb{R}} \mathbb{C}$ be the complex Clifford algebra. In the even-dimensional case, $\mathcal{C}\ell_{2m}^{\mathbb{C}}$ is generated by complex vectors $(\epsilon_j, \overline{\epsilon}_j)$, related with the generators (e_j) by the formula (2.1), which verify

$$\epsilon_r \cdot \epsilon_s + \epsilon_s \cdot \epsilon_r = \overline{\epsilon}_r \cdot \overline{\epsilon}_s + \overline{\epsilon}_s \cdot \overline{\epsilon}_r = 0, \qquad \epsilon_r \cdot \overline{\epsilon}_s + \overline{\epsilon}_s \cdot \epsilon_r = -2\delta_{rs}. \tag{2.7}$$

Finally, we recall that the $\mathrm{Spin}^{\mathbb{C}}$-*group* is the Lie group $\mathrm{Spin}_n^{\mathbb{C}} := \mathrm{Spin}_n \times_{\mathbb{Z}_2} S^1$. It is a 2-fold covering of $\mathrm{SO}_n \times S^1$ by means of the map

$$(\tau_n, \varrho_n) : \mathrm{Spin}_n^{\mathbb{C}} \longrightarrow \mathrm{SO}_n \times S^1,$$

where

$$\begin{aligned} \tau_n &: \mathrm{Spin}_n^{\mathbb{C}} \longrightarrow \mathrm{SO}_n, & \tau_n([g,z]) &:= \tau_n(g), \\ \varrho_n &: \mathrm{Spin}_n^{\mathbb{C}} \longrightarrow S^1, & \varrho_n([g,z]) &:= z^2, \end{aligned}$$

and admits a representation on $\mathcal{S}_n$, again denoted by κ_n, defined by

$$\kappa_n : \mathrm{Spin}_n^{\mathbb{C}} \longrightarrow \mathrm{SU}(\mathcal{S}_n), \qquad \kappa_n([g,z]) := z\kappa_n(g). \tag{2.8}$$

2.3. $\mathrm{spin}^{\mathbb{C}}$ **structures on Hermitian manifolds.** Let (M, g) be an oriented Riemannian manifold with oriented orthonormal frame bundle

$$\pi : \mathrm{SO}_g(M) \longrightarrow M.$$

A $\mathrm{spin}^{\mathbb{C}}$ *structure* on (M, g) is a $\mathrm{Spin}_n^{\mathbb{C}}$-bundle $\widehat{\pi} : \mathcal{P} \to M$ together with an equivariant bundle morphism $\varpi : \mathcal{P} \to \mathrm{SO}_g(M)$ such that $\widehat{\pi} = \pi \circ \varpi$. Given a $\mathrm{spin}^{\mathbb{C}}$ structure $\mathcal{P}$, the corresponding *spinor bundle* is the associated bundle $\mathcal{S}M := \mathcal{P} \times_{\kappa_n} \mathcal{S}_n$. The space of its global sections is indicated with $\mathfrak{S}(M)$.

Most, but not all, orientable Riemannian manifolds admit a $\mathrm{spin}^{\mathbb{C}}$ structure, see [**8,** page 393]. A crucial property of the subclass of Hermitian manifolds is that all of them have two very natural $\mathrm{spin}^{\mathbb{C}}$ structures. In fact, the homomorphisms

$$f_{\pm} : \mathrm{U}_n \longrightarrow \mathrm{SO}_{2n} \times S^1, \qquad f_{\pm}(A) := (\imath_{\mathrm{U}_n}(A), \det(A)^{\pm 1}) \tag{2.9}$$

where $\imath_{\mathrm{U}_n} : \mathrm{U}_n \hookrightarrow \mathrm{SO}_{2n}$ is the canonical immersion of U_n into SO_{2n} and can be uniquely lifted to two group homomorphisms $F_{\pm} : \mathrm{U}_n \to \mathrm{Spin}_{2n}^{\mathbb{C}}$ in such a way that the diagram

$$\begin{tikzcd} & \mathrm{Spin}^{\mathbb{C}}_{2n} \arrow[d, "(\tau_{2n},\varrho_{2n})"] \\ \mathrm{U}_n \arrow[ur, "F_\pm"] \arrow[r, "f_\pm"'] & \mathrm{SO}_{2n} \times S^1 \end{tikzcd}$$

commutes.

Definition 2.1. Let (M, g, J) be an Hermitian $2n$-manifold. Its *canonical* spin$^{\mathbb{C}}$ *structure* is the bundle

$$\mathcal{P}^{\uparrow}(M) := \mathrm{U}_{g,J}(M) \times_{F_+} \mathrm{Spin}^{\mathbb{C}}_{2n} .$$

Similarly, its *anticanonical* spin$^{\mathbb{C}}$ *structure* is

$$\mathcal{P}^{\downarrow}(M) := \mathrm{U}_{g,J}(M) \times_{F_-} \mathrm{Spin}^{\mathbb{C}}_{2n} .$$

If $\mathcal{S}M$ is a spinor bundle on (M, g, J) associated with a spin$^{\mathbb{C}}$ structure, it is known that the Kähler form $\omega = \omega_{g,J}$ acts on $\mathcal{S}M$ by Clifford multiplication as a bundle endomorphism. Its eigenvalues are the imaginary numbers $(2k-n)i$, $0 \leq k \leq n$, and, in each fibre, the corresponding eigenspaces

$$\begin{gathered} \mathcal{S}^k_x M := \{\psi \in \mathcal{S}_x M : \omega_x \cdot \psi = (2k-n)i\,\psi\}, \\ 0 \leq k \leq n, \quad x \in M, \end{gathered}$$

have dimension $\binom{n}{k}$. It may also be directly verified that

$$\begin{aligned} \mathcal{S}^0_x M &= \{\psi \in \mathcal{S}_x M : \overline{v} \cdot \psi = 0 \text{ for every } \overline{v} \in T^{01}_x M\}, \\ \mathcal{S}^n_x M &= \{\psi \in \mathcal{S}_x M : v \cdot \psi = 0 \text{ for every } v \in T^{10}_x M\}. \end{aligned} \tag{2.10}$$

Furthermore, it is known that there exist Hermitian metrics on $\mathcal{S}M$ invariant under Clifford multiplication by tangent vectors. With respect to one such metric, for every $0 \leq k \leq n$, the maps

(2.11)

$$\begin{aligned} \alpha^k &: \Lambda^{0,k}(T^*M) \otimes \mathcal{S}^0 M \longrightarrow \mathcal{S}^k M, \qquad \alpha^k(\overline{\mu} \otimes \psi) := \frac{1}{2^{k/2}}\,\overline{\mu} \cdot \psi, \\ \beta^k &: \Lambda^{k,0}(T^*M) \otimes \mathcal{S}^n M \longrightarrow \mathcal{S}^{n-k} M, \quad \beta^k(\nu \otimes \psi) := \frac{1}{2^{k/2}}\,\nu \cdot \psi \end{aligned}$$

are $\mathbb{C}$-linear isometries, and their sums give rise to global isometries

$$\alpha : \Lambda^{0,\cdot}(T^*M) \otimes \mathcal{S}^0 M \xrightarrow{\simeq} \mathcal{S}M$$
$$\beta : \Lambda^{\cdot,0}(T^*M) \otimes \mathcal{S}^n M \xrightarrow{\simeq} \mathcal{S}M.$$

3. Chern-Dirac bundles.

3.1. Chern-Dirac bundles and partial Chern-Dirac operators. Given an Hermitian $2n$-manifold (M, g, J), we can always consider the *complex Clifford bundle* over M, defined by $\mathcal{C}\ell^{\mathbb{C}} M := \mathrm{U}_{g,J}(M) \times_{\mathrm{U}_n} \mathcal{C}\ell^{\mathbb{C}}_{2n}$, where the group U_n acts on $\mathcal{C}\ell^{\mathbb{C}}_{2n}$ in the standard way. It is the complex analogue of the (real) Clifford bundles considered in [**8,** subsection I.3]. In full analogy with the notion of the (real) *Dirac bundle*, see e.g., [**8,** subsection I.5]), it is convenient to introduce the following

Definition 3.1. A *Chern-Dirac bundle* over (M, g, J) is a complex vector bundle $\pi : E \to M$ endowed with an Hermitian metric h, a covariant derivative D which preserves the metric and a structure of complex left $\mathcal{C}\ell^{\mathbb{C}} M$-modules

$$c : \mathcal{C}\ell^{\mathbb{C}} M \longrightarrow \mathfrak{gl}(E),$$

satisfying the conditions:

(i) for every $v \in T^{\mathbb{C}}M$, $\sigma_1, \sigma_2 \in E$,

$$h(c(v)\sigma_1, \sigma_2) + h(\sigma_1, c(\overline{v})\sigma_2) = 0; \tag{3.1}$$

(ii) for every $X \in \mathfrak{X}(M)$, for sections w of $\mathcal{C}\ell^{\mathbb{C}} M$ and σ of E,

$$D_X(c(w)\sigma) = c\big(D^{\mathcal{C}}_X w\big)\sigma + c(w)D_X\sigma. \tag{3.2}$$

Note that, if M is Kähler, then $D^{\mathcal{C}} = D^{LC}$ and, consequently, any Chern-Dirac bundle over M is a Dirac bundle in the usual sense.

The main results of this paper are based upon the following differential operators on Chern-Dirac bundles, which are natural analogues of Dirac operators.

Definition 3.2. Let E be a Chern-Dirac bundle over (M, g, J). The *partial Chern-Dirac operators* on section E are the maps $\eth'$ and $\eth''$

that transform any section σ of E into the sections defined for every $x \in M$ by

$$(3.3)\qquad \begin{aligned} \partial\!\!\!/'\sigma|_x &:= \sum_{j=1}^{n} c(\bar\epsilon_j) D_{\epsilon_j}\sigma - \frac{1}{2}\sum_{r<s} c(\bar\epsilon_r \cdot \bar\epsilon_s \cdot T_{rs})\sigma_x, \\ \partial\!\!\!/''\sigma|_x &:= \sum_{j=1}^{n} c(\epsilon_j) D_{\bar\epsilon_j}\sigma - \frac{1}{2}\sum_{r<s} c(\epsilon_r \cdot \epsilon_s \cdot T_{\bar r\bar s})\sigma_x, \end{aligned}$$

where $(\epsilon_j, \bar\epsilon_j)$ is the normalized complex basis (2.1) associated with a unitary basis $(e_j) \subset T_xM$, T is the torsion of $D^{\mathcal{C}}$ and $T_{rs} := T(\epsilon_r, \epsilon_s)$, $T_{\bar r\bar s} := T(\bar\epsilon_r, \bar\epsilon_s)$. The sum of these operators gives what we call the *Chern-Dirac operator*

$$(3.4)\qquad D\!\!\!\!/ := \partial\!\!\!/' + \partial\!\!\!/''.$$

It can be directly verified that the formulas (3.3) define two global operators on the entire manifold, i.e., they are coordinate invariant. Indeed, if (e'_j) and (e_k) are unitary frames at $x \in M$ with $e'_j = e_k A^k_j$, then the associated normalized complex frames are related by

$$(3.5)\qquad \epsilon'_j = \epsilon_k \alpha^k_j, \quad \bar\epsilon'{}_j = \bar\epsilon_k \bar\alpha^k_j$$

with $\alpha^k_j := A^{2k-1}_{2j-1} - iA^{2k-1}_{2j}$. Since the complex coefficients α^k_j verify $\sum_j \alpha^k_j\,\bar\alpha^m_j = \delta^{km}$, then

$$\begin{aligned} \sum_{j=1}^{n} c(\bar\epsilon'_j) D_{\epsilon'_j}\sigma &- \frac{1}{2}\sum_{r<s} c(\bar\epsilon'_r \cdot \bar\epsilon'_s \cdot T(\epsilon'_r, \epsilon'_s))\sigma_x \\ &= \sum_{j=1}^{n} \bar\alpha^k_j \alpha^h_j c(\bar\epsilon_k) D_{\epsilon_h}\sigma \\ &\quad - \frac{1}{2}\sum_{r<s} \bar\alpha^\ell_r\,\alpha^p_r\,\bar\alpha^m_s\alpha^q_s\, c(\bar\epsilon_\ell \cdot \bar\epsilon_m \cdot T(\epsilon_p, \epsilon_q))\sigma_x \\ &= \sum_{k=1}^{n} c(\bar\epsilon_k) D_{\epsilon_k}\sigma - \frac{1}{2}\sum_{\ell<m} c(\bar\epsilon_\ell \cdot \bar\epsilon_m \cdot T(\epsilon_\ell, \epsilon_m))\sigma_x, \end{aligned}$$

and analogously for $\partial\!\!\!/''$.

It follows from the definition that the Chern-Dirac operator $D\!\!\!\!/$ is a first-order elliptic operator. Moreover, it turns out that the operators

(3.3) are formal adjoints of one another and, consequently, $\not{D}$ is formal self-adjoint. In fact:

Proposition 3.3. *Let E be a Chern-Dirac bundle over M and σ_1, σ_2 two sections of E. Then*:

$$h(\not{\partial}'\sigma_1, \sigma_2) - h(\sigma_1, \not{\partial}''\sigma_2) = \mathrm{Div}(V^{\sigma_1,\sigma_2}), \tag{3.6}$$

where V^{σ_1,σ_2} is the unique complex vector field which satisfies

$$g(V^{\sigma_1,\sigma_2}, X) = -h(\sigma_1, c(X^{10})\sigma_2) \quad \textit{for every } X \in \mathfrak{X}(M).$$

Consequently, if σ_1 and σ_2 are compactly supported, then

$$\int_M h(\not{\partial}'\sigma_1, \sigma_2)\, d\,\mathrm{vol}_g = \int_M h(\sigma_1, \not{\partial}''\sigma_2)\, d\,\mathrm{vol}_g\,.$$

Proof. Let (e_j) be a unitary frame field defined on some open subset $\mathcal{U} \subset M$ and ϵ_j, $\bar{\epsilon}_j$ the associated normalized complex vector fields defined in (2.1). It may be directly verified that the vector field V^{σ_1,σ_2} takes values in $T^{01}M$. By the properties of Levi-Civita connection, this implies that

$$\begin{aligned}
&\mathrm{Div}(V^{\sigma_1,\sigma_2}) \\
&\quad= \sum_j \big(g\big(D^{LC}_{\epsilon_j}V^{\sigma_1,\sigma_2}, \bar{\epsilon}_j\big) + g\big(D^{LC}_{\bar{\epsilon}_j}V^{\sigma_1,\sigma_2}, \epsilon_j\big)\big) \\
&\quad= \sum_j \big(- g\big(V^{\sigma_1,\sigma_2}, D^{LC}_{\epsilon_j}\bar{\epsilon}_j\big) + \bar{\epsilon}_j\big(g(V^{\sigma_1,\sigma_2}, \epsilon_j)\big) - g(V^{\sigma_1,\sigma_2}, D^{LC}_{\bar{\epsilon}_j}\epsilon_j)\big) \\
&\quad= \sum_j \big(- \bar{\epsilon}_j\big(h(\sigma_1, c(\epsilon_j)\sigma_2)\big) - \mathrm{Div}(\bar{\epsilon}_j)h(\sigma_1, c(\epsilon_j)\sigma_2)\big).
\end{aligned} \tag{3.7}$$

On the other hand, we also have that

$$\sum_j D^{\mathcal{C}}_{\bar{\epsilon}_j}\epsilon_j = \sum_j \big(- \mathrm{Div}(\bar{\epsilon}_j) + \vartheta(\bar{\epsilon}_j)\big)\epsilon_j. \tag{3.8}$$

Using (3.7) and (3.8), we obtain that

$$\begin{aligned}
h(\not{\partial}'\sigma_1, \sigma_2) &= \sum_j h(c(\bar{\epsilon}_j)D_{\epsilon_j}\sigma_1, \sigma_2) - \frac{1}{2}\sum_{r<s} h(c(\bar{\epsilon}_r)c(\bar{\epsilon}_s)c(T_{rs})\sigma_1, \sigma_2) \\
&= \sum_j -h(D_{\epsilon_j}\sigma_1, c(\epsilon_j)\sigma_2) + \frac{1}{2}\sum_{r<s} h(\sigma_1, c(T_{\bar{r}\bar{s}})c(\epsilon_s)c(\epsilon_r)\sigma_1, \sigma_2)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j (-\overline{\epsilon}_j (h(\sigma_1, c(\epsilon_j)\sigma_2)) + h(\sigma_1, D_{\overline{\epsilon}_j}(c(\epsilon_j)\sigma_2))) \\
&\quad + \frac{1}{2}\sum_{r<s} h(\sigma_1, c(T_{\overline{r}\overline{s}})c(\epsilon_s)c(\epsilon_r)\sigma_1, \sigma_2) \\
&= \sum_j \big(-\overline{\epsilon}_j (h(\sigma_1, c(\epsilon_j)\sigma_2)) + h\big(\sigma_1, c\big(D^{\mathcal{C}}_{\overline{\epsilon}_j}\epsilon_j\big)\sigma_2\big) \\
&\qquad\quad + h(\sigma_1, c(\epsilon_j)D_{\overline{\epsilon}_j}\sigma_2)\big) \\
&\quad + \frac{1}{2}\sum_{r<s} h(\sigma_1, c(T_{\overline{r}\overline{s}})c(\epsilon_s)c(\epsilon_r)\sigma_1, \sigma_2) \\
&\overset{(3.8)}{=} \sum_j (-\overline{\epsilon}_j (h(\sigma_1, c(\epsilon_j)\sigma_2)) - \mathrm{Div}(\overline{\epsilon}_j)h(\sigma_1, c(\epsilon_j)\sigma_2) \\
&\qquad\quad + \vartheta(\overline{\epsilon}_j)h(\sigma_1, c(\epsilon_j)\sigma_2) \\
&\quad + h(\sigma_1, c(\epsilon_j)D_{\overline{\epsilon}_j}\sigma_2)) + \frac{1}{2}\sum_{r<s} h(\sigma_1, c(T_{\overline{r}\overline{s}})c(\epsilon_s)c(\epsilon_r)\sigma_1, \sigma_2) \\
&\overset{(3.7)}{=} \mathrm{Div}(V^{\sigma_1,\sigma_2}) + \sum_j h(\sigma_1, c(\epsilon_j)D_{\overline{\epsilon}_j}\sigma_2) \\
&\quad + \sum_j \vartheta(\overline{\epsilon}_j)h(\sigma_1, c(\epsilon_j)\sigma_2) \\
&\quad + \frac{1}{2}\sum_{r<s} h(\sigma_1, c(T_{\overline{r}\overline{s}})c(\epsilon_s)c(\epsilon_r)\sigma_1, \sigma_2) \\
&= \mathrm{Div}(V^{\sigma_1,\sigma_2}) + \sum_j h(\sigma_1, c(\epsilon_j)D_{\overline{\epsilon}_j}\sigma_2) \\
&\quad - \frac{1}{2}\sum_{r<s} h(\sigma_1, c(\epsilon_r)c(\epsilon_s)c(T_{\overline{r}\overline{s}})\sigma_2) \\
&= \mathrm{Div}(V^{\sigma_1,\sigma_2}) + h(\sigma_1, \eth''\sigma_2),
\end{aligned}
$$

so that (3.6) holds. The last assertion follows from Stokes' theorem. $\square$

3.2. Bochner-type formulas for Chern-Dirac operators. Let E be a Chern-Dirac bundle over an Hermitian $2n$-manifold (M, g, J). We now determine Bochner-type formulas for the squares of $\eth'$, $\eth''$ and of the Chern-Dirac operator $\not{D}$. For this, we must introduce a few operators on sections of E, determined by the curvature and the torsion of Chern connection.

First, we consider the action of the curvature R of D on sections σ of E

$$R_{XY}\sigma := D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma \quad \text{for every } X, Y \in \mathfrak{X}(M).$$

Second, for each section σ, we define

(3.9)
$$\mathcal{R}^{2,0}\sigma|_x := \sum_{j<k} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k) R_{\epsilon_j \epsilon_k}\sigma, \quad \mathcal{R}^{0,2}\sigma|_x := \sum_{j<k} c(\epsilon_j \cdot \epsilon_k) R_{\bar{\epsilon}_j \bar{\epsilon}_k}\sigma,$$
$$\mathcal{R}^{1,1}\sigma|_x := \frac{1}{2}\sum_{j,k} \left(c(\bar{\epsilon}_j \cdot \epsilon_k) R_{\epsilon_j \bar{\epsilon}_k}\sigma + c(\epsilon_j \cdot \bar{\epsilon}_k) R_{\bar{\epsilon}_j \epsilon_k}\sigma \right),$$
$$\mathcal{R}\sigma := \mathcal{R}^{2,0}\sigma + \mathcal{R}^{1,1}\sigma + \mathcal{R}^{0,2}\sigma,$$

(3.10)
$$\mathcal{T}_1\sigma|_x := \sum_{\substack{j<k \\ r<s}} c\big(\epsilon_j \cdot \epsilon_k \cdot \bar{\epsilon}_r \cdot \bar{\epsilon}_s \cdot T_{\bar{j}\bar{k}} \cdot T_{rs} + \bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \epsilon_r \cdot \epsilon_s \cdot T_{jk} \cdot T_{\bar{r}\bar{s}}\big)\sigma$$
$$\mathcal{T}_2\sigma|_x := \sum_{j\neq k} c\big(\epsilon_j \cdot (D^{\mathcal{C}}{}_{\epsilon_k} T)_{\bar{j}\bar{k}} + \bar{\epsilon}_j \cdot (D^{\mathcal{C}}_{\bar{\epsilon}_k} T)_{jk}\big)\sigma,$$

where $(\epsilon_j, \bar{\epsilon}_j)$ is the usual normalized complex frame (2.1) determined by a unitary frame (e_j) for T_xM. Third, we define as $\mathcal{Q}$ the first order differential operator on sections of E by

$$\mathcal{Q}\sigma|_x := \sum_{j\neq k} (c(\epsilon_k \cdot T_{\bar{j}\bar{k}}) D_{\epsilon_j}\sigma + c(\bar{\epsilon}_k \cdot T_{jk}) D_{\bar{\epsilon}_j}\sigma). \tag{3.11}$$

A first Bochner-type formula is the following.

Theorem 3.4. *On each Chern-Dirac bundle E over M, we have*

$$(\not{\partial}')^2 = \mathcal{R}^{2,0}, \qquad (\not{\partial}'')^2 = \mathcal{R}^{0,2}. \tag{3.12}$$

Proof. Consider the decompositions of partial Chern-Dirac operators into sums of differential operators of order 1 and 0, namely,

$$\not{\partial}' = A' - \frac{1}{2}B', \qquad \not{\partial}'' = A'' - \frac{1}{2}B'', \tag{3.13}$$

where

$$A' := \sum_k c(\bar{\epsilon}_k) D_{\epsilon_k}, \qquad B' := \sum_{r<s} c(\bar{\epsilon}_r \cdot \bar{\epsilon}_s \cdot T_{rs}),$$
$$A'' := \sum_j c(\epsilon_j) D_{\bar{\epsilon}_j}, \qquad B'' := \sum_{r<s} c(\epsilon_r \cdot \epsilon_s \cdot T_{\bar{r}\bar{s}}).$$

Using (2.7) and standard properties of metric connections, with some tedious but straightforward computations, we obtain that, for each section σ of E:

(i) $$(A')^2\sigma = -\sum_{j<k} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k) D_{T_{jk}}\sigma + \mathcal{R}^{2,0}\sigma;$$

(ii)

$$\begin{aligned} A'B'\sigma = &-\sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot T([\epsilon_j, \epsilon_k], \epsilon_m))\sigma \\ &- \sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot T(T_{jk}, \epsilon_m))\sigma \\ &+ \sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot D^{\mathcal{C}}_{\epsilon_m} T_{jk})\sigma \\ &+ \sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot T_{jk}) D_{\epsilon_m}\sigma; \end{aligned}$$

(iii) $$B'A'\sigma = \sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot T_{jk} \cdot \bar{\epsilon}_m) D_{\epsilon_m}\sigma \ ;$$

(iv) $$(B')^2\sigma = -2\sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot T(T_{jk}, \epsilon_m))\sigma \ .$$

With similar computations, we may also obtain:

$$\sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot \bar{\epsilon}_m \cdot T_{jk}) D_{\epsilon_m}\sigma + \sum_{\substack{j<k\\ m}} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k \cdot T_{jk} \cdot \bar{\epsilon}_m) D_{\epsilon_m}\sigma = -2\sum_{j<k} c(\bar{\epsilon}_j \cdot \bar{\epsilon}_k) D_{T_{jk}}\sigma. \tag{3.14}$$

From (i)–(iv) and (3.14), it easily follows that

$$
\begin{aligned}
(\partial\!\!\!/')^2\sigma = & (A')^2\sigma - \frac{1}{2}(A'B' + B'A')\sigma + \frac{1}{4}(B')^2\sigma \\
= & \mathcal{R}^{2,0}\sigma - \sum_{j<k} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k) D_{T_{jk}}\sigma + \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot T([\epsilon_j,\epsilon_k],\epsilon_m))\sigma \\
& + \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot T(T_{jk},\epsilon_m))\sigma - \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot D^{\mathcal{C}}_{\epsilon_m} T_{jk})\sigma \\
& - \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot T_{jk}) D_{\epsilon_m}\sigma - \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot T_{jk} \cdot \overline{\epsilon}_m) D_{\epsilon_m}\sigma \\
& - \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot T(T_{jk},\epsilon_m))\sigma \\
= & \mathcal{R}^{2,0}\sigma + \frac{1}{2}\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot (T([\epsilon_j,\epsilon_k],\epsilon_m) - D^{\mathcal{C}}_{\epsilon_m} T_{jk}))\sigma \\
= & \mathcal{R}^{2,0}\sigma + \frac{1}{2}\sum_{j<k<m} c\Big(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot \Big(T([\epsilon_j,\epsilon_k],\epsilon_m) + T([\epsilon_k,\epsilon_m],\epsilon_j) \\
& \qquad + T([\epsilon_m,\epsilon_j],\epsilon_k) - D^{\mathcal{C}}_{\epsilon_m} T_{jk} - D^{\mathcal{C}}_{\epsilon_j} T_{km} - D^{\mathcal{C}}_{\epsilon_k} T_{mj}\Big)\Big)\sigma.
\end{aligned}
$$

Here, the second term vanishes due to the first Bianchi identity. Indeed,

$$
\begin{aligned}
D^{\mathcal{C}}_{\epsilon_j} T_{km} + D^{\mathcal{C}}_{\epsilon_m} T_{jk} + D^{\mathcal{C}}_{\epsilon_k} T_{mj} = & \big(D^{\mathcal{C}}_{\epsilon_j} T\big)(\epsilon_k,\epsilon_m) + \big(D^{\mathcal{C}}_{\epsilon_k} T\big)(\epsilon_m,\epsilon_j) \\
& + \big(D^{\mathcal{C}}_{\epsilon_m} T\big)(\epsilon_j,\epsilon_k) + T(D^{\mathcal{C}}_{\epsilon_j}\epsilon_k,\epsilon_m) + T(\epsilon_k, D^{\mathcal{C}}_{\epsilon_j}\epsilon_m) + T(D^{\mathcal{C}}_{\epsilon_k}\epsilon_m,\epsilon_j) \\
& + T(\epsilon_m, D^{\mathcal{C}}_{\epsilon_k}\epsilon_j) + T(D^{\mathcal{C}}_{\epsilon_m}\epsilon_j,\epsilon_k) + T(\epsilon_j, D^{\mathcal{C}}_{\epsilon_m}\epsilon_k) \\
= & -T(T_{jk},\epsilon_m) - T(T_{km},\epsilon_j) - T(T_{mj},\epsilon_k) + T(D^{\mathcal{C}}_{\epsilon_j}\epsilon_k - D^{\mathcal{C}}_{\epsilon_k}\epsilon_j,\epsilon_m) \\
& + T(D^{\mathcal{C}}_{\epsilon_k}\epsilon_m - D^{\mathcal{C}}_{\epsilon_m}\epsilon_k,\epsilon_j) + T(D^{\mathcal{C}}_{\epsilon_m}\epsilon_j - D^{\mathcal{C}}_{\epsilon_j}\epsilon_m,\epsilon_k) \\
= & T([\epsilon_j,\epsilon_k],\epsilon_m) + T([\epsilon_k,\epsilon_m],\epsilon_j) + T([\epsilon_m,\epsilon_j],\epsilon_k).
\end{aligned}
$$

This proves that $(\partial\!\!\!/')^2 = \mathcal{R}^{2,0}$. The identity $(\partial\!\!\!/'')^2 = \mathcal{R}^{0,2}$ is similarly proven. $\square$

A formula for the square of the Chern-Dirac operator is given as follows.

Theorem 3.5. *The Chern-Dirac operator of E verifies*

$$\not{D}^2 = \Delta + \mathcal{Q} + \mathcal{R} + \frac{1}{4}\mathcal{T}_1 - \frac{1}{2}\mathcal{T}_2, \tag{3.15}$$

where Δ is the rough Laplacian of D defined locally by

$$\Delta\sigma := -\sum_{j=1}^{2n}(D_{e_j}D_{e_j}\sigma - D_{D^{\mathcal{C}}_{e_j}e_j}\sigma),$$

and $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are the operators defined in (3.9)–(3.11).

Proof. As in the previous proof, (e_j) is a locally defined unitary frame field and $(\epsilon_j, \bar{\epsilon}_j)$ the corresponding normalized complex frame field. Consider the decompositions of $\not{D}$ into a sum of differential operators of order 1 and 0, respectively, namely,

$$\not{D} = A - \frac{1}{2}B, \tag{3.16}$$

where

$$A := \sum_k c(e_k)D_{e_k},$$
$$B := \sum_{r<s} c(e_r \cdot e_s \cdot T_{e_r e_s}).$$

Then, with computations very similar to those of the previous proof, we obtain

(i) $A^2\sigma = \Delta\sigma + \mathcal{R}\sigma - \sum_{j<k} c(e_j \cdot e_k)D_{T_{e_j e_k}}\sigma$;

(ii) $(AB + BA)\sigma$

$$\begin{aligned} &= -\sum_{\substack{j<k\\m}} c(e_j \cdot e_k \cdot e_m \cdot T([e_j, e_k], e_m))\sigma \\ &\quad - \sum_{\substack{j<k\\m}} c(e_j \cdot e_k \cdot e_m \cdot T(T_{e_j e_k}, e_m))\sigma \\ &\quad + \sum_{\substack{j<k\\m}} c\big(e_m \cdot e_j \cdot e_k \cdot D^{\mathcal{C}}_{e_m} T_{e_j e_k}\big)\sigma - 2\sum_{j<k} c(e_j \cdot e_k)D_{T_{e_j e_k}}\sigma \\ &\quad + \sum_{r,s} c\big(e_s \cdot T(D^{\mathcal{C}}_{e_r} e_r, e_s)\big)\sigma - 2\sum_{r,s} c(e_s \cdot T_{e_r e_s})D_{e_r}\sigma; \end{aligned}$$

(iii)
$$B^2\sigma = \sum_{\substack{j<k\\ r<s}} c(e_j \cdot e_k \cdot e_r \cdot e_s \cdot T_{e_je_k} \cdot T_{e_re_s})\sigma \\ - 2\sum_{\substack{j<k\\ m}} c(\overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \overline{\epsilon}_m \cdot T(T_{jk}, \epsilon_m))\sigma.$$

Combining (i), (ii) and (iii), we have

$$\begin{aligned}
\not{D}^2\sigma &= A^2\sigma - \frac{1}{2}(AB + BA)\sigma + \frac{1}{4}B^2\sigma \\
&= \Delta\sigma + \mathcal{R}\sigma + \frac{1}{2}\sum_{\substack{j<k\\ m}} c(e_j \cdot e_k \cdot e_m \cdot T([e_j, e_k], e_m))\sigma \\
&\quad - \frac{1}{2}\sum_{\substack{j<k\\ m}} c\big(e_m \cdot e_j \cdot e_k \cdot D^{\mathcal{C}}_{e_m} T_{e_je_k}\big)\sigma \\
&\quad - \frac{1}{2}\sum_{r,s} c\big(e_s \cdot T(D^{\mathcal{C}}_{e_r} e_r, e_s)\big)\sigma + \sum_{r,s} c(e_s \cdot T_{e_re_s}) D_{e_r}\sigma \\
&\quad + \frac{1}{4}\sum_{\substack{j<k\\ r<s}} c(e_j \cdot e_k \cdot e_r \cdot e_s \cdot T_{e_je_k} \cdot T_{e_re_s})\sigma \\
&= \Delta\sigma + \mathcal{R}\sigma + \frac{1}{2}\sum_{r,s} c\big(e_s \cdot \big(D^{\mathcal{C}}_{e_r} T_{e_re_s} + T([e_r, e_s], e_r) \\
&\qquad\qquad - T(D^{\mathcal{C}}_{e_r} e_r, e_s)\big)\big)\sigma + \mathcal{Q}\sigma + \frac{1}{4}\mathcal{T}_1\sigma \\
&= \Delta\sigma + \mathcal{R}\sigma + \mathcal{Q}\sigma + \frac{1}{4}\mathcal{T}_1\sigma + \frac{1}{2}\sum_{r,s} c\big(e_s \cdot (D^{\mathcal{C}}_{e_r} T)_{e_re_s}\big)\sigma \\
&= \Delta\sigma + \mathcal{R}\sigma + \mathcal{Q}\sigma + \frac{1}{4}\mathcal{T}_1\sigma - \frac{1}{2}\mathcal{T}_2\sigma. \qquad \square
\end{aligned}$$

Formula (3.15) contains the first-order term $\mathcal{Q}$, and hence, is difficult to handle. For this reason, it is convenient to define a new covariant derivative $\widehat{D}$ on E by

$$\widehat{D}_X\sigma := D_X\sigma - \frac{1}{2}\sum_j c\big(e_j \cdot T(X, e_j)\big)\sigma.$$

With straightforward computation, it can be directly verified that the rough Laplacian $\widehat{\Delta}$ of $\widehat{D}$, defined locally by

$$\widehat{\Delta}\sigma := -\sum_{j=1}^{2n} \big(\widehat{D}_{e_j}\widehat{D}_{e_j}\sigma - \widehat{D}_{D^{\mathcal{C}}_{e_j}e_j}\sigma\big),$$

satisfies

$$\widehat{\Delta} = \Delta + \mathcal{Q} - \frac{1}{2}\mathcal{T}_2 + \frac{1}{4}\sum_{j,k,m} c(e_j \cdot e_k \cdot T_{e_m,e_j} \cdot T_{e_m,e_k}),$$

and thus, we obtain the following.

Theorem 3.6. *The Chern-Dirac operator of E verifies*

$$\not{D}^2 = \widehat{\Delta} + \mathcal{R} - \frac{1}{2}P - \frac{1}{8}|T|^2, \tag{3.17}$$

where P is defined by

$$P := \sum_{j<k<r<s} (g(T_{e_j,e_k}, T_{e_r,e_s}) + g(T_{e_j,e_s}, T_{e_k,e_r}) + g(T_{e_j,e_r}, T_{e_k,e_s}))c(e_j \cdot e_k \cdot e_r \cdot e_s)$$

and

$$|T|^2 = \sum_{j,k,r} T(e_j, e_k, e_r)^2.$$

Moreover, if the complex dimension of M is $n = 2$, then $P = 0$ and $|T|^2 = 2|\vartheta|^2$. Thus, we obtain the following.

Corollary 3.7. *The Chern-Dirac operator over a complex surface verifies*

$$\not{D}^2 = \widehat{\Delta} + \mathcal{R} - \frac{1}{4}|\vartheta|^2. \tag{3.18}$$

Following the same arguments of classical Bochner type theorems, Theorems 3.4 and 3.6 can be used to determine vanishing properties for solutions of equations of the form $\not{\partial}'\sigma = 0$ or $\not{D}\sigma = 0$, provided that appropriate conditions on curvature and torsion are imposed. Here, we do not investigate such possibilities.

In the following sections, we prove the existence of some important Chern-Dirac bundles, canonically associated with any Hermitian manifold (not necessarily Kähler), to which all results determined so far immediately apply.

4. $\mathcal{V}$-spinors and cohomology of Hermitian manifolds.

4.1. Canonical and anticanonical spinor bundle on Hermitian manifolds. Let (M, g, J) be an Hermitian $2n$-manifold. We indicate with $\mathcal{S}^{\uparrow}M$ and $\mathcal{S}^{\downarrow}M$ the spinor bundles on M associated with the canonical and anticanonical spin$^{\mathbb{C}}$ structures, respectively. In these cases, it can be directly verified that the eigen-subbundles $\mathcal{S}^{\uparrow 0}M$ and $\mathcal{S}^{\downarrow n}M$ are trivial line bundles. Hence, we may fix:

(a) two nowhere vanishing global sections

$$\psi^0 : M \longrightarrow \mathcal{S}^{\uparrow 0}M, \qquad \varphi^0 : M \longrightarrow \mathcal{S}^{\downarrow n}M; \tag{4.1}$$

(b) two Hermitian metrics $h^{\uparrow}$, $h^{\downarrow}$ on $\mathcal{S}^{\uparrow}M$, $\mathcal{S}^{\downarrow}M$, respectively, which are invariant under the Clifford multiplication by tangent vectors and such that

$$h^{\uparrow}(\psi^0, \psi^0) = 1 = h^{\downarrow}(\varphi^0, \varphi^0). \tag{4.2}$$

Tensoring with sections ψ^0 and φ^0, we may identify $\Lambda^{0,q}(T^*M) \simeq \Lambda^{0,q}(T^*M) \otimes \mathcal{S}^{\uparrow 0}M$ and $\Lambda^{p,0}(T^*M) \simeq \Lambda^{p,0}(T^*M) \otimes \mathcal{S}^{\downarrow n}M$, so that the maps (2.11) determine isometries

$$\begin{aligned} \alpha^{\uparrow k} &: \Lambda^{0,k}(T^*M) \simeq \Lambda^{0,k}(T^*M) \otimes \mathcal{S}^{\uparrow 0}M \longrightarrow \mathcal{S}^{\uparrow k}M, \\ \beta^{\downarrow k} &: \Lambda^{k,0}(T^*M) \simeq \Lambda^{k,0}(T^*M) \otimes \mathcal{S}^{\downarrow n}M \longrightarrow \mathcal{S}^{\downarrow n-k}M. \end{aligned} \tag{4.3}$$

Moreover, from (2.5), (2.6) and (2.10), the following useful lemma can be immediately proven.

Lemma 4.1. *Let λ be a 1-form. Then, for every $\nu \in \Omega^{p,0}(M)$ and $\overline{\mu} \in \Omega^{0,q}(M)$, we have*

$$\begin{aligned} \lambda \cdot \overline{\mu} \cdot \psi^0 &= (\lambda^{01} \wedge \overline{\mu}) \cdot \psi^0 - 2\big((\lambda^{\sharp})^{01} \lrcorner\, \overline{\mu}\big) \cdot \psi^0, \\ \lambda \cdot \nu \cdot \varphi^0 &= (\lambda^{10} \wedge \nu) \cdot \varphi^0 - 2\big((\lambda^{\sharp})^{10} \lrcorner\, \nu\big) \cdot \varphi^0. \end{aligned} \tag{4.4}$$

Considering the action of the Kähler form ω on the dual bundle $\mathcal{S}^{\uparrow *}M$ given by

$$(\omega \cdot L)(\psi) \ := \ -L(\omega \cdot \psi),$$

we obtain a corresponding split

$$\mathcal{S}^{\uparrow *}M = \mathcal{S}^{\uparrow *0}M \oplus \cdots \oplus \mathcal{S}^{\uparrow *n}M.$$

Note that the $\mathbb{C}$-linear maps

$$\delta^k : \mathcal{S}^{\downarrow n-k}M \longrightarrow \mathcal{S}^{\uparrow *k}M, \qquad \delta^k(\varphi) := h^{\uparrow}(\alpha^{\uparrow k}(\overline{(\beta^{\downarrow k})^{-1}(\varphi)}), \cdot)$$

are actually isometries. Moreover, if $\varphi = \nu \cdot \varphi^0 \in \mathcal{S}^{\downarrow n-k}M$ and $\psi = \overline{\mu} \cdot \psi^0 \in \mathcal{S}^{\uparrow k}M$, then we have that

$$\varphi(\psi) := \delta^k(\varphi)(\psi) = h^{\uparrow}(\overline{\nu} \cdot \psi^0, \overline{\mu} \cdot \psi^0) = g(\nu, \overline{\mu}). \tag{4.5}$$

From now on, we tacitly use such maps to identify $\mathcal{S}^{\downarrow}M \simeq \mathcal{S}^{\uparrow *}M$.

Denote by $\mathfrak{S}^{\uparrow}(M)$ and $\mathfrak{S}^{\downarrow}(M)$ the spaces of global sections of $\mathcal{S}^{\uparrow}M$ and $\mathcal{S}^{\uparrow}M$, respectively. Since the unitary frame bundle $\mathrm{U}_{g,J}(M)$ is a U_n-reduction of both $\mathcal{P}^{\downarrow}(M)$ and $\mathcal{P}^{\uparrow}(M)$ (see Definition 2.1), we obtain the following.

Proposition 4.2. *The Hermitian bundles* $(\mathcal{S}^{\uparrow}M, h^{\uparrow})$, $(\mathcal{S}^{\downarrow}M, h^{\downarrow})$ *are Chern-Dirac bundles with respect to the action of the Chern connection of M and the standard Clifford multiplication, which are isometric to* $\Lambda^{0,\cdot}(T^*M)$ *and* $\Lambda^{\cdot,0}(T^*M)$*, respectively, by the maps* (4.3)*.*

Proof. From the very definition of $\mathcal{P}^{\uparrow}(M)$, it follows that

$$\mathcal{S}^{\uparrow}M = \mathrm{U}_{g,J}(M) \times_{(\kappa_{2n} \circ F_+)} \mathcal{S}_{2n},$$

and thus, the Chern connection $\omega^{\mathcal{C}}$ on $\mathrm{U}_{g,J}(M)$ defines a covariant derivative of the sections of $\mathcal{S}^{\uparrow}M$ by

$$D^{\mathcal{C}}_X \psi := d^{\mathcal{C}}\psi(\widehat{X}) = d\psi(\widehat{X}) + (\kappa_{2n} \circ F_+)_*\big(\omega^{\mathcal{C}}(\widehat{X})\big)\psi, \tag{4.6}$$

where:

(a) $\psi \in \mathfrak{S}^{\uparrow}(M)$ is identified with a function $\psi : \mathrm{U}_{g,J}(M) \to \mathcal{S}_{2n}$ such that

$$\psi(uA) = \kappa_{2n}\big(F_+(A^{-1})\big)\psi(u) \quad \text{for every } A \in \mathrm{U}_n;$$

(b) $\widehat{X}$ is the horizontal lift of X on $T\mathrm{U}_{g,J}(M)$ determined by the connection form $\omega^{\mathcal{C}}$;

(c) $d^{\mathcal{C}}$ is the exterior covariant derivative on $\mathrm{U}_{g,J}(M)$ determined by $\omega^{\mathcal{C}}$.

Using (4.6) and the fact that the differential of F_+ is merely

$$(F_+)_* : \mathfrak{u}_n \longrightarrow \mathfrak{so}_{2n} \oplus i\mathbb{R}, \qquad (F_+)_*(A) = (\tau_{2n})_*^{-1}(A) + \frac{1}{2}\operatorname{Tr}(A),$$

we get that

$$D^{\mathcal{C}}_X\big(Y \cdot \psi\big) = \big(D^{\mathcal{C}}_X Y\big) \cdot \psi + Y \cdot D^{\mathcal{C}}_X \psi$$
$$\text{for every } X, Y \in \mathfrak{X}(M),\ \psi \in \mathfrak{S}^{\uparrow}(M).$$

Since the representation κ_{2n} of $\mathrm{Spin}^{\mathbb{C}}_{2n}$ is unitary, it follows that $D^{\mathcal{C}}\, h^{\uparrow} = 0$ and, furthermore, it may be verified that (3.1) holds. The same arguments, mutatis mutandis, determine a covariant derivative $D^{\mathcal{C}}$ on spinors $\varphi \in \mathfrak{S}^{\downarrow}(M)$ which fulfill the necessary conditions. □

4.2. A fundamental example of Chern-Dirac bundle: The $\mathcal{V}$-spinors. Given an Hermitian $2n$-manifold (M, g, J), with canonical and anticanonical spinor bundles $\mathcal{S}^{\uparrow}M$, $\mathcal{S}^{\downarrow}M$, we define as the $\mathcal{V}$-*spinor bundle of* M the vector bundle

$$\mathcal{V}M := \mathcal{S}^{\downarrow}M \otimes \mathcal{S}^{\uparrow}M.$$

Note that $\mathcal{V}M$ is equipped with the Hermitian metric $\check{h} := h^{\downarrow} \otimes h^{\uparrow}$, where $h^{\downarrow}$ and $h^{\uparrow}$ are defined in (4.2), and with the bigradation given by the subbundles

$$\mathcal{V}^{\,p,q}M := \mathcal{S}^{\downarrow n-p}M \otimes \mathcal{S}^{\uparrow q}M \simeq \mathcal{S}^{\uparrow * p}M \otimes \mathcal{S}^{\uparrow q}M, \quad 0 \le p,\ q \le n.$$

In addition, note that, since $\mathcal{S}^{\uparrow 0}M$ and $\mathcal{S}^{\downarrow n}M$ are trivial, the subbundles $\mathcal{V}^{\,\cdot,0}M$ and $\mathcal{V}^{\,0,\cdot}M$ are isomorphic to $\mathcal{S}^{\downarrow}M$ and $\mathcal{S}^{\uparrow}M$, respectively, so that $\mathcal{V}M$ can be considered as a bundle which naturally includes both the canonical and anticanonical spinor bundle. Let us denote the space of global sections of $\mathcal{V}M$ by $\mathfrak{V}(M)$.

From the proof of Proposition 4.2, we obtain that the Chern connection defines a covariant derivative along the vector fields of M of the sections of $\mathcal{V}M$:

$$D^{\mathcal{C}} \colon \mathfrak{X}(M) \otimes \mathfrak{V}(M) \longrightarrow \mathfrak{V}(M).$$

Let φ^0 and ψ^0 be the global sections in (4.1), satisfying (4.2), and ξ^0 the distinguished section $\xi^0 := \varphi^0 \otimes \psi^0 \in \mathfrak{V}(M)$. Clearly, ξ^0 is a nowhere vanishing global section of $\mathcal{V}^{\,0,0}M$ such that $\check{h}(\xi^0, \xi^0) = 1$. Note that the action of $D^{\mathcal{C}}$ on $\mathfrak{S}^{\uparrow 0}(M) \simeq \mathcal{C}^{\infty}(M; \mathbb{C})$ coincides with the Chern covariant derivative of the trivial holomorphic Hermitian line bundle $\big(\mathcal{S}^{\uparrow 0}M,\, h^{\uparrow}|_{\mathcal{S}^{\uparrow 0}M\otimes\mathcal{S}^{\uparrow 0}M}, \psi^0\big)$ and therefore corresponds to the trivial covariant derivative on this bundle. The same holds on $\mathfrak{S}^{\downarrow n}(M) \simeq \mathcal{C}^{\infty}(M;\mathbb{C})$. Thus, it follows that the action of $D^{\mathcal{C}}$ on sections of $\mathcal{V}^{0,0}M$ is trivial as well.

Interest in the bundle of $\mathcal{V}$-spinors comes from the fact that there are *two structures of complex left $\mathcal{C}\ell^{\mathbb{C}} M$-modules on $\mathcal{V}M$ which makes it a Chern-Dirac bundle isomorphic with the bundle of complex forms of M.* In order to see this, let

$$\cdot_L, \cdot_R : \mathcal{C}\ell^{\mathbb{C}} M \otimes \mathcal{V}M \longrightarrow \mathcal{V}M \tag{4.7}$$

be the unique $\mathbb{C}$-linear operations which transform each pair, given by an element $w \in \mathcal{C}\ell^{\mathbb{C}}_x M$ and a homogeneous decomposable $\mathcal{V}$-spinor $\varphi\otimes\psi \in \mathcal{V}^{p,q}_x M$ at some $x\in M$, into the $\mathcal{V}$-spinors

$$w \cdot_L (\varphi\otimes\psi) := (w\cdot\varphi)\otimes\psi, \qquad w \cdot_R (\varphi\otimes\psi) := (-1)^p \varphi\otimes(w\cdot\psi).$$

As mentioned above, the following properties hold.

Proposition 4.3.

(i) *The bundle $\mathcal{V}M$ is a Chern-Dirac bundle with respect to* (4.7).
(ii) *There exists an isometry $\varsigma : \Lambda^{\cdot}(T^{*\mathbb{C}}M)\to\mathcal{V}M$ such that $D^{\mathcal{C}}_X \circ \varsigma = \varsigma\circ D^{\mathcal{C}}_X$ for every $X\in\mathfrak{X}(M)$.*

Proof. The first claim follows directly from Proposition 4.2. For the second one, let

$$\begin{aligned}\jmath^{p,q} : (\Lambda^{p,0}(T^*M)\otimes\mathcal{S}^{\downarrow n}M)\otimes(\Lambda^{0,q}(T^*M)\otimes\mathcal{S}^{\uparrow 0}M)&\\ \longrightarrow \Lambda^{p,q}(T^*M)\otimes\mathcal{V}^{0,0}M&\end{aligned}$$

be the vector bundle isomorphism which transforms decomposable elements into

$$(\jmath^{p,q})_x\big((\nu\otimes\varphi^0_x)\otimes(\overline{\mu}\otimes\psi^0_x)\big) := (\nu\wedge\overline{\mu})\otimes\xi^0_x, \quad x\in M.$$

Also, let

$$\varsigma^{p,q} : \Lambda^{p,q}(T^*M)\simeq\Lambda^{p,q}(T^*M)\otimes\mathcal{V}^{0,0}M \longrightarrow \mathcal{V}^{p,q}M, \tag{4.8}$$

$$\varsigma^{p,q} := \big(\beta^{\downarrow p}\otimes\alpha^{\uparrow q}\big)\circ(\jmath^{p,q})^{-1},$$

where $\alpha^{\uparrow q}$ and $\beta^{\downarrow p}$ are the isometries defined in (4.3). By construction, each isomorphism $\varsigma^{p,q}$ is actually an isometry of Hermitian bundles, and they all combine into a global isometry

$$\varsigma : \Lambda^{\cdot}(T^{*\mathbb{C}}M) \simeq \Lambda^{\cdot}(T^{*\mathbb{C}}M) \otimes \mathcal{V}^{0,0}M \longrightarrow \mathcal{V}M.$$

The fact that $D^{\mathcal{C}}_X \circ \varsigma = \varsigma \circ D^{\mathcal{C}}_X$ follows from the previously mentioned property $D^{\mathcal{C}}\psi^0 = D^{\mathcal{C}}\varphi^0 = 0$ yields that, for every $\eta = \nu \wedge \overline{\mu} \in \Omega^{p,q}(M)$ and $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
D^{\mathcal{C}}_X \varsigma^{p,q}(\eta) &= \frac{1}{2^{(p+q)/2}} D^{\mathcal{C}}_X\big((\nu \cdot \varphi^0) \otimes (\overline{\mu} \cdot \psi^0)\big) \\
&= \frac{1}{2^{(p+q)/2}} \big((D^{\mathcal{C}}_X\nu \cdot \varphi^0) \otimes (\overline{\mu} \cdot \psi^0) + (\nu \cdot \varphi^0) \otimes (D^{\mathcal{C}}_X\overline{\mu} \cdot \psi^0)\big) \\
&= \big(\beta^{\downarrow p} \otimes \alpha^{\uparrow q}\big)\big((D^{\mathcal{C}}_X\nu \otimes \varphi^0) \otimes (\overline{\mu} \otimes \psi^0) + (\nu \otimes \varphi^0) \\
&\qquad\qquad\qquad\qquad \otimes (D^{\mathcal{C}}_X\overline{\mu} \otimes \psi^0)\big) \\
&= \big(\beta^{\downarrow p} \otimes \alpha^{\uparrow q}\big)\big((\jmath^{p,q})^{-1}\big((D^{\mathcal{C}}_X\nu \wedge \overline{\mu}) \otimes \xi^0 + (\nu \wedge D^{\mathcal{C}}_X\overline{\mu}) \otimes \xi^0\big)\big) \\
&= \varsigma^{p,q}\big(D^{\mathcal{C}}_X\eta\big),
\end{aligned}$$

and this completes the proof. □

The components $\varsigma^{p,q}$ of the isometry ς defined in (4.8) play an important role in the following discussion. It is therefore convenient to introduce the notation

$$\widehat{\cdot}\,|_{\Lambda^{p,q}(T^*M)\otimes\mathcal{V}^{0,0}M} := \Big(\big(\,\cdot\,|_{\Lambda^{p,0}(T^*M)\otimes\mathcal{S}^{\downarrow n}M}\big) \otimes \big(\,\cdot\,|_{\Lambda^{0,q}(T^*M)\otimes\mathcal{S}^{\uparrow 0}M}\big)\Big) \circ (\jmath^{p,q})^{-1},$$

which allows setting the map $\varsigma^{p,q}$ as

$$\varsigma^{p,q}(\eta) = \frac{1}{2^{(p+q)/2}}\,\eta \,\widehat{\cdot}\, \xi^0 \quad \text{for every } \eta \in \Omega^{p,q}(M). \tag{4.9}$$

4.3. The algebraic structure of $\mathcal{V}$-spinors. Let (M, g, J) be an Hermitian $2n$-manifold and $\mathcal{C}\ell^{\mathbb{C}}M$ its complex Clifford bundle. Since $\mathcal{C}\ell^{\mathbb{C}}_{2n} \simeq \mathfrak{gl}(\mathcal{S}_{2n}) = \mathcal{M}_{2^n}(\mathbb{C})$, there exists a $\mathbb{C}$-linear isomorphism

$$\chi : \mathcal{V}M \longrightarrow \mathcal{C}\ell^{\mathbb{C}}M \simeq \mathfrak{gl}(\mathcal{S}^{\uparrow}M), \tag{4.10}$$

which maps each decomposable $\mathcal{V}$-spinor $\varphi \otimes \psi \in \mathcal{V}M$ into the unique element $w := \chi(\varphi\otimes\psi) \in \mathcal{C}\ell^{\mathbb{C}}M$, defined by the condition

$$w \cdot \psi' = \varphi(\psi')\psi \quad \text{for every } \psi' \in \mathcal{S}^{\uparrow}M.$$

This gives rise to the following $\mathbb{C}$-linear isomorphism between $\Lambda^{\cdot}(T^{*\mathbb{C}}M)$ and $\mathcal{C}\ell^{\mathbb{C}}M$:

$$\chi\circ\varsigma:\Lambda^{\cdot}(T^{*\mathbb{C}}M)\longrightarrow\mathcal{C}\ell^{\mathbb{C}}M. \tag{4.11}$$

The reader should nonetheless be aware that *such a map is in general different from the canonical isomorphism* $\mathcal{C}\ell^{\mathbb{C}}M\simeq\Lambda^{\cdot}(T^{\mathbb{C}}M)\overset{g}{\simeq}\Lambda^{\cdot}(T^{*\mathbb{C}}M)$ *described in* (2.6). Furthermore, the isomorphism (4.11) induces a bigradation on $\mathcal{C}\ell^{\mathbb{C}}M$, defined by

$$\mathcal{C}\ell^{p,q}M:=\chi(\mathcal{V}^{p,q}M)=(\chi\circ\varsigma)(\Lambda^{p,q}(T^{*}M)), \tag{4.12}$$

which is different from that considered by Michelsohn in [**10,** Section 2.B].

We can finally define a structure of bundles of algebras on $\mathcal{V}M$ by setting

$$\xi_1\cdot\xi_2:=\chi^{-1}(\chi(\xi_1)\cdot\chi(\xi_2))\quad\text{for every }\xi_1,\xi_2\in\mathcal{V}M.$$

Note that, if $\xi_1=\varphi_1\otimes\psi_1$ and $\xi_2=\varphi_2\otimes\psi_2$ are homogeneous decomposable $\mathcal{V}$-spinors, with $\varphi_j=\nu_j\cdot\varphi^0\in\mathcal{S}^{\downarrow n-p_j}M$ and $\psi_j=\overline{\mu}_j\cdot\psi^0\in\mathcal{S}^{\uparrow q_j}M$, from (4.5) and (4.8) it follows that such a product is merely

$$\begin{aligned}\xi_1\cdot\xi_2&=(\varphi_1\otimes\psi_1)\cdot(\varphi_2\otimes\psi_2)=\varphi_1(\psi_2)(\varphi_2\otimes\psi_1)\\&=2^{(p_2+q_1)/2}g(\nu_1,\overline{\mu}_2)\,\varsigma(\nu_2\wedge\overline{\mu}_1).\end{aligned}$$

4.4. Partial Chern-Dirac operators on $\mathcal{V}$-spinors. Let (M,g,J) be an Hermitian $2n$-manifold. The main result of this section consists in the proof that the partial Chern-Dirac operators on $\mathcal{V}M$ correspond to the standard operators ∂, ∂^* and $\overline{\partial}^*$, $\overline{\partial}$ on differential forms.

We indicate by $\not{\partial}'^{(L)}$, $\not{\partial}''^{(L)}$ and $\not{\partial}'^{(R)}$, $\not{\partial}''^{(R)}$ the partial Chern-Dirac operators on sections of $\mathcal{V}M$ determined in (3.3) using the Clifford multiplications $c=\cdot_L$ and $c=\cdot_R$, respectively. In full analogy with [**10,** Theorem 4.1], we prove the following.

Theorem 4.4. *The operators* $\not{\partial}'^{(L)}$ *and* $\not{\partial}''^{(L)}$ (*respectively,* $\not{\partial}'^{(R)}$ *and* $\not{\partial}''^{(R)}$) *are formal adjoints of each other and squares satisfy*

$$(\not{\partial}'^{(L)})^2=0=(\not{\partial}''^{(L)})^2,\qquad(\not{\partial}'^{(R)})^2=0=(\not{\partial}''^{(R)})^2. \tag{4.13}$$

Moreover, the cochain complexes

$$\begin{aligned}\mathfrak{V}^{0,q}(M) &\xrightarrow{\not{\partial}'^{(L)}} \mathfrak{V}^{1,q}(M) \xrightarrow{\not{\partial}'^{(L)}} \cdots \xrightarrow{\not{\partial}'^{(L)}} \mathfrak{V}^{n,q}(M),\\ \mathfrak{V}^{p,0}(M) &\xrightarrow{\not{\partial}''^{(R)}} \mathfrak{V}^{p,1}(M) \xrightarrow{\not{\partial}''^{(R)}} \cdots \xrightarrow{\not{\partial}''^{(R)}} \mathfrak{V}^{p,n}(M),\end{aligned} \tag{4.14}$$

are elliptic.

Proof. The first claim follows directly from Proposition 3.3, while (4.13) follows from (3.12) and properties of the curvature of the Chern connection. In order to prove that $\not{\partial}'^{(L)}\big(\mathfrak{V}^{p,q}(M)\big) \subset \mathfrak{V}^{p+1,q}(M)$, consider a unitary frame field defined in an open subset $\mathcal{U} \subset M$ and the associated normalized complex frame $(\epsilon_j, \overline{\epsilon}_j)$ defined in (2.1). It may be directly verified that

(i) $\displaystyle\sum_j \omega \cdot \overline{\epsilon}_j = \sum_j (\overline{\epsilon}_j \cdot \omega - 2i\,\overline{\epsilon}_j)$,

(ii) $\displaystyle\sum_{r<s} \omega \cdot \overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs} = \sum_{r<s} (\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs} \cdot \omega - 2i\,\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs})$.

Now, set a homogeneous section $\xi \in \mathfrak{V}^{p,q}(M)$. Since $D^{\mathcal{C}}\omega = 0$, from (i) and (ii), we obtain:

$$\begin{aligned}\omega \cdot_L \not{\partial}'^{(L)}\xi &= \sum_j (\omega \cdot \overline{\epsilon}_j) \cdot_L D^{\mathcal{C}}_{\epsilon_j}\xi - \frac{1}{2}\sum_{r<s}(\omega \cdot \overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L \xi\\ &= (n-2p-2)\, i \sum_j \overline{\epsilon}_j \cdot_L D^{\mathcal{C}}_{\epsilon_j}\xi\\ &\quad - \frac{1}{2}(n-2p-2)\, i \sum_{r<s} (\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L \xi\\ &= (n-2(p+1))\, i\, \not{\partial}'^{(L)}\xi,\end{aligned}$$

and then it is immediate to verify that $\omega \cdot_R \not{\partial}'^{(L)}\xi = (-1)^{p+1}(2q - n)i\,\not{\partial}'^{(L)}\xi$. The inclusion $\not{\partial}''^{(R)}\big(\mathfrak{V}^{p,q}(M)\big) \subset \mathfrak{V}^{p,q+1}(M)$ can be similarly shown. It remains to prove that equations (4.14) are elliptic. In order to see this, consider a real covector $\lambda \in TM$, and observe that the principal symbols of the four partial Chern-Dirac operators are

$$\sigma(\not{\partial}'^{(L)})(\lambda) = (\lambda^\sharp)^{10} \cdot_L, \qquad \sigma(\not{\partial}''^{(L)})(\lambda) = (\lambda^\sharp)^{01} \cdot_L,$$
$$\sigma(\not{\partial}''^{(R)})(\lambda) = (\lambda^\sharp)^{01} \cdot_R, \qquad \sigma(\not{\partial}'^{(R)})(\lambda) = (\lambda^\sharp)^{10} \cdot_R .$$

Ellipticity now follows from the relation

$$(\lambda^\sharp)^{10} \cdot (\lambda^\sharp)^{01} + (\lambda^\sharp)^{01} \cdot (\lambda^\sharp)^{10} = -g(\lambda, \lambda),$$

and this completes the proof. □

The next theorem establishes a crucial relation between the partial Chern-Dirac operators on $\mathcal{V}$-spinors and the standard operators ∂, ∂^* and $\overline{\partial}^*$, $\overline{\partial}$ on differential forms. It can immediately be seen that this relation is much simpler than the analogous result on the operators $\mathcal{D}$, $\overline{\mathcal{D}}$, considered by Michelsohn for Clifford bundles on Kähler manifolds (see [**10,** Proposition 5.1]).

Theorem 4.5. *Let $\varsigma : \Lambda^\cdot(T^{*\mathbb{C}}M) \to \mathcal{V}M$ be the isometry defined in Proposition* 4.3*. Then, the partial Chern-Dirac operators verify*

$$\varsigma^{-1} \circ \not{\partial}'^{(L)} \circ \varsigma = \sqrt{2}\partial, \qquad \varsigma^{-1} \circ \not{\partial}''^{(L)} \circ \varsigma = \sqrt{2}\partial^*,$$
$$\varsigma^{-1} \circ \not{\partial}'^{(R)} \circ \varsigma = \sqrt{2}\bar{\partial}^*, \qquad \varsigma^{-1} \circ \not{\partial}''^{(R)} \circ \varsigma = \sqrt{2}\bar{\partial}.$$

For the proof, we need a preparatory lemma.

Lemma 4.6. *Let $x \in M$ and $\eta \in \Omega^{p,q}(M)$. Then*:

$$\begin{aligned}
\partial\eta|_x &= \sum_j \big(\epsilon^j \wedge D^{\mathcal{C}}_{\epsilon_j}\eta + \epsilon^j(T) \wedge \big(\epsilon_j \lrcorner \eta\big)\big),\\
\overline{\partial}\eta|_x &= \sum_j \big(\bar{\epsilon}^j \wedge D^{\mathcal{C}}_{\bar{\epsilon}_j}\eta + \bar{\epsilon}^j(T) \wedge \big(\bar{\epsilon}_j \lrcorner \eta\big)\big),\\
\partial^*\eta|_x &= -\sum_j \epsilon_j \lrcorner D^{\mathcal{C}}_{\bar{\epsilon}_j}\eta - \sum_{r<s} \epsilon_r \lrcorner \epsilon_s \lrcorner \big((T_{\bar{r}\bar{s}})^\flat \wedge \eta\big),\\
\overline{\partial}^*\eta|_x &= -\sum_j \bar{\epsilon}_j \lrcorner D^{\mathcal{C}}_{\epsilon_j}\eta - \sum_{r<s} \bar{\epsilon}_r \lrcorner \bar{\epsilon}_s \lrcorner \big((T_{rs})^\flat \wedge \eta\big),
\end{aligned}$$

where $(\epsilon_j, \bar{\epsilon}_j)$ is the standard normalized complex frame (2.1) *determined by a unitary basis for T_xM.*

Proof. Since $S = D^{\mathcal{C}} - D^{LC}$, the contorsion tensor acts in a natural manner on forms. In addition, it can be proven by induction that its action is such that

$$S_X\big(\Omega^{p,q}(M)\big) \subset \Omega^{p+1,q-1}(M) \,\oplus\, \Omega^{p,q}(M) \,\oplus\, \Omega^{p-1,q+1}(M),$$

$0 \leq p,\ q \leq n$, for every $X \in \mathfrak{X}(M)$. From standard properties of the Levi-Civita connection and (2.2), given a unitary basis $(e_j) \subset T_xM$, we obtain

$$\begin{aligned} d\eta|_x &= \sum_{s=1}^{2n} \big(e^s \wedge D^{LC}_{e_s}\eta\big) \\ &= \sum_j \big(\epsilon^j \wedge D^{\mathcal{C}}_{\epsilon_j}\eta + \overline{\epsilon}^j \wedge D^{\mathcal{C}}_{\overline{\epsilon}_j}\eta - \epsilon^j \wedge S_{\epsilon_j}\eta - \overline{\epsilon}^j \wedge S_{\overline{\epsilon}_j}\eta\big), \end{aligned}$$

and thus,

$$\begin{aligned} \partial\eta|_x &= \big(d\eta|_x\big)^{p+1,q} \\ &= \sum_j \big(\epsilon^j \wedge D^{\mathcal{C}}_{\epsilon_j}\eta - \epsilon^j \wedge \big(S_{\epsilon_j}\eta\big)^{p,q} - \overline{\epsilon}^j \wedge \big(S_{\overline{\epsilon}_j}\eta\big)^{p+1,q-1}\big), \\ \overline{\partial}\eta|_x &= \big(d\eta|_x\big)^{p,q+1} \\ &= \sum_j \big(\overline{\epsilon}^j \wedge D^{\mathcal{C}}_{\overline{\epsilon}_j}\eta - \epsilon^j \wedge \big(S_{\epsilon_j}\eta\big)^{p-1,q+1} - \overline{\epsilon}^j \wedge \big(S_{\overline{\epsilon}_j}\eta\big)^{p,q}\big). \end{aligned}$$

Furthermore, it can be directly checked that

$$\begin{aligned} \sum_j \big(- \epsilon^j \wedge \big(S_{\epsilon_j}\eta\big)^{p,q} - \overline{\epsilon}^j \wedge \big(S_{\overline{\epsilon}_j}\eta\big)^{p+1,q-1}\big) &= \sum_j \epsilon^j(T) \wedge \big(\epsilon_j \lrcorner \eta\big), \\ \sum_j \big(- \epsilon^j \wedge \big(S_{\epsilon_j}\eta\big)^{p-1,q+1} - \overline{\epsilon}^j \wedge \big(S_{\overline{\epsilon}_j}\eta\big)^{p,q}\big) &= \sum_j \overline{\epsilon}^j(T) \wedge \big(\overline{\epsilon}_j \lrcorner \eta\big). \end{aligned}$$

The remaining identities can be proven in a similar manner. □

We may now proceed with the proof of Theorem 4.5.

Proof of Theorem 4.5. Consider a unitary frame field $(e_j) : \mathcal{U} \subset M \to \mathrm{U}_{g,J}(M)$, the normalized complex vectors ϵ_j, $\overline{\epsilon}_j$ defined in (2.1) and a form $\eta = \nu \wedge \overline{\mu} \in \Omega^{p,q}(M)$. Then, using notation (4.9), from (4.4) and Lemma 4.6, it follows that

$$\begin{aligned} \not{\partial}'^{(L)}(\eta \mathbin{\widehat{\cdot}} \xi^0) &= \sum_j \overline{\epsilon}_j \cdot_L D^{\mathcal{C}}_{\epsilon_j}(\eta \mathbin{\widehat{\cdot}} \xi^0) - \frac{1}{2}\sum_{r<s}(\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0) \\ &= \sum_j \big(\overline{\epsilon}_j \cdot_L \big((D^{\mathcal{C}}_{\epsilon_j}\nu \wedge \overline{\mu}) \mathbin{\widehat{\cdot}} \xi^0\big) + \overline{\epsilon}_j \cdot_L \big((\nu \wedge D^{\mathcal{C}}_{\epsilon_j}\overline{\mu}) \mathbin{\widehat{\cdot}} \xi^0\big)\big) \end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{2}\sum_{r<s}(\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0) \\
&= \sum_j \big((\epsilon^j \cdot D^{\mathcal{C}}_{\epsilon_j}\nu \cdot \varphi^0) \otimes (\overline{\mu} \cdot \psi^0) \\
&\qquad\quad + (\epsilon^j \cdot \nu \cdot \varphi^0) \otimes (D^{\mathcal{C}}_{\epsilon_j}\overline{\mu} \cdot \psi^0)\big) \\
&\quad -\frac{1}{2}\sum_{\substack{r<s\\ m}} T_{rs\overline{m}}(\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot \epsilon_m \cdot \nu \cdot \varphi^0) \otimes (\overline{\mu} \cdot \psi^0) \\
&= \sum_j \big(\big((\epsilon^j \wedge D^{\mathcal{C}}_{\epsilon_j}\nu) \cdot \varphi^0\big) \otimes (\overline{\mu} \cdot \psi^0) \\
&\qquad\quad + \big((\epsilon^j \wedge \nu) \cdot \varphi^0\big) \otimes (D^{\mathcal{C}}_{\epsilon_j}\overline{\mu} \cdot \psi^0)\big) \\
&\quad + \sum_{\substack{r<s\\ m}} T_{rs\overline{m}}\big((\epsilon^r \wedge \epsilon^s \wedge (\epsilon_m \lrcorner \nu)) \cdot \varphi^0\big) \otimes (\overline{\mu} \cdot \psi^0) \\
&= \sum_j (\epsilon^j \wedge D^{\mathcal{C}}_{\epsilon_j}\eta) \mathbin{\widehat{\cdot}} \xi^0 + \sum_{\substack{r<s\\ m}} T_{rs\overline{m}}\,(\epsilon^r \wedge \epsilon^s \wedge (\epsilon_m \lrcorner \eta)) \mathbin{\widehat{\cdot}} \xi^0 \\
&= \partial\eta \mathbin{\widehat{\cdot}} \xi^0.
\end{aligned}
$$

Thus, we obtain:

$$
\big(\varsigma^{-1} \circ \not{\partial}'^{(L)} \circ \varsigma\big)(\eta) = \frac{1}{2^{p+q/2}}\varsigma^{-1}\big(\partial\eta \mathbin{\widehat{\cdot}} \xi^0\big) = \frac{2^{(p+q+1)/2}}{2^{(p+q)/2}}\partial\eta = \sqrt{2}\partial\eta.
$$

Proceeding in a similar manner, we obtain

$$
\begin{aligned}
\not{\partial}''^{(L)}(\eta \mathbin{\widehat{\cdot}} \xi^0) &= \sum_j \epsilon_j \cdot_L D^{\mathcal{C}}_{\overline{\epsilon}_j}(\eta \mathbin{\widehat{\cdot}} \xi^0) - \frac{1}{2}\sum_{r<s}(\epsilon_r \cdot \epsilon_s \cdot T_{\overline{r}\overline{s}}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0) \\
&= \sum_{j=1}^n \big(\epsilon_j \cdot_L \big((D^{\mathcal{C}}_{\overline{\epsilon}_j}\nu \wedge \overline{\mu}) \mathbin{\widehat{\cdot}} \xi^0\big) + \epsilon_j \cdot_L \big((\nu \wedge D^{\mathcal{C}}_{\overline{\epsilon}_j}\overline{\mu}) \mathbin{\widehat{\cdot}} \xi^0\big)\big) \\
&\quad - \frac{1}{2}\sum_{r<s}(\epsilon_r \cdot \epsilon_s \cdot T_{\overline{r}\overline{s}}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0) \\
&= \sum_{j=1}^n \big((\overline{\epsilon}^j \cdot D^{\mathcal{C}}_{\overline{\epsilon}_j}\nu \cdot \varphi^0) \otimes (\overline{\mu} \cdot \psi^0) \\
&\qquad\quad + (\overline{\epsilon}^j \cdot \nu \cdot \varphi^0) \otimes (D^{\mathcal{C}}_{\overline{\epsilon}_j}\overline{\mu} \cdot \psi^0)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} \sum_{\substack{r<s \\ m}} T_{\bar{r}\bar{s}m} \left(\epsilon_r \cdot \epsilon_s \cdot \bar{\epsilon}_m \cdot \nu \cdot \varphi^0\right) \otimes \left(\overline{\mu} \cdot \psi^0\right) \\
= &\sum_{j=1}^{n} \Big(\big(-2(\epsilon_j \lrcorner D^{\mathcal{C}}_{\bar{\epsilon}_j} \nu) \cdot \varphi^0\big) \otimes \left(\overline{\mu} \cdot \psi^0\right) \\
&\qquad + \big(-2(\epsilon_j \lrcorner \nu) \cdot \varphi^0\big) \otimes \left(D^{\mathcal{C}}_{\bar{\epsilon}_j} \overline{\mu} \cdot \psi^0\right)\Big) \\
&+ \sum_{\substack{r<s \\ m}} T_{\bar{r}\bar{s}m} \big(-2(\epsilon_r \lrcorner \epsilon_s \lrcorner (\epsilon^m \wedge \nu)) \cdot \varphi^0\big) \otimes \left(\overline{\mu} \cdot \psi^0\right) \\
= &\sum_{j} \big(-2(\epsilon_j \lrcorner D^{\mathcal{C}}_{\bar{\epsilon}_j} \eta) \,\widehat{\cdot}\, \xi^0\big) \\
&+ \sum_{\substack{r<s \\ m}} T_{\bar{r}\bar{s}m} \big(-2(\epsilon_r \lrcorner \epsilon_s \lrcorner (\epsilon^m \wedge \eta)) \,\widehat{\cdot}\, \xi^0\big) \\
= &\, 2\Big(-\sum_{j} \epsilon_j \lrcorner D^{\mathcal{C}}_{\bar{\epsilon}_j} \eta - \sum_{r<s} \epsilon_r \lrcorner \epsilon_s \lrcorner \big((T_{\bar{r}\bar{s}})^\flat \wedge \eta\big)\Big) \,\widehat{\cdot}\, \xi^0 \\
= &\, 2\partial^* \eta \,\widehat{\cdot}\, \xi^0,
\end{aligned}
$$

and thus,

$$
\left(\varsigma^{-1} \circ \partial\!\!\!/\,''^{(L)} \circ \varsigma\right)(\eta) = \frac{1}{2^{(p+q)/2}} \varsigma^{-1}\left(2\partial^* \eta \,\widehat{\cdot}\, \xi^0\right) = 2\frac{2^{(p+q-1)/2}}{2^{(p+q)/2}} \partial^* \eta = \sqrt{2}\partial^* \eta.
$$

The remaining two cases are perfectly analogous. □

4.5. Harmonic $\mathcal{V}$-spinors. Now, assume that (M, g, J) is a compact Hermitian $2n$-manifold, and consider the Chern-Dirac operators on $\mathcal{V}M$ determined by (3.4) using the Clifford multiplications $c = \cdot_L$ and $c = \cdot_R$, i.e.,

$$
D\!\!\!\!/^{(L)} = \partial\!\!\!/\,'^{(L)} + \partial\!\!\!/\,''^{(L)}, \qquad D\!\!\!\!/^{(R)} = \partial\!\!\!/\,'^{(R)} + \partial\!\!\!/\,''^{(R)}, \tag{4.15}
$$

called *left Chern-Dirac operator* and *right Chern-Dirac operator on $\mathcal{V}$-spinors*, respectively. As we pointed out in Section 4, they are both first order elliptic operators and, since M is compact, they are also self-adjoint. We denote by *total-harmonic $\mathcal{V}$-spinors* the sections of $\mathcal{V}M$ which are in the kernel of $D\!\!\!\!/^{(L)} + D\!\!\!\!/^{(R)}$ and *right-* (respectively, *left-*)*harmonic $\mathcal{V}$-spinors* the sections of $\mathcal{V}M$ which are in the kernel of $D\!\!\!\!/^{(R)}$ (respectively, $D\!\!\!\!/^{(L)}$). From Theorem 4.5, we obtain the following isomorphism between spaces of harmonic $\mathcal{V}$-spinors and cohomology groups.

Theorem 4.7. *Let (M, g, J) be a compact Hermitian $2n$-manifold. Then*:

$$\ker\left(\not{D}^{(L)} + \not{D}^{(R)}\right) \simeq \bigoplus_{k=0}^{2n} H_d^k(M;\mathbb{C}), \qquad \ker \not{D}^{(R)} \simeq \bigoplus_{p,q=0}^{n} H_{\bar\partial}^{p,q}(M),$$

where $H_d^k(M;\mathbb{C})$ and $H_{\bar\partial}^{p,q}(M)$ are the usual De Rham and Dolbeault cohomology groups of M.

Proof. Since, for every $0 \le k \le 2n$, the spaces $d\big(\Omega^{k-1}(M;\mathbb{C})\big)$ and $d^*\left(\Omega^{k+1}(M;\mathbb{C})\right)$ are orthogonal, from Theorem 4.5 and standard Hodge theory, it follows that

$$\begin{aligned}\ker\left(\not{D}^{(L)} + \not{D}^{(R)}\right) &\overset{\varsigma}{\simeq} \left\{\eta \in \Omega^{\cdot}(M;\mathbb{C}) : d\eta + d^*\eta = 0\right\} \\ &= \bigoplus_{k=0}^{2n} \left\{\eta \in \Omega^k(M;\mathbb{C}) : d\eta = d^*\eta = 0\right\} \\ &\simeq \bigoplus_{k=0}^{2n} H_d^k(M;\mathbb{C}).\end{aligned}$$

The proof of the second isomorphism is similar. □

Remark 4.8. We recall that the canonical spinor bundle of M is naturally included in $\mathcal{V}M$ (see subsection 4.2). Due to this, Theorem 4.7 can be considered as a generalization of the well-known isomorphism between harmonic spinors and cohomology classes $H_{\bar\partial}^{0,\cdot}(M)$ on compact Kähler manifolds [**7,** Theorem 2.1].

We conclude this section by showing how the partial Chern-Dirac operators on $\mathcal{V}$-spinors can be used to provide a spinorial characterization for the *Bott-Chern* and *Aeppli cohomologies*

$$H_{BC}^{\cdot,\cdot}(M) := \frac{\ker\partial \cap \ker\bar\partial}{\operatorname{Im}\partial\bar\partial}, \qquad H_{A}^{\cdot,\cdot}(M) := \frac{\ker\partial\bar\partial}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}. \tag{4.16}$$

We recall that both cohomologies coincide with the usual Dolbeault cohomology in the Kähler case, and they are an important tool in studies of non-Kähler Hermitian manifolds. For an introduction, see e.g., [**2, 3**].

There is a Hodge theory for these two cohomologies. In fact, they satisfy

$$H^{\bullet,\bullet}_{BC}(M) \simeq \ker \Delta_{BC}, \qquad H^{\bullet,\bullet}_{A}(M) \simeq \ker \Delta_{A}, \tag{4.17}$$

where Δ_{BC} and Δ_A are the fourth order elliptic self-adjoint operators, defined by

$$\begin{aligned}\Delta_{BC} &:= \big(\partial\overline{\partial}\big)\big(\partial\overline{\partial}\big)^* + \big(\partial\overline{\partial}\big)^*\big(\partial\overline{\partial}\big) + \big(\overline{\partial}^*\partial\big)\big(\overline{\partial}^*\partial\big)^* \\ &\quad + \big(\overline{\partial}^*\partial\big)^*\big(\overline{\partial}^*\partial\big) + \overline{\partial}^*\overline{\partial} + \partial^*\partial, \\ \Delta_{A} &:= \big(\overline{\partial}\partial^*\big)\big(\overline{\partial}\partial^*\big)^* + \big(\overline{\partial}\partial^*\big)^*\big(\overline{\partial}\partial^*\big) + \big(\partial\overline{\partial}\big)\big(\partial\overline{\partial}\big)^* \\ &\quad + \big(\partial\overline{\partial}\big)^*\big(\partial\overline{\partial}\big) + \overline{\partial}\overline{\partial}^* + \partial\partial^*.\end{aligned}$$

Due to the fact that, for every complex form η, we have

$$\begin{aligned}\Delta_{BC}\eta = 0 \quad &\text{if and only if} \quad \partial\eta = \overline{\partial}\eta = \overline{\partial}^*\partial^*\eta = 0, \\ \Delta_{A}\eta = 0 \quad &\text{if and only if} \quad \partial^*\eta = \overline{\partial}^*\eta = \partial\overline{\partial}\eta = 0,\end{aligned} \tag{4.18}$$

it is natural to consider the operators

$$\begin{gathered}\not{D}_{BC},\ \not{D}_{A} : \mathfrak{V}(M) \longrightarrow \mathfrak{V}(M), \\ \not{D}_{BC} := \not{\partial}'^{(L)} + \not{\partial}''^{(R)} + \not{\partial}'^{(R)} \circ \not{\partial}''^{(L)}, \\ \not{D}_{A} := \not{\partial}''^{(L)} + \not{\partial}'^{(R)} + \not{\partial}'^{(L)} \circ \not{\partial}''^{(R)},\end{gathered}$$

which we call the *Bott-Chern-Dirac operator* and the *Aeppli-Dirac operator on* $\mathcal{V}$*-spinors*, respectively. From (4.17), (4.18) and Theorem 4.5, the following becomes obvious.

Proposition 4.9. *Let (M, g, J) be a compact Hermitian $2n$-manifold. Then, for every $0 \leq p,\ q \leq n$, the kernels of $\not{D}_{BC}$ and $\not{D}_{A}$ satisfy*

$$\begin{aligned}H^{p,q}_{BC}(M) &\simeq \ker \not{D}_{BC} \cap \mathfrak{V}^{p,q}(M), \\ H^{p,q}_{A}(M) &\simeq \ker \not{D}_{A} \cap \mathfrak{V}^{p,q}(M),\end{aligned}$$

and thus, there exist injective homomorphisms

$$\begin{aligned}\bigoplus_{p,q=0}^{n} H^{p,q}_{BC}(M) &\hookrightarrow \ker \not{D}_{BC}, \\ \bigoplus_{p,q=0}^{n} H^{p,q}_{A}(M) &\hookrightarrow \ker \not{D}_{A}.\end{aligned}$$

5. Twisted cohomology of Hermitian manifolds and $\mathcal{V}$-spinors.

5.1. Twisted $\mathcal{V}$-spinor bundle with respect to an Hermitian bundle. Let (M, g, J) be an Hermitian $2n$-manifold, and let E be a Chern-Dirac bundle over M with Clifford multiplication

$$c : \mathcal{C}\ell^{\mathbb{C}} M \longrightarrow \mathfrak{gl}(E),$$

Hermitian metric h and covariant derivative D. Also, let $\left(W, h^W\right)$ be an Hermitian bundle over M, endowed with a metric covariant derivative D^W. We can trivially extend c to the tensor product bundle $E \otimes W$ by

$$c(w)(\sigma \otimes s) := (c(w)\sigma) \otimes s \tag{5.1}$$

and define

$$\widetilde{h} := h \otimes h^W, \qquad \widetilde{D} := D \otimes \mathrm{Id}_W + \mathrm{Id}_E \otimes D^W. \tag{5.2}$$

With straightforward computation, the following can immediately be verified.

Proposition 5.1. *The tensor product bundle $(E \otimes W, \widetilde{h}, \widetilde{D})$ is a Chern-Dirac bundle with respect to the Clifford multiplication* (5.1)*.*

We now focus on the case in which $E = \mathcal{V}M$. We call $\mathcal{V}M \otimes W$ the *W-twisted $\mathcal{V}$-spinor bundle*. It is naturally endowed with the four *twisted partial Chern-Dirac operators*

$$\widetilde{\partial\!\!\!/\,'^{(L)}}, \quad \widetilde{\partial\!\!\!/\,''^{(L)}}, \quad \widetilde{\partial\!\!\!/\,'^{(R)}}, \quad \widetilde{\partial\!\!\!/\,''^{(R)}}, \tag{5.3}$$

defined in (3.3). On the other hand, we recall that the covariant derivative D^W defines the exterior derivative $d^W \colon \Omega^k(M; W) \to \Omega^{k+1}(M; W)$ on W-valued differential forms

(5.4)
$$\begin{aligned}\left(d^W\zeta\right)(X_1, \ldots, X_{k+1}) := &\sum_j (-1)^{j+1} D^W_{X_j}\left(\zeta(X_1, \ldots, \widehat{X_j}, \ldots, X_{k+1})\right)\\ &+ \sum_{r<s} (-1)^{r+s} \zeta([X_r, X_s], X_1, \ldots, \widehat{X_r}, \ldots, \widehat{X_s}, \ldots, X_{k+1}).\end{aligned}$$

This exterior derivative splits into the sum $d^W = \partial^W + \overline{\partial}^W$, with

$$\partial^W : \Omega^{p,q}(M; W) \longrightarrow \Omega^{p+1,q}(M; W),$$

$$\overline{\partial}^W : \Omega^{p,q}(M;W) \longrightarrow \Omega^{p,q+1}(M;W).$$

Finally, consider the extension of the isometry $\varsigma : \Lambda^\cdot(T^{*\mathbb{C}}M) \to \mathcal{V}M$ defined in Proposition 4.3 to

$$\tag{5.5} \begin{gathered} \varsigma^W : \Lambda^\cdot(T^{*\mathbb{C}}M) \otimes W \longrightarrow \mathcal{V}M \otimes W, \\ \varsigma^W := \varsigma \otimes \mathrm{Id}_W . \end{gathered}$$

In order to have simple notation for ς^W, for each decomposable element $\eta \otimes s \in \Lambda^\cdot(T^{*\mathbb{C}}M) \otimes W$, it is convenient to define

$$(\eta \otimes s) \mathbin{\widehat{\cdot}} \xi^0 := (\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s$$

so that we can simply write

$$\varsigma^W(\zeta) = \frac{1}{2^{(p+q)/2}} \zeta \mathbin{\widehat{\cdot}} \xi^0 \quad \text{for every } \zeta \in \Omega^{p,q}(M;W).$$

In analogy with Theorem 4.5, we have the following important identities for the W-twisted Chern-Dirac operators.

Theorem 5.2. *Let $\varsigma^W : \Lambda^\cdot(T^{*\mathbb{C}}M) \otimes W \to \mathcal{V}M \otimes W$ be the isometry* (5.5)*. Then, the partial Chern-Dirac operators* (5.3) *verify*

$$\begin{aligned} (\varsigma^W)^{-1} \circ \widetilde{\partial\!\!\!/\,'^{(L)}} \circ \varsigma^W &= \sqrt{2}\partial^W, & (\varsigma^W)^{-1} \circ \widetilde{\partial\!\!\!/\,''^{(L)}} \circ \varsigma^W &= \sqrt{2}\partial^{W*}, \\ (\varsigma^W)^{-1} \circ \widetilde{\partial\!\!\!/\,'^{(R)}} \circ \varsigma^W &= \sqrt{2}\,\overline{\partial}^{W*}, & (\varsigma^W)^{-1} \circ \widetilde{\partial\!\!\!/\,''^{(R)}} \circ \varsigma^W &= \sqrt{2}\,\overline{\partial}^{W}. \end{aligned}$$

Proof. Consider a unitary frame field $(e_j) : \mathcal{U} \subset M \to \mathrm{U}_{g,J}(M)$, the associated normalized complex vectors ϵ_j, $\overline{\epsilon}_j$ and a decomposable element $\zeta = \eta \otimes s \in \Omega^{p,q}(M;W)$. Following the same arguments of the proof of Theorem 4.5, we have

$$\begin{aligned} \widetilde{\partial\!\!\!/\,'^{(L)}}((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) &= \sum_j \overline{\epsilon}_j \cdot_L \widetilde{D}_{\epsilon_j}((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) \\ &\qquad - \frac{1}{2} \sum_{r<s} (\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L ((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) \\ &= \sum_j (\overline{\epsilon}_j \cdot_L (D^{\mathcal{C}}_{\epsilon_j} \eta \mathbin{\widehat{\cdot}} \xi^0)) \otimes s \\ &\qquad - \frac{1}{2} \sum_{r<s} ((\overline{\epsilon}_r \cdot \overline{\epsilon}_s \cdot T_{rs}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0)) \otimes s \end{aligned}$$

$$
\begin{aligned}
&+\sum_j (\bar\epsilon_j \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0)) \otimes D^W_{\epsilon_j} s \\
&= (\partial\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s + \sum_j ((\epsilon^j \wedge \eta) \mathbin{\widehat{\cdot}} \xi^0) \otimes D^W_{\epsilon_j} s \\
&= \Big(\partial\eta \otimes s + \sum_j (\epsilon^j \wedge \eta) \otimes D^W_{\epsilon_j} s\Big) \mathbin{\widehat{\cdot}} \xi^0 \\
&= \partial^W(\eta \otimes s) \mathbin{\widehat{\cdot}} \xi^0.
\end{aligned}
$$

Hence,

$$
((\varsigma^W)^{-1} \circ \widetilde{\not\partial'^{(L)}} \circ \varsigma^W)(\eta \otimes s) = \frac{2^{(p+q+1)/2}}{2^{(p+q)/2}} \partial^W(\eta \otimes s) = \sqrt{2}\partial^W(\eta \otimes s).
$$

Proceeding in a similar manner, we obtain

$$
\begin{aligned}
\widetilde{\not\partial''^{(L)}}((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) &= \sum_j \epsilon_j \cdot_L \widetilde{D}_{\bar\epsilon_j}((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) \\
&\quad - \frac{1}{2}\sum_{r<s} (\epsilon_r \cdot \epsilon_s \cdot T_{\bar r \bar s}) \cdot_L ((\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s) \\
&= \sum_j \big(\epsilon_j \cdot_L (D^{\mathcal{C}}_{\bar\epsilon_j} \eta \mathbin{\widehat{\cdot}} \xi^0)\big) \otimes s \\
&\quad - \frac{1}{2}\sum_{r<s} ((\epsilon_r \cdot \epsilon_s \cdot T_{\bar r \bar s}) \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0)) \otimes s \\
&\quad + \sum_j \big(\epsilon_j \cdot_L (\eta \mathbin{\widehat{\cdot}} \xi^0)\big) \otimes D^W_{\bar\epsilon_j} s \\
&= 2(\partial^*\eta \mathbin{\widehat{\cdot}} \xi^0) \otimes s + \sum_j \big(-2(\epsilon_j \lrcorner \eta) \mathbin{\widehat{\cdot}} \xi^0\big) \otimes D^W_{\bar\epsilon_j} s \\
&= 2\Big(\partial^*\eta \otimes s - \sum_j (\epsilon_j \lrcorner \eta) \otimes D^W_{\bar\epsilon_j} s\Big) \mathbin{\widehat{\cdot}} \xi^0 \\
&= 2\,\partial^{W*}(\eta \otimes s) \mathbin{\widehat{\cdot}} \xi^0
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\Big((\varsigma^W)^{-1} \circ \widetilde{\not\partial''^{(L)}} \circ \varsigma^W\Big)(\eta \otimes s) &= 2\,\frac{2^{(p+q-1)/2}}{2^{(p+q)/2}}\,\partial^{W*}(\eta \otimes s) \\
&= \sqrt{2}\partial^{W*}(\eta \otimes s).
\end{aligned}
$$

The remaining two cases are perfectly analogous. □

Finally, if we consider the *twisted left Chern-Dirac operator* and the *twisted right Chern-Dirac operator*, namely, the operators

$$\widetilde{\not{D}^{(L)}} = \widetilde{\not{\partial}'^{(L)}} + \widetilde{\not{\partial}''^{(L)}}, \qquad \widetilde{\not{D}^{(R)}} = \widetilde{\not{\partial}'^{(R)}} + \widetilde{\not{\partial}''^{(R)}}, \tag{5.6}$$

by the same line of arguments in Theorem 4.7, we obtain

Theorem 5.3. *Let (M, g, J) be a compact Hermitian $2n$-manifold and W an Hermitian bundle over M endowed with a fixed metric covariant derivative D^W. Then:*

$$\ker\left(\widetilde{\not{D}^{(L)}} + \widetilde{\not{D}^{(R)}}\right) \simeq \bigoplus_{k=0}^{2n} H^k_{d^W}(M; W),$$

$$\ker \widetilde{\not{D}^{(R)}} \simeq \bigoplus_{p,q=0}^{n} H^{p,q}_{\bar{\partial}^W}(M; W),$$

where $H^k_{d^W}(M; W)$ and $H^{p,q}_{\bar{\partial}^W}(M; W)$ are the W-valued De Rham and Dolbeault cohomology groups.

5.2. θ-twisted $\mathcal{V}$-spinor bundles. The results of the previous section have immediate applications to the case of twisted cohomology groups $H^{\cdot}_{d_\theta}(M; \mathbb{C})$ and $H^{\cdot,\cdot}_{\bar{\partial}_\theta}(M)$ of Hermitian manifolds [**4**].

Let (M, g, J) be an Hermitian $2n$-manifold with a fixed closed 1-form θ. Consider the trivial complex line bundle $\mathcal{L}_\theta$ on M endowed with the flat covariant derivative D^θ, defined for every global section s of $\mathcal{L}_\theta$ by

$$D^\theta s := ds + \theta \otimes s.$$

Fixing an open covering $\{\mathcal{U}_j\}$ of M such that $\theta|_{\mathcal{U}_j} = df_j$, we obtain a holomorphic trivialization $\{(\mathcal{U}_j, e^{-f_j})\}$ of $\mathcal{L}_\theta$, with transition functions $e^{f_j - f_k}$ on $\mathcal{U}_j \cap \mathcal{U}_k$, with respect to which $s^0 = (\mathcal{U}_j, e^{f_j})$ is a parallel nowhere vanishing section. This gives rise to an Hermitian metric h^θ on $\mathcal{L}_\theta$ with $h^\theta(s^0, s^0) = 1$ so that D^θ is metric with respect to h^θ.

Definition 5.4. The *θ-twisted $\mathcal{V}$-spinor bundle* is the tensor product bundle

$$\mathcal{V}_\theta M := \mathcal{V}M \otimes \mathcal{L}_{-\theta}$$

endowed with the Hermitian metric $\check{h}^\theta := \check{h} \otimes h^\theta$ and the covariant derivative $\widetilde{D^\theta} := D^{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{L}_\theta} + \mathrm{Id}_{\mathcal{V}M} \otimes D^\theta$ (see subsection 5.1 for the definitions of $\check{h}$ and $D^{\mathcal{C}}$).

From Proposition 5.1, $\mathcal{V}_\theta M$ is a Chern-Dirac bundle. The corresponding twisted partial Chern-Dirac operators $\partial\!\!\!/_\theta^{\prime(L)}$, $\partial\!\!\!/_\theta^{\prime\prime(L)}$, $\partial\!\!\!/_\theta^{\prime(R)}$, $\partial\!\!\!/_\theta^{\prime\prime(R)}$ are called *θ-twisted partial Chern-Dirac operators.* Their sums

$$D\!\!\!\!/_\theta^{(L)} = \partial\!\!\!/_\theta^{(L)} + \partial\!\!\!/_\theta^{\prime\prime(L)}, \qquad D\!\!\!\!/_\theta^{(R)} = \partial\!\!\!/_\theta^{\prime(L)} + \partial\!\!\!/_\theta^{\prime\prime(L)}$$

are called the *θ-twisted left Chern-Dirac operator* and the *θ-twisted right Chern-Dirac operator.* By means of the isomorphism between the De Rham complex of differential forms with values in $\mathcal{L}_{-\theta}$ and the Lichnerowicz-Novikov complex

$$\cdots \xrightarrow{d_\theta} \Omega^{k-1}(M;\mathbb{C}) \xrightarrow{d_\theta} \Omega^{k-1}(M;\mathbb{C}) \xrightarrow{d_\theta} \cdots,$$
$$d_\theta := d - \theta\wedge,$$

from Theorems 5.2 and 5.3, we immediately obtain the following.

Theorem 5.5. *Let (M,g,J) be an Hermitian $2n$-manifold and*

$$\varsigma^{\mathcal{L}_{-\theta}} : \Lambda^{\cdot}(T^{*\mathbb{C}}M) \otimes \mathcal{L}_{-\theta} \longrightarrow \mathcal{V}_\theta M$$

the isometry defined in (5.5) *with $W = \mathcal{L}_\theta$. Then, the θ-twisted partial Chern-Dirac operators satisfy*

$$(\varsigma^{\mathcal{L}_\theta})^{-1} \circ \partial\!\!\!/_\theta^{\prime(L)} \circ \varsigma^{\mathcal{L}_{-\theta}} = \sqrt{2}\partial_\theta,$$
$$(\varsigma^{\mathcal{L}_\theta})^{-1} \circ \partial\!\!\!/_\theta^{\prime\prime(L)} \circ \varsigma^{\mathcal{L}_{-\theta}} = \sqrt{2}\partial_\theta^*,$$
$$(\varsigma^{\mathcal{L}_\theta})^{-1} \circ \partial\!\!\!/_\theta^{\prime(R)} \circ \varsigma^{\mathcal{L}_{-\theta}} = \sqrt{2}\,\overline{\partial}_\theta^*,$$
$$(\varsigma^{\mathcal{L}_\theta})^{-1} \circ \partial\!\!\!/_\theta^{\prime\prime(R)} \circ \varsigma^{\mathcal{L}_{-\theta}} = \sqrt{2}\,\overline{\partial}_\theta.$$

In particular, if M is compact, then

$$\ker(D\!\!\!\!/_\theta^{(L)} + D\!\!\!\!/_\theta^{(R)}) \simeq \bigoplus_{k=0}^{2n} H^k_{d_\theta}(M;\mathbb{C}),$$
$$\ker D\!\!\!\!/_\theta^{(R)} \simeq \bigoplus_{p,q=0}^{n} H^{p,q}_{\bar\partial_\theta}(M),$$

where $H^k_{d_\theta}(M;\mathbb{C})$ *and* $H^{p,q}_{\bar\partial_\theta}(M)$ *are the* θ*-twisted De Rham and Dolbeault cohomology groups of* M*.*

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