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NUMERICAL RANGES OF NORMAL WEIGHTED COMPOSITION OPERATORS ON $\ell^2(\mathbb{N})$

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ABSTRACT. In this paper, we obtain numerical ranges of normal weighted composition operators on $\ell^2(\mathbb{N})$.

1. Introduction. Let $\mathbb{N}$ denote the set of natural numbers, and let $\ell^2(\mathbb{N})$ be the Hilbert space of square summable sequences of complex numbers. The set $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis for $\ell^2(\mathbb{N})$, where $e_k(m) = \delta_{km}$ is the Kronecker delta. Suppose that $\theta : \mathbb{N} \to \mathbb{C}$ and $\phi : \mathbb{N} \to \mathbb{N}$ are two mappings. Let $F(\mathbb{N}, \mathbb{C})$ be the linear space of all sequences of complex numbers. Then, a linear transformation $C_{\theta,\phi} : \ell^2(\mathbb{N}) \to F(\mathbb{N}, \mathbb{C})$, defined by $C_{\theta,\phi} f = \theta \cdot f \circ \phi$ for every $f \in \ell^2(\mathbb{N})$, is known as a weighted composition transformation. If $C_{\theta,\phi}$ is bounded and ran $C_{\theta,\phi} \subset \ell^2(\mathbb{N})$, we shall call $C_{\theta,\phi}$ a *weighted composition operator* induced by θ and ϕ. It is easy to see that $C_{\theta,\phi}$ is a bounded operator if and only if there exists an M > 0 such that

$$\sum_{m\in\phi^{-1}(n)} |\theta(m)|^2 \leq M,$$

for every $n \in \mathbb{N}$. If $\phi(n) = n$ for every $n \in \mathbb{N}$, then $C_{\theta,\phi} = M_\theta$ is the multiplication operator induced by θ. In the case where $\theta(n) = 1$ for every $n \in \mathbb{N}$, $C_{\theta,\phi} = C_\phi$ is the composition operator induced by ϕ. The adjoint of $C_{\theta,\phi}$ is given by

$$(C^*_{\theta,\phi} f)(n) = \begin{cases} \displaystyle\sum_{m\in\phi^{-1}(n)} \overline{\theta(m)} f(m) & \text{if } \phi^{-1}(n) \neq \emptyset, \\ 0 & \text{if } \phi^{-1}(n) = \emptyset. \end{cases}$$

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A mapping $\phi : \mathbb{N} \to \mathbb{N}$ is said to be antiperiodic at n $\in \mathbb{N}$ if $\phi^m(n) \neq$ n for every m $\in \mathbb{N}$. If ϕ is not antiperiodic at n, then we say that ϕ is periodic at n. If ϕ is periodic at every n $\in \mathbb{N}$, then we say that ϕ is a periodic mapping. If ϕ is periodic at n $\in \mathbb{N}$, then the integer $m_n = \inf\{m : \phi^m(n) = n\}$ is called the *period* of ϕ at n. The set $\{m_n :$ n $\in \mathbb{N}\}$ of all periods of ϕ is denoted by P (ϕ). If ϕ is antiperiodic at every n $\in \mathbb{N}$, then we say that ϕ is an antiperiodic mapping. For n $\in \mathbb{N}$, the orbit of n with respect to ϕ is defined as

$$O_\phi(n) = \{m \in \mathbb{N} : \phi^r(m) = \phi^s(n) \text{ for some r, s} \in \mathbb{N}\}.$$

By the symbol #(E) we shall denote the cardinality of the set E, and by χ_{E} we denote the characteristic function of E. The Banach algebra of all bounded linear operators from a Hilbert space H into itself is denoted by $B(H)$. For $A \in B(H)$, the spectrum of A is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

The smallest convex set containing the set E $\subset H$ is called the *convex hull* of E, and we shall denote it by $C_0(E)$.

A complex number λ is called an eigenvalue of an operator A if there exists a nonzero vector f $\in$ H such that Af $= \lambda$f. The set of all eigenvalues of A is called the point spectrum of A, and it is denoted by Π_0(A). For G $\subset \mathbb{N}$, let

$$\ell^2(G) = \{\text{f} \in \ell^2(\mathbb{N}) : \text{f(m)} = 0 \text{ for every m} \notin \text{G}\}.$$

The symbol $C_T|_{\ell^2(G)}$ denotes the restriction of C_T to $\ell^2(G)$. The numerical range of A $\in B(H)$ is defined as

$$W(A) = \{\langle Ax, x\rangle : \text{x} \in \text{H and } \|x\| = 1\}.$$

By the symbol $\|\theta\|_\infty$, we shall mean $\sup\{|\theta(n)| : n \in \mathbb{N}\}$.

Weighted composition operators have been the subject matter of systematic study over the past several decades. For more information regarding weighted composition operators and numerical ranges of operators, the reader is referred to [**1, 2, 4, 5, 7, 9, 10, 12, 13**], etc.

Numerical ranges and their generalizations were studied due to their connections and applications to several branches of the mathematical sciences. Some of the more well-known results about numerical ranges of operators are presented in the following proposition.

Proposition 1.1. *Let* $\mathrm{A} \in \mathrm{B(H)}$. *Then*:

(a) $\mathrm{W(A)}$ *lies in the closed disc of radius* $\|\mathrm{A}\|$ *centered at the origin.*

(b) $\mathrm{W(A)}$ *is always convex.*

(c) $\mathrm{W}(\alpha\mathrm{A}+\beta\mathrm{I}) = \alpha\mathrm{W(A)}+\beta$, *where* α *and* β *are complex numbers.*

(d) $\mathrm{W(A)}$ *is invariant under a unitary transformation.*

(e) *The numerical range of the unilateral shift is the open unit disc centered at the origin.*

(f) *The closure of numerical range of a normal operator is the convex hull of its spectrum.*

(g) $W(\mathrm{A}^*) = \overline{W(\mathrm{A})} = \{\overline{\lambda} : \lambda \in W(\mathrm{A})\}$, *where* A^* *denotes the adjoint of the operator* A, *and* $\overline{\lambda}$ *is the conjugate of complex number* λ.

The main purpose of the present paper is to compute the numerical ranges of normal weighted composition operators on $\ell^2(\mathbb{N})$.

2. Numerical ranges of normal weighted composition operators. In this section, we shall obtain the numerical ranges of normal weighted composition operators. Let $\phi : \mathbb{N} \to \mathbb{N}$ be an invertible map. Define a relation $\equiv$ on $\mathbb{N}$ as follows: for $\mathrm{m}, \mathrm{n} \in \mathbb{N}, \mathrm{m} \equiv \mathrm{n}$ if m and n are in the same orbit of ϕ. This is an equivalence relation in $\mathbb{N}$ and will partition $\mathbb{N}$ into disjoint equivalence classes, say $O_\phi(n_i)$ for $\mathrm{i} = 1, 2, \ldots, p$, where p is the total number of distinct equivalence classes. Clearly, $1 \leq p \leq \infty$. Then,

$$\mathbb{N} = \bigcup_{i=1}^{p} O_\phi(n_i).$$

Let

$$\mathrm{G} = \{\# O_\phi(n_i) : O_\phi(n_i) \text{ is a finite set}\}.$$

For each $k \in \mathrm{G}$, let q(k) be the number of distinct equivalence classes, each of cardinality k. For $k \in \mathrm{G}$, let $n_1^k, n_2^k, n_3^k, \ldots, n_{q(k)}^k$ be positive integers such that $\# O_\phi(n_j^k) = k$, $1 \leq j \leq q(k)$. Denote the set $O_\phi(n_j^k)$ by E_j^k for $1 \leq j \leq q(k)$. Let

$$E^{(k)} = \bigcup_{j=1}^{q(k)} E_j^k \quad \text{and} \quad E_1 = \bigcup_{k \in G} E^{(k)}.$$

Let m be the number of distinct equivalence classes of infinite orbits, and set

$$E_2 = \bigcup_{i=1}^{\mathrm{m}} \{O_\phi(r_i) : \#O_\phi(r_i) = \infty\}.$$

Then, $\mathbb{N} = E_1 \bigcup E_2$.

Theorem 2.1. *Let $\theta : \mathbb{N} \to \mathbb{C}\backslash\{0\}$ and $C_{\theta,\phi} \in B(\ell^2(\mathbb{N}))$. Then, $C_{\theta,\phi}$ is a normal operator if and only if ϕ is invertible and $|\theta| = |\theta \circ \phi|$* **[3]**.

Proof. Suppose first that $C_{\theta,\phi}$ is a normal operator. If ϕ is not surjective, then there exists an $n_0 \in \mathbb{N}$ such that $n_0 \notin \phi(\mathbb{N})$. Let $e_{n_0} = (0, 0, \ldots, 1_{n_0^{\text{th}}\text{place}}, 0, \ldots)$. Then, $C^*_{\theta,\phi} C_{\theta,\phi} e_{n_0} = 0$ and

$$C_{\theta,\phi} C^*_{\theta,\phi} e_{n_0} = \overline{\theta(\phi(n_0))} \sum_{m \in \phi^{-1}(\phi(n_0))} \theta(m) e_m \neq 0.$$

This contradicts the fact that $C_{\theta,\phi}$ is a normal operator.

Again, if ϕ is not injective, then there exist two distinct positive integers n_1 and n_2 such that $\phi(n_1) = \phi(n_2)$. Simple computation shows that

$$C^*_{\theta,\phi} C_{\theta,\phi} e_{n_1} = \sum_{m \in \phi^{-1}(n_1)} |\theta(m)|^2 e_{n_1} \tag{2.1}$$

and

$$\begin{aligned} C_{\theta,\phi} C^*_{\theta,\phi} e_{n_1} &= \overline{\theta(n_1)} \Big[\sum_{m \in \phi^{-1}(\phi(n_1))} \theta(m) e_m \Big] \\ &= \overline{\theta(n_1)} \theta(n_1) e_{n_1} + \overline{\theta(n_1)} \theta(n_2) e_{n_2} \\ &\quad + \overline{\theta(n_1)} \sum_{m \in \phi^{-1}(\phi(n_1)) \backslash \{n_1, n_2\}} \theta(m) e_m. \end{aligned} \tag{2.2}$$

Taking the values of the functions in (2.1) and (2.2) at the point n_2, we find that the value of the function in (2.1) at n_2 is zero, whereas the value of the function in (2.2) at $n_2 = \overline{\theta(n_1)}\theta(n_2)$. This, again, contradicts the normality of $C_{\theta,\phi}$. Hence, ϕ must be injective. This proves that ϕ is invertible.

Finally, if ϕ is invertible, then the value of the function in (2.1) at n_1 is $|\theta(m)|^2$, where $m \in \phi^{-1}(n_1)$, and the value of the function in

(2.2) at n_1 is $\overline{\theta(n_1)}\theta(n_1)$. Thus, $|\theta(m)|^2 = |\theta(n_1)|^2 = |\theta(\phi(m))|^2$, or equivalently, $|\theta(m)| = |\theta(\phi(m))|$. This can be proven for every $m \in \mathbb{N}$. Hence, $|\theta| = |\theta \circ \phi|$. Conversely, it is clear from equations (2.1) and (2.2) that $C_{\theta,\phi}C^*_{\theta,\phi} = C^*_{\theta,\phi}C_{\theta,\phi}$. This completes the proof. □

Theorem 2.2. *Suppose that $\theta : \mathbb{N} \to \mathbb{C} \setminus \{0\}$, and let $C_{\theta,\phi} \in B(\ell^2(\mathbb{N}))$ be a normal operator. Then*

$$\overline{W(C_{\theta,\phi})} = C_0\Bigg(\overline{\bigcup_{k\in G}\bigcup_{j=1}^{q(k)}\{\lambda \in \mathbb{C} : \lambda^k = |\theta(n_j^k)|^k \prod_{j=1}^{k} B_j\}\Bigg)} \cup \overline{\bigcup_{i=1}^{m}\{|\theta(r_i)|\lambda : |\lambda| < 1\}},$$

where $B : \mathbb{N} \to \mathbb{C} \setminus \{0\}$ is defined by $B(j) = B_j = e^{\iota\beta_j}$, $e^{\iota\beta} = \cos\beta + \iota\sin\beta$, where $\iota^2 = -1$, and β_j is the principal argument of complex number $\theta(j)$.

Proof. Suppose that $C_{\theta,\phi}$ is a normal operator. Then, from Theorem 2.1, ϕ is invertible and $|\theta| = |\theta \circ \phi|$. The equation $|\theta| = |\theta \circ \phi|$ implies that, for each $n \in \mathbb{N}$, $|\theta|$ is constant in $O_\phi(n)$, in other words, $|\theta(m)| = |\theta(n)|$ for all $m \in O_\phi(n)$. Now, for $m \in O_\phi(n)$, we have $\theta(m) = |\theta(m)|e^{\iota\beta_m} = |\theta(n)|e^{\iota\beta_m}$, where β_m is the principal argument of the complex number $\theta(m)$. Let $n_0 \in \mathbb{N}$ and $\#(O_\phi(n_0)) = k$. Simple computation shows that

$$\begin{aligned}\sigma(C_{\theta,\phi}|_{\ell^2(O_\phi(n_0))}) &= \Pi_0(C_{\theta,\phi}|_{\ell^2(O_\phi(n_0))})\\ &= \Bigg\{\lambda \in \mathbb{C} : \frac{\lambda^k}{|\theta(n_0)|^k} = \prod_{j=1}^{k} B_j\Bigg\}\\ &= \Bigg\{\lambda \in \mathbb{C} : \lambda^k = |\theta(n_0)|^k \prod_{j=1}^{k} B_j\Bigg\}.\end{aligned}$$

Since $\mathbb{N} = E_1 \bigcup E_2$,

$$\begin{aligned}\ell^2(\mathbb{N}) &= \ell^2(E_1) \oplus \ell^2(E_2)\\ &= \Bigg(\sum_{k\in G} \oplus\, \ell^2(E^{(k)})\Bigg) \oplus \Bigg(\sum_{i=1}^{m} \oplus\, \ell^2(O_\phi(r_i))\Bigg)\end{aligned}$$

and

$$\ell^2(E^{(k)}) = \sum_{j=1}^{q(k)} \oplus \, \ell^2(O_\phi(n_j^k)).$$

Therefore,

$$\begin{aligned} C_{\theta,\phi} &= C_{\theta,\phi}|_{\ell^2(E_1)} \oplus C_{\theta,\phi}|_{\ell^2(E_2)} \\ &= \sum_{k\in G} \oplus \, C_{\theta,\phi}|_{\ell^2(E^{(k)})} \oplus \sum_{i=1}^{m} \oplus \, C_{\theta,\phi}|_{\ell^2(O_\phi(r_i))} \end{aligned}$$

and

$$C_{\theta,\phi}|_{\ell^2(E^{(k)})} = \sum_{j=1}^{q(k)} \oplus \, C_{\theta,\phi}|_{\ell^2(O_\phi(n_j^k))}.$$

It follows that

$$\sigma(C_{\theta,\phi}|_{\ell^2(E_1)}) = \overline{\bigcup_{k\in G} \bigcup_{j=1}^{q(k)} \sigma(C_{\theta,\phi}|_{\ell^2(O_\phi(n_j^k))})}.$$

Now, $C_{\theta,\phi}$ is normal. Therefore, by Proposition 1.1 (f),
(2.3)

$$\overline{W(C_{\theta,\phi}|_{\ell^2(E_1)})} = C_0\overline{\left(\bigcup_{k\in G} \bigcup_{j=1}^{q(k)} \left\{ \lambda \in \mathbb{C} : \lambda^k = |\theta(n_j^k)|^k \prod_{j=1}^{k} B_j \right\} \right)}.$$

Let $r_1 \in \mathbb{N}$ be such that $\#O_\phi(r_1) = \infty$. Choose $n_0 \in O_\phi(r_1)$. Then, $O_\phi(r_1) = O_\phi(n_0)$. Write $\phi^k(n_0) = n_k$ and $(\phi^k)^{-1}(n_0) = n_{-k}$. Then, $O_\phi(n_0) = \{n_k : k \in \mathbb{Z}\}$. We can easily show that $(C_{\theta,\phi}|_{\ell^2(O_{\phi(n)})})f = |\theta(n)|(C_{B,\phi}|_{\ell^2(O_{\phi(n)})})f$, and $C_{B,\phi}$ is a normal operator. Define

$$A : \ell^2(O_\phi(n_0)) \longrightarrow \ell^2(O_\phi(n_0))$$

by $Ae_{n_k} = \alpha_{n_k} e_{n_k}$, where $\alpha_{n_0} = 1$ and $\alpha_{n_{k+1}} = \alpha_{n_k} B_{n_k}$, and B_{n_k} is as previously defined. Clearly,

$$AC^*_{B,\phi} e_{n_k} = A(\overline{B_{n_k}} e_{n_{k+1}}) = \overline{B_{n_k}} \alpha_{n_{k+1}} e_{n_{k+1}} = \alpha_{n_k} e_{n_{k+1}}$$

and

$$C^*_\phi A e_{n_k} = C^*_\phi \alpha_{n_k} e_{n_k} = \alpha_{n_k} e_{n_{k+1}}.$$

Thus, $AC^*_{B,\phi}|_{\ell^2(O_\phi(n_0))} = C^*_\phi A|_{\ell^2(O_\phi(n_0))}$. It can be seen that $|\alpha_{n_k}| = 1$ for every $k \in \mathbb{Z}$ so that A is a unitary operator. Hence, $C^*_{B,\phi}|_{\ell^2(O_\phi(n_0))}$

is unitarily equivalent to $C_\phi^*|_{\ell^2(O_\phi(n_0))}$. Consequently, $C_{B,\phi}|_{\ell^2(O_\phi(n_0))}$ is unitarily equivalent to $C_\phi|_{\ell^2(O_\phi(n_0))}$. From [**9,** Proposition 1.1 (d), Lemma 2.1], we obtain

$$W(C_{B,\phi}|_{\ell^2(O_\phi(n_0))}) = W(C_\phi|_{\ell^2(O_\phi(n_0))}) = \{\lambda : |\lambda| < 1\}.$$

Hence,

$$\begin{aligned} W\big(C_{\theta,\phi}|_{\ell^2(O_\phi(n_0))}\big) &= W\big(|\theta(n_0)|C_{B,\phi}|_{\ell^2(O_\phi(n_0))}\big) \\ &= |\theta(n_0)|W\big(C_{B,\phi}|_{\ell^2(O_\phi(n_0))}\big) \\ &= |\theta(r_1)|W\big(C_\phi|_{\ell^2(O_\phi(r_1))}\big) \\ &= \{|\theta(r_1)|\lambda : |\lambda| < 1\}. \end{aligned} \tag{2.4}$$

Finally, from equations (2.3) and (2.4), we can conclude that

$$\begin{aligned} \overline{W(C_{\theta,\phi})} = C_0 &\overline{\bigg(\bigcup_{k\in G} \bigcup_{j=1}^{q(k)} \bigg\{ \lambda \in \mathbb{C} : \lambda^k = |\theta(n_j^k)|^k \prod_{j=1}^{k} B_j \bigg\} \bigg)} \\ &\cup \overline{\bigcup_{i=1}^{m} \{|\theta(r_i)|\lambda : |\lambda| < 1\}}. \end{aligned}$$

□

Example 2.3. For each $n \in \mathbb{N}$, set

$$E_n = \bigg[\frac{n(n-1)}{2} + 1, \frac{n(n+1)}{2}\bigg) \quad \text{and} \quad F_n = \bigg\{\frac{n(n+1)}{2}\bigg\}.$$

Write $E = \bigcup_{n=2}^{\infty} E_n$ and $F = \bigcup_{n=2}^{\infty} F_n$. Clearly, $\mathbb{N} = E_1 \cup E \cup F$. For $n \geq 2$, let $G_n = E_n \cup F_n$. For every $n \in \mathbb{N}$, define $\phi : \mathbb{N} \to \mathbb{N}$ as

$$\phi(m) = \begin{cases} 1 & \text{if } m \in E_1, \\ m+1 & \text{if } m \in E, \\ m-1 & \text{if } m \in F. \end{cases} \tag{2.5}$$

Let $\theta : \mathbb{N} \to \mathbb{C}$ be defined as

$$\theta(m) = \begin{cases} e^{\iota(2\pi/n)} & \text{if } m \in G_n, \\ 8 & \text{if } m \in E_1. \end{cases} \tag{2.6}$$

Then, $|\theta(1)| = 8$ and $|\theta(m)| = 1$ for every $m \in G_n$ for $n \geq 2$. Now,

$$\Pi_0(C_{\theta,\phi}|_{\ell^2(E_1)}) = \{8\}$$

and

$$\Pi_0(C_{\theta,\phi}|_{\ell^2(G_n)}) = \{\lambda \in \mathbb{C} : \lambda^n = \theta_{n(n-1)/2+1} \cdot \theta_{n(n-1)/2+2} \cdot \ldots \cdot \theta_{n(n+1)/2-1} \cdot \theta_{n(n+1)/2}\}.$$

Since $|\theta(m)| = |\theta(\phi(m)|$ for $m \in G_n$, $n \in \mathbb{N}$, and $\phi|_{G_n} : G_n \to G_n$ is invertible, thus, in view of Theorem 2.1, $C_{\theta,\phi}|_{\ell^2(G_n)}$ is normal.

Therefore, by Theorem 2.2, for $n \geq 2$,

$$\begin{aligned} W\big(C_{\theta,\phi}|_{\ell^2(G_n)}\big) &= C_0\big(\sigma\big(C_{\theta,\phi}|_{\ell^2(G_n)}\big)\big) \\ &= C_0\left\{\lambda \in \mathbb{C} : \lambda^n = \left(e^{\iota}\frac{2\pi}{n}\right)^n\right\} \\ &= C_0\{\lambda \in \mathbb{C} : \lambda^n = 1\} \end{aligned}$$

and, for n = 1, $W(C_{\theta,\phi}|_{\ell^2(E_1)}) = \{8\}$. Since

$$C_{\theta,\phi} = \left(\sum_{n\in\mathbb{N}} \oplus C_{\theta,\phi}|_{\ell^2(G_n)}\right)$$

and

$$W(C_{\theta,\phi}) = C_0\left(\sigma\left(\sum_{n=1}^{\infty} \oplus C_{\theta,\phi}|_{\ell^2(G_n)}\right)\right),$$

it follows that

$$\overline{W(C_{\theta,\phi})} = \overline{C_0\left(\bigcup_{n=1}^{\infty}\{\lambda \in \mathbb{C} : \lambda^n = 1\}\bigcup\{8\}\right)}.$$

The numerical range of $C_{\theta,\phi}$ is as shown in Figure 1.

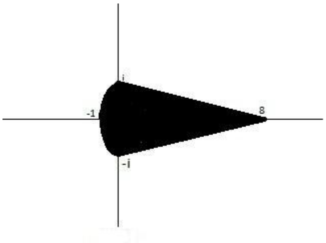

FIGURE 1.

Example 2.4. Define $\phi : \mathbb{N} \to \mathbb{N}$ by

$$(2.7) \qquad \phi(m) = \begin{cases} m+2 & \text{if } m \equiv 1 \text{ or } m \equiv 2, \\ m-2 & \text{if } m \equiv 3 \text{ or } m \equiv 4. \end{cases} \qquad (\text{mod } 4)$$

Let $\theta : \mathbb{N} \to \mathbb{C}$ be defined by

$$(2.8) \qquad \theta(m) = \begin{cases} 2e^{\iota\pi/4m} & \text{if } m \equiv 3 \text{ or } m \equiv 1, \\ 3e^{\iota\pi/2m} & \text{if } m \equiv 4 \text{ or } m \equiv 2. \end{cases} \qquad (\text{mod } 4)$$

Clearly, $|\theta| = |\theta \circ \phi|$, and ϕ is invertible. Hence, $C_{\theta,\phi}$ is normal. In view of Theorem 2.2,

$$\begin{aligned} W\big(C_{\theta,\phi}\big) &= W\big(C_{\theta,\phi}|_{\ell^2(O_\phi(1))}\big) \bigcup W\big(C_{\theta,\phi}|_{\ell^2(O_\phi(2))}\big) \\ &= \big\{\lambda \in \mathbb{C} : |\lambda| < 2\big\} \bigcup \big\{\lambda \in \mathbb{C} : |\lambda| < 3\big\} \\ &= \big\{\lambda \in \mathbb{C} : |\lambda| < 1\big\}. \end{aligned}$$

3. Numerical ranges of weighted composition operators induced by antiperiodic mappings. In this section, we obtain the numerical ranges of weighted composition operators induced by antiperiodic mappings.

Theorem 3.1. *Let $\phi : \mathbb{N} \to \mathbb{N}$ be an antiperiodic injection, and let $\theta : \mathbb{N} \to \mathbb{N}$ be such that $\lim_{n\to\infty}\theta_n = \|\theta\|_\infty$ and $C_{\theta,\phi} \in B(\ell^2(\mathbb{N}))$. Then,*

$$W(C_{\theta,\phi}) = \{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\}.$$

Proof. Let $E = \{n \in \mathbb{N} : \#(\phi^{-1}(n)) = 0\}$,

$$G = \bigcup_{n \in E} O_\phi(n)$$

and $H = \mathbb{N} \setminus G$. If $E = \emptyset$, then $H = \mathbb{N}$. Next, if $E \neq \emptyset$, choose $n_1 \in E$. Write $\phi^k(n_1) = n_{k+1}$ for all $k \in \mathbb{N}$. Let $E_k = \{n_k : k \in \mathbb{N}\}$. Define $S : \ell^2(E_k) \to \ell^2(\mathbb{N})$ by $S(e_{n_k}) = e_k$ for every $k \in \mathbb{N}$. Then, S is a unitary operator and $C^*_{\theta,\phi}|_{\ell^2(E_k)} = S^{-1}US$, where U is the unilateral weighted shift with weights $\{\overline{\theta(n_k)}\}$. Hence, in view of [**13,** Theorem 1 (i)],

$$W\big(C^*_{\theta,\phi}|_{\ell^2(E_k)}\big) = W(U) = \big\{\lambda \in \mathbb{C} : \big|\lambda\big| < \|\theta|_{E_k}\|_\infty\big\}.$$

If $H \neq \emptyset$, choose $n_0 \in H$. Define $\phi^k(n_0) = n_k$ and $(\phi^k)^{-1}(n_0) = n_{-k}$. Let $F_k = \{n_k : k \in \mathbb{Z}\}$. Then, $C^*_{\theta,\phi}|_{\ell^2(F_k)}$ is the bilateral weighted shift B with weights $\{\overline{\theta(n_k)}\}$. Hence, again in view of **[13,** Theorem 1 (ii)],

$$W\big(C^*_{\theta,\phi}|_{\ell^2(F_k)}\big) = W(B) = \big\{\lambda \in \mathbb{C} : |\lambda| < \|\theta|_{F_k}\|_\infty\big\}.$$

Now, $\ell^2(\mathbb{N}) = \ell^2(G) \oplus \ell^2(H)$. However,

$$\ell^2(G) = \sum_{k\in E} \oplus\, \ell^2(E_k) \quad \text{and} \quad \ell^2(H) = \sum_{k\in H} \oplus\, \ell^2(F_k),$$

where $E_j \cap E_k = \emptyset$, $F_j \cap F_k = \emptyset$ for $j \neq k$ and

$$C^*_{\theta,\phi} = \bigg(\sum_{k\in E} \oplus\, C^*_{\theta,\phi}|_{\ell^2(E_k)}\bigg) \oplus \bigg(\sum_{k\in H} \oplus\, C^*_{\theta,\phi}|_{\ell^2(F_k)}\bigg).$$

Therefore,

$$W(C^*_{\theta,\phi}) = C_0\bigg(\bigcup_{k\in E}\{\lambda \in \mathbb{C} : |\lambda| < \|\theta|_{E_k}\|_\infty\} \cup \bigcup_{k\in H}\{\lambda \in \mathbb{C} : |\lambda| < \|\theta|_{F_k}\|_\infty\}\bigg),$$

which yields that $W(C^*_{\theta,\phi}) = \{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\} = W(C_{\theta,\phi})$. □

Theorem 3.2. *Suppose that $\theta : \mathbb{N} \to \mathbb{R}_+$ is bounded away from zero, where $\mathbb{R}_+$ is the set of non-negative real numbers. Then, $0 \in W(C_{\theta,\phi})$ if and only if $\phi \neq I$.*

Proof. Suppose that $\phi \neq I$. Then, $\phi(n_0) \neq n_0$ for some $n_0 \in \mathbb{N}$. Consider $\langle C_{\theta,\phi}\ e_{n_0}, e_{n_0}\rangle = \langle \theta \cdot \chi_{\phi^{-1}(n_0)}, e_{n_0}\rangle = 0$ as $n_0 \notin \phi^{-1}(n_0)$. Therefore, $0 \in W(C_{\theta,\phi})$. Conversely, suppose that $0 \in W(C_{\theta,\phi})$. Since θ is bounded away from zero, there exists an $\epsilon > 0$ such that $|\theta(n)| \geq \epsilon$, that is, $\theta(n) \geq c$. We must show that $\phi \neq I$. Suppose, on the contrary, that $\phi = I$. Then

$$\begin{aligned}\langle C_{\theta,\phi} f, f\rangle = \langle \theta \cdot f \circ \phi, f\rangle &= \sum_{n=1}^{\infty} \theta(n) f(\phi(n))\overline{f(n)} \\ &\geq \epsilon\big[|f(1)|^2 + |f(2)|^2 + |f(3)|^2 + \cdots\big] = \epsilon,\end{aligned}$$

which implies that $0 \notin W(C_{\theta,\phi})$, a contradiction. Hence, $\phi \neq I$. □

Note 3.3. Theorem 3.2 fails if θ is a complex-valued function. Let $\theta : \mathbb{N} \to \mathbb{C}$ be defined by $\theta(n) = \iota^n$. Then, $|\theta(n)| = 1$ for every $n \in \mathbb{N}$. Suppose that $\phi = I$. Then, $C_{\theta,\phi} = M_\theta$, which is a normal operator. From Proposition 1.1 (f), $\overline{W(M_\theta)} = C_0(\sigma(M_\theta)) = C_0(\overline{\operatorname{ran}\theta}) = C_0\{1, -1, \ \iota, -\iota\}$. Clearly, $0 \in W(C_{\theta,\phi})$.

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