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ON THE ALGEBRA OF WCE OPERATORS

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ABSTRACT. In this paper, we consider the algebra of WCE operators on L^p-spaces, and we investigate some algebraic properties of it. For instance, we show that the set of normal WCE operators is a unital finite Von Neumann algebra, and we obtain the spectral measure of a normal WCE operator on $L^2(\mathcal{F})$. Then, we specify the form of projections in the Von Neumann algebra of normal WCE operators, and we obtain that, if the underlying measure space is purely atomic, then all projections are minimal. In the non-atomic case, there is no minimal projection. Also, we give a non-commutative operator algebra on which the spectral map is subadditive and submultiplicative. As a consequence, we obtain that the set of quasinilpotents is an ideal, and we get a relation between quasinilpotents and commutators. Moreover, we give some sufficient conditions for an algebra of WCE operators to be triangularizable, and consequently, that its quotient space over its quasinilpotents is commutative.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a complete σ-finite measure space. All sets and function statements are to be interpreted as holding up to sets of measure zero. We denote the collection of (equivalence classes modulo sets of zero measure of) $\mathcal{F}$-measurable complex-valued functions on X by $L^0(\mathcal{F})$. For a σ-subalgebra $\mathcal{A}$ of $\mathcal{F}$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping

$$f \longrightarrow E^{\mathcal{A}} f,$$

defined for all non-negative functions f as well as for all

$$f \in L^p(\mathcal{F}) = L^p(X, \mathcal{F}, \mu), \quad 1 \leq p \leq \infty,$$

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where $E^{\mathcal{A}}f$ is the unique $\mathcal{A}$-measurable function, satisfying

$$\int_A (E^{\mathcal{A}}f)\, d\mu = \int_A f d\mu, \quad \text{for all } A \in \mathcal{A}.$$

Throughout, we write E for $E^{\mathcal{A}}$. This operator will play a major role in our work, and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- If $f \geq 0$, then $E(f) \geq 0$; if $E(|f|) = 0$, then $f = 0$.
- $|E(fg)| \leq (E(|f|^p))^{1/p}(E(|g|^{p'}))^{1/p'}$; $p^{-1} + p'^{-1} = 1$.
- For each $f \geq 0$, $S(E(f))$ is the smallest $\mathcal{A}$-set containing $S(f)$, where $S(f) = \{x \in X : f(x) \neq 0\}$.

A detailed discussion and verification of most of these properties may be found in **[10]**. We are concerned here with linear operators acting on $L^p(\mathcal{F}) = L^p(X, \mathcal{F}, \mu)$, especially on $L^2(\mathcal{F})$. We recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that, for each $F \in \mathcal{A}$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \mathcal{F}, \mu)$ with no atoms is called a *non-atomic measure space*. It is a well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be uniquely partitioned as

$$X = \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cup B,$$

where $\{A_n\}_{n \in \mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see **[11]**).

We continue our investigation regarding the class of bounded linear operators on L^p-spaces having the form $M_w E M_u$, where E is the conditional expectation operator, M_w and M_u are, possibly unbounded, multiplication operators, and called a *weighted conditional expectation operator* (WCE operator). Our interest in WCE operators stems from the fact that such forms tend to appear often in the study of those operators related to conditional expectation. WCE operators appeared in **[2]**, where it was shown that every contractive projection on certain L^1-spaces can be decomposed into an operator of the form $M_w E M_u$ and a nilpotent operator. For stronger results concerning WCE operators, the reader may refer to **[1, 5, 7, 8]**. In these papers, it can easily be seen that large classes of operators are of the form of WCE operators.

In Section 2, we consider the algebra of WCE operators on L^p-spaces, and we investigate some of its algebraic properties. For instance, we show that the set of normal WCE operators is a unital finite Von Neumann algebra, and we obtain the spectral measure of a normal WCE operator on $L^2(\mathcal{F})$. In addition, we show that the Von Neumann algebra of normal WCE operators is generated by the spectral projections of its elements. Then, we compute and specify the form of the inverse of $\lambda I+T$ for WCE operator T, and we show that the set of operators of the form $\lambda I + T$ is inversely closed. Moreover, we give a non-commutative operator algebra on which the spectral map is subadditive and submultiplicative. As a consequence, we obtain that the set of quasinilpotents is an ideal, and we get a relation between quasinilpotents and commutators.

In Section 3, first we give an equivalence condition for a WCE operator to be quasinilpotent. In addition, we obtain a non-commutative operator algebra of WCE operators on which the spectral map is subadditive and submultiplicative. As a consequence, we obtain that the set of quasinilpotents is an ideal, and we obtain a relation between quasinilpotents and commutators. Moreover, we give some sufficient conditions for an algebra of WCE operators to be triangularizable, and consequently, that its quotient space over its quasinilpotents is commutative.

2. Some operator algebra structures. In this section, first we give the definition of weighted conditional expectation operators on L^p-spaces.

Definition 2.1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, and let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{F}$ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite. Let E be the conditional expectation operator relative to $\mathcal{A}$. If $1 \leq p,\ q \leq \infty$ and $u, w \in L^0(\mathcal{F})$ such that uf is conditionable and

$$wE(uf) \in L^q(\mathcal{F}) \quad \text{for all } f \in \mathcal{D} \subseteq L^p(\mathcal{F}),$$

where $\mathcal{D}$ is a linear subspace, then the corresponding weighted conditional expectation (or briefly WCE) operator is the *linear transformation*

$$M_w E M_u : \mathcal{D} \longrightarrow L^q(\mathcal{F})$$

defined by

$$f \longrightarrow wE(uf).$$

As was proven in [**4**], the WCE operator M_wEM_u on $L^p(\mathcal{F})$ is bounded if and only if $(E(|u|^{p'}))^{1/p'}(E(|w|^p))^{1/p} \in L^\infty(\mathcal{A})$, where $1/p + 1/p' = 1$. Also, M_wEM_u on $L^1(\mathcal{F})$ is bounded if and only if $uE(|w|) \in L^\infty(\mathcal{F})$.

Now, we define some notation. Let $1 < p < \infty$ and

$$\begin{aligned}\mathcal{W}_p &= \mathcal{W}_p(\mathcal{A})\\ &= \{M_wEM_u : (E(|u|^{p'}))^{1/p'}(E(|w|^p))^{1/p} \in L^\infty(\mathcal{A})\ u, w \in \mathcal{D}(E)\}.\end{aligned}$$

In addition, for $u, w \in \mathcal{D}(E)$, we set

$$\begin{aligned}\mathcal{W}_{w,p} &= \{M_wEM_u : (E(|w|^p))^{1/p}(E(|u|^{p'}))^{1/p'} \in L^\infty(\mathcal{A})\},\\ \mathcal{W}_{p,u} &= \{M_wEM_u : (E(|w|^p))^{1/p}(E(|u|^{p'}))^{1/p'} \in L^\infty(\mathcal{A})\}.\end{aligned}$$

Hence, if we suppose $\mathbf{1}$ as a constant function, then

$$\begin{aligned}\mathcal{W}_{\mathbf{1},p} &= \{EM_u : (E(|u|^{p'}))^{1/p'} \in L^\infty(\mathcal{A})\},\\ \mathcal{W}_{p,\mathbf{1}} &= \{M_wE : (E(|w|^{p'}))^{1/p'} \in L^\infty(\mathcal{A})\}.\end{aligned}$$

Similarly, for $p = 1$, we have

$$\begin{aligned}\mathcal{W}_1 &= \{M_wEM_u : uE(|w|) \in L^\infty(\mathcal{F})\ u, w \in \mathcal{D}(E)\},\\ \mathcal{W}_{w,1} &= \{M_wEM_u : uE(|w|) \in L^\infty(\mathcal{F})\},\\ \mathcal{W}_{1,u} &= \{M_wEM_u : uE(|w|) \in L^\infty(\mathcal{F})\}.\end{aligned}$$

These observations and the reflexivity of $L^p(\mathcal{F})$ for $1 < p < \infty$ show that $\mathcal{W}_{w,p}$ is the adjoint of $\mathcal{W}_{p,w}$ as a subset of the Banach algebra $\mathcal{B}(L^p(\mathcal{F}))$ (the algebra of all bounded linear operators on $L^p(\mathcal{F})$), where $w \in \mathcal{D}(E)$. In addition, $(\mathcal{W}_{w,p})^{**} = \mathcal{W}_{w,p}$ and $(\mathcal{W}_{p,w})^{**} = \mathcal{W}_{p,w}$. For case $p = 1$, we have $\mathcal{W}^*_{w,1} = \mathcal{W}_{1,w}$ and $\mathcal{W}^*_{1,w} = \mathcal{W}_{w,1}$; however, $\mathcal{W}^{**}_{1,w} \neq \mathcal{W}_{1,w}$ and $\mathcal{W}^{**}_{w,1} \neq \mathcal{W}_{w,1}$, since $L^1(\mathcal{F})$ is irreflexive. Here, we have the conditional-type Minkowski inequality as follows:

Lemma 2.2. *For measurable functions $f, g \in \mathcal{D}(E)$ and $1 \leq p < \infty$, we have*

$$(E(|f + g|^p))^{1/p} \leq (E(|f|^p))^{1/p} + (E(|g|^p))^{1/p}.$$

Proof. Let $1 < p,\ p' < \infty$ be such that $p^{-1} + p'^{-1} = 1$ (or $p'(p-1) = p$). Suppose that $f > 0$, $g > 0$ almost everywhere. Then, by using the conditional-type Hölder inequality, we have

$$\begin{aligned} E((f+g)^p) &= E((f+g)^{p-1}(f+g)) \\ &\leq E((f+g)^{p-1}f) + E((f+g)^{p-1}g) \\ &\leq (E((f+g)^{p'(p-1)}))^{1/p'}(E(f^p))^{1/p} \\ &\quad + (E((f+g)^{p'(p-1)}))^{1/p'}(E(g^p))^{1/p}. \end{aligned}$$

Since $f+g>0$ almost everywhere, then $E(f+g)>0$ almost everywhere. Hence,

$$(E(f+g)^p)^{1/p} \leq (E(f^p))^{1/p} + (E(g^p))^{1/p}.$$

Case $p=1$ is clear. □

Let $u, u', w, w' \in \mathcal{D}(E)$ and $\alpha \in \mathbb{C}$. Then, we have $\alpha w, wE(uw') \in \mathcal{D}(E)$ and the following:

$$M_w E M_u \circ M_{w'} E M_{u'} = M_{wE(uw')} E M_{u'},$$

$$M_w E M_u + M_w E M_{u'} = M_w E M_{u+u'},$$

$$\alpha M_w E M_u = M_{\alpha w} E M_u = M_w E M_{\alpha u}.$$

Now, by using conditional-type Hölder and Minkowski inequalities for $1<p<\infty$, we have the following:

$$(E(|\alpha w|^p))^{1/p}(E(|u|^{p'}))^{1/p'} = |\alpha|(E(|w|^p))^{1/p}(E(|u|^{p'}))^{1/p'},$$

$$\begin{aligned} &(E(|w|^p))^{1/p}(E(|u+u'|^{p'}))^{1/p'} \\ &\quad \leq (E(|w|^p))^{1/p}[(E(|u|^{p'}))^{1/p'} + (E(|u'|^{p'}))^{1/p'}], \end{aligned}$$

$$\begin{aligned} &(E(|wE(uw')|^p))^{1/p}(E(|u'|^{p'}))^{1/p'} \\ &\quad \leq (E(|u|^{p'}))^{1/p'}(E(|w|^p))^{1/p}(E(|w'|^p))^{1/p}(E(|u'|^{p'}))^{1/p'}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|M_{wE(uw')}EM_{u'}\| \\ &\quad = \|(E(|wE(uw')|^p))^{1/p}(E(|u'|^{p'}))^{1/p'}\|_\infty \\ &\quad \leq \|(E(|u|^{p'}))^{1/p'}(E(|w|^p))^{1/p}(E(|w'|^p))^{1/p}(E(|u'|^{p'}))^{1/p'}\|_\infty \\ &\quad \leq \|(E(|u|^{p'}))^{1/p'}(E(|w|^p))^{1/p}\|_\infty \|(E(|w'|^p))^{1/p}(E(|u'|^{p'}))^{1/p'}\|_\infty, \end{aligned}$$

and

$$\begin{aligned}\|M_wEM_{u+u'}\| &= \|E(|w|^p))^{1/p}(E(|u+u'|^{p'}))^{1/p'}\|_\infty \\ &\leq \|(E(|w|^p))^{1/p}(E(|u|^{p'}))^{1/p'}\|_\infty \\ &\quad + \|(E(|w|^p))^{1/p}(E(|u'|^{p'}))^{1/p'}\|_\infty.\end{aligned}$$

Also, if $uE(|w|), u'E(|w'|) \in L^\infty(\mathcal{F})$, then

$$\|u'E(|wE(uw')|)\|_\infty \leq \|uE(|w|)\|_\infty \|u'E(|w'|)\|_\infty,$$

since $E(|wE(uw')|) \leq E(|u|E(|w|)|w'|)$. These observations imply that:

- $\mathcal{W}_p$ is closed with respect to the scalar and operator product for all $1 \leq p < \infty$.
- The spaces $\mathcal{W}_{w,p}$ and $\mathcal{W}_{p,u}$ are operator algebras, for all $w, u \in \mathcal{D}(E)$ and $1 \leq p < \infty$.

If $\mathcal{V}$ is an algebra of bounded operators, then its commutant $\mathcal{V}'$ is the set of all bounded operators which commute with every element in $\mathcal{V}$. In symbols (and in the text of WCE operators):

$$(\mathrm{Alg}(\mathcal{W}_2))' = \mathcal{W}_2' = \{T \in \mathcal{B}(L^2(\mathcal{F})) : ST = TS \text{ for all } S \in \mathcal{W}_2\},$$

in which, $\mathrm{Alg}(\mathcal{W}_2)$ is the operator algebra generated by $\mathcal{W}_2$.

Now, we recall that $(\mathcal{W}_{\mathbf{1},2})' = \mathcal{L}^\infty(\mathcal{A})$ where

$$\mathcal{L}^\infty(\mathcal{A}) = \{M_a : a \in L^\infty(\mathcal{A})\}.$$

In the next proposition, we obtain the commutant of $\mathrm{Alg}(\mathcal{W}_2)$.

Proposition 2.3. $\mathrm{Alg}(\mathcal{W}_2)' = \mathcal{L}^\infty(\mathcal{A})$.

Proof. Let $v \in L^\infty(\mathcal{A})$. Then, for $M_wEM_u \in \mathcal{W}_2$ and $f \in L^2(\mathcal{F})$, we have

$$M_vM_wEM_u(f) = vwE(uf) = wE(uvf) = M_wEM_uM_vf.$$

Hence, $M_vM_wEM_u = M_wEM_uM_v$, and thus, $\mathcal{L}^\infty(\mathcal{A}) \subseteq \mathcal{W}_2'$. In addition, since $\mathcal{W}_{\mathbf{1},2} \subseteq \mathcal{W}_2$, then

$$\mathcal{W}_2' \subseteq (\mathcal{W}_{\mathbf{1},2})' = \mathcal{L}^\infty(\mathcal{A}).$$

Therefore, $\mathcal{L}^\infty(\mathcal{A}) = \mathcal{W}_2' = \mathrm{Alg}(\mathcal{W}_2)'$. $\square$

From Proposition 2.3, we obtain that $\mathcal{L}^\infty(\mathcal{A})' = \mathcal{W}_2''$, and thus, $\mathcal{L}^\infty(\mathcal{A})' = \mathrm{Alg}(\mathcal{W}_2)''$. Hence, using [**7,** Theorem 3.2, Corollary 3.3], we have the next proposition that characterizes the double commutant of $\mathcal{W}_2$.

Proposition 2.4. *Let T be a continuous linear transformation on $L^2(\mathcal{F})$. Then the following are equivalent*:

- $T \in \mathcal{W}_2'' = \mathrm{Alg}(\mathcal{W}_2)''$;
- *there is a constant C such that, for every $f \in L^2(\mathcal{F})$*

$$E(|Tf|^2) \leq C \cdot E(|f|^2) \text{ almost everywhere};$$

- *for each $f \in L^2(\mathcal{F})$, there is a constant C_f such that*

$$E(|Tf|^2) \leq C_f \cdot E(|f|^2) \text{ almost everywhere};$$

- *for each $f \in L^2(\mathcal{F})$, $S(Tf) \subseteq S(E(|f|))$*;
- *for each $f \in L^2(\mathcal{F})$, define the measure μ_f on $\mathcal{A}$ by*

$$d\mu_f = |f|^2 d\mu|_{\mathcal{A}}.$$

Then, for all f, $\mu_{Tf} \ll \mu_f$.

From Proposition 2.4, we obtain that $S(T(\chi_A f)) \subseteq S(E(\chi_A f))$ for $A \in \mathcal{A}$, $f \in L^2(\mathcal{F})$ and $T \in \mathcal{W}_2''$. Since $\mathcal{W}_2 \subseteq \mathcal{W}_2''$, then, for $M_wEM_u \in \mathcal{W}_2$, we have

$$S(M_wEM_u(\chi_A f)) \subseteq S(E(\chi_A f)) = S(\chi_A E(f)) \subseteq A.$$

This implies that $L^2(A) = L^2(A, \mathcal{F}_A, \mu_{\mathcal{F}_A})$ is invariant under M_wEM_u, where $\mathcal{F}_A = \{A \cap C : C \in \mathcal{F}\}$. Thus, $\mathrm{Alg}(\mathcal{W}_2)$ is a reducible operator algebra. If $T = M_wEM_u \in \mathcal{W}_2$, then $T^* = M_{\overline{u}}EM_{\overline{w}}$. Therefore, $T^* \in \mathcal{W}_2$ and $(\mathcal{W}_2)^* = \mathcal{W}_2$. Thus, $L^2(A)$ is also invariant under the adjoint of M_wEM_u. Hence, the set

$$\{L^2(A) : A \in \mathcal{A}\}$$

is a subset of $\mathrm{Lat}(\mathcal{W}_2)$, i.e., it is a collection of reducing subspaces for $\mathcal{W}_2$. We easily obtain $(\mathcal{W}_{\mathbf{1},2})^* = \mathcal{W}_{2,\mathbf{1}}$. Let $\mathcal{W}_{\mathbf{1},2}^N = \{EM_u : u \in L^\infty(\mathcal{A})\}$.

Here, we recall a fundamental lemma from general operator theory.

Lemma 2.5. *Let $T \in \mathcal{B}(L^2(\mathcal{F}))$, and let S be a closed operator on $L^2(\mathcal{F})$. If $T = S$ on a dense subset of $L^2(\mathcal{F})$, then S is bounded and $T = S$.*

A $*$-subalgebra of $\mathcal{B}(L^2(\mathcal{F}))$ is called a *Von Neumann algebra* on the Hilbert space $L^2(\mathcal{F})$ if it is closed in strong operator topology (SOT).

In the next theorem, we prove that $\mathcal{W}^N_{\mathbf{1},2}$ is a unital commutative Von Neumann algebra.

Theorem 2.6. *If $(X, \mathcal{F}, \mu)$ is a finite measure space and $\mathcal{A} \subset \mathcal{F}$ is a σ-subalgebra, then $\mathcal{W}^N_{\mathbf{1},2}$ is a unital commutative Von Neumann algebra with unit E.*

Proof. It is easy to see that $\mathcal{W}^N_{\mathbf{1},2}$ is a self-adjoint operator subalgebra of $\mathcal{B}(H)$. Then, we only need prove that $\mathcal{W}^N_{\mathbf{1},2}$ is strongly closed. Let $\{EM_{u_\alpha}\}_\alpha \subseteq \mathcal{W}^N_{\mathbf{1},2}$ and $T \in \mathcal{B}(L^2(\mathcal{F}))$ be such that

$$\|E(u_\alpha f) - T(f)\|_{L^2} \longrightarrow 0 \quad \text{for all } f \in L^2(\mathcal{F}).$$

Hence, for the constant function $\mathbf{1}$, we have

$$\|u_\alpha - T(\mathbf{1})\|_{L^2} \longrightarrow 0,$$

and thus, $T(\mathbf{1})$ is $\mathcal{A}$-measurable. In addition, for every $f \in L^\infty(\mathcal{F})$ and α, we have

$$\begin{aligned}
&\|T(\mathbf{1})E(f) - T(f)\|_{L^2} \\
&\quad \leq \|T(\mathbf{1})E(f) - E(u_\alpha f)\|_{L^2} + \|E(u_\alpha f) - T(f)\|_{L^2} \\
&\quad \leq \|T(\mathbf{1}) - u_\alpha\|_{L^2}\|f\|_\infty + \|E(u_\alpha f) - T(f)\|_{L^2}.
\end{aligned}$$

This implies that $T = EM_{T(\mathbf{1})}$ on $L^\infty(\mathcal{F})$. Since $L^\infty(\mathcal{F})$ is dense in $L^2(\mathcal{F})$ and $EM_{T(\mathbf{1})}$ is closed, then, by Lemma 2.5, we get that $EM_{T(\mathbf{1})}$ is bounded and $T = EM_{T(\mathbf{1})}$. Therefore, $\mathcal{W}^N_{\mathbf{1},2}$ is strongly closed and, consequently, is a unital commutative Von Neumann algebra with unit E. $\square$

Therefore, $\mathcal{W}^N_{\mathbf{1},2}$ is a finite Von Neumann algebra. Let EM_{u_α} be the approximate unit for $\mathcal{W}^N_{1,2}$. Then, we obtain that

$$\|u_\alpha - \mathbf{1}\|_\infty = \|EM_{u_\alpha - \mathbf{1}}\| = \|EM_{u_\alpha} - E\| \longrightarrow 0.$$

Thus, a net EM_{u_α} of $\mathcal{W}^N_{\mathbf{1},2}$ is an approximate unit if and only if

$$\|u_\alpha - \mathbf{1}\|_\infty \longrightarrow 0.$$

In the next proposition, we specify the form of the projections and minimal projections in $\mathcal{W}^N_{\mathbf{1},2}(\mathcal{A})$.

Proposition 2.7. *An element $P \in \mathcal{W}^N_{\mathbf{1},2}(\mathcal{A})$ is a projection if and only if $P = EM_{\chi_A}$ for some $A \in \mathcal{A}$. The projection P is an atom (minimal projection) if and only if $P = EM_{\chi_A}$, in which A is an $\mathcal{A}$-atom. For each projection P of $\mathcal{W}^N_{\mathbf{1},2}$, we have $P \leq E$, which means that E is a maximal projection in $\mathcal{W}^N_{\mathbf{1},2}$. If $(X, \mathcal{A}, \mu)$ is a non-atomic measure space, then there is no minimal projection in $\mathcal{W}^N_{\mathbf{1},2}$ and, if $(X, \mathcal{A}, \mu)$ is a purely atomic measure space, then all projections in $\mathcal{W}^N_{\mathbf{1},2}$ are minimal.*

Proof. Let $A \in \mathcal{A}$. Then, it is easy to see that EM_{χ_A} is a projection. Also, if $P \in \mathcal{W}^N_{\mathbf{1},2}$ is a projection, then we obtain that $u \in L^\infty(\mathcal{A})$ is real valued and $P = EM_{S(u)}$. For every $A \in \mathcal{A}$, we have $EM_{\chi_A} \leq E$; hence, E is the maximal projection. In addition, if $A \in \mathcal{A}$ is an atom, then there is no $\mathcal{A}$-measurable subset of A with positive measure. Thus, EM_{χ_A} is a minimal projection. If $B \in \mathcal{A}$ is a non-atomic measurable set, then, for every $0 < \alpha < \mu(B)$, there is a measurable subset C of B with $\mu(C) = \alpha$. This implies that EM_{χ_B} cannot be a minimal projection. $\square$

It is known that, if a, b are elements of a unital algebra A, then $1-ab$ is invertible if and only if $1 - ba$ is invertible. A consequence of this equivalence is that

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

Now, in the next theorem, we compute the spectrum of M_wEM_u, and we also give a formula for the inverse of $\lambda I - M_wEM_u$.

Theorem 2.8. *Let $M_wEM_u \in \mathcal{W}_p$ for $1 \leq p < \infty$ or $M_wEM_u \in \mathcal{U}$. Then,*

$$\sigma(M_wEM_u) \setminus \{0\} = \text{ess range}(E(uw)) \setminus \{0\}.$$

Moreover, for each $\lambda \in \mathbb{C} \setminus \sigma(M_wEM_u) \cup \{0\}$, we have

$$(\lambda I - M_wEM_u)^{-1} = \frac{1}{\lambda} I - M_{(w/\lambda(E(wu)-\lambda))} EM_u.$$

Proof. We know that $\sigma(EM_u) \setminus \{0\} = \text{ess range}(E(u)) \setminus \{0\}$ **[3]**. Since the operator $M_w EM_u$ is the composition of two operators M_w and EM_u in the algebra of linear operators on L^p-spaces, then from the previously recalled information, we obtain that

$$\begin{aligned}\sigma(M_w EM_u) \setminus \{0\} &= \sigma(EM_u M_w) \setminus \{0\} \\ &= \sigma(EM_{uw}) \setminus \{0\} \\ &= \text{ess range}(E(uw)) \setminus \{0\}.\end{aligned}$$

Therefore, we have $\sigma(M_w EM_u) \setminus \{0\} = \text{ess range}(E(uw)) \setminus \{0\}$. Suppose that $\lambda \in \mathbb{C} \setminus \sigma(M_w EM_u) \cup \{0\}$, and consider the linear transformation T defined by

$$Tf = \frac{f}{\lambda} - \left(\frac{w}{\lambda(E(wu) - \lambda)}\right) E(uf)$$

for any $f \in L^p(\mathcal{F})$. Since $\lambda \notin \text{ess range}(E(uw))$, then $1/(E(wu) - \lambda) \in L^\infty(\mathcal{A})$, and we easily obtain that

$$\|Tf\| \leq \frac{1}{|\lambda|}(\|(E(wu) - \lambda)^{-1}\|_\infty \|M_w EM_u\| + 1)\|f\|_p.$$

Thus, T is a bounded operator on $L^p(\mathcal{F})$. Also, direct computations show that $T(\lambda I - M_w EM_u) = (\lambda I - M_w EM_u)T = I$. Hence, $T = (\lambda I - M_w EM_u)^{-1}$. □

From Theorem 2.8, we obtain, for $1 \leq p < \infty$, the collection

$$\mathfrak{W}_p = \{\lambda I + T : 0 \neq \lambda \in \mathbb{C},\ T \in \mathcal{W}_p\}$$

is inversely closed. In addition, the spaces

$$\mathfrak{W}_{w,p} = \{\lambda I + T : 0 \neq \lambda \in \mathbb{C},\ T \in \mathcal{W}_{w,p}\}$$

and

$$\mathfrak{W}_{p,u} = \{\lambda I + T : 0 \neq \lambda \in \mathbb{C},\ T \in \mathcal{W}_{p,u}\}$$

are inverse closed operator algebras.

Now, we recall the definition of the spectral measure with respect to a measurable space and a Hilbert space.

Definition 2.9. If X is a set, $\mathcal{F}$ a σ-algebra of subsets of X and H a Hilbert space, then a *spectral measure* for $(X, \mathcal{F}, H)$ is a function $\mathcal{E} : \mathcal{F} \to \mathcal{B}(H)$ having the following properties.

(i) $\mathcal{E}(S)$ is a projection.
(ii) $\mathcal{E}(\emptyset) = 0$ and $\mathcal{E}(X) = I$.
(iii) For each $S_1, S_2 \in \mathcal{F}$, $\mathcal{E}(S_1 \cap S_2) = \mathcal{E}(S_1)\mathcal{E}(S_2)$.
(iv) If $\{S_n\}_{n=0}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{F}$, then

$$\mathcal{E}\left(\bigcup_{n=0}^{\infty} S_n\right) = \sum_{n=0}^{\infty} \mathcal{E}(S_n),$$

where the right hand side converges in strong operator topology.

The spectral theorem states that, for every normal operator T on a Hilbert space H, there is a unique spectral measure $\mathcal{E}$ relative to $(\sigma(T), H)$ such that

$$T = \int_{\sigma(T)} z \, d\mathcal{E},$$

where z is the inclusion map of $\sigma(T)$ in $\mathbb{C}$. Recall that, $T = EM_u \in \mathcal{W}_{\mathbf{1},2}$ is normal if and only if $u \in L^{\infty}(\mathcal{A})$. As is known, for every $f \in L^{\infty}(\mathcal{F})$, we have $\sigma(f) = \text{ess range}(f)$, and we also have

$$\sigma(EM_u) \setminus \{0\} = \text{ess range}(E(u)) \setminus \{0\}.$$

Hence, if we define the operator Θ as:

$$\Theta : L^{\infty}(\mathcal{A}) \longrightarrow \mathcal{W}_{\mathbf{1},2}, \qquad \Theta(u) = EM_u,$$

then we easily obtain that Θ is a unital isometric $*$-isomorphism that preserves the spectrum.

If $T = EM_u$ is a normal operator on $L^2(\mathcal{F})$, then $T^n = M_{u^n}E$ and $(T^*)^n = M_{\overline{u}^n}E$. Thus,

$$(T^*)^n T^m = M_{(\overline{u})^n u^m} E$$

and

$$p(T, T^*) = p(u, \overline{u})E = Ep(u, \overline{u}),$$

where

$$p(z, t) = \sum_{n,m=0}^{N,M} \alpha_{n,m} z^m t^n.$$

From the Weierstrass approximation theorem, we have $f(T) = M_{f(u)}E$, for all $f \in C(\text{ess range}(u))$. Therefore, we obtain the following:

1. Let EM_u be a normal operator on $L^2(\mathcal{F})$. Then, the map

$$\phi : C(\text{ess range}(u) \cup \{0\}) \longrightarrow C^*(EM_u, I),$$

defined by $\phi(f) = M_{f(u)}E$, is a unital $*$-homomorphism. Moreover, by [**9,** Theorem 2.1.13], ϕ is also a unique $*$-isomorphism such that $\phi(z) = EM_u$, where

$$z : \text{ess range}(u) \cup \{0\} \longrightarrow \mathbb{C}$$

is the inclusion map.

2. The map

$$\Theta : L^\infty(\mathcal{A}) \longrightarrow \mathcal{B}(L^2(\mathcal{F}))$$

defined by $\Theta(u) = EM_u$ is a $*$-homomorphism, i.e., the pair $(\Theta, L^2(\mathcal{F}))$ is a representation for $L^\infty(\mathcal{A})$ as a C^*-algebra.

For $S \in \mathcal{A}$, let

$$\mathcal{E}(S) : L^2(\mathcal{F}) \longrightarrow L^2(\mathcal{F})$$

be defined by

$$\mathcal{E}(S) = E^{\mathcal{A}} M_{\chi_S}.$$

Then, $\mathcal{E}$ is a spectral measure on $(X, \mathcal{A}, L^2(\mathcal{F}))$. Hence, we obtain the following theorem.

Theorem 2.10. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $\mathcal{A} \subset \mathcal{F}$ a σ-subalgebra and u in $L^\infty(\mathcal{A})$. Consider the operator $E^{\mathcal{A}} M_u$ on $L^2(\mathcal{F})$. Then, the set function $\mathcal{E}(S) = E^{\mathcal{A}} M_{\chi_S}$ is a spectral measure on $(X, \mathcal{A}, L^2(\mathcal{F}))$. In addition, $\mathcal{E}$ has compact support and*

$$E^{\mathcal{A}} M_u = \int z \, d\mathcal{E}.$$

From Theorem 2.10, we have that the Von Neumann algebra $\mathcal{W}^N_{1,2}(\mathcal{A})$ is generated by projections of the form of $\mathcal{E}(S) = E^{\mathcal{A}} M_{\chi_S}$ where $S \in \mathcal{A}$.

In the next theorem, we prove that the weak* convergence of the net $\{u_\alpha\}_\alpha$ in $L^\infty(\mathcal{A})$ is equivalent to weak operator convergence, or in weak operator topology (WOT), of the net $\{\Theta(u_\alpha)\}_\alpha$.

Theorem 2.11. *If $(X, \mathcal{F}, \mu)$ is a σ-finite measure space and $\{u_\alpha\}_\alpha$ is a net in $L^\infty(\mathcal{A})$, then $u_\alpha \to 0$ weak* in $L^\infty(\mathcal{A})$ if and only if*

$$\Theta(u_\alpha) \longrightarrow 0 \text{ WOT}.$$

Proof. Assume that $u_\alpha \to 0$ weak* in $L^\infty(\mathcal{A}) = (L^1(\mathcal{A}))^*$. If $f, g \in L^2(\mathcal{F})$, then $E(f), E(g) \in L^2(\mathcal{A})$ and $E(f)E(g) \in L^1(\mathcal{A})$. Thus,

$$\langle \Theta(u_\alpha)(f), g\rangle = \int_X u_\alpha E(f)E(g)\, d\mu \longrightarrow 0.$$

Conversely, assume $\Theta(u_\alpha) \to 0$ WOT in $\mathcal{B}(L^2(\mathcal{F}))$. If $h \in L^1(\mathcal{A})$, then $h = E(f)E(g)$, where $f, g \in L^2(\mathcal{F})$. Thus,

$$\int_X E(u_\alpha f)E(g)\, d\mu = \int_X E(u_\alpha f) g\, d\mu = \langle \Theta(u_\alpha)(f), \overline{g}\rangle \longrightarrow 0. \qquad \square$$

3. Simultaneous triangularizablity. Let A be a Banach algebra. An element a of A is called *quasinilpotent* if $r(a) = 0$, and we denote

$$\mathcal{QN}(A) = \{a \in A : r(a) = 0\},$$

where $r(a)$ is the spectral radius of a. The Jacobson radical A is often denoted by $\mathcal{R}(A)$ and is equal to

$$\{a \in A : ab \in \mathcal{QN}(A) \text{ for all } b \in A\}.$$

The algebra A is called *semisimple* if $\mathcal{R}(A) = \{0\}$. And, if $\mathcal{R}(A) = A$, then A is called a *radical algebra*. It is obvious that $\mathcal{R}(A) \subseteq \mathcal{QN}(A)$. In light of Theorem 2.8, we have the following remark.

Remark 3.1. If

$$T = M_w E M_u : L^p(\Sigma) \longrightarrow L^p(\Sigma), \quad 1 \le p < \infty,$$

then the following are equivalent:

(a) $M_w E M_u$ is quasinilpotent;

(b) $E(uw) \equiv 0$;

(c) $M_{wE(uw)} E M_u \equiv 0$.

Let

$$J(\mathcal{W}_p) = \{T \in A : 1 \notin \sigma(ST) \text{ for all } S \in \mathcal{W}_p\}.$$

Then, for $1 \leq p < \infty$, we have

1. $J(\mathcal{W}_p) \subseteq \mathcal{NP}(\mathcal{W}_p)$.
2. For every $u, w \in \mathcal{D}(E)$ we have

$$J(\mathcal{W}_{w,p}) = \mathcal{NP}(\mathcal{W}_{w,p}), \qquad J(\mathcal{W}_{p,u}) = \mathcal{NP}(\mathcal{W}_{p,u}).$$

Let

$$r : \mathcal{W}_{w,p} \longrightarrow \mathbb{R}$$

be the spectral map i,e., r maps M_wEM_u to $r(M_wEM_u)$. As is known, in general, the spectral map is not subadditive and submultiplicative. Indeed, for elements a, b of an arbitrary unital normed algebra with $ab = ba$, we have $r(a + b) \leq r(a) + r(b)$, $r(ab) \leq r(a)r(b)$. In the next lemma, we obtain that r is subadditive and submultiplicative on the algebra $W_{w,p}$.

Lemma 3.2. *Let $1 \leq p < \infty$, $w \in \mathcal{D}(E)$ and $S, T \in \mathcal{W}_{w,p}$. Then, we have*

$$r(S + T) \leq r(T) + r(S), \qquad r(ST) \leq r(S)r(T).$$

Proof. Let $u, v, w \in \mathcal{D}(E)$ be such that $M_wEM_u, M_wEM_v \in \mathcal{W}_{w,p}$. Hence, $E(uw), E(wv) \in L^\infty(\mathcal{A})$, and thus, by Theorem 2.8, we obtain the proof. □

In the next proposition, we have that every quasinilpotent element of $\mathcal{W}_{w,p}$ is in $\mathcal{R}(\mathcal{W}_{w,p})$.

Proposition 3.3. *If $1 \leq p < \infty$, $w \in \mathcal{D}(E)$, then $\mathcal{R}(\mathcal{W}_{w,p}) = \mathcal{QN}(\mathcal{W}_{w,p})$. Indeed, $\mathcal{QN}(\mathcal{W}_{w,p})$ is a two-sided ideal in $\mathcal{W}_{w,p}$.*

Proof. It is a direct consequence of Lemma 3.2. □

Here, we recall that the commutator of two operators T and S is the operator $[T, S]$, defined by

$$[T, S] = TS - ST.$$

In the next proposition, we give a relation between commutators and quasi-nilpotent operators in $\mathcal{W}_p$.

Proposition 3.4. *Let $w, u \in \mathcal{D}(E)$ and $p > 1$. Then, we have*

$$[\mathcal{W}_{w,p}, \mathcal{W}_{w,p}] \subseteq \mathcal{QN}(\mathcal{W}_{w,p})$$

and

$$[\mathcal{W}_{p,u}, \mathcal{W}_{p,u}] \subseteq \mathcal{QN}(\mathcal{W}_{p,u}).$$

Proof. Let $T = M_w E M_u$ and $S = M_{w'} E M_{u'}$. Then, for any $f \in L^p(\mathcal{F})$, we have

$$TS(f) = M_{wE(uw')} E M_{u'}(f),$$
$$ST(f) = M_{w'E(u'w)} E M_u(f).$$

Combining these relations gives:

$$[T, S] = M_v E M_{u'} - M_{v'} E M_u,$$

such that $E(vu') - E(v'u) = 0$, in which $v = wE(uw')$ and $v' = w'E(u'w)$. This means that

$$\begin{aligned}[\mathcal{W}_p, \mathcal{W}_p] &= \{[T, S], T, S \in \mathcal{W}_p\} \\ &\subseteq \{M_w E M_u - M_{w'} E M_{u'} : E(uw) = E(u'w')\}.\end{aligned}$$

Specifically, we have

$$[\mathcal{W}_{w,p}, \mathcal{W}_{w,p}] \subseteq \{M_w E M_u : E(uw) = 0, M_w E M_u \in \mathcal{W}_{w,p}\}$$

and

$$[\mathcal{W}_{p,u}, \mathcal{W}_{p,u}] \subseteq \{M_w E M_u : E(uw) = 0, M_w E M_u \in \mathcal{W}_{p,u}\}.$$

Consequently, by using Remark 3.1, we get that

$$[\mathcal{W}_{w,p}, \mathcal{W}_{w,p}] \subseteq \mathcal{QN}(\mathcal{W}_{w,p})$$

and

$$[\mathcal{W}_{p,u}, \mathcal{W}_{p,u}] \subseteq \mathcal{QN}(\mathcal{W}_{p,u}). \qquad \square$$

A collection $\mathcal{W}$ of bounded operators on a Banach space X is called simultaneously triangularizable if there is a maximal totally ordered complete set of (closed) subspaces of X (a maximal nest) which are $\mathcal{W}$-invariant.

Now, we recall an assertion of [**4**] for compactness of a WCE operator on $L^p(\mathcal{F})$.

Theorem 3.5 ([**4**])**.** *A* WCE *operator M_wEM_u is a compact operator on $L^p(\mathcal{F})$ if and only if, for every $\epsilon > 0$, the set N_ϵ consists of finitely many $\mathcal{A}$-atoms, in which*

$$N_\epsilon = \{x \in X : (E(|w|^p))^{1/p}(x)(E(|u|^{p'}))^{1/p'}(x) \geq \epsilon\}.$$

It is not difficult to see that $\mathcal{W}_{w,p} \subseteq \mathcal{K}(L^p(\mathcal{F}))$ if and only if M_wE is a compact operator on $L^p(\mathcal{F})$. Hence, we have the next proposition.

Proposition 3.6. *If, for every $\epsilon > 0$, the set $N_{\epsilon,1}$ consists of finitely many $\mathcal{A}$-atoms, then $\mathcal{W}_{w,p}$ is a subalgebra of $\mathcal{K}(L^p(\mathcal{F}))$ (the ideal of compact operators on $L^p(\mathcal{F})$), where $N_{\epsilon,1} = \{x \in X : E(|w|^p)(x) \geq \epsilon\}$.*

Proof. Since the WCE operator M_wEM_u is the composition of two operators M_wE and EM_u, then, by Theorem 3.5, we obtain that the operator M_wE is compact, and thus, M_wEM_u is compact. $\square$

In the next theorem, we give a sufficient condition under which the operator algebra $\mathcal{W}_{w,p}$ is simultaneously triangularizable.

Theorem 3.7. *If, for every $\epsilon > 0$, the set $N_{\epsilon,1}$ consists of finitely many $\mathcal{A}$-atoms, then $\mathcal{W}_{w,p}$ is simultaneously triangularizable and*

$$\frac{\mathcal{W}_{w,p}}{\mathcal{NP}(\mathcal{W}_{w,p})}$$

is commutative.

Proof. By our assumption and Proposition 3.6, we obtain that $\mathcal{W}_{w,p} \subseteq \mathcal{K}(L^p(\mathcal{F}))$. Therefore, by [**6,** Theorem 1] and Proposition 3.4, we obtain that $\mathcal{W}_{w,p}$ is simultaneously triangularizable. By [**6,** Theorem 2] we get that $\mathcal{W}_{w,p}/(\mathcal{NP}(\mathcal{W}_{w,p}))$ is commutative. $\square$

Now, using Theorem 3.5, Proposition 3.6 and Theorem 3.7, we have the next corollary.

Corollary 3.8.

(a) *The operator algebra* $\mathcal{W}_{w,p} \cap \mathcal{K}(L^p(\mathcal{F}))$ *is simultaneously triangularizable for all* $w \in \mathcal{D}(E)$ *and* $1 \leq p < \infty$.
(b) *If* $\mathcal{W}_{w,p} \cap \mathcal{K}(L^p(\mathcal{F})) \neq \emptyset$, *then* $\mathcal{W}_{w,p} \subseteq \mathcal{K}(L^p(\mathcal{F}))$.
(c) *If* $\mathcal{W}_{w,p} \cap \mathcal{K}(L^p(\mathcal{F})) \neq \emptyset$, *then* $\mathcal{W}_{w,p}$ *is simultaneously triangularizable and* $\mathcal{W}_{w,p}/(\mathcal{NP}(\mathcal{W}_{w,p}))$ *is commutative.*

REFERENCES

1. P.G. Dodds, C.B. Huijsmans and B. De Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.

2. R.G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math. **15** (1965), 443–462.

3. Y. Estaremi, *Unbounded weighted conditional expectation operators*, Compl. Anal. Oper. Th. **10** (2016), 567-580.

4. Y. Estaremi and M.R. Jabbarzadeh, *Weighted Lambert type operators on L^p-spaces*, Oper. Matrices **1** (2013), 101–116.

5. J.J. Grobler and B. de Pagter, *Operators representable as multiplication-conditional expectation operators*, J. Oper. Th. **48** (2002), 15–40.

6. A. Katavolos and H. Radjavi, *Simultaneous triangularization of operators on a Banach space*, J. Lond. Math. Soc. **41** (1990) 547–554.

7. A. Lambert, *Conditional expectation related characterizations of the commutant of an abelian W^*-algebra*, Far East J. Math. Sci. **2** (1994), 1–7.

8. Shu-Teh Chen Moy, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math. **4** (1954), 47–63.

9. G.J. Murphy, C*-*algebras and operator theory*, Academic Press, Boston, 1990.

10. M.M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

11. A.C. Zaanen, *Integration*, North-Holland, Amsterdam, 1967.

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