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REDUCTION GRAPH AND ITS APPLICATION ON ALGEBRAIC GRAPHS

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ABSTRACT. Evans, et al., **[9]** defined an equivalence relation $\sim$ on the set of vertices of a simple graph G by taking $a \sim b$ if and only if their open neighborhoods are the same. They introduced a new graph $G_{\mathrm{red}} = G/{\sim}$, reduction graph of G, as follows. The vertices are $V(G_{\mathrm{red}}) = \{[a] : a \in V(G)\}$, and two equivalence classes $[a]$ and $[b]$ are adjacent if and only if a and b are adjacent in G. Recently, Anderson and LaGrange **[4]** defined some equivalence relations on the set of vertices of the zero-divisor graph of a commutative ring, one of which yields the reduction graph of the zero-divisor graph. In this paper, we state some basic graph theoretic properties of G_{red} and study the relations between some properties of graph G and its subgraph, G_{red}, such as the chromatic number, clique number, girth and diameter. Moreover, we study the reduction graph of some algebraic graphs, such as the comaximal graph, zero-divisor graph and Cayley graph of a commutative ring. Among other results, we show that, for every commutative ring R, $\Gamma_2(R)_{\mathrm{red}} \simeq \Gamma_1(\mathbb{Z}_2^n)$, where $\Gamma_1(\mathbb{Z}_2^n)$ is the zero-divisor graph of the Boolean ring $\mathbb{Z}_2^n$, $\Gamma_2(R)$ is the comaximal graph of R and $n = |\operatorname{Max}(R)|$.

1. Introduction. Let G be a graph and $\sim$ an equivalence relation on $V(G)$. The *quotient graph* of G with respect to $\sim$ is a graph whose vertex set is the quotient set $V(G)/{\sim}$, and equivalence class $[a]$ is adjacent to $[b]$ if there is an element in $[a]$ which is adjacent to some element in $[b]$, see **[11,** page 24].

Special cases of this concept have been studied separately in many different contexts. Mulay **[19,** Section 3] and Spiroff and Wickham **[23]**, introduced the graph of equivalence classes of the zero-divisor

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graph of a commutative ring, using the concept of an annihilator. Recently, this graph was investigated by Anderson and LaGrange [**4, 5**] under the title of *compressed graph*. In addition, Aalipour and Akbari [**2**] defined another special case of the quotient graph, using a particular equivalence relation on the vertex set of a graph G. Moreover, they stated some results on the quotient graph of the Cayley graph of the additive group of a zero-dimensional semilocal ring.

The main purpose of this paper is the investigation of the *reduction graph* as a special case of the quotient graph. As is known, the reduction graph of a graph G was originally introduced in [**9**]. The authors used this concept for studying representation of graphs modulo n. Also, this concept has been studied by Evans, et al., [**10**]. In fact, the reduction graph agrees with the graph corresponding to one of the equivalence relations defined in [**4, 5**]. Akhtar, et al., [**3**] considered the reduction graph of the Cayley graph of a ring. They showed that the reduction graph of $\mathrm{Cay}(R^+, U(R))$ is isomorphic to $\mathrm{Cay}(\overline{R}^+, U(\overline{R}))$, where R^+ is the additive group of R, $\overline{R} = R/\mathrm{nil}(R)$, $\mathrm{nil}(R)$ and $U(R)$ are the nilradical and the set of unit elements of the artinian ring R, respectively. Furthermore, Meng and Zhang [**18**] investigated the reduction graph of a k-regular vertex transitive graph.

Throughout this paper, R is a commutative ring with unit, G is a graph and all graphs are simple. Two vertices a and b of a graph G are said to be equivalent if their open neighborhoods are the same, i.e., $a \sim b$ if and only if $N_G(a) = N_G(b)$, where $N_G(a) = \{c \in V(G) : a \text{ and } c \text{ are adjacent}\}$. We will use the symbol $N(a)$ to denote $N_G(a)$ and we set $N(G) = \{N(a) : a \in G\}$. In addition, for a graph G, we will use the symbol $a - c$ to denote adjacent vertices a and c.

The *reduction graph* of G, denoted by G_{red}, is a simple graph whose vertices are the equivalence classes of graph G, and each pair of distinct classes is joined by an edge if and only if representatives of the classes are adjacent. The remarkable thing is that G_{red} can be considered as a subgraph of G, and it can inherit many properties of G. In particular, in many cases, some graph theoretic properties of G and G_{red} are the same, such as the chromatic number, clique number, girth and diameter. In particular, Theorem 2.5 shows that, if G is not a complete or a complete r-partite graph, then $\mathrm{diam}(G) = \mathrm{diam}(G_{\mathrm{red}})$. For equalities of clique, chromatic number and girth of G and G_{red}, see Corollary 3.2. Since G_{red} is usually smaller than G, its investigation is much easier. A graph is called *reduced* if it is isomorphic to its reduction. Inasmuch as G_{red} is a subgraph of G, then G is a reduced graph if and only if $G = G_{\mathrm{red}}$. We note that the number of connected components of G and G_{red} are the same. In particular, G is a connected graph if and only if G_{red} is a connected graph. Thus, without loss of generality, we can assume that G is a connected graph.

Recall that $\mathrm{d}(a,b)$ is the length of the shortest path from a to b for two vertices $a, b \in V(G)$. If G is a connected graph, the *diameter* of G, denoted $\mathrm{diam}(G)$, is $\sup\{\mathrm{d}(a,b) : a, b \in G\}$. The *girth* of G, denoted $\mathrm{gr}(G)$, is the length of a minimal cycle in G. If G contains no cycle, then $\mathrm{gr}(G) = \infty$. A subset C of the vertex set of G is called a *clique* if any two distinct vertices of C are adjacent. The *clique number* $\omega(G)$ is the supremum of the size of the cliques. A *proper vertex coloring* is a coloring of the vertices in such a way that any two adjacent vertices have different colors. By $\chi(G)$, we denote the *chromatic number* of G, that is, the minimum number of colors with which the vertices of G may be colored. Graph G is called *perfect* whenever $\omega(H) = \chi(H)$, for all induced subgraphs H of G. Suppose that $A \subseteq V(G)$. If, for every vertex a of G, either $a \in A$ or a is adjacent to an element in A, then A is called a *dominating set*. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set of G. $B \subseteq V(G)$ is called an *independent set* if any two distinct vertices of B are not adjacent. The *independence number*, most commonly denoted by $\alpha(G)$, is the cardinality of the largest independent set.

The *line graph* of G is the graph $L(G)$ with the edges of G as its vertices and where two edges of G are adjacent in $L(G)$ if and only if they are incident in G. The *Cartesian product* $G \times H$ of graphs G and H is a graph such that its vertex set is the Cartesian product $V(G) \times V(H)$, and two vertices (a_1, a_2) and (b_1, b_2) are adjacent if and only if either $a_1 = b_1$ and a_2 is adjacent with b_2 in H or $a_2 = b_2$ and a_1 is adjacent with b_1 in G. A graph is called a *complete bipartite graph* if there is a partition of its vertex set into two subsets $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ such that a_i is adjacent to b_j for all pairs (i,j), but no two elements of the same subset are adjacent. We use the symbol $K_{m,n}$ for the complete bipartite graph. More generally, a graph is *complete r-partite* if the vertices can be partitioned into r disjoint subsets such that each element of a subset is adjacent to every element which is not in the same subset, but no two elements of the same subset are adjacent. An *r-partite graph* is one whose vertex set can be partitioned into r disjoint subsets so that no edge has both ends in any one subset. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We use K_n for the complete graph with n vertices. A *regular graph* is a graph where each vertex has the same degree. A *strongly regular graph* with parameters (n, k, λ, μ) (for short, $\mathrm{srg}(n, k, \lambda, \mu)$) is a graph on n vertices which is regular with valency k and has the following properties:

(1) any two adjacent vertices have exactly λ common neighbors;
(2) any two non-adjacent vertices have exactly μ common neighbors.

Let i, k, ν be fixed positive integers with $\nu \geq k \geq i$. Let Λ be a fixed set of size ν, and define the graph $j(\nu, k, i)$ as follows: the vertices of $j(\nu, k, i)$ are the subsets of Λ with size k, where two subsets are adjacent if their intersection has size i. For $\nu \geq 2k$, the graphs $j(\nu, k, k-1)$ are known as the *Johnson graphs*. Let H be a group with identity element e and S a generating set for H such that $e \notin S$ and $S^{-1} = \{s^{-1} : s \in S\} = S$. The *Cayley graph* $\Gamma = \mathrm{Cay}(H, S)$ is the graph with vertex set H, and two distinct vertices a and b are adjacent if and only if $ab^{-1} \in S$.

In recent years, researchers have assigned a graph to a ring or a group and considered the relationship between graph theoretic properties and algebraic properties that could produce significant results on the ring or the group.

In Section 2, we state more results on reduction graphs. Among other results, we investigate some conditions under which G will be Hamiltonian or Eulerian if G_{red} has the corresponding property. Section 3 is devoted to the investigation of reduced subgraphs and determination of some common properties between G and G_{red} such as chromatic number, clique number, girth, diameter and perfectness. Furthermore, we consider the relation between the reduction of a Cartesian product of graphs and its components. In Section 4, we show that the reduction of the Cayley graph of an abelian group is again a Cayley graph. Moreover, we give a characterization of a semisimple ring R in terms of its Cayley graph, see Corollary 4.4. In Section 5, we will also obtain some new results on the reduction graph of the comaximal graph and the zero-divisor graph of a ring. For example, we show that the reduction graph of the comaximal graph of a ring R is isomorphic to the zero-divisor graph of a Boolean ring. Finally, we provide a simpler proof for [**13,** Main theorem].

2. Reduction of a graph. Let G be a simple graph. The relation $\sim$ on $V(G)$, defined by

$$a \sim b \Longleftrightarrow N(a) = N(b)$$

is an equivalence relation.

Definition 2.1. Let G be a simple graph. The reduction graph of G, denoted by G_{red}, is the simple graph whose vertex set is $V(G_{\mathrm{red}}) = \{[a] : a \in V(G)\}$, and two distinct equivalence classes $[a]$ and $[b]$ are adjacent in G_{red} if a and b are adjacent in G.

Define $\pi : G \to G_{\mathrm{red}}$ by $\pi(a) = [a]$. The map π is an onto graph homomorphism. It can be seen that $G_{\mathrm{red}} = G/\pi$, where G/π is the quotient graph of G, see [**7,** Definition 7.4.1].

In this section, we will compare some properties of graphs G and G_{red}, such as diameter, completeness and (vertex, edge and arc) transitivity. It is clear that, if two graphs G and K are isomorphic, then so are G_{red} and K_{red}. However, the converse does not necessarily follow.

Theorem 2.2. *Let $f : G_{\mathrm{red}} \to K_{\mathrm{red}}$ be a graph isomorphism such that $|[a]| = |[f(a)]|$, $a \in G$. Then, $G \simeq K$.*

Proof. Suppose that $G_{\mathrm{red}} = \{[a_i] : i \in I\}$, $f([a_i]) = [b_i]$ and $K_{\mathrm{red}} = \{[b_i] : i \in I\}$. Also, let $[a_i] = \{a_{i\lambda} : \lambda \in \Lambda\}$ and $[b_i] = \{b_{i\lambda} : \lambda \in \Lambda\}$. It may be verified that $\overline{f} : G \to K$ by the rule that $\overline{f}(a_{i\lambda}) = b_{i\lambda}$ is a graph isomorphism. $\square$

The hypothesis of the above theorem fails if $G = K_{1,m}$ and $K = K_{1,n}$, where $m \neq n$.

Proposition 2.3. *Let $V(G_{\mathrm{red}}) = \{[a_i] : i \in I\}$. Then,*

(1) *The map $f : V(G_{\mathrm{red}}) \to N(G)$, defined by $f([a]) = N(a)$, is a bijection.*

(2) $V(G) = \bigcup_{i \in I} N(a_i)$.

(3) *G_{red} is a star graph if and only if $|I| = 2$.*

Proof.

(1) For any $[a],[b] \in V(G_{\mathrm{red}})$,

$$[a]=[b] \Longleftrightarrow a\sim b \Longleftrightarrow N(a)=N(b) \Longleftrightarrow f([a])=f([b]).$$

Hence, f is well defined and injective. Moreover, surjectivity of f is obvious.

(2) $V(G)$ is the union of all of its neighborhoods. Thus, by (1), the assertion holds.

(3) Suppose that G_{red} is a star graph. Then, there exists an $[a_i] \in V(G_{\mathrm{red}})$ which is adjacent to every other vertex of G_{red}. Therefore, for every $i', i'' \in I\setminus\{i\}$, $N(a_{i'}) = N(a_{i''}) = [a_i]$, i.e., $[a_{i'}] = [a_{i''}]$. It follows that $|I| = 2$. The converse is straightforward. $\square$

Proposition 2.4. *Let $|V(G_{\mathrm{red}})| > 2$. The following statements are equivalent.*

(1) *G_{red} is a complete graph K_r.*

(2) *G is a complete r-partite graph.*

(3) *G_{red} is a complete r-partite graph.*

(4) *For each distinct vertex $[a],[b],[c] \in V(G_{\mathrm{red}})$,*

$$N(a) \subseteq N(b)\cup N(c).$$

(5) *For each $a \in V(G)$,*

$$[a]=\bigcap_b N(b),$$

where b ranges over $V(G)\setminus [a]$.

Proof.

(1) $\Rightarrow$ (2). Suppose that $V(G_{\mathrm{red}}) = \{[a_1],\dots,[a_r]\}$. From the assumption, a_i is adjacent to a_j for every i,j, where $1\le i<j\le r$. Hence, G is a complete r-partite graph and each $[a_i]$ is one of the r-parts in G.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Since every complete r-partite graph has exactly r distinct neighborhoods, by Proposition 2.3 (1), $|V(G_{\mathrm{red}})| = |N(G)| = r$. Hence, $G_{\mathrm{red}} \simeq K_r$.

(1) $\Rightarrow$ (4). Suppose that $d \in N(a) \setminus N(b)$. From the assumption, $[d]$ and $[c]$ are adjacent for all $[c] \in V(G_{\mathrm{red}})$ with $[d] \neq [c]$. Hence, $d \in N(c)$ as required.

(4) $\Rightarrow$ (5). Suppose that $c \in [a]$ and $b \in V(G)\setminus[a]$. Then, there exists $[d] \in V(G)$ such that $[a]$ and $[d]$ are adjacent. From the assumption, $a \in N(d) \subseteq N(c) \cup N(b)$. Since $a \notin N(a) = N(c)$, $a \in N(b)$, and thus, $c \in N(b)$. The converse is trivial.

(5) $\Rightarrow$ (1). From the assumption, for every two distinct vertices $[a]$, $[b] \in V(G_{\mathrm{red}})$, $[a] \subseteq N(b)$. It follows that $[a]$ and $[b]$ are adjacent, i.e., G_{red} is a complete graph. $\square$

The next theorem compares the concepts of distance and diameter in graphs G and G_{red}.

Theorem 2.5. *The following statements hold.*

(1) *For distinct vertices* $a, b \in V(G)$*,*

$$d(a,b) = \begin{cases} d([a],[b]) & [a] \neq [b], \\ 2 & [a] = [b]. \end{cases}$$

(2) *Let G be a graph that is not complete. It follows that* $\mathrm{diam}(G) = \mathrm{diam}(G_{\mathrm{red}})$ *if and only if G is not a complete r-partite graph. Furthermore, if G is a complete r-partite graph, then* $\mathrm{diam}(G) = 2$ *and* $\mathrm{diam}(G_{\mathrm{red}}) = 1$.

Proof.

(1) If $[a] = [b]$, then there is a $c \in N(a) = N(b)$ and also $d(a,b) = 2$. Now, suppose that $[a] \neq [b]$ and $d(a,b) = k$. There is a path of length k between a and b, such as

$$a = a_0 - a_1 - \cdots - a_j - a_{j+1} - \cdots - a_{k-1} - a_k = b.$$

Thus, we obtain the walk

$$[a] - [a_1] - \cdots - [a_j] - [a_{j+1}] - \cdots - [a_{k-1}] - [b].$$

If there exist i, j, where $0 \leqslant i < j \leqslant k$, such that $[a_i] = [a_j]$, then we have

$$[a] - [a_1] - \cdots - [a_i] - [a_{j+1}] - \cdots - [b].$$

This yields the walk

$$a - \cdots - a_i - a_{j+1} - \cdots - b$$

between a and b, a contradiction.

(2) Suppose that G is not a complete r-partite graph and $\operatorname{diam}(G) = t$. Then, there are $a, b \in V(G)$ such that $d(a, b) = t$. If $[a] \neq [b]$, applying part (1), we obtain $d([a], [b]) = d(a, b) = t$. It follows that $\operatorname{diam}(G) = \operatorname{diam}(G_{\text{red}})$. Now, suppose that $[a] = [b]$. From (1), $\operatorname{diam}(G) = 2$. We know that $\operatorname{diam}(G_{\text{red}}) \leq \operatorname{diam}(G)$. If $\operatorname{diam}(G_{\text{red}}) = 1$, then, by Proposition 2.4, G is a complete r-partite graph, a contradiction. Hence, $\operatorname{diam}(G_{\text{red}}) = 2$. The converse is easy to verify. $\square$

Definition 2.6. Suppose that $\sim$ is an equivalence relation on a set A and $B \subseteq A$. We say that B is a *saturated subset* of A if $[a] \subseteq B$ for all $a \in B$, i.e., $B = \bigcup_{a \in B} [a]$.

Proposition 2.7. *Let $B \subseteq V(G)$.*

(1) *If B is a maximal independent set, then B is a saturated subset of $V(G)$.*

(2) *B is a maximal independent set of $V(G)$ if and only if $B/{\sim}$ is a maximal independent set of $V(G_{\text{red}})$.*

(3) *If $\alpha(G) = |B|$, then $\alpha(G_{\text{red}}) = |B/{\sim}|$.*

Proof.

(1) For any $a \in B$, $B \cup [a]$ is an independent set of $V(G)$. From the maximality of B, $B = B \cup [a]$, and thus, $[a] \subseteq B$.

(2)

($\Rightarrow$). Clearly, $B/{\sim}$ is an independent set of $V(G_{\text{red}})$. In order to establish maximality of $B/{\sim}$, suppose, for some $a \in V(G)$, that $B/{\sim} \cup \{[a]\}$ is an independent set. If $[a] \notin B/{\sim}$, then $a \notin B$, and thus, $B \cup \{a\}$ is not an independent set. Therefore, there exists a $b \in B$ such that b and a are adjacent. This implies that $[b]$ and $[a]$ are adjacent and $[b] \in B/{\sim}$, which is a contradiction to the independence of $B/{\sim} \cup \{[a]\}$.

($\Leftarrow$). Assume that $B' = \bigcup_{[a] \in B/\sim} [a]$ and $b \in V(G) \setminus B'$. Clearly, B' is an independent set of $V(G)$. We claim that B' is a maximal independent set. In order to establish the claim, we show that $B' \cup \{b\}$

is not an independent set of $V(G)$. Since $[b] \notin B/\sim$, by the assumption, $B/\sim \cup \{[b]\}$ is not an independent set. Thus, there is an $[a] \in B/\sim$ such that $[b]$ and $[a]$ are adjacent. Therefore, b and a are adjacent, and $a \in B'$. This implies that $B' \cup \{b\}$ is not an independent set.

(3) Follows from (2). □

Proposition 2.8. *B is a dominating set in graph G if and only if $B/\sim$ is a dominating set in graph G_{red}.*

Proof. B is a dominating set in graph G if and only if, for every $a \in V(G)$, either $a \in B$ or there exists a $b \in B$ such that a and b are adjacent if and only if, for every $[a] \in V(G_{\text{red}})$, either $[a] \in B/\sim$ or there exists a $[b] \in B/\sim$ such that $[a]$ and $[b]$ are adjacent if and only if $B/\sim$ is a dominating set in G_{red}. □

For any $a \in V(G)$, the *reduction number* of a, $\text{red}(a)$, is defined as $|[a]|$, and the *reduction number* of G, $\text{red}(G)$, as $\text{Max}\{\text{red}(a) : a \in V(G)\}$. If $\text{red}(a) = t$ for all vertices a of G, then G is said to have the *constant reduction number t.*

Recall that a graph containing a cycle passing through all vertices of the graph is called a *Hamiltonian graph.* Also, a finite connected graph is *Eulerian*, if there exists a closed trail containing every edge. It is known that a connected graph G is Eulerian if and only if each vertex of G has even degree, see [**12,** Theorem 7.1].

Theorem 2.9. *Let G_{red} be a Hamiltonian graph and $|V(G_{\text{red}})| = r$. If G has a constant reduction number t, then G is a Hamiltonian graph.*

Proof. Suppose that G_{red} is a Hamiltonian graph with a Hamiltonian cycle $[a_1] - [a_2] - \cdots - [a_r] - [a_1]$, and set $[a_i] = \{a_{i1}, a_{i2}, \ldots, a_{it}\}$, $1 \leqslant i \leqslant r$. Then,

$$a_{11} - a_{21} - \cdots - a_{r1} - a_{12} - a_{22} - \cdots - a_{r2} - \cdots - a_{1t} - \cdots - a_{rt} - a_{11}$$

is a Hamiltonian cycle in G. □

It must be noted that, if $G = K_{n,n}$, then $G_{\text{red}} = K_{1,1}$ so that the converse of the preceding theorem does not hold.

Theorem 2.10. *Let G be a graph, and let I be the set of reduction numbers of all vertices of G. Suppose that I consists entirely of either odd or even numbers. If G_{red} is Eulerian, then G is Eulerian also.*

Proof. Let $V(G_{\text{red}}) = \{[a_i] : i \in I\}$. Suppose that there are $m_i \in \mathbb{N}$ such that

$$\deg[a_i] = 2m_i, N([a_i]) = \{[b_1], \ldots, [b_{2m_i}]\}.$$

Hence,

$$N(a_i) = \{b_{11}, \ldots, b_{1(2t_1-1)}, b_{21}, \ldots, b_{2(2t_2-1)}, \ldots, b_{2m_i(2t_{2m_i}-1)}\},$$

where $\text{red}(b_i) = 2t_i - 1$, $t_i \in \mathbb{N}$ and $[b_i] = \{b_{i1}, \ldots, b_{i(2t_i-1)}\}$. This shows that the degree of a_i is $\sum_{i=1}^{2m_i} 2t_i - 1$, which is even. Consequently, G is Eulerian. Similarly, we can prove that, if the reduction number of every vertex of G is even, then G is Eulerian. $\square$

Lemma 2.11. *Let $f \in \text{Aut}(G)$, and let $\overline{f} : V(G_{\text{red}}) \to V(G_{\text{red}})$ be the function given by*

$$\overline{f}([a]) = [f(a)].$$

Then, $\overline{f} \in \text{Aut}(G_{\text{red}})$.

Proof. For any $a \in V(G)$,

$$f(N(a)) = \{f(b) : b - a\} = \{f(b) : f(b) - f(a)\} = N(f(a)).$$

We first note that $\overline{f}$ is well-defined and injective since, for every $[a_1], [a_2] \in V(G_{\text{red}})$,

$$\begin{aligned}[a_1] = [a_2] &\Longleftrightarrow N(a_1) = N(a_2) \Longleftrightarrow f(N(a_1)) \\ &= f(N(a_2)) \Longleftrightarrow N(f(a_1)) = N(f(a_2)) \Longleftrightarrow [f(a_1)] \\ &= [f(a_2)] \Longleftrightarrow \overline{f}([a_1]) = \overline{f}([a_2]).\end{aligned}$$

Moreover, $\overline{f}$ is clearly surjective. Finally,

$$\begin{aligned}[a_1] - [a_2] \Longleftrightarrow a_1 - a_2 \Longleftrightarrow f(a_1) - f(a_2)f \Longleftrightarrow [f(a_1)] \\ = [f(a_2)] \Longleftrightarrow \overline{f}([a_1]) = \overline{f}([a_2]).\end{aligned}$$

Hence, the assertion holds. $\square$

A graph G is *vertex transitive* (*edge transitive, respectively*) if, for any two vertices a and b (two edges e and e'), there exists a $\rho \in \mathrm{Aut}(G)$ such that $\rho(a) = b$, ($\rho(e) = e'$, respectively). Also, a graph is *arc transitive* if its automorphism group acts transitively upon ordered pairs of adjacent vertices.

The following corollary is an immediate consequence of Lemma 2.11.

Corollary 2.12. *For the graph G, the following statements hold.*

(1) *If G is vertex transitive, then so is G_{red}.*

(2) *If G is edge transitive, then so is G_{red}.*

(3) *If G is arc transitive, then so is G_{red}.*

Theorem 2.13. *Let G be a graph with the constant reduction number t. Then,*

(1) *G is vertex transitive if and only if G_{red} is vertex transitive.*

(2) *G is edge transitive if and only if G_{red} is edge transitive.*

(3) *G is arc transitive if and only if G_{red} is arc transitive.*

Proof.

(1) Suppose that G_{red} is vertex transitive and

$$V(G_{\mathrm{red}}) = \{[a_1], \ldots, [a_r]\},$$

where $[a_i] = \{a_{i1}, a_{i2}, \ldots, a_{it}\}$. Then, $V(G) = \{a_{ij} : 1 \leq i \leq r, 1 \leq j \leq t\}$. For every $a_{ij}, a_{kl} \in V(G)$, there exists a $\theta \in S_r$ such that

$$\overline{\theta}([a_i]) = [a_{\theta(i)}] = [a_k].$$

Now, we set $\mu = (j, l)$, where $j, l \in \{1, 2, \ldots, t\}$ with $j \neq l$. Define

$$\overline{\omega} : V(G) \longrightarrow V(G)$$

by

$$\overline{\omega}(a_{mn}) = a_{\theta(m)\mu(n)}.$$

It is easy to see that $\overline{\omega} \in \mathrm{Aut}(G)$. The converse follows from Corollary 2.12 (1).

Statements (2) and (3) can be proven by similar arguments as presented in the proof of (1). $\square$

3. Reduced graphs. Let G be a graph. We recall that G is said to be *reduced* if $G = G_{\mathrm{red}}$. In this section, we investigate the reduced graphs and determine certain common properties between G and G_{red} such as the chromatic number, the clique number, the girth and the perfectness.

Suppose that H is an induced subgraph of G. For every $a \in V(H)$, we define

$$[a]_H = \{b \in H : N_H(a) = N_H(b)\}, \text{ where } N_H(a) = \{b \in H : a - b\}.$$

In this case, $N_H(a) = N(a) \cap V(H)$.

The following theorem shows that reduced subgraphs of G and G_{red} coincide.

Theorem 3.1. *Let H be a subgraph of graph G.*

(1) *There is a graph monomorphism $f : H_{\mathrm{red}} \to G_{\mathrm{red}}$ such that*

$$f(H_{\mathrm{red}}) \simeq H_{\mathrm{red}}.$$

(2) *If H is a reduced graph, then H is isomorphic to a subgraph of G_{red}.*

(3) $(G_{\mathrm{red}})_{\mathrm{red}} = G_{\mathrm{red}}$.

Proof.

(1) For every $[a]_H \in V(H_{\mathrm{red}})$, we set

$$A_{[a]_H} = \{[b] \in V(G_{\mathrm{red}}) : [a]_H = [b]_H\}.$$

Since $A_{[a]_H} \neq \varnothing$, a choice function $f : H_{\mathrm{red}} \to G_{\mathrm{red}}$ may be constructed such that $f([a]_H) \in A_{[a]_H}$. Moreover, for every $[a]_H, [b]_H \in V(H_{\mathrm{red}})$ with

$$f([a]_H) = [a'] \quad \text{and} \quad f([b]_H) = [b'],$$

we have

$$f([a]_H) - f([b]_H) \Longleftrightarrow a' - b' \Longleftrightarrow [a']_H - [b']_H \Longleftrightarrow [a]_H - [b]_H.$$

Hence f is a graph homomorphism and $f(H_{\mathrm{red}}) \simeq H_{\mathrm{red}}$.

(2) Clearly, $H = H_{\mathrm{red}} \simeq f(H_{\mathrm{red}})$.

(3) It is clear. $\square$

In the following corollary, we show that there are several numerical invariants of both G and G_{red} that are the same.

Corollary 3.2. *The following statements hold.*

(1) $\omega(G) = \omega(G_{\mathrm{red}})$.

(2) $\chi(G) = \chi(G_{\mathrm{red}})$.

(3) *If* $\mathrm{gr}(G) \neq 4$, *then* $\mathrm{gr}(G_{\mathrm{red}}) = \mathrm{gr}(G)$.

(4) G *is perfect if and only if* G_{red} *is perfect.*

Proof.

(1) $C = \{a_1, \ldots, a_n\}$ is a clique in G if and only if $[a_i]$ and $[a_j]$ are adjacent for all $1 \leqslant i < j \leqslant n$. However, the latter occurs precisely when $C' = \{[a_1], \ldots, [a_n]\}$ is a clique in G_{red}.

(2) Suppose that $\chi(G_{\mathrm{red}}) = m$. Then, there is a surjective map

$$f : V(G_{\mathrm{red}}) \longrightarrow \{1, 2, \ldots, m\}$$

such that every two adjacent vertices have different colors. Define the function

$$\overline{f} : V(G) \longrightarrow \{1, 2, \ldots, m\}$$

by $\overline{f}(a) = f([a])$. Clearly, $\overline{f}$ is surjective. Moreover, for every $a, b \in V(G)$, $[a]$ and $[b]$ are adjacent whenever a and b are adjacent. Thus, by our assumption, $f([a]) \neq f([b])$, and thus, $\overline{f}(a) \neq \overline{f}(b)$. It now follows that a and b have different colors and $\chi(G) = m$.

(3) It is well known that the cyclic graph $C_n, n \neq 4$ is a reduced graph. Therefore, the result follows from Theorem 3.1 (2).

(4) The assertion follows by applying parts (1) and (2) to subgraphs of G and G_{red}. $\square$

The next theorem determines the relation between the reduction of Cartesian product of two graphs and its components.

Theorem 3.3. *Let G_1 and G_2 be connected graphs with more than two vertices. Then, the graph $G_1 \times G_2$ is reduced.*

Proof. Suppose that, for distinct vertices $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$,

$$N(a_1, a_2) = N(b_1, b_2).$$

Then,

$$(\{a_1\} \times N(a_2)) \cup (N(a_1) \times \{a_2\}) = (\{b_1\} \times N(b_2)) \cup (N(b_1) \times \{b_2\}).$$

Now, if $N(b_1) = \{a_1\}$ and $N(a_2) = \{b_2\}$, then $N(a_1) = \{b_1\}$ and $N(b_2) = \{a_2\}$. However, this would contradict the connectivity of G_i, $i = 1, 2$. Therefore, we must have $\{a_1\} = \{b_1\}$ and $\{a_2\} = \{b_2\}$, which again is a contradiction since (a_1, a_2) and (b_1, b_2) were chosen to be distinct. This completes the proof. $\square$

We finish this section by providing many examples of reduced graphs.

Example 3.4. Each of the following graphs are reduced.

(1) The complete graph K_n.

(2) The Johnson graph $j(\nu, k, k-1)$, where $(\nu, k, k-1) \neq (4, 2, 1)$: suppose that $A, B \in V(G)$ are non-adjacent, and thus, $|A \cap B| \leq k-2$. First, assume that $\Lambda \neq A \cup B$, and take $c \in \Lambda \setminus (A \cup B)$. Set $C = (B \setminus \{b\}) \cup \{c\}$, where $b \in B \setminus A$. Hence, $B \cap C = B \setminus \{b\}$ and $|A \cap C| \leq k-2$. This implies that B and C are adjacent, but A and C are not. Now, suppose that $\Lambda = A \cup B$, and so, $A \cap B = \varnothing$. Set $C = (B \setminus \{b\}) \cup \{a\}$, where $a \in A$ and $b \in B$. Thus, $B \cap C = B \setminus \{b\}$ and $|A \cap C| \leq k-2$; this means that C and B are adjacent, but C and A are not, i.e., $N(A) \neq N(B)$.

(3) The cyclic graph C_n, where $n \neq 4$.

(4) The line graph of G, where $|V(G)| > 4$: for every five distinct vertices $a, b, c, d, e \in V(G)$, we have $\{a, e\} \in N_{L(G)}(\{a, b\}) \setminus N_{L(G)}(\{c, d\})$. In other words, $N_{L(G)}(\{a, b\}) \neq N_{L(G)}(\{c, d\})$. In particular, if G is a reduced graph and $|V(G)| > 4$, then $L(G)_{\mathrm{red}} = L(G_{\mathrm{red}})$.

(5) The n-cube Q_n: we know $Q_n = \mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Also, for every $a \in \mathbb{Z}_2^n \setminus \{0\}$, $a + S \neq S$. Hence, for the canonical map $\pi : V(Q_n) \to V((Q_n)_{\mathrm{red}})$,

$$\ker \pi = \{a \in \mathbb{Z}_2^n : [a] = [0]\} = \{a \in \mathbb{Z}_2^n : a + S = S\} = \{0\},$$

that is, Q_n is a reduced graph, see Theorem 4.1.

(6) The strongly regular graph, $\mathrm{srg}(n, k, \lambda, \mu)$, where $\mu < k$.

4. Reduction of the Cayley graph of a ring. In this section, for the Cayley graph $\Gamma = \mathrm{Cay}(H, S)$, we show that $V(\Gamma_{\mathrm{red}})$ is an additive abelian group and the canonical map $\pi : V(\Gamma) \to V(\Gamma_{\mathrm{red}})$, defined by

$$\pi(a) = [a], \quad a \in \Gamma, \tag{4.1}$$

is a group epimorphism. In fact, we will see that the reduction graph of the Cayley graph of an abelian group is again a Cayley graph. Furthermore, we give a characterization of a semisimple ring R in terms of its Cayley graph.

Theorem 4.1. *Let H be an abelian group with identity element e, let $S \subseteq H$ be a generating set with $e \notin S$, and let $S^{-1} = S$ and $\Gamma = \mathrm{Cay}(H, S)$. Then, the vertex set*

$$V(\Gamma_{\mathrm{red}}) = \{[a] : a \in H\}$$

forms an abelian group under the addition $[a] + [b] = [a + b]$ and

$$\mathrm{Cay}(H, S)_{\mathrm{red}} = \mathrm{Cay}(V(\Gamma_{\mathrm{red}}), \pi(S)),$$

where π is the canonical map defined in (4.1). *Moreover, π is a group epimorphism with the kernel*

$$H_S = \{a \in H : a + S = S\}.$$

Proof. Suppose that $[a_1] = [a_2]$ and $[b_1] = [b_2]$. Then, $N(a_1) = N(a_2)$ and $N(b_1) = N(b_2)$, and thus, $a_1 + S = a_2 + S$ and $b_1 + S = b_2 + S$. Hence,

$$\begin{aligned} N(a_1 + b_1) &= a_1 + b_1 + S = a_1 + b_2 + S \\ &= b_2 + a_1 + S = b_2 + a_2 + S = N(a_2 + b_2). \end{aligned}$$

Therefore, $[a_1] + [b_1] = [a_2] + [b_2]$.

We know that vertices $[a]$ and $[b]$ are adjacent in $\mathrm{Cay}(H, S)_{\mathrm{red}}$ if and only if a and b are adjacent in $\mathrm{Cay}(H, S)$, where the latter occurs if and only if $a - b = s$, for some $s \in S$. On the other hand, $[a]$ and $[b]$ are adjacent in $\mathrm{Cay}(V(\Gamma_{\mathrm{red}}), \pi(S))$ if and only if $[a] - [b] \in \pi(S)$, which occurs precisely when $[a - b] = [s]$, for some $s \in S$. Therefore, it follows that $[a]$ and $[b]$ are adjacent in $\mathrm{Cay}(H, S)_{\mathrm{red}}$ if and only if a and b are adjacent in $\mathrm{Cay}(V(\Gamma_{\mathrm{red}}), \pi(S))$.

Finally, we note that it is routine to verify that π is a group epimorphism. Moreover,

$$\begin{aligned}\ker\pi &= \{a \in H : [a] = [0]\} = \{a \in H : N_\Gamma(a) = N_\Gamma(0)\} \\ &= \{a \in H : a + S = S\} = H_S. \qquad \square\end{aligned}$$

Let R be a commutative ring with unity, and let R^+ be the additive group of R. We denote by $U(R), Z(R), Z^*(R), J(R)$ and $\mathrm{Max}(R)$, the set of unit elements, zero-divisors, non zero zero-divisors, Jacobson radical and the set of maximal ideals of R, respectively. We set $M(a) = \{M \in \mathrm{Max}(R) : a \in M\}$ and $D(a) = \mathrm{Max}(R) \setminus M(a)$.

The next lemma is necessary for the proof of the subsequent theorem.

Lemma 4.2. *Let R be a zero-dimensional semilocal ring and $a \in R$. The following assertions are equivalent.*

(1) $a + Z(R) = Z(R)$.

(2) $a \in J(R)$.

(3) $a + U(R) = U(R)$.

Proof.

(1) $\Rightarrow$ (2). By hypothesis, $R = U(R) \cup Z(R)$. Now, suppose that $a \notin J(R)$. Then, $D(a) \neq \varnothing$, and thus, there is a $b \in \cap D(a) \setminus \cup M(a)$. Therefore, $b \in Z(R)$ and $M(a+b) = \varnothing$. This implies that $a+b \in U(R)$; however, by the assumption, $a + b \in Z(R)$, a contradiction.

(2) $\Rightarrow$ (1). For any $b \in Z(R)$, there is an $M \in \mathrm{Max}(R)$ such that $a + b \in M \subset Z(R)$. On the other hand, if $b \in Z(R)$, then there is an $M \in \mathrm{Max}(R)$ such that $b - a = m \in M$. Therefore, $b \in a + Z(R)$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (2). Similar to the proof of (1) $\Rightarrow$ (2). $\square$

Recall that the *unitary Cayley graph* of R is the Cayley graph $\Gamma = \mathrm{Cay}(R^+, U(R))$. For more information regarding Γ, refer to [**3, 14, 16**]. Akhtar, et al., [**3,** Proposition 2.5] showed that the reduction graph of $\mathrm{Cay}(R^+, U(R))$ is isomorphic to

$$\mathrm{Cay}\left(\frac{R^+}{J(R)}, U\left(\frac{R}{J(R)}\right)\right),$$

where R is an Artinian ring. In the following theorem, we provide a generalization of this result with a shorter proof.

Theorem 4.3. *Let R be a zero-dimensional semilocal ring. Then,*

(1) $\mathrm{Cay}(R^+, U(R))_{\mathrm{red}} \simeq \mathrm{Cay}(R^+/J(R), U(R/J(R)))$.

(2) *If R is the m-Boolean ring $\mathbb{Z}_2^m$, then* $\mathrm{Cay}(R^+, Z^*(R))_{\mathrm{red}}$ *is isomorphic to the complete graph $K_{2^{m-1}}$. Otherwise,* $\mathrm{Cay}(R^+, Z^*(R))$ *is a reduced graph.*

Proof.

(1) Let π be the map defined in (4.1). We have that

$$\pi(U(R)) = \{a + J(R) : a \in U(R)\} = U\left(\frac{R}{J(R)}\right).$$

On the other hand, by Lemma 4.2,

$$\begin{aligned}\ker \pi &= \{a \in R^+ : a + U(R) = U(R)\} \\ &= \{a \in R^+ : a \in J(R)\} = J(R).\end{aligned}$$

The statement now follows from Theorem 4.1.

(2) Suppose that $\Gamma' = \mathrm{Cay}(R^+, Z^*(R))$. If R is a finite Boolean ring, then $R \simeq \mathbb{Z}_2^m$, for some $m \geq 1$ and $N_{\Gamma'}(1) = 1 + Z^*(R) = Z^*(R) = N_{\Gamma'}(0)$. Consequently, $\mathrm{Cay}(R^+, Z^*(R))$ is not a reduced graph. On the other hand, since

$$\ker \pi = \{a \in R : a + Z^*(R) = Z^*(R)\} = \{0, 1\},$$

we have

$$V(\Gamma'_{\mathrm{red}}) \simeq \frac{R^+}{\ker \pi} \simeq \frac{\mathbb{Z}_2^m}{\mathbb{Z}_2} \simeq \mathbb{Z}_2^{m-1}.$$

This, together with Theorem 4.1, implies that

$$\mathrm{Cay}(R^+, Z^*(R))_{\mathrm{red}} = \mathrm{Cay}(\mathbb{Z}_2^m, Z^*(\mathbb{Z}_2^m))_{\mathrm{red}} \simeq \mathrm{Cay}(\mathbb{Z}_2^{m-1}, \mathbb{Z}_2^{m-1} \backslash \{0\}) \simeq k_{2^{m-1}}.$$

Now, suppose that R is not a Boolean ring and $a \in \ker \pi$. Then,

$$a + Z^*(R) = Z^*(R).$$

Assume, toward contradiction, that $a \neq 0$. Then, it is easy to see that $a \in U(R)$ and

$$\begin{aligned} Z^*(R) &= a^{-1}Z^*(R) = a^{-1}(a + Z^*(R)) \\ &= 1 + a^{-1}Z^*(R) = 1 + Z^*(R). \end{aligned}$$

If $J(R) \neq 0$, then there is an $M \in \mathrm{Max}(R)$ such that $a + (J(R) \setminus \{0\}) \subseteq M$. Consequently, $a \in M$, which is impossible. Therefore, $J(R) = 0$, and thus, $R \simeq \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$, where each $\mathbb{F}_i$ is a field. On the other hand, we must have $|\mathbb{F}_i| = 2$, for all i. In order to see this, assume that $|\mathbb{F}_1| > 2$. Then, there is a $b = (c, 0, \ldots, 0) \in R$ such that $c \in \mathbb{F}_1 \setminus \{0, 1\}$ and

$$1 + b = (1 + c, 1, \ldots, 1) \in 1 + Z^*(R) = Z^*(R).$$

However, this is impossible since $1 + b \in U(R)$. Hence, $|\mathbb{F}_i| = 2$, which contradicts the assumption that R is not a Boolean ring. Consequently, $a = 0$, and thus, $\ker \pi = \{0\}$. This implies that $\mathrm{Cay}(R^+, Z^*(R))$ is a reduced graph. $\square$

The following corollary, whose proof follows routinely from Theorem 4.3, provides an important relation between the semisimplicity of a zero-dimensional semilocal ring and the reducibility of its Cayley graph.

Corollary 4.4. *Let R be a zero-dimensional semilocal ring. The following assertions are equivalent.*

(1) *R is a semisimple ring.*

(2) $\mathrm{Cay}(R^+, U(R))$ *is a reduced graph.*

5. Reduction of the comaximal graph of a ring. Sharma and Bhatwadekar in **[22]** defined the *comaximal graph*, $\Gamma(R)$, of a commutative ring R, as a graph whose vertices are elements of R and edges are pairs of distinct vertices a and b such that $Ra + Rb = R$. Obviously, each $a \in U(R)$ is adjacent to every vertex of $\Gamma(R)$, and each $a \in J(R)$ is an isolated vertex of $\Gamma(R)$. Thus, the main part of the graph $\Gamma(R)$ is the subgraph $\Gamma_2(R)$ induced by $V(\Gamma_2(R)) = R \setminus (U(R) \cup J(R))$, which is also called the comaximal graph of R.

In this section, we study and investigate the reduction of the comaximal graph $\Gamma_2(R)$. As in Section 2, two vertices $[a]$ and $[b]$ are adjacent in $\Gamma_2(R)_{\rm red}$ if and only if $Ra + Rb = R$. This is also equivalent to the assertion that $M(a) \cap M(b) = \varnothing$, where $M(a) = \{M \in \mathrm{Max}(R) : a \in M\}$, see [**21,** Proposition 2.2].

Lemma 5.1. *For each* $[a]$, $[b] \in \Gamma_2(R)_{\rm red}$, $[a] = [b]$ *if and only if* $M(a) = M(b)$.

Proof. Suppose that $[a] = [b]$ and $M \in M(a) \setminus M(b)$. There exists a $c \in M \setminus \cup M(b)$ such that $M(b) \cap M(c) = \varnothing$ and $M(a) \cap M(c) \neq \varnothing$. Hence, $c \in N(b)$ and $c \notin N(a)$, a contradiction. Conversely, if $M(a) = M(b)$, then

$$N(a) = \cup D(a) \setminus \cup M(a) = \cup D(b) \setminus \cup M(b) = N(b). \qquad \square$$

The *zero-divisor graph* of a ring R, denoted by $\Gamma_1(R)$, is defined to be the graph with elements of $Z^*(R) = Z(R) \setminus \{0\}$ as the vertices and distinct vertices a and b are adjacent if and only if $ab = 0$, see [**6**].

As noted in [**4**], there is an equivalence relation $\sim$ on vertices of the graph $\Gamma_1(R)$ such that for two vertices $a, b \in Z^*(R)$, $a \sim b \Leftrightarrow \mathrm{Ann}(a) = \mathrm{Ann}(b)$. We denote $\overline{a}$ to be the equivalence class of a. The *compressed* graph of the zero-divisor graph $\Gamma_1(R)$, denoted by $\Gamma_{1E}(R)$, is the graph with the vertex set $V(\Gamma_{1E}(R)) = \{\overline{a} : a \in Z^*(R)\}$ and each pair of distinct classes $\overline{a}$ and $\overline{b}$ are adjacent in $\Gamma_{1E}(R)$ if and only if vertices a and b are adjacent in $\Gamma_1(R)$. If R is a reduced ring, then the compressed graph and the reduction graph of $\Gamma_1(R)$ coincide as

$$\begin{aligned}[a] &= \{b \in Z^*(R) : \mathrm{Ann}(a) \setminus \{a\} = \mathrm{Ann}(b) \setminus \{b\}\} \\ &= \{b \in Z^*(R) : \mathrm{Ann}(a) = \mathrm{Ann}(b)\} = \overline{a}.\end{aligned}$$

In [**5,** Question 3.6], for the compressed graph of the zero-divisor graph of a commutative ring R, the following question was asked:

Question 5.2. *Let* R *be a commutative ring with* $1 \neq 0$ *such that* $\Gamma_{1E}(R)$ *is finite. Then, is there a finite commutative ring* S *with* $1 \neq 0$ *such that* $\Gamma_{1E}(R) \simeq \Gamma_{1E}(S)$?

A similar question can be asked for the comaximal graphs: for a commutative ring R, is there a finite commutative ring S such that $\Gamma_2(R)_{\rm red} \simeq \Gamma_2(S)_{\rm red}$?

In the following theorem, we answer these questions in the case where $|\operatorname{Max}(R)| < \infty$. Moreover, we show that, in this case, the reduction graph of this ring is isomorphic to the zero-divisor graph of a finite Boolean ring.

Theorem 5.3. *Let R be a ring such that $\operatorname{Max}(R) = \{M_1, \ldots, M_n\}$, where $n \geqslant 3$. Then, $\Gamma_2(R)_{\mathrm{red}} \simeq \Gamma_1(\mathbb{Z}_2^n) \simeq \Gamma_2(\mathbb{Z}_2^n) \simeq \Gamma_2(\mathbb{Z}_2^n)_{\mathrm{red}}$.*

Proof. Define the function $f : V(\Gamma_2(R)_{\mathrm{red}}) \to V(\Gamma_1(\mathbb{Z}_2^n))$ by

$$f([a]) = (\chi_{M_1}(a), \ldots, \chi_{M_n}(a)),$$

where χ_{M_i} is the characteristic function of M_i. We claim that f is a graph isomorphism. By Lemma 5.1, f is well defined and injective. In order to show the surjectivity of f, suppose that $x = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_2^n \setminus \{0, 1\}$ and

$$\Lambda = \{i \in \{1, 2, \ldots, n\} : a_i \neq 0\}.$$

Since $\bigcap_{i\in\Lambda} M_i \nsubseteq \bigcup_{i\notin\Lambda} M_i$, there is an element $a \in R$ such that

$$M(a) = \{M_i : i \in \Lambda\}.$$

This implies that $f([a]) = x$. Finally, for $[a], [b] \in \Gamma_2(R)_{\mathrm{red}}$,

$$\begin{aligned} [a] - [b] &\Longleftrightarrow M(a) \cap M(b) = \varnothing \\ &\Longleftrightarrow \chi_{M_i}(a) = 0 \vee \chi_{M_i}(b) = 0, \\ &\Longleftrightarrow (\chi_{M_1}(a), \ldots, \chi_{M_n}(a)) - (\chi_{M_1}(b), \ldots, \chi_{M_n}(b)) \\ &\Longleftrightarrow f([a]) - f([b]), \quad 1 \leqslant i \leqslant n. \end{aligned}$$

Hence, f is a graph isomorphism. For the second isomorphism, it is routine to verify that the function $g : V(\Gamma_1(\mathbb{Z}_2^n)) \to V(\Gamma_2(\mathbb{Z}_2^n))$, given by

$$g(a_1, a_2, \ldots, a_n) = (1 - a_1, 1 - a_2, \ldots, 1 - a_n)$$

is a graph isomorphism. Hence, the assertion holds. The third isomorphism is clear. $\square$

Corollary 5.4. *Let R be a ring, and let S be a reduced Noetherian ring. If*

$$|\operatorname{Max}(R)| = |\operatorname{Min}(S)| = n \geqslant 3,$$

then $\Gamma_2(R)_{\mathrm{red}} \simeq \Gamma_1(S)_{\mathrm{red}}$.

Proof. Follows from Theorem 5.3 and [**4,** Theorem 1.1]. □

Corollary 5.5. *Let R be a reduced finite ring such that* $\mathrm{Max}(R) = \{M_1, \ldots, M_n\}$*, where $n \geqslant 3$. Then, $\Gamma_2(R)$ is a reduced graph if and only if $R \simeq \mathbb{Z}_2^n$.*

Proof. The "only if" part holds trivially. Conversely, by our assumption, $R \simeq \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$, where $\mathbb{F}_i$ is a field. Since $\Gamma_2(R)$ is a reduced graph, $|\mathbb{F}_i| = 2$. Hence $R \simeq \mathbb{Z}_2^n$. □

Corollary 5.6. *Let R be a ring such that* $\mathrm{Max}(R) = \{M_1, \ldots, M_n\}$*, where $n \geqslant 3$. Then, $\omega(\Gamma_2(R)) = \chi(\Gamma_2(R)) = |\mathrm{Max}(R)| = n$.*

Proof. We note that $\omega(\Gamma_1(\mathbb{Z}_2^n)) = \chi(\Gamma_1(\mathbb{Z}_2^n)) = n$, see [**8,** Theorem 6.13]. Now, the result follows from Corollary 3.2 and Theorem 5.3. □

Recall that a graph G is called a *split graph* if its vertex set can be partitioned into a clique C and an independent set I. It is easy to see that, if G is a split graph, then so is G_{red}.

The following theorem is the main result of [**13**], where a thorough proof is provided. Below, we provide a simplified proof using the reduction graph concept.

Theorem 5.7. *Let $\Gamma(R)$ be a split graph with a clique C and an independent set I. Then, R satisfies one of the following conditions*:

(1) *R is a local ring.*

(2) *$R \simeq \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*

(3) *$R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. It suffices to prove the result for $\Gamma_2(R)$. Suppose that R is not a local ring. We consider two cases:

Case 1. $\mathrm{Max}(R) = \{M_1, M_2\}$. By our assumption and [**17,** Theorem 2.2], $\Gamma_2(R)$ is a star graph. Thus, by [**24,** Corollary 3.6], there is a field $\mathbb{F}$ such that $R \simeq \mathbb{Z}_2 \times \mathbb{F}$.

Case 2. $|\mathrm{Max}(R)| = n \geqslant 3$. We first claim that $n = 3$. Suppose otherwise, and pick four maximal ideals M_1, M_2, M_3 and M_4 in R.

Then, clearly, M_1M_2 and M_3M_4 are comaximal ideals. Hence, there exist $a \in M_1M_2$ and $b \in M_3M_4$ such that $a+b=1$. If $a \in C$, then $C \cap M_1 = C \cap M_2 = \{a\}$, which is impossible. Hence, $a \in I$. Similarly, we can show that $b \in I$. However, this is impossible since $a+b=1$. Thus, R must have precisely three maximal ideals. From Corollary 5.6, $\omega(\Gamma_2(R))=3$. Hence, $\Gamma_2(R)=C \cup I$, where $C=\{x_1,x_2,x_3\}$ and $M(x_i) = \{M_i\}$, for each $i=1,2,3$. If $0 \neq a \in J(R)$, then the vertices $a+x_1$ and $a+x_2$ are adjacent as elements of I, which is a contradiction. Therefore, $J(R)=0$, and thus, by the Chinese remainder theorem, $R \simeq \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where each $\mathbb{F}_i$ is a field. Now, if $|\mathbb{F}_1| > 2$, then the clique C of size three in graph $\Gamma_2(R)$ contains the vertices $(1,0,1)$, $(0,1,1)$ and $(1,1,0)$. If there is an $a_1' \in \mathbb{F}_1 \setminus \{0,1\}$, then vertices $(a',1,0)$, $(1,0,a') \in I$ are adjacent, and we obtain a contradiction. Similarly, we can show that $|\mathbb{F}_i|=2$, $i=2,3$. Thus, $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

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