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CONSTRUCTION OF GLOBALIZATIONS FOR PARTIAL ACTIONS ON RINGS, ALGEBRAS, C*-ALGEBRAS AND HILBERT BIMODULES

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ABSTRACT. We give a necessary condition for a partial action on a ring to have globalization. We also show that every partial action on a C*-algebra satisfying this condition admits a globalization and, finally, we use the linking algebra of a Hilbert module to translate our condition to the realm of partial actions on Hilbert modules.

1. Introduction. Among the simplest examples of partial actions on C*-algebras **[7, 8, 10]**, we find restrictions of actions to non invariant C*-ideals **[1]**. Many objects associated to a global action, β, and a restriction it, α, are closely related. For example, the partial crossed product of α is a hereditary C*-subalgebra of the crossed product of β **[1, 2]**. Therefore, it is interesting to know which partial actions can be described as the restriction of a global action (the globalization). This problem was stated in **[1]**, where it was solved for commutative C*-algebras.

Partial actions and globalizations can be defined in other categories, such as topological spaces, rings, algebras and Hilbert modules **[1, 3, 5, 6]**. Here, we will work with rings, algebras, *-algebras, C*-algebras and Hilbert modules, mainly because C*-algebras can be viewed in any of these categories. The respective notions of partial (global) action are r-, a-, *-, C*- and Hb-partial (global) actions, and we are interested in determining when a given partial action in any of these categories has a globalization (in the respective category).

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This work is organized as follows. In Section 2, we study the problem of constructing a globalization of r-partial actions. Our intention is to study C*-partial actions from the point of view of ring theory, which amounts to eliminating all of the topological and vector space structure while keeping the ring theoretic properties we need. Hence, when necessary, we make some assumptions on rings which are known to hold for C*-algebras. Under these assumptions, we give a necessary and sufficient condition for the existence of an r-globalization. Then, we gradually add more structure: first, scalar multiplication, then involution and, finally, a C*-norm. At that point, we show that a C*-partial action has a C*-globalization if and only if it has an r-globalization. Finally, in the last section, we give a necessary and sufficient condition for the existence of a globalization of an Hb-partial action. We specifically show that an Hb-partial action has a globalization if and only if its linking partial action [**1**] has a C*-globalization.

2. Partial actions on rings. The definition of r-partial actions we adopt is [**6,** Definition 2.1]. In case the partial action under consideration is global, we substitute the term *partial* by *global*. The reader is referred to [**9**] for the definition of a partial action on a set.

A fundamental operation in the theory of partial actions is the restriction of an action to a set; however, partial actions can also be restricted. Suppose $\sigma = (\{X_t\}_{t\in G}, \{\sigma\}_{t\in G})$ is a partial action of the group G on the set X. Given a set $U \subset X$ and $t \in G$, define $U_t := U \cap \sigma_t(X_{t^{-1}} \cap U)$. It is obvious that $U_{t^{-1}} \subset X_{t^{-1}}$ and $\sigma_t(U_{t^{-1}}) \subset U_t$; hence, it makes sense to define

$$\kappa_t \colon U_{t^{-1}} \longrightarrow U_t$$

as

$$\kappa_t(x) = \sigma_t(x).$$

Straightforward arguments imply that the *restriction* of σ to U, defined as

$$\sigma|_U := (\{U_t\}_{t\in G}, \{\kappa_t\}_{t\in G}),$$

is a partial action of G on U. Recall [**9,** Definition 2.9] that U is said to be σ-invariant if $\sigma_t(X_{t^{-1}} \cap U) \subset U$, for all $t \in G$.

Remark 2.1. If $U \subset V \subset X$, then $\sigma|_U = \sigma|_V|_U$. In addition, if σ is global, $\sigma|_U$ is global if and only if U is σ-invariant.

The next two examples will be used frequently. We remind the reader that, by an ideal of a ring, we mean a bilateral ideal.

Example 2.2. Let A be a ring and I an ideal of it. The r-partial action of $\mathbb{Z}_2 = \{0, 1\}$ on A determined by I is $\alpha^{AI} := (\{A_0, A_1\}, \{\alpha_0, \alpha_1\})$, where $A_0 = A$, $A_1 = I$, $\alpha_0 = \mathrm{id}_A$ and $\alpha_1 = \mathrm{id}_I$.

Example 2.3. Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be an r-partial action. The opposite ring of A, A^{op}, is the ring obtained from A by changing the product to $a \cdot b := ba$. For each $t \in G$, A_t^{op} is an ideal of A^{op}, and we write α_t^{op} instead of α_t when we think of α_t as a map from $A_{t^{-1}}^{\mathrm{op}}$ to A_t^{op}. Then, $\alpha^{\mathrm{op}} := (\{A_t^{\mathrm{op}}\}_{t\in G}, \{\alpha_t^{\mathrm{op}}\}_{t\in G})$ is an r-partial action, called the *opposite* of α.

We now briefly recall some terminology. An annihilator of a ring A is an element $a \in A$ such that $ab = ba = 0$ for all $b \in A$. The set formed by all annihilators will be denoted $\mathrm{Ann}(A)$, and we say A is non degenerate if $\mathrm{Ann}(A) = \{0\}$. Recall that a ring homomorphism is an additive and multiplicative function between two rings and that an automorphism of A is a ring isomorphism from A to A. The set of automorphisms of A will be denoted $\mathrm{Aut}(A)$.

Let α and β be r-partial actions of G on A and B, respectively. A morphism from α to β is a homomorphism $f\colon A \to B$ which is G-equivariant in the sense of [**9,** Definition 2.7]. The identity associated to α is merely the identity of A, and the composition of morphisms is the composition of functions. This determines the notion of isomorphism.

In order to facilitate the exposition we rephrase Definition 2.2 of [**6**].

Definition 2.4. An r-globalization of α is a 4-tuple $\Xi = (B, \beta, I, \iota)$, where β is an r-global action of G on B, I is an ideal of B, $\iota\colon \alpha \to \beta|_I$ is an isomorphism and $[\beta I] := \sum_{t\in G} \beta_t(I)$ equals B. We say that Ξ is non degenerate if B is non degenerate.

Given another r-globalization of α, $\Sigma = (C, \gamma, J, \kappa)$, we say $\pi\colon \Xi \to \Sigma$ is a morphism if $\pi\colon \beta \to \gamma$ is a morphism and $\pi \circ \iota = \kappa$. Note that every morphism between r-globalizations is surjective as a function and that there is at most one morphism between any two r-globalization of α.

Example 2.5. Consider the partial action of Example 2.2. Assume that there exists an ideal J of A such that $A = I \oplus J$ (the direct sum of rings). Let β be the action of $\mathbb{Z}_2$ on $B := I \oplus J \oplus J$ given by $\beta_1(a \oplus b \oplus c) = a \oplus c \oplus b$. In addition, let $\iota\colon A \to B$ be defined as $\iota(a \oplus b) = a \oplus b \oplus 0$. Then, $(B, \beta, I \oplus J \oplus 0, \iota)$ is an r-globalization of α^{AI}.

Remark 2.6. If $\Xi = (B, \beta, I, \iota)$ is an r-globalization of α, then $\Xi^{\mathrm{op}} := (B^{\mathrm{op}}, \beta^{\mathrm{op}}, I^{\mathrm{op}}, \iota^{\mathrm{op}})$ is an r-globalization of α^{op}.

Partial actions on degenerate rings may have more than one isomorphism class of globalizations.

Example 2.7. For an abelian group V we denote V_0 the ring obtained by considering on V the null multiplication. Let α be the partial action of $\mathbb{Z}_3 = \{0, 1, 2\}$ (with additive notation) on $\mathbb{R}_0$ such that $\alpha_0 = \mathrm{id}_{\mathbb{R}_0}$ and $\alpha_1 = \alpha_2 = \mathrm{id}_{\{0\}}$. Two globalizations of α are $(\mathbb{R}_0^2, \beta, I, \iota)$ and $(\mathbb{R}_0^3, \gamma, J, \kappa)$, where I and J are the x-axis, $\iota(x) = (x, 0)$, $\kappa(x) = (x, 0, 0)$, β_1 is the rotation by an angle of $2\pi/3$ and $\gamma_1(x, y, z) = (z, x, y)$. These globalizations are not isomorphic since $(1, 0) + \beta_1(1, 0) + \beta_2(1, 0) = 0$; however, $(1, 0, 0) + \gamma_1(1, 0, 0) + \gamma_2(1, 0, 0) \neq 0$.

Assume that $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is an r-partial action of G on A. The main idea of this section is to use an r-globalization of α to construct a new r-globalization in such a way that the latter can be described in terms of α. This description will make no reference to the first globalization and will be interpreted as the building instructions of a globalization. Then, we will give conditions (on α) under which the construction can be performed and produces a globalization. We will need some facts regarding centralizers which we now recall, see **[4]** for a detailed exposition.

A double centralizer of A is a pair $\mu = (L, R)$ such that L and R are additive functions from A to A and, for all $a, b \in A$, $L(ab) = L(a)b$,

$R(ab) = aR(b)$ and $aL(b) = R(a)b$. It is standard to write $\mu a = L(a)$ and $a\mu = R(a)$. The set formed by all of the double centralizers of A, $M(A)$, is a unital ring with the operations $(L, R) + (L', R') = (L + L', R + R')$ and $(L, R)(L', R') = (L \circ L', R' \circ R)$. The function $\tau \colon A \to M(A)$, where $\tau(a)b = ab$ and $b\tau(a) = ba$, is a ring homomorphism which is injective if and only if A is non degenerate.

The ring of functions from G to A (with point wise operations) will be denoted A^G. Given $t \in G$, we define

$$\theta_t \colon A^G \longrightarrow A^G$$

as $\theta_t(f)|_t := f|_{rt}$.

Definition 2.8. The canonical (global) action of G on $M(A^G)$, Θ, is given by the map $\Theta \colon G \to \mathrm{Aut}(M(A^G))$,

$$\Theta_t(L, R) = (\theta_t \circ L \circ \theta_{t^{-1}}, \theta_t \circ R \circ \theta_{t^{-1}}).$$

Definition 2.9. Assume that $\Xi = (B, \beta, I, \iota)$ is an r-globalization of α. The canonical morphism associated to Ξ, denoted π_Ξ or simply π, is the map

$$\pi \colon B \longrightarrow M(A^G)$$

given by

$$\pi(b)f|_r = \iota^{-1}(\beta_r(b)\iota(f|_r)),$$
$$f\pi(b)|_r = \iota^{-1}(\iota(f|_r)\beta_r(b)),$$

for all $b \in B$, $f \in A^G$ and $r \in G$.

It is left to the reader to verify the fact that

$$\pi \colon \beta \longrightarrow \Theta$$

is a morphism. After this is done, we can see that $\pi(B)$ is a Θ-invariant subring and $\pi(I)$ is an ideal of $\pi(B)$. Moreover, $\Theta|_{\pi(B)}$ is an r-global action, $[\Theta\pi(I)] = \pi(B)$, and Remark 2.1 implies $\Theta|_{\pi(B)}|_{\pi(I)} = \Theta|_{\pi(I)}$. The 4-tuple

$$\Xi_\Theta := (\pi(B), \Theta|_{\pi(B)}, \pi(I), \pi \circ \iota)$$

is an r-globalization if and only if $\pi \circ \iota$ is injective and $\pi \circ \iota(A_t) = \pi(I) \cap \Theta_t(\pi(I))$, for all $t \in G$. These two conditions seem to be unrelated, even if we require additional structure on the ring (see Example 3.2).

Lemma 2.10. *Let* $\Xi = (B, \beta, I, \iota)$ *be an* r*-globalization of the* r*-partial action* α *(of* G *on* A*). If* π *is the canonical morphism associated to* Ξ*, then* $\ker(\pi) = \mathrm{Ann}(B)$*. Moreover, if* A *is non degenerate, then* $\pi \circ \iota$ *is injective and* $\pi(B)$ *is non degenerate.*

Proof. For $a \in A$ and $r \in G$, we denote δ_r^a as the function

$$G \longrightarrow A, \qquad t \longmapsto a\delta_{r,t},$$

where $\delta_{r,t}$ is the Kronecker delta. If $b \in \ker(\pi)$, then $0 = \iota(\pi(b)\delta_r^a|_r) = \beta_r(b)\iota(a)$, whence (by Remark 2.6) $0 = \iota(a)\beta_r(b)$. Thus, $\ker(\pi) \subset \mathrm{Ann}(B)$. Now, assume that $b \in \mathrm{Ann}(B)$ and take $f \in A^G$. Clearly, $\beta_r(b) \in \mathrm{Ann}(B)$, for all $r \in G$, and this implies $b \in \ker(\pi)$ since

$$\pi(b)f|_r = \iota^{-1}(\beta_r(b)\iota(f|_r)) = 0 = \iota^{-1}(\iota(f|_r)\beta_r(b)) = f\pi(b)|_r.$$

In the case where A is non degenerate and $a \in \ker(\pi \circ \iota)$ it is evident that $\iota(a) \in \mathrm{Ann}(B)$. Hence, $a \in \mathrm{Ann}(A)$, and this implies $a = 0$.

Finally, assume that $\pi(b) \in \mathrm{Ann}(\pi(B))$. Since $\pi(B)$ is Θ-invariant, for all $r \in G$, we have $\pi(\beta_r(b)) = \Theta_r(\pi(b)) \in \mathrm{Ann}(\pi(B))$. Then, for all $r \in G$ and $c, d \in A$,

$$0 = \pi(\beta_r(b)\iota(c))\delta_e^d|_e = \iota^{-1}(\beta_r(b)\iota(c)\iota(d)) = \iota^{-1}(\beta_r(b)\iota(c))d.$$
$$0 = \delta_e^d\pi(\beta_r(b)\iota(c))|_e = \iota^{-1}(\iota(d)\beta_r(b)\iota(c)) = d\iota^{-1}(\beta_r(b)\iota(c)).$$

In other words, $\beta_r(d)\iota(c) = 0$ for all $r \in G$ and $c \in A$. Using β^{op}, we conclude that $0 = \beta_r(b)\iota(c) = \iota(c)\beta_r(b)$, for all $r \in G$ and $c \in A$, that is to say, $\pi(b) = 0$. ☐

In order to give a sufficient condition for $\pi \circ \iota$ to be an isomorphism between α and $\Theta|_{\pi(I)}$, we introduce the concept of assimilative sets. Before doing so, we recall that the product of two subsets, U and V, of a ring is $UV := \{uv\colon u \in U,\ v \in V\}$.

Definition 2.11. A subset S of the ring C is *assimilative* (in C) if, given $c \in C$ such that $cC \cup Cc \subset S$, then $c \in S$.

Recall **[6]** that a ring A is left s-unital if $a \in Aa$, for all $a \in A$. Evidently, every left s-unital ring is non degenerate and any subset of a left s-unital ring is assimilative in the entire ring. Any sum of left s-unital ideals is left s-unital [**6,** Remark 2.5].

Proposition 2.12. *Let α be an r-partial action of G on A with A_t non degenerate (as a ring), for all $t \in G$. Then,*

(a) *for any two r-globalizations of α, Ξ and Σ with Σ non degenerate, there exists a morphism*

$$_\Xi\rho_\Sigma \colon \Xi \longrightarrow \Sigma.$$

Furthermore, the following are equivalent:

(i) *$_\Xi\rho_\Sigma$ is an isomorphism.*
(ii) *Ξ is non degenerate.*
(iii) *The canonical morphism associated to Ξ is injective.*

(b) *If A_t is assimilative in A for all $t \in G$, then α has an r-globalization if and only if it has a non degenerate r-globalization.*

(c) *If, for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in G$, the ideal $A_{t_1} + \cdots + A_{t_n}$ is assimilative in A, then every r-globalization of α is non degenerate.*

Proof. Suppose the r-globalizations Ξ and Σ are (B, β, I, ι) and (C, γ, J, κ), respectively. For each $(t, a, b) \in G \times A \times A$, define $u_t(a, b) := \iota^{-1}(\beta_t(\iota(a))\iota(b))$. The motivation for this definition comes from [**1,** Lemma 2.2]. We want to prove that, for all $c \in A_t$

$$cu_t(a, b) = \alpha_t(\alpha_{t^{-1}}(c)a)b, \tag{2.1}$$

$$u_t(a, b)c = \alpha_t(a\alpha_{t^{-1}}(bc)). \tag{2.2}$$

We proceed to prove equation (2.1) and leave (2.2) to the reader. Since $\iota \colon \alpha \to \beta|_I$ is an isomorphism and $\iota(A_t) = I \cap \beta_t(I)$ an ideal of B, it follows that $u_t(a, b) \in A_t$ and

$$\begin{aligned} cu_t(a, b) &= \iota^{-1}(\iota(c)\beta_t(\iota(a))\iota(b)) = \iota^{-1}(\beta_t(\iota(\alpha_{t^{-1}}(c)))\beta_t(\iota(a))\iota(b)) \\ &= \iota^{-1} \circ \iota(\alpha_t(\alpha_{t^{-1}}(c)a)b) = \alpha_t(\alpha_{t^{-1}}(c)a)b. \end{aligned}$$

Note that $u_t(a, b)$ is completely determined by the α and (t, a, b) since it is an element of the non degenerate ideal A_t that satisfies (2.1) and (2.2) for all $c \in A_t$. Thus, $\iota^{-1}(\beta_t(\iota(a))\iota(b)) = u_t(a, b) =$

$\kappa^{-1}(\gamma_t(\kappa(a))\kappa(b))$. This gives $\pi_\Xi \circ \iota(a)f = \pi_\Sigma \circ \kappa(a)f$ for all $f \in A^G$. If α is now replaced by α^{op} and β by β^{op}, then we obtain

$$f\pi_\Xi \circ \iota(a) = f\pi_\Sigma \circ \kappa(a)$$

for all $f \in A^G$. Combining, we obtain $\pi_\Xi \circ \iota = \pi_\Sigma \circ \kappa$. Hence,

$$\mathrm{Im}(\pi_\Xi) = [\Theta\pi_\Xi(I)] = [\Theta\pi_\Sigma(J)] = \mathrm{Im}(\pi_\Sigma).$$

Clearly, Lemma 2.10 implies π_Σ is injective. Then, we can define ${}_\Sigma\rho_\Xi := \pi_\Sigma^{-1} \circ \pi_\Xi$, and it is clear that ${}_\Sigma\rho_\Xi \circ \iota = \kappa$. For $a \in A$ and $t \in G$, we have

$$\pi_\Xi(\beta_t(\iota(a))) = \Theta_t(\pi_\Xi \circ \iota(a)) = \Theta_t(\pi_\Sigma \circ \kappa(a)) = \pi_\Sigma(\gamma_t(\kappa(a))).$$

Then, ${}_\Sigma\rho_\Xi(\beta_t(\iota(a))) = \gamma_t(\kappa(a))$, and ${}_\Sigma\rho_\Xi \colon \Xi \to \Sigma$ is a morphism.

Observe that Lemma 2.10 implies (a)(ii) is equivalent to (a)(iii). Also, observe that (a)(i) implies B and C are isomorphic; thus, (a)(i) implies (a)(ii) since C is non degenerate. If (a)(iii) holds, then $\pi_\Xi^{-1} \circ \pi_\Sigma$ is the inverse of ${}_\Sigma\rho_\Xi$ and (a)(i) holds.

Now we show (b). Assume that α has an r-globalization $\Xi = (B, \beta, I, \iota)$, and let π be the canonical morphism associated to Ξ. It is evident that $\pi(I_t) \subset \pi(I) \cap \Theta_t(\pi(I))$. In order to prove $\pi(I) \cap \Theta_t(\pi(I)) \subset \pi(I_t)$, take $a \in I$ such that $\pi(a) = \Theta_t(\pi(b))$ for some $b \in I$. Note that, if $c \in I$, then

$$\begin{aligned} ac &= \iota(\pi(a)\delta_e^{\iota^{-1}(c)}|_e) = \iota(\Theta_t(\pi(a))\delta_e^{\iota^{-1}(c)}|_e) = \beta_t(a)\iota(c) \in I_t, \\ ca &= \iota(\delta_e^{\iota^{-1}(c)}|_e\pi(a)|_e) = \iota(\delta_e^{\iota^{-1}(c)}\Theta_t(\pi(a))|_e) = c\beta_t(a) \in I_t. \end{aligned}$$

Hence, $a \in I_t$ since I_t is assimilative in I.

From Lemma 2.10 and the comments preceding it, we know

$$(\pi(B), \Theta|_{\pi(B)}, \pi(I), \pi \circ \iota)$$

is a non degenerate r-globalization of α. Thus, we have proved (b).

In order to prove the last claim, observe first that, for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in G$, the ideal $I_{t_1} + \cdots + I_{t_n}$ is assimilative in I. Let $b \in \mathrm{Ann}(B)$. From Definition 2.4, there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in G$ and

$b_1, \ldots, b_n \in I$ such that

$$b = \sum_{j=1}^{n} \beta_{t_j}(b_j).$$

We want to prove that $b = 0$ by induction on n.

If $n = 1$, it follows that $Ib_1 = b_1 I = \{0\}$, and this implies $b = 0$ since I is non degenerate. In general, $n > 1$,

$$\sum_{j=1}^{n} \beta_{t_n^{-1}t_j}(b_j)c = \beta_{t_n^{-1}}(b)c = 0,$$

for all $c \in I$ and $r \in G$. Then, we have

$$b_n c = -\sum_{j=1}^{n-1} \beta_{t_n^{-1}t_j}(b_j)c \in \sum_{j=1}^{n-1} I_{t_n^{-1}t_j},$$

for all $c \in I$. In a similar manner, it can be shown that

$$cb_n \in \sum_{j=1}^{n-1} I_{t_n^{-1}t_j},$$

for all $c \in I$. Thus, $b_n = b'_1 + \cdots + b'_{n-1}$ with $b'_j \in I_{t_n^{-1}t_j}$, $j = 1, \ldots, n-1$. Furthermore,

$$\beta_{t_n}(b'_j) \in \beta_{t_n}(I_{t_n^{-1}t_j}) \subset \beta_{t_n} \circ \beta_{t_n^{-1}t_j}(I) = \beta_{t_j}(I).$$

Hence, there exists $b''_j \in I$ $(j = 1, \ldots, n-1)$ such that

$$\beta_{t_n}(b_n) = \sum_{j=1}^{n-1} \beta_{t_j}(b''_j).$$

This gives

$$b = \sum_{j=1}^{n-1} \beta_{t_j}(b_j + b''_j)$$

with $b_j + b''_j \in I$. Therefore, $b = 0$ by the induction hypothesis. □

We are almost ready to present the main theorem of this section; to clarify its proof, we state the following.

Lemma 2.13. *Let A be a ring, I and J ideals of A, $a \in I$ and $b \in J$ such that $I \cap J$ and $I + J$ are non degenerate,*

$$aI \cup Ia \subset J, \qquad bJ \cup Jb \subset I,$$

$ca = cb$ and $ac = bc$ for all $c \in I \cap J$. Then, $a = b$.

Proof. For all $z \in I \cap J$ and $x \in I + J$, we have $(a-b)x \in I \cap J$ and $(a-b)xz = 0 = z(a-b)x$. Since $I \cap J$ is non degenerate, $(a-b)x = 0$ for all $x \in I + J$. If we now replace A by A^{op}, we obtain $x(a-b) = 0$ for all $x \in I + J$. Hence, $a = b$. □

The next result is a combination of [**5,** Theorem 4.5] and [**6,** Theorem 3.1].

Theorem 2.14. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be an r-partial action of G on A, and consider the conditions*:

(a) *α has an r-globalization.*
(b) *For all $(t,a,b) \in G \times A \times A$, there exists a $u \in A_t$ such that, for all $c \in A_t$, $cu = \alpha_t(\alpha_{t^{-1}}(c)a)b$ and $uc = \alpha_t(a\alpha_{t^{-1}}(bc))$.*

Then, (a) *implies* (b)*. If $A_t + A_s$ and $A_t \cap A_s$ are non degenerate and A_t is assimilative in A, for all $s, t \in G$, then* (b) *implies* (a)*. Moreover, in this last case, α has a non degenerate r-globalization.*

Proof. The direct implication is implicit in the first part of the proof of Proposition 2.12. For the converse, note that the element u of (b) is uniquely determined by (t,a,b) since A_t is non degenerate; thus, it will be denoted $u_t(a,b)$.

We begin by showing that there exists a ring homomorphism

$$\pi\colon A \longrightarrow M(A^G)$$

such that

$$\begin{aligned} \pi(a)f|_t &= u_t(a, f|_t), \\ f\pi(a)|_t &= \alpha_t(u_{t^{-1}}(f|_t, a)). \end{aligned} \tag{2.3}$$

In order to avoid repetition, we show that α^{op} satisfies condition (b) write $f\pi(a)|_t$ in terms of α^{op}. For $(t,a,b) \in G \times A^{\mathrm{op}} \times A^{\mathrm{op}}$, let

$u_t^{\mathrm{op}}(a,b) := \alpha_t(u_{t^{-1}}(b,a))$. Clearly, $f\pi(a)|_t = u_t^{\mathrm{op}}(a, f|_t)$. Furthermore, for all $c \in A_t^{\mathrm{op}}$,

$$\begin{aligned} c \cdot u_t^{\mathrm{op}}(a,b) &= \alpha_t(u_{t^{-1}}(b,a))c = \alpha_t(u_{t^{-1}}(b,a)\alpha_{t^{-1}}(c)) \\ &= \alpha_t\left(\alpha_{t^{-1}}\left[b\alpha_t(a\alpha_{t^{-1}}(c))\right]\right) \\ &= b\alpha_t(a\alpha_{t^{-1}}(c)) = \alpha_t^{\mathrm{op}}(\alpha_{t^{-1}}^{\mathrm{op}}(c) \cdot a) \cdot b \end{aligned}$$

and

$$\begin{aligned} u_t^{\mathrm{op}}(a,b) \cdot c &= c\alpha_t(u_{t^{-1}}(b,a)) = \alpha_t(\alpha_{t^{-1}}(c)u_{t^{-1}}(b,a)) \\ &= \alpha_t(\alpha_{t^{-1}}(cb)a) = \alpha_t^{\mathrm{op}}(a \cdot \alpha_{t^{-1}}^{\mathrm{op}}(b \cdot c)). \end{aligned}$$

Note that the uniqueness of the element $u_t(a,b)$ with respect to (t,a,b) implies that $a \mapsto u_t(a,b)$ and $b \mapsto u_t(a,b)$ are additive. Then, $f \mapsto \pi(a)f$ and $f \mapsto f\pi(a)$ are both additive. Moreover, π is additive.

In order to see that $[\pi(a)f]g = \pi(a)[fg]$, it suffices to prove that $u_t(a,bc) = u_t(a,b)c$, for all $a,b,c \in A$ and $t \in G$. For all $d \in A_t$, we have

$$\begin{aligned} du_t(a,bc) &= \alpha_t(\alpha_{t^{-1}}(d)a)bc = du_t(a,b)c, \\ u_t(a,bc)d &= \alpha_t(a\alpha_{t^{-1}}(bcd)) = u_t(a,b)cd. \end{aligned}$$

Since A_t is non degenerate, $u_t(a,bc) \in A_t$ and $u_t(a,b)c \in A_t$ we conclude that $u_t(a,bc) = u_t(a,b)c$. Thus,

$$\begin{aligned} g[f\pi(a)]|_t &= g|_t u_t^{\mathrm{op}}(a, f|_t) = u_t^{\mathrm{op}}(a, f|_t) \cdot g|_t \\ &= u_t^{\mathrm{op}}(a, f|_t \cdot g|_t) \\ &= u_t^{\mathrm{op}}(a, gf|_t) = [gf\pi(a)]|_t. \end{aligned}$$

The identity $[f\pi(a)]g = f[\pi(a)g]$ is equivalent to $au_t(b,c) = \alpha_t(u_{t^{-1}}(a,b))c$. In order to prove the latter, note that $au_t(b,c)$ and $\alpha_t(u_{t^{-1}}(a,b))c$ belong to the non degenerate ideal A_t. The desired identity follows from the fact that, for all $d \in A_t$,

$$dau_t(b,c) = \alpha_t(\alpha_{t^{-1}}(da)b)c = \alpha_t(\alpha_{t^{-1}}(d)u_{t^{-1}}(a,b))c = d\alpha_t(u_{t^{-1}}(a,b))c.$$

and

$$au_t(b,c)d = d \cdot \alpha_t^{\mathrm{op}}(u_{t^{-1}}^{\mathrm{op}}(c,b)) \cdot a = d \cdot c \cdot u_t^{\mathrm{op}}(b,a) = \alpha_t(u_{t^{-1}}(a,b))cd.$$

At this point, we have shown that $\pi(a) \in M(A^G)$ and that π is additive.

In order to prove that $\pi(a)\pi(b)f = \pi(ab)f$, for all $f \in A^G$ and $a \in A$, it suffices to show that $u_t(a, u_t(b,c)) = u_t(ab,c)$, for all $t \in G$ and $a,b,c \in A$. Once again, we use that A_t is non degenerate. In this case, $u_t(a,u_t(b,c)), u_t(ab,c) \in A_t$ and, for all $d \in A_t$:

$$\begin{aligned} du_t(a,u_t(b,c)) &= \alpha_t(\alpha_{t^{-1}}(d)a)u_t(b,c) = \alpha_t(\alpha_{t^{-1}}(d)ab)c = du_t(ab,c);\\ u_t(a,u_t(b,c))d &= \alpha_t(a\alpha_{t^{-1}}(u_t(b,c)d)) = \alpha_t(a\alpha_{t^{-1}}(\alpha_t[b\alpha_{t^{-1}}(cd)]))\\ &= \alpha_t(ab\alpha_{t^{-1}}(cd)) = u_t(ab,c)d. \end{aligned}$$

Hence, the identity follows and $f\pi(a)\pi(b) = f\pi(ab)$ since

$$f\pi(a)\pi(b)|_t = u_t^{\mathrm{op}}(b, f\pi(a)|_t) = u_t^{\mathrm{op}}(b, u_t^{\mathrm{op}}(a, f|_t)) = u_t^{\mathrm{op}}(b\cdot a, f|_t) = f\pi(ab)|_t.$$

In order to see that π is injective, first observe that $u_e(a,b) = ab$. If $\pi(a) = 0$, then for all $b \in A$, we have $ba = \delta_e^b\pi(a)|_e = 0$ and $ab = \pi(a)\delta_e^b|_e = 0$. Thus, $a = 0$.

Let Θ be the canonical action of G on $M(A^G)$, and define the ring B as the minimal Θ-invariant ring containing $\pi(A)$. It is evident that $B = \sum_{t\in G}\Theta_t(\pi(A))$ and that $\beta := \Theta|_B$ is an r-global action. The proof will be complete once we show that $\pi(A)$ is an ideal of B, that $\pi\colon \alpha \to \beta|_{\pi(A)}$ is an isomorphism and that B is non degenerate.

We claim that $\pi\colon \alpha \to \beta$ is G-equivariant. Fix $a \in A_{t^{-1}}$ and note that, to prove $\pi(\alpha_t(a)) = \Theta_t(\pi(a))$, it suffices to show that $u_r(\alpha_t(a),b) = u_{rt}(a,b)$ and $u_r(b,\alpha_t(a)) = u_{rt}(b,a)$, for all $r,t \in G$ and $b,c \in A$. It is easy to verify that the second equality follows from the first by replacing α by α^{op}. We can use Lemma 2.13 to conclude that $u_r(\alpha_t(a),b) = u_{rt}(a,b)$ since $u_r(\alpha_t(a),b), u_{rt}(a,b) \in A_r + A_{rt}$,

$$\begin{aligned} A_ru_r(\alpha_t(a),b) &= \alpha_r(A_{r^{-1}}\alpha_t(a))b \in A_r \cap A_{rt},\\ u_r(\alpha_t(a),b)A_r &= \alpha_r(\alpha_{t^{-1}}(a)\alpha_{r^{-1}}(bA_r)) \in A_r \cap A_{rt},\\ A_{rt}u_{rt}(a,b) &= \alpha_{rt}(A_{t^{-1}r^{-1}}a)b \in A_r \cap A_{rt},\\ u_{rt}(a,b)A_{rt} &= \alpha_{rt}(a\alpha_{t^{-1}r^{-1}}(bA_{rt})) \in A_r \cap A_{rt} \end{aligned}$$

and, for all $z \in A_r \cap A_{rt}$,

$$\begin{aligned} zu_r(\alpha_t(a),b) &= \alpha_r(\alpha_{r^{-1}}(z)\alpha_t(a))b = \alpha_r(\alpha_t(\alpha_{t^{-1}r^{-1}}(z)a))b\\ &= \alpha_{rt}(\alpha_{t^{-1}r^{-1}}(z)a)b = zu_{rt}(a,b);\\ u_r(\alpha_t(a),b)z &= \alpha_r(\alpha_t(a)\alpha_{r^{-1}}(bz)) = \alpha_r(\alpha_t(a\alpha_{t^{-1}r^{-1}}(bz)))\\ &= \alpha_{rt}(a\alpha_{t^{-1}r^{-1}}(bz)) = u_{rt}(a,b)z. \end{aligned}$$

As previously mentioned, we must show that $\pi(A)$ is an ideal of B. In order to do so, we prove that $\beta_t(\pi(A))\pi(A) \subset \pi(A_t)$ for all $t \in G$, which is accomplished by showing that $\beta_t(\pi(a))\pi(b) = \pi(u_t(a,b))$. From the definitions of β and π, we obtain

$$\begin{aligned}\beta_t(\pi(a))\pi(b)f|_t &= u_{rt}(a, u_r(b, f|_r)),\\ \pi(u_t(a,b))f|_r &= u_r(u_t(a,b), f|_r).\end{aligned}$$

Then, it suffices to show that $u_{rt}(a, u_r(b,c)) = u_r(u_t(a,b),c)$. We now use Lemma 2.13 once more. Note that $u_{rt}(a, u_r(b,c)) \in A_{rt}$, $u_r(u_t(a,b),c) \in A_r$,

$$\begin{aligned}A_{rt}u_{rt}(a, u_r(b,c)) &= \alpha_{rt}(A_{t^{-1}r^{-1}}a)u_r(b,c) \in A_{rt} \cap A_r,\\ u_{rt}(a, u_r(b,c))A_{rt} &= \alpha_{rt}(a\alpha_{t^{-1}r^{-1}}(u_r(b,c)A_{rt})) \in A_{rt} \cap A_r,\\ A_r u_r(u_t(a,b),c) &= \alpha_r(A_{r^{-1}}u_t(a,b))c \in \alpha_r(A_{r^{-1}} \cap A_t) = A_r \cap A_{rt},\\ u_r(u_t(a,b),c)A_r &= \alpha_r(u_t(a,b)\alpha_{r^{-1}}(cA_r)) \in A_r \cap A_{rt}.\end{aligned}$$

Furthermore, for all $z \in A_{rt} \cap A_r$, it follows that

$$\begin{aligned}zu_{rt}(a, u_r(b,c)) &= \alpha_{rt}(\alpha_{t^{-1}r^{-1}}(z)a)u_r(b,c)\\ &= \alpha_r(\alpha_t(\alpha_{t^{-1}r^{-1}}(z)a)b)c\\ &= \alpha_r(\alpha_{r^{-1}}(z)u_t(a,b))c\\ &= zu_r(u_t(a,b),c)\end{aligned}$$

and

$$\begin{aligned}u_{rt}(a, u_r(b,c))z &= \alpha_{rt}(a\alpha_{t^{-1}r^{-1}}(u_r(b,c)z))\\ &= \alpha_{rt}(a\alpha_{t^{-1}r^{-1}}(\alpha_r(b\alpha_{r^{-1}}(cz))))\\ &= \alpha_{rt}(a\alpha_{t^{-1}}(b\alpha_{r^{-1}}(cz)))\\ &= \alpha_r \circ \alpha_t(a\alpha_{t^{-1}}(b\alpha_{r^{-1}}(cz)))\\ &= \alpha_r(u_t(a, b\alpha_{r^{-1}}(cz)))\\ &= \alpha_r(u_t(a,b)\alpha_{r^{-1}}(cz))\\ &= u_r(u_t(a,b),c)z.\end{aligned}$$

Then, Lemma 2.13 implies $u_{rt}(a, u_r(b,c)) = u_r(u_t(a,b),c)$. Thus,

$$\begin{aligned}f\beta_t(\pi(a))\pi(b)|_r &= u_r^{\mathrm{op}}(b, f\beta_t(\pi(a))|_r) = u_r^{\mathrm{op}}(b, u_{rt}^{\mathrm{op}}(a, f|_r))\\ &= u_{rt}^{\mathrm{op}}(u_{t^{-1}}^{\mathrm{op}}(b,a), f|_r) = u_{rt}^{\mathrm{op}}(u_{t^{-1}}^{\mathrm{op}}(b,a), \theta_{t^{-1}}(f)|_{rt})\end{aligned}$$

$$= f\beta_t(\pi(u^{\mathrm{op}}_{t^{-1}}(b,a))|_r = f\beta_t(\pi(\alpha_{t^{-1}}(u_t(a,b)))|_r$$
$$= f\pi(u_t(a,b))|_r.$$

Hence, $\beta_t(\pi(a))\pi(b) = \pi(u_t(a,b))$.

If π^{op} is the map π constructed using α^{op} instead of α, then β^{op} is (isomorphic to) the action β constructed from α^{op}. Thus,

$$\begin{aligned}\pi(a)\beta_t(\pi(b)) &= \beta^{\mathrm{op}}_t(\pi^{\mathrm{op}}(b)) \cdot \pi^{\mathrm{op}}(a)\\ &= \pi^{\mathrm{op}}(u^{\mathrm{op}}_t(b,a))\\ &= \pi(\alpha_t(u_{t^{-1}}(a,b))) \in \pi(A_t).\end{aligned}$$

In order to see that $\pi(A)$ is an ideal of B, simply note that

$$B\pi(A) = \sum_{t\in G} \operatorname{span} \beta_t(\pi(A))\pi(A) \subset \sum_{t\in G} \pi(A_t) = \pi(A)$$
$$\pi(A)B = \sum_{t\in G} \operatorname{span} \pi(A)\beta_t(\pi(A)) \subset \sum_{t\in G} \pi(A_t) = \pi(A).$$

The conclusion of the proof is at hand. We will know that

$$\pi\colon \alpha \longrightarrow \beta|_{\pi(A)}$$

is an isomorphism after we show that $\pi(A_t) = \pi(A) \cap \beta_t(\pi(A))$. Note that $\pi(A_t) = \pi(\alpha_t(A_{t^{-1}})) = \beta_t(\pi(A_{t^{-1}})) \subset \pi(A) \cap \beta_t(\pi(A))$. Now, fix $\mu \in \pi(A) \cap \beta_t(\pi(A))$. There exist $a, b \in A$ such that $\mu = \pi(a) = \beta_t(\pi(b))$. Thus, for all $c \in A$,

$$ac = u_e(a,c) = \pi(a)\delta^c_e|_e = \beta_t(\pi(a))\delta^c_e|_e = u_t(a,c) \in A_t.$$

An analogous argument implies that $ca \in A_t$. Using that A_t is assimilative in A, we conclude that $a \in A_t$.

Up to this point, we have shown that $\Xi := (B, \Theta|_B, \pi(A), \pi)$ is an r-globalization of α. Note that the canonical morphism associated to Ξ,

$$\pi_\Xi\colon B \longrightarrow M(A^G),$$

is simply the natural inclusion $\rho\colon B \to M(A^G)$ since ρ is G-equivariant and $\rho\circ\pi = \pi$. Then, Lemma 2.10 implies that B is non degenerate. □

With the previous theorem, we can generalize Example 2.5. Recall that an orthogonal complement for the ideal I of ring A is an ideal

$J \subset A$ such that $A = I \oplus J$. In this situation, $IJ = JI = \{0\}$ since

$$IJ \subset I \cap J = \{0\}.$$

Several remarks are in order. Firstly, every non degenerate ideal has at most one orthogonal complement. Secondly, if J is an orthogonal complement for I, then the following are equivalent:

- A and I are non degenerate.
- I and J are non degenerate.
- A and J are non degenerate.

Finally, I is assimilative in A each time A and I are non degenerate, and I has an orthogonal complement.

Corollary 2.15. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be an r-partial action of G on A. If, for all $s, t \in G$, the ideals $A_t + A_s$ and $A_t \cap A_s$ are non degenerate and A_t has an orthogonal complement in A, then α has a non degenerate r-globalization.*

Proof. It suffices to verify that α satisfies the hypotheses of Theorem 2.14 (b). Since $A_{t^{-1}}$ has an orthogonal complement, for every $(t, a, b) \in G \times A \times A$, there exists an $a' \in A_{t^{-1}}$ such that $ca = ca'$ and $ac = a'c$, for all $c \in A_{t^{-1}}$. With $u \in A_t$ defined as $\alpha_t(a')b$, we have that, for all $c \in A_t$, $cu = \alpha_t(\alpha_{t^{-1}}(c)a')b = \alpha_t(\alpha_{t^{-1}}(c)a)b$ and $uc = \alpha_t(a'\alpha_{t^{-1}}(bc)) = \alpha_t(a\alpha_{t^{-1}}(bc))$. $\square$

The next example shows that the existence of an orthogonal complement for the domains A_t is not a necessary condition for the existence of an r-globalization.

Example 2.16. Let $C_0(\mathbb{R})$ be the C*-algebra of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ vanishing at $\pm\infty$. In addition, let β be the action of $\mathbb{R}$ on $C_0(\mathbb{R})$ given by $\beta_t(f)(r) = f(r - t)$, and define A as the ideal

$$C_0(0, +\infty) = \{f \in C_0(\mathbb{R})\colon f|_{\mathbb{R}\setminus(0,+\infty)} \equiv 0\}.$$

Thus,

$$B := \sum_{t\in\mathbb{R}} \beta_t(A)$$

is a β-invariant ring and, if $\iota\colon A \to B$ is the canonical inclusion, then $(B, \beta|_B, A, \iota)$ is an r-globalization of $\beta|_A$. Every ideal of A is non degenerate, but, if $t > 0$, $A_t = A \cap \beta_t(A) = C_0(t, +\infty)$ does not have an orthogonal complement in A.

In the context of r-partial actions [**1,** Proposition 2.1] becomes

Proposition 2.17. *Assume that α is an r-partial action of G on the commutative ring A and that $\Xi = (B, \beta, I, \iota)$ is an r-globalization of α. If A_t is non degenerate for all $t \in G$, then B is commutative.*

Proof. It suffices to prove that $a\beta_t(b) = \beta_t(b)a$, for all $a, b \in I$ and $t \in G$. Note that I is commutative, $I_t := \iota(A_t)$ non degenerate and $a\beta_t(b)$, $\beta_t(b)a \in I \cap \beta_t(I) = I_t$. Then $a\beta_t(b) = \beta_t(b)a$ if and only if $ca\beta_t(b) = c\beta_t(b)a$, for all $c \in I_t$. Fix $c \in I_t$. Since, clearly, $a, b, c, \beta_{t^{-1}}(ac)$, $\beta_{t^{-1}}(c)$, $\beta_t(b)a \in I$, we have that

$$ca\beta_t(b) = \beta_t(\beta_{t^{-1}}(ac)b) = \beta_t(b\beta_{t^{-1}}(ac)) = \beta_t(b)ac = c\beta_t(b)a. \qquad \square$$

3. Partial actions on algebras and *-algebras. By an *a-partial action*, we mean a partial action on an algebra in the sense of [**6**]. We define morphisms between a-partial actions as linear morphisms of r-partial actions, and the definition of a-globalizations is obtained from Definition 2.4 by replacing "r-" by "a-."

Recall that a *-algebra is a complex algebra A with a conjugate linear and anti-multiplicative involution $A \to A$, $a \mapsto a^*$. A *-homomorphism between two *-algebras, $\phi\colon A \to B$, is a morphism of algebras such that $\phi(a^*) = \phi(a)^*$. By an ideal of a *-algebra, we mean an ideal (in the algebraic sense) closed by involution. It is then natural to say that $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a *-partial action if it is an a-partial action on the *-algebra A, A_t is an ideal of A for all $t \in G$ and α_t is a *-homomorphism for all $t \in G$. In the context of *-partial actions, morphisms are morphisms of a-partial actions which are also *-homomorphisms. Once again, the definition of *-globalization is obtained from Definition 2.4 by replacing "r-" by "*-."

In the same manner, *-algebras are rings with extra structure; *-globalizations are r-globalizations with an additional structure.

Proposition 3.1. *If $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is an a-partial (*-partial) action with A_t non degenerate for all $t \in G$, then every non degenerate r-globalization of α is obtained by forgetting the vector space structure (vector space structure and involution) of an a-globalization (*-globalization). Moreover, this a-globalization (*-globalization) is uniquely determined by the r-globalization.*

Proof. Let $\Xi = (B, \beta, I, \iota)$ be a non degenerate r-globalization of α, and let $\mathbb{F}$ be the field of scalars of A. For every $(\lambda, b) \in \mathbb{F} \times B$, there are $a_1, \dots, a_n \in A$ and $t_1, \dots, t_n \in G$, such that

$$b = \sum_{j=1}^{n} \beta_{t_j}(\iota(a_j)).$$

Thus, we are forced to define

$$\lambda b := \sum_{j=1}^{n} \beta_{t_j}(\iota(\lambda a_j)).$$

In order to show the operation

$$(\lambda, b) \longmapsto \lambda b$$

is defined, it suffices to prove that

$$c := \sum_{j=1}^{n} \beta_{t_j}(\iota(\lambda a_j))$$

is zero every time $b = 0$.

Suppose $b = 0$. Recall that the elements $u_t(x, y)$ constructed in the proof of Proposition 2.12 are uniquely determined by equations (2.1) and (2.2), and observe that uniqueness implies

$$(x, y) \longmapsto u_t(x, y)$$

is bilinear. Then, for all $d \in A$ and $r \in G$, we have

$$\sum_{j=1}^{n} \beta_{r^{-1}t_j}(\iota(a_j))\iota(d) = \beta_{r^{-1}}(b\beta_r(\iota(d))) = 0,$$

and

$$\begin{aligned}
c\beta_r(\iota(d)) &= \sum_{j=1}^{n} \beta_r(\beta_{r^{-1}t_j}(\iota(\lambda a_j))\iota(d)) \\
&= \beta_r \circ \iota\left(\sum_{j=1}^{n} u_{r^{-1}t_j}(\lambda a_j, d)\right) \\
&= \beta_r \circ \iota\left(\lambda \sum_{j=1}^{n} u_{r^{-1}t_j}(a_j, d)\right) \\
&= \beta_r \circ \iota\left(\lambda \iota^{-1}\left(\sum_{j=1}^{n} \beta_{r^{-1}t_j}(\iota(a_j))\iota(d)\right)\right) = 0.
\end{aligned}$$

Clearly, this implies $cB = \sum_{t\in G} c\beta_t(I) = \{0\}$, and, by symmetry (Remark 2.6), we obtain $Bc = \{0\}$. Hence, $c = 0$.

The manner in which we have defined scalar multiplication ensures it is compatible with the sum. Furthermore, ι and the automorphisms β_t are linear. In order to prove that the product of B is bilinear, it suffices to show that

$$\lambda[\beta_t(\iota(a))\beta_r(\iota(b))] = \beta_t(\iota(\lambda a))\beta_r(\iota(b)),$$

which follows from the fact that

$$\begin{aligned}
\lambda[\beta_t(\iota(a))\beta_r(\iota(b))] &= \lambda\beta_r(\beta_{r^{-1}t}(\iota(a))\iota(b)) \\
&= \lambda\beta_r(\iota(u_{r^{-1}t}(a, b))) \\
&= \beta_r(\iota(u_{r^{-1}t}(\lambda a, b)) \\
&= \beta_t(\iota(\lambda a))\beta_r(\iota(b)).
\end{aligned}$$

At this point, it is clear that B is an algebra over $\mathbb{F}$ and that Ξ is an a-globalization of α. Now, we deal with *-algebras, so from here on, we assume that α is a *-partial action.

The unique involution of B making all the β_t and ι *-homomorphisms should be given by

$$\sum_{j=1}^{n} \beta_{t_j}(\iota(a_j)) \longmapsto \sum_{j=1}^{n} \beta_{t_j}(\iota(a_j^*)).$$

In order to show this defines an involution we must show that, if

$$b = \sum_{j=1}^{n} \beta_{t_j}(\iota(a_j))$$

is zero, then

$$c := \sum_{j=1}^{n} \beta_{t_j}(\iota(a_j^*))$$

is zero. For every $d \in A$ and $r \in G$,

$$c\beta_r(\iota(d)) = \beta_r \circ \iota\Bigg(\sum_{j=1}^{n} u_{r^{-1}t_j}(a_j^*, d) \Bigg) \tag{3.1}$$

We will need the identity $u_s(x,y)^* = \alpha_s(u_{s^{-1}}(y^*, x^*))$. Note that both terms of the last equality belong to the non degenerate ideal A_s and that, for all $z \in A_s$:

$$\begin{aligned} zu_s(x,y)^* &= [u_s(x,y)z^*]^* = [\alpha_s(x\alpha_{s^{-1}}(yz^*))]^* \\ &= \alpha_s(\alpha_{s^{-1}}(zy^*)x^*) = \alpha_s(\alpha_{s^{-1}}(z)u_{s^{-1}}(y^*, x^*)) \\ &= z\alpha_s(u_{s^{-1}}(y^*, x^*)) \end{aligned}$$

and, also,

$$\begin{aligned} u_s(x,y)^*z &= [z^*u_s(x,y)]^* = [\alpha_s(\alpha_{s^{-1}}(z^*)x)y]^* = y^*\alpha_s(x^*\alpha_{s^{-1}}(z)) \\ &= \alpha_s(\alpha_{s^{-1}}(y^*\alpha_s(x^*\alpha_{s^{-1}}(z)))) = \alpha_s(u_{s^{-1}}(y^*, x^*)\alpha_{s^{-1}}(z)) \\ &= \alpha_s(u_{s^{-1}}(y^*, x^*))z. \end{aligned}$$

This implies $u_s(x,y)^* = \alpha_s(u_{s^{-1}}(y^*, x^*))$, and, together with (3.1), yields

$$\begin{aligned} c\beta_r(\iota(d)) = 0 \iff \sum_{j=1}^{n} u_{r^{-1}t_j}(a_j^*, d) = 0 \iff \sum_{j=1}^{n} u_{r^{-1}t_j}(a_j^*, d)^* = 0 \\ \iff \beta_r \circ \iota(\sum_{j=1}^{n} \alpha_{r^{-1}t_j}(u_{t_j^{-1}r}(d^*, a_j))) = 0 \\ \iff \sum_{j=1}^{n} \beta_{t_j} \circ \iota(u_{t_j^{-1}r}(d^*, a_j)) = 0 \iff \beta_r(\iota(d^*))b = 0. \end{aligned}$$

The last condition is true due to the assumption $b = 0$. By symmetry, we obtain $cB = Bc = \{0\}$, and this implies $c = 0$.

The involution is anti-multiplicative since, for all $a, b \in A$ and $r, t \in G$,

$$\begin{aligned}
[\beta_t(\iota(a))\beta_r(\iota(b))]^* &= \beta_r(\iota(u_{r^{-1}t}(a,b)))^* \\
&= \beta_r(\iota(\alpha_{r^{-1}t}(u_{t^{-1}r}(b^*,a^*)))) \\
&= \beta_t(\iota(u_{t^{-1}r}(b^*,a^*))) \\
&= \beta_t(\beta_{t^{-1}r}(\iota(b^*))\iota(a^*)) \\
&= \beta_r(\iota(b))^*\beta_t(\iota(a))^*.
\end{aligned}$$

The remaining facts, such as the conjugate linearity of the involution, are left to the reader. □

Proposition 3.1 can be used in conjunction with Theorem 2.14 and Proposition 2.12 to decide whether a given a-partial action (*-partial action) has an a-globalization (*-globalization). In fact, all of the results we have obtained for r-partial actions hold for a- and *-partial actions.

We close this section with an example where we construct a degenerate *-globalization but not a non degenerate r-globalization. The idea is to produce an example without enough assimilative A_ts (see Theorem 2.14).

Example 3.2. Let $U \in \mathbb{M}_4(\mathbb{C})$ be the matrix corresponding to the permutation of rows (1 4)(2 3) (written as a product of cycles). Define an involution in $\mathbb{M}_4(\mathbb{C})$ by the formula $a^* := u\overline{a}^t u$, where

$$a \longmapsto \overline{a}$$

is the entrywise complex conjugation and

$$a \longmapsto a^t$$

is the usual matrix transposition. Note that a^* is obtained from $\overline{a}$ by performing a symmetry with respect to the antidiagonal.

Let A be the *-subalgebra of $\mathbb{M}_4(\mathbb{C})$, whose elements are matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

Then, $I := \{a \in A\colon a_{23} = 0\}$ is a *-ideal of A. Note that both I and A are non degenerate since the condition $aI = \{0\}$ implies $a = 0$. However, I is not assimilative since $AA \subset I$ and $A \neq I$.

Consider the partial action $\alpha := \alpha^{AI}$ described in Example 2.2. Condition (b) from Theorem 2.14 is satisfied if we define $u_t(a, b) = ab$, for all $(t, a, b) \in \mathbb{Z}_2 \times A \times A$. However, this does not ensure that α has an r-globalization since $I = A_1$ is not assimilative in A.

Suppose that α has a *-globalization $\Xi = (B, \beta, J, \iota)$. For convenience, we set $J = A$, $\iota = \mathrm{id}_A$ and $\alpha = \beta|_J$. Since $B = A + \beta_1(A)$ and $\dim(A \cap \beta_1(A)) = \dim(I) = 7$, we see that $\dim(B) = \dim(A) + 1 = 9$. Let $v \in A$ be such that $v_{23} = 1$ and $v_{ij} = 0$ if $i \neq 2$ or $j \neq 3$. Note that $v \notin I$; thus, $\beta_1(v) \notin A$ and $B = A \oplus \mathbb{C}\beta_1(v)$ (as vector spaces). From now on, we think $A \oplus \mathbb{C} = B$ and $a \oplus \mu = a + \mu\beta_1(v)$.

In order to compute $\beta_1(a \oplus \mu)$, note that $a_I := a - a_{23}v \in I$, $\beta_1(a_I) = \alpha_1(a_I) = a_I$ and

$$\beta_1(a \oplus \mu) = \beta_1(a_I + a_{23}v + \mu\beta_1(v)) = (a_I + \mu v) \oplus a_{23}. \tag{3.2}$$

We claim that the *-algebra structure of $A \oplus \mathbb{C}$ is completely determined by α. On one hand, $(a \oplus \lambda)^* = a^* \oplus \overline{\lambda}$ since β_1 is a *-homomorphism and $v^* = v$. On the other hand, using that $v^2 = 0$ and recalling the relation between $u_t(a, b)$ and β described in the proof of Proposition 2.12, it follows that

$$\begin{aligned}(a \oplus \mu)(b \oplus \nu) &= ab + \mu\beta_1(v)b + \nu a\beta_1(v) + \mu\nu\beta_1(v^2) \\ &= ab + \mu u_1(v, b) + \nu\alpha_1(u_1(a, v)) \\ &= (ab + \mu vb + \nu av) \oplus 0.\end{aligned}$$

Clearly, equation (3.2) and the formulas for the product and involution of $A \oplus \mathbb{C}$ determine Ξ up to isomorphisms (of *-globalizations). Furthermore, the reader may verify that $A \oplus \mathbb{C}$ actually is a *-algebra with those operations, β is a *-global action and $\beta|_{A\oplus 0} = \alpha$. Thus, α

has a unique *-globalization, which is degenerate since $\mathrm{Ann}(A \oplus \mathbb{C}) = \mathbb{C}(v \oplus -1)$. Hence, Proposition 3.1 implies that α does not have a non degenerate r-globalization. However, we can still produce a *-globalization out of an r-globalization.

4. Partial actions on C*-algebras. Most, if not all, theory regarding partial actions on C*-algebras has been developed for locally compact and Hausdorff groups. Nevertheless, we consider topological groups in general since we want to study the effect the group's topology has on the existence of globalizations.

Our definition of C*-partial action is that of [**1**] (which is a simplified version of [**8**]), except that we make no assumptions about the group's topology. To be precise, assume that A is a C*-algebra and G a topological group. We say that $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a C*-partial action if it is a *-partial action and

- A_t is closed, i.e., A_t is a C*-ideal of A, for all $t \in G$.
- $\{A_t\}_{t\in G}$ is a continuous family [**8,** Definition 3.1].
- The function $\{(t,a) \in G \times A \colon a \in A_{t^{-1}}\} \to A,\ (t,a) \mapsto \alpha_t(a)$, is continuous.

For future reference, we group several well-known facts about C*-algebras in the following:

Remark 4.1. Every *-homomorphism between C*-algebras is contractive and it is injective if and only if it is an isometry [**11,** Theorem 1.5.7]. In addition, every C*-algebra is non degenerate, and every C*-ideal of a C*-algebra is assimilative since C*-algebras have approximate units [**11,** Theorem 1.4.2]. Finally, the algebraic sum of finitely many C*-ideals is closed and hence a C*-ideal [**11,** Corollary 1.5.8].

It is clear, then, that the algebraic and topological structure of C*-partial actions are so intimately related that every morphism of *-partial actions between two C*-partial actions is, automatically, continuous. Thus, a morphism of C*-partial actions is merely a morphism of *-partial actions. We will further exploit the interplay between the topological and algebraic structures, especially when constructing C*-globalizations (enveloping actions in [**1**]).

Definition 4.2. A C*-globalization of the C*-partial action α, of G on A, is a 4-tuple $\Xi = (B, \beta, I, \iota)$, where B is a C*-algebra, β is a C*-global action of G on B, I is a C*-ideal of B, $\iota\colon \alpha \to \beta|_I$ is an isomorphism of *-partial actions and

$$[\beta I] := \sum_{t\in G} \beta_t(I)$$

is dense in B.

Suppose we are given a C*-globalization, $\Xi = (B, \beta, I, \iota)$, of the C*-partial action α (of G on A). Then, $[\beta I]$ is a β-invariant *-subalgebra of B and $[\Xi] := ([\beta I], \beta|_{[\beta I]}, I, \iota)$ is a *-globalization of α. From Lemma 2.2 and Theorem 2.1 of [**1**], there is only one way of reversing the process $\Xi \rightsquigarrow [\Xi]$. Moreover, we can regard the proofs of those results as the method for constructing a C*-globalization out of a *-globalization. In addition, Propositions 2.12 and 3.1, Theorem 2.14 and Remark 4.1 imply that α has a (nondegenerate) *-globalization if and only if it satisfies Theorem 2.14 (b). Moreover, α has at most one *-globalization.

Up to this point, we have focused our discussion on the construction of β (regarded as a *-partial action). At some point, topology, i.e., continuity, must come into play, and it is not clear at all that condition (b) of Theorem 2.14 relates to topology. In fact, it is the continuity of α itself that implies the continuity of β.

Lemma 4.3. *Suppose that G is a topological group, and write G^{dis} when regarding G as a discrete group. Let β be a* C**-global action of G^{dis} on B and I an ideal of B such that $[\beta I]$ is dense in B. Then, β is a* C**-global action of G if and only if $\alpha := \beta|_I$ is a* C**-partial action of G.*

Proof. The direct implication is [**1,** Example 2.1]. For the converse, it suffices to show that $t \mapsto \beta_t(a)$ is continuous at e (the group's unit) for all $a \in I$.

Let a and t belong to I and G, respectively. From [**1,** Lemma 2.1], it follows that

$$\begin{aligned}\|\beta_t(a) - a\| &= \sup\{\|(\beta_t(a) - a)\beta_r(b)\| \colon r \in G,\ b \in I,\ \|b\| \le 1\}\\ &= \sup\{\|(\beta_{r^{-1}t}(a) - \beta_{r^{-1}}(a))b\| \colon r \in G,\ b \in I,\ \|b\| \le 1\}\\ &= \sup\{\|(\beta_{rt}(a) - \beta_r(a))b\| \colon r \in G,\ b \in I,\ \|b\| \le 1\}.\end{aligned}$$

Given $r \in G$ and $b \in I$, $\|b\| \le 1$, we have

$$(\beta_{rt}(a) - \beta_r(a))b \in \beta_{rt}(I)I + \beta_r(I)I = I_{rt} + I_r.$$

Thus, for every approximate unit $\{v_\lambda\}_{\lambda\in\Lambda}$ of $I_{rt} + I_r$, we have

$$\|(\beta_{rt}(a) - \beta_r(a))b\| = \lim_\lambda \|(\beta_{rt}(a) - \beta_r(a))bv_\lambda\|.$$

By **[11,** Proposition 1.5.9], for every $\lambda \in \Lambda$, there exist $c \in I_{rt}^+$ and $d \in I_r^+$ such that $v_\lambda = c + d$. Thus, $\|c\|, \|d\| \le \|v_\lambda\| \le 1$. If $\{w_\mu\}_{\mu\in M}$ is an approximate unit of I_t, then

$$\begin{aligned}
&\|(\beta_{rt}(a) - \beta_r(a))bc\| \\
&\quad = \|a\beta_{t^{-1}r^{-1}}(bc) - \beta_{t^{-1}}(a)\beta_{t^{-1}r^{-1}}(bc)\| \\
&\quad = \lim_\mu \|a\beta_{(rt)^{-1}}(bc) - \beta_{t^{-1}}(w_\mu)\beta_{t^{-1}}(a)\beta_{(rt)^{-1}}(bc)\| \\
&\quad \le \limsup_\mu \|a - \beta_{t^{-1}}(w_\mu a)\| \le \limsup_\mu \|a - \alpha_{t^{-1}}(w_\mu a)\|.
\end{aligned}$$

Recall that the canonical approximate unit of I_t is $\{\mu\}_{\mu\in M_t}$, where $M_t := \{c \in I_t^+ \colon \|a\| < 1\}$ is a directed set with its natural order. According to previous calculations,

$$\|(\beta_{rt}(a) - \beta_r(a))bc\| \le \lim_{\mu\in M_t} \sup\{\|a - \alpha_{t^{-1}}(\nu a)\| : \nu \in M_t, \nu \ge \mu\}.$$

In order to simplify our notation, we denote by $C(t)$ the limit on the right-hand side of the previous inequality. With $s = rt$, we obtain

$$\|(\beta_{rt}(a) - \beta_r(a))bd\| = \|(\beta_{st^{-1}}(a) - \beta_s(a))bd\| \le C(t^{-1}).$$

Altogether, we obtain $\|\beta_t(a) - a\| \le C(t) + C(t^{-1})$; therefore, all we need to show is that $\lim_{t\to e} C(t) = 0$.

Fix $\varepsilon > 0$. Since α is a partial action of G on I, there are neighborhoods $V \subset G$ and $W \subset B$ of e and a, respectively, such that $\|\alpha_s(b) - a\| < \varepsilon/2$, for all $s \in V$ and $b \in I_{s^{-1}} \cap W$. Take $\delta > 0$ such that the ball of center a and radius δ, $B(a, \delta)$, is contained in W. Then,

$$U := \{r \in G \colon B(a, \delta/2) \cap I_r \ne \emptyset\}$$

is an open set containing e. For every $r \in U \cap V^{-1}$, there exists a $b \in I_r \cap B(a, \delta/2)$; thus, $\lim_{\mu\in M_r} \|a - \mu a\| = \mathrm{dist}(a, I_r) \le \delta/2$. Take $\mu_r \in M_r$ such that $\|a - \nu a\| < \delta$ for all $\nu \in M_r$ with $\nu \ge \mu_r$. Then, for every $r \in U \cap V^{-1}$ we have $C(r) < \varepsilon$ since, for all $\nu \in M_r$ with $\nu \geq \mu_r$, the inequality $\|a - \alpha_{r^{-1}}(\nu a)\| \leq \varepsilon/2$ holds and implies

$$\sup\{\|a - \alpha_{r^{-1}}(\nu' a)\| \colon \nu' \in M_r,\ \nu' \geq \nu\} < \varepsilon. \qquad \square$$

Corollary 4.4. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a* C**-partial action, and use the expression α^{dis} to denote α, when considered as a* C**-partial action of G^{dis}. Then, α has a* C**-globalization if and only if α^{dis} has a* C**-globalization.*

Proof. As mentioned above, the direct implication is immediate. For the converse, assume that α^{dis} has a C*-globalization $\Xi = (B, \beta, I, \iota)$. Lemma 4.3 implies that Ξ is a C*-globalization since $\beta|_I$ is isomorphic (as a *-partial action) to the C*-partial action α and *-homomorphisms between C*-algebras are homeomorphisms (Remark 4.1). $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.5. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a* C**-partial action. Then the following are equivalent:*

(a) *α has a* C**-globalization.*
(b) *α has an r-globalization.*
(c) *For all $(t, a, b) \in G \times A \times A$, there exists a $u \in A_t$ such that, for all $c \in A_t$, $cu = \alpha_t(\alpha_{t^{-1}}(c)a)b$.*

Proof. We already know (a) implies (b), which in turn, implies (c) (Theorem 2.14). It is clear from the discussion preceding Lemma 4.3 that, to show (c) implies (b), it suffices to prove that $uc = \alpha_t(a\alpha_{t^{-1}}(bc))$, with a, b, c and t as in (c). If $\{v_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit to A_t, then

$$du = \lim_\lambda \alpha_t(\alpha_{t^{-1}}(v_\lambda)a\alpha_{t^{-1}}(bc)) = \alpha_t(a\alpha_{t^{-1}}(bc)).$$

In order to show that (b) implies (a), we assume, without loss of generality, that G is discrete. From Proposition 3.1, we conclude that α has a nondegenerate *-globalization $\Xi = (B, \beta, I, \iota)$. Since essentially this is the unique *-globalization, we may assume $A = I$, ι is the identity and $\alpha = \beta|_A$.

The ring of centralizers of A, $M(A)$, is, in fact, the multiplier C*-algebra of A, which we view as the C*-algebra of adjointable operators of the right Hilbert module A (with the usual inner product $\langle a, b\rangle = a^*b$). For every $t \in G$ and $b \in B$, the function $A \to A$, $a \mapsto \beta_t(b)a$, is an adjointable operator with adjoint

$$a \longmapsto \beta_t(b^*)a.$$

Thus, for every $t \in G$, there exists a *-homomorphism

$$\psi_t \colon B \longrightarrow M(A)$$

such that $\psi_t(b)a = \beta_t(b)a$. According to [**1,** proof of Theorem 2.1] the C*-norm of B should be

$$\|b\| := \sup\{\|\psi_t(b)\| \colon t \in G\}. \tag{4.1}$$

In order to prove the supremum is finite, take $t_1, \ldots, t_n \in G$ and $a_1, \ldots, a_n \in A$ such that

$$b = \sum_{j=1}^{n} \beta_{t_j}(a_j).$$

For every $r \in G$ and $c \in A$, we have $u := \beta_{rt_j}(a_j)c \in A_{rt_j}$ and

$$\|u\|^2 = \|u^*\beta_{rt_j}(a_j)c\| = \|\alpha_{rt_j}(\alpha_{t_j^{-1}r^{-1}}(u^*)a_j)c\| \leq \|u\|\|a_j\|\|c\|.$$

Thus,

$$\|\psi_r(b)c\| = \left\| \sum_{j=1}^{n} \beta_{rt_j}(a_j)c \right\| \leq \|c\| \sum_{j=1}^{n} \|a_j\|,$$

and $\sum_{j=1}^{n} \|a_j\|$ is an upper bound of $\{\|\psi_t(b)\| \colon t \in G\}$.

The function $b \mapsto \|b\|$ is a seminorm since it is the supremum of the seminorm norms $b \mapsto \|\psi_r(b)\|$. Moreover, for all $r \in G$ and $a, b \in B$, we have $\|ab\| \leq \|a\|\|b\|$ since $\|\psi_r(ab)\| \leq \|\psi_r(a)\|\|\psi_r(b)\|$. The C*-identity $\|b^*b\| = \|b\|^2$ easily follows due to

$$\|\psi_t(b)\|^2 = \|\psi_t(b)^*\psi_t(b)\| = \|\psi_t(b^*b)\|.$$

Assume that $\|b\| = 0$. Then, $\beta_r(b)a = 0$ for all $r \in G$ and $a \in A$. Since, clearly, $\psi_r(b^*) = 0$, we conclude that

$$(a\beta_r(b))(a\beta_r(b))^* = (a\beta_r(b))\beta_r(b^*)a^* = 0.$$

Hence, the C*-identity of A implies $a\beta_r(b) = 0$ for all $r \in G$ and $a \in A$. Note that, if π is the canonical morphism associated to Ξ, then $\pi(b) = 0$. This implies $b = 0$ since B is non degenerate (Lemma 2.10).

Let $\overline{B}$ be the completion of B with respect to $\|\ \|$, which is a C*-algebra. Each one of the *-homomorphisms $\beta_t \in \mathrm{Aut}(B)$ is an isometry since $\psi_r(\beta_t(b)) = \psi_{rt}(b)$; thus, there exists a unique *-homomorphism $\overline{\beta}_t \in \mathrm{Aut}(\overline{B})$ extending β_t. From uniqueness of the extension, it follows that $\overline{\beta}\colon G \to \mathrm{Aut}(\overline{B})$ is a C*-global action such that $\overline{\beta}|_B = \beta$. Thus, Remark 2.1 implies

$$\overline{\beta}|_A = \overline{\beta}|_B|_A = \beta|_A = \alpha.$$

By construction, $\overline{[\beta A]} = \overline{B}$; $(\overline{B}, \overline{\beta}, A, \iota)$ is a C*-globalization of α if and only if A is a C*-ideal of $\overline{B}$. The inclusion of A into $\overline{B}$ is an isometry since it is an injective *-homomorphism between two C*-algebras, meaning that A is closed in $\overline{B}$. Then, A is a C*-ideal of $\overline{B}$ due to the fact that

$$A\overline{B} \subset \overline{AB} \subset A \supset \overline{BA} \supset \overline{B}A. \qquad \square$$

Theorem 4.5 (c) asserts the existence of a certain element u for each $(t, a, b) \in G \times A \times A$. That element is the limit of the net given in (b), below.

Proposition 4.6. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a C*-partial action. Then, the following are equivalent.*

(a) *α has a C*-globalization.*
(b) *There exist $U, V \subset A$ such that:*

(i) *$\overline{\operatorname{span}}\, AU = \overline{\operatorname{span}}\, VA = A$, and*
(ii) *for all $(t, a, b) \in G \times U \times V$, there exists an approximate unit of $A_{t^{-1}}$, $\{v_\lambda\}_{\lambda\in\Lambda}$, such that $\{\alpha_t(v_\lambda a)b\}_{\lambda\in\Lambda}$ converges.*

Proof. Suppose (a) holds. Define $U = V := A$ and take $(t, a, b) \in G \times A \times A$. Let u be the element given in Theorem 4.5 (c). Then, for every approximate unit of $A_{t^{-1}}$, $\{v_\lambda\}_{\lambda\in\Lambda}$, the net $\{\alpha_t(v_\lambda a)b\}_{\lambda\in\Lambda}$ converges to u since $\{\alpha_t(v_\lambda)\}_{\lambda\in\Lambda}$ is an approximate unit of A_t and $\alpha_t(v_\lambda)u = \alpha_t(v_\lambda a)b$.

Now, assume (b) is true. Fix $(t,a,b) \in G \times A \times A$ for which there exists an approximate unit $\{v_\lambda\}_{\lambda\in\Lambda}$ such that $\{\alpha_t(v_\lambda a)b\}_{\lambda\in\Lambda}$ converges to an element u. Note that u belongs to A_t since it is the limit of a net contained in A_t. We claim that, given every other approximate unit of $A_{t^{-1}}$, $\{w_\mu\}_{\mu\in M}$, the net $\{\alpha_t(w_\mu a)b\}_{\mu\in M}$ converges to u. Indeed, since $\{\alpha_t(w_\mu)\}_{\mu\in M}$ is an approximate unit of A_t

$$u = \lim_\mu \lim_\lambda \alpha_t(w_\mu)\alpha_t(v_\lambda a)b = \lim_\mu \lim_\lambda \alpha_t(w_\mu v_\lambda a)b = \lim_\mu \alpha_t(w_\mu a)b.$$

Thus, u is completely determined by (t,a,b), and we denote it $u_t(a,b)$. Moreover, if $c \in A_{t^{-1}}$ then

$$\alpha_t(c)u_t(a,b) = \lim_\lambda \alpha_t(c)\alpha_t(v_\lambda a)b = \lim_\lambda \alpha_t(cv_\lambda a)b = \alpha_t(ca)b.$$

We claim that (b) holds if we consider AU and VA instead of U and V, respectively. Indeed, take $a \in U$, $b \in V$, $c,d \in A$ and an approximate unit of $A_{t^{-1}}$, $\{v_\lambda\}_{\lambda\in\Lambda}$. Hence, $\{\alpha_t(v_\lambda ca)bd\}_{\lambda\in\Lambda}$ converges to $\alpha_t(c\alpha_{t^{-1}}(u_t(a,b)d))$ since, for all $\lambda \in \Lambda$,

$$\alpha_t(v_\lambda ca)bd = \alpha_t(v_\lambda c)u_t(a,b)d = \alpha_t(v_\lambda c\alpha_{t^{-1}}(u_t(a,b)d))$$

and $\lim_\lambda \alpha_t(v_\lambda c\alpha_{t^{-1}}(u_t(a,b)d)) = \alpha_t(c\alpha_{t^{-1}}(u_t(a,b)d))$. From now on, we assume that $U = AU$ and $V = VA$.

We can easily justify the existence, for all $t \in G$, of a unique bilinear function

$$u_t\colon \operatorname{span} U \times \operatorname{span} V \longrightarrow A_t$$

such that $\alpha_t(c)u_t(a,b) = \alpha_t(ca)b$ for all $a \in \operatorname{span} U$, $b \in \operatorname{span} V$ and $c \in A_{t^{-1}}$. Note also that

$$\|u_t(a,b)\|^2 = \|\alpha_t(\alpha_{t^{-1}}(u_t(a,b)^*)a)b\| \le \|u_t(a,b)\|\|a\|\|b\|,$$

whence there exists a unique continuous bilinear function

$$v_t\colon A \times A \longrightarrow A_t$$

extending u_t.

Given $(t,a,b) \in G \times A \times A$ and $c \in A_{t^{-1}}$, take sequences

$$\{a_n\}_{n\in\mathbb{N}} \subset \operatorname{span} U \quad \text{and} \quad \{b_n\}_{n\in\mathbb{N}} \subset \operatorname{span} V$$

converging to a and b, respectively. Then, α satisfies Theorem 4.5 (c) since, for all $c \in A_{t^{-1}}$,

$$\alpha_t(c)v_t(a,b) = \lim_n \alpha_t(c)u_t(a_n,b_n) = \lim_n \alpha_t(ca_n)b_n = \alpha_t(ca)b. \qquad \square$$

Combining Theorem 4.5 with Corollary 2.15, we obtain:

Corollary 4.7. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a* C*-*partial action. If A_t has an orthogonal complement for all $t \in G$, then α has a* C*-*globalization.*

Corollary 4.8. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a* C*-*partial action with A unital. Then, α has a* C*-*globalization if and only if A_t is a unital algebra, for all $t \in G$.*

Proof. The proof follows directly from [**5,** Theorem 4.5] and Theorem 4.5. $\square$

For partial actions on locally compact and Hausdorff spaces, the next result can be shown using [**1,** Proposition 2.1] instead of Theorem 4.5.

Corollary 4.9. *Let*

$$\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G}) \quad \textit{and} \quad \beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$$

be C*-*partial actions. If α has a* C*-*globalization and there exists a* *-*homomorphism $\phi\colon A \to M(B)$ such that*:

(a) $\overline{\operatorname{span}}\, \phi(A_t)B = B_t$ *for all $t \in G$, and*
(b) $\phi(\alpha_t(a))\beta_t(b) = \beta_t(\phi(a)b)$ *for all $t \in G$, $a \in A_{t^{-1}}$ and $b \in B_{t^{-1}}$,*

then β has a C*-*globalization.*

Proof. It suffices to prove that β satisfies Theorem 4.5 (c). First, observe that, by Cohen-Hewitt's theorem, $B_t = \phi(A_t)B$ for all $t \in G$. The same theorem implies $A_t = A_tA_t$; thus, $B_t = \phi(A_t)B_t$.

Given $(t, b_1, b_2) \in G \times B \times B$, take $a_1, a_2 \in A$ and $c_1, c_2 \in B$ such that $b_1^* = \phi(a_1^*)c_1^*$ and $b_2 = \phi(a_2)c_2$. Let u be the element given for α and (t, a_1, a_2) in Theorem 4.5 (c). Then, $c_1\phi(u)c_2 \in B_t$ since $u \in A_t$.

It suffices to show that $\beta_t(d)c_1\phi(u)c_2 = \beta_t(db_1)b_2$ for all $d \in B_{t^{-1}}$. Given $d \in B_{t^{-1}}$, take $d' \in B_{t^{-1}}$ and $e \in A_{t^{-1}}$ such that $\beta_{t^{-1}}(c_1^*\beta_t(d^*)) = \phi(e^*)d'^*$. Then,

$$\begin{aligned}\beta_t(d)c_1\phi(u)c_2 &= \beta_t(d')\phi(\alpha_t(e)u)c_2 = \beta_t(d')\phi(\alpha_t(ea_1)a_2)c_2\\ &= \beta_t(d')\phi(\alpha_t(ea_1))\phi(a_2)c_2 = \beta_t(d'\phi(e)\phi(a_1))b_2\\ &= \beta_t(db_1)b_2. \qquad \square\end{aligned}$$

We conclude this section with an example showing that condition (a) above cannot be weakened to (a′) $B_t = \overline{\operatorname{span}}\{\phi(a)b\colon a \in A_t,\ b \in B_t\}$ for all $t \in G$.

Consider $G := \mathbb{Z}_2$ and $A = B := C[0,1]$. Let α be the trivial global action of G on A, and define β in such a way that β_1 is the identity on the C*-ideal $C_0([0,1))$. If we consider the identity map,

$$\phi\colon C[0,1] \longrightarrow M(C[0,1]) = C[0,1],$$

then conditions (a′) and (b) are satisfied and α has a C*-globalization but β does not.

5. Partial action on equivalence bimodules. In [**1**], Abadie defined Morita equivalence of C*-partial actions using partial actions on positive C*-trings. Recall that positive C*-trings are precisely equivalence bimodules [**1, 12, 13**].

In this section, we give a necessary and sufficient condition for the existence of a globalization of a partial action on an equivalence bimodule.

We adopt the terminology of [**12**] and agree that "${}_A\mathcal{X}_B$ is an equivalence bimodule" means "$\mathcal{X}$ is an $A{-}B$-equivalence bimodule." Let ${}_A\mathcal{X}_B$ and ${}_C\mathcal{Y}_D$ be equivalence bimodules. A function $\phi\colon \mathcal{X} \to \mathcal{Y}$ is an Hb-homomorphism if it is linear and

$$\phi(x\langle y,z\rangle_B) = \phi(x)\langle\phi(y),\phi(z)\rangle_D$$

for all $x,y,z \in \mathcal{X}$. Such functions are contractive and, with the previous notation, there exist [**1**] unique *-homomorphisms

$${}^l\phi\colon A \longrightarrow C \quad \text{and} \quad \phi^r\colon B \longrightarrow D$$

such that

$$ {}^{l}\phi({}_A\langle x, y\rangle) = {}_C\langle \phi(x), \phi(y)\rangle \quad \text{and} \quad \phi^r(\langle x, y\rangle_B) = \langle \phi(x), \phi(y)\rangle_D $$

for all $x, y \in \mathcal{X}$.

Given a C*-ideal I of A, by Cohen-Hewitt's theorem,

$$ I\mathcal{X} := \{ax \colon a \in I,\ x \in \mathcal{X}\} $$

is a closed submodule of $\mathcal{X}$. We denote ${}_IB$ the C*-ideal of B induced by I (or $I\mathcal{X}$) through $\mathcal{X}$, that is,

$$ {}_IB = \overline{\operatorname{span}}\,\{\langle u, v\rangle_B \colon u, v \in I\mathcal{X}\}. $$

In a similar way, we define, for a C*-ideal J of B, $\mathcal{X}J$ and A_J. Recall that $A_{{}_IB} = I$ and $J = {}_{A_J}B$.

For a closed subspace $\mathcal{Z}$ of $\mathcal{X}$, the following are equivalent:

(i) there exists a C*-ideal I of A such that $\mathcal{Z} = I\mathcal{X}$,
(ii) there exists a C*-ideal J of B such that $\mathcal{Z} = \mathcal{X}J$,
(iii) $\mathcal{X}\langle \mathcal{Z}, \mathcal{X}\rangle_B \subset \mathcal{Z}$, and
(iv) ${}_A\langle \mathcal{X}, \mathcal{Z}\rangle\mathcal{X} \subset \mathcal{Z}$.

If these conditions are satisfied, then $A_{\mathcal{Z}} := \overline{\operatorname{span}}\,{}_A\langle \mathcal{Z}, \mathcal{Z}\rangle$ and ${}_{\mathcal{Z}}B$ are C*-ideals,

$$ \mathcal{Z} = A_{\mathcal{Z}}\mathcal{X} = \mathcal{X}{}_{\mathcal{Z}}B, $$

and we say that $\mathcal{Z}$ is an ideal of $\mathcal{X}$. Every ideal $\mathcal{Z}$ of $\mathcal{X}$ is an $A_{\mathcal{Z}} - {}_{\mathcal{Z}}B$-equivalence bimodule.

Definition 5.1. We say $\gamma = (\{\gamma_t\}_{t\in G}, \{\mathcal{X}_t\}_{t\in G})$ is an Hb-*partial action* if:

- $\mathcal{X}$ is an $(A{-}B$-)equivalence bimodule and G a topological group.
- γ is a set theoretic partial action of G on $\mathcal{X}$.
- $\{\mathcal{X}_t\}_{t\in G}$ is a continuous family of ideals of $\mathcal{X}$.
- $\gamma_t \colon \mathcal{X}_{t^{-1}} \to \mathcal{X}_t$ is an Hb-homomorphism for all $t \in G$.
- The function

  $$ \{(t, x) \in G \times \mathcal{X} \colon x \in \mathcal{X}_{t^{-1}}\} \longrightarrow \mathcal{X}, \quad (t, x) \mapsto \gamma_t(x), $$

  is continuous.

Assume that δ is an Hb-partial action G on ${}_C\mathcal{Y}_D$. We say that

$$\phi\colon \gamma \longrightarrow \delta$$

is a morphism if it is an Hb-homomorphism from $\mathcal{X}$ to $\mathcal{Y}$ which is also G-equivariant.

From [**1**], we know that, for every Hb-partial action γ of G on ${}_A\mathcal{X}_B$, there are unique C*-partial actions,

$$\alpha = (\{\alpha_t\}_{t\in G}, \{A_t\}_{t\in G})$$

and

$$\beta = (\{\beta_t\}_{t\in G}, \{B_t\}_{t\in G}),$$

such that

- $A_t = A_{\mathcal{X}_t}$ and $B_t = {}_{\mathcal{X}_t}B$ for all $t \in G$.
- $\alpha_t({}_A\langle x, y\rangle) = {}_A\langle \gamma_t(x), \gamma_t(y)\rangle$ and $\beta_t(\langle x, y\rangle_B) = \langle \gamma_t(x), \gamma_t(y)\rangle_B$ for all $t \in G$ and $x, y \in \mathcal{X}_{t^{-1}}$.

We will call α the left side of γ and β the right side of γ, and we will denote them as ${}^l\gamma$ and γ^r, respectively.

Example 5.2. Every C*-partial action α on a C*-algebra A is an Hb-partial action on ${}_AA_A$. In addition, $\alpha = {}^l\alpha = \alpha^r$.

Example 5.3. Given an Hb-global action of G on ${}_A\mathcal{X}_B$, γ, and an ideal $\mathcal{Y}$ of $\mathcal{X}$, the restriction $\gamma|_\mathcal{Y}$ is an Hb-partial action on the $A_\mathcal{Y} - {}_\mathcal{Y}B$-equivalence bimodule $\mathcal{Y}$. In this case,

$${}^l(\gamma|_\mathcal{Y}) = ({}^l\gamma)|_{A_\mathcal{Y}} \quad \text{and} \quad (\gamma|_\mathcal{Y})^r = \gamma^r|_{{}_\mathcal{Y}B}.$$

Definition 5.4. Let γ be an Hb-partial action of G on $\mathcal{X}$. By an Hb-globalization of γ, we mean a 4-tuple $\Xi = (\mathcal{Y}, \delta, \mathcal{Z}, \iota)$ such that $\mathcal{Y}$ is an equivalence bimodule, δ is an Hb-global action of G on $\mathcal{Y}$, $\mathcal{Z}$ is an ideal of $\mathcal{Y}$, $\iota\colon \gamma \to \delta|_\mathcal{Z}$ is an isomorphism of Hb-partial actions and $[\beta\mathcal{Z}]$ is dense in $\mathcal{Y}$.

The nexus between Hb-partial actions and C*-partial actions is the linking partial action [**1**]. In order to describe this action, we begin with an Hb-partial action of a group G on ${}_A\mathcal{X}_B$, γ, and set $\alpha := {}^l\gamma$

and $\beta := \gamma^r$. The linking algebra of $\mathcal{X}$ is the algebra of generalized compact operators of the A-Hilbert module $\mathcal{X} \oplus A$, $\mathbb{L}(\mathcal{X}) = \mathbb{K}(\mathcal{X} \oplus A)$. In matrix representation,

$$\mathbb{L}(\mathcal{X}) = \begin{pmatrix} A & \mathcal{X} \\ \widetilde{\mathcal{X}} & B \end{pmatrix},$$

with $\widetilde{\mathcal{X}}$ being the B-A-equivalence bimodule adjoint to $\mathcal{X}$.

The linking partial action of γ,

$$\mathbb{L}(\gamma) = (\{\mathbb{L}(\gamma)_t\}_{t \in G}, \{\mathbb{L}(\mathcal{X})_t\}_{t \in G}),$$

is the unique C*-partial action such that, for all $t \in G$: $\mathbb{L}(\mathcal{X})_t = \mathbb{L}(\mathcal{X}_t)$ and

$$\mathbb{L}(\gamma)_t \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} = \begin{pmatrix} \alpha_t(a) & \gamma_t(x) \\ \widetilde{\gamma_t(y)} & \beta_t(b) \end{pmatrix}$$

for all $x, y \in X_{t^{-1}}$, $a \in A_{t^{-1}}$ and $b \in B_{t^{-1}}$.

In the case where δ is an Hb-partial action of G on $\mathcal{Y}$ and $\pi \colon \gamma \to \delta$ is an isomorphism, the morphism

$$\mathbb{L}(\pi) \colon \mathbb{L}(\gamma) \longrightarrow \mathbb{L}(\delta),$$

defined as

$$\mathbb{L}(\pi) \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} = \begin{pmatrix} {}^l\pi(a) & \pi(x) \\ \widetilde{\pi(y)} & \pi^r(b) \end{pmatrix},$$

is an isomorphism with inverse $\mathbb{L}(\pi^{-1})$.

Proposition 5.5. *If* $\Xi = ({}_C\mathcal{Y}_D, \delta, \mathcal{Z}, \iota)$ *is an* Hb-*globalization of* γ, *then*

- $(\mathbb{L}(\mathcal{Y}), \mathbb{L}(\delta), \mathbb{L}(\mathcal{Z}), \mathbb{L}(\iota))$ *is a* C*-*globalization of* $\mathbb{L}(\gamma)$.
- $(C, {}^l\delta, C_{\mathcal{Z}}, {}^l\iota)$ *is a* C*-*globalization of* ${}^l\gamma$.
- $(D, \delta^r, {}_{\mathcal{Z}}D, \iota^r)$ *is a* C*-*globalization of* γ^r.

Proof. Straightforward and is therefore left to the reader. $\square$

In the same manner in which equivalence bimodules are constructed from C*-algebras and projections, Hb-partial actions are constructed from C*-partial actions and equivariant projections.

Theorem 5.6. *Let α be a* C^**-partial action of G on A. Also, let $p \in M(A)$ be a projection ($p^*p = p$) such that $\overline{\mathrm{span}}\, ApA = A$ and $\alpha_t(pa) = p\alpha_t(a)$ for all $t \in G$ and $a \in A_{t^{-1}}$. If $\mathcal{X} := (1-p)Ap$, $C := (1-p)A(1-p)$ and $D := pAp$, then*:

(a) *$\mathcal{X}$ is a C–D-equivalence bimodule,*
(b) *$\mathcal{X}$ is α-invariant,*
(c) *the restriction of α to $\mathcal{X}$, γ, is an* Hb*-partial action.*
(d) *C and D are α-invariant, ${}^l\gamma = \alpha|_C$ and $\gamma^r = \alpha|_D$.*

Moreover, α is isomorphic (as a C^**-partial action) to $\mathbb{L}(\gamma)$, and α has a* C^**-globalization if and only if γ has an* Hb*-globalization.*

Proof. The proof of claims (a)–(d) are left to the reader. The standard identification of A with $\mathbb{L}(\mathcal{X})$ is an isomorphism of C^*-partial actions between α and $\mathbb{L}(\gamma)$. Then, in the case where γ has an Hb-globalization, $\mathbb{L}(\gamma)$ has a C^*-globalization, and this implies that α has a C^*-globalization.

In order to prove the converse, assume that α has a C^*-globalization. Then, we can assume, without loss of generality, that there exists a C^*-partial action, β of G on B, such that A is an ideal of B, $\alpha = \beta|_A$ and $\overline{[\beta A]} = B$.

We claim that there exists a projection $\overline{p} \in M(B)$ such that

$$\beta_t(\overline{p}b) = \overline{p}\beta_t(b) \quad \text{and} \quad \overline{p}a = pa,$$

for all $t \in G$, $b \in B$ and $a \in A$. In order to prove this, it suffices to show that, for all $t_1, \dots, t_n \in G$ and $a_1, \dots, a_n \in A$,

$$\Big\| \sum_{j=1}^{n} \beta_{t_j}(pa_j) \Big\| \le \Big\| \sum_{j=1}^{n} \beta_{t_j}(a_j) \Big\|. \tag{5.1}$$

From **[1,** Lemma 2.1], we conclude that it is sufficient to show that for all $r \in G$ and $b \in A$ with $\|b\| < 1$,

$$\Big\| \sum_{j=1}^{n} \beta_{t_j}(pa_j)\beta_r(b) \Big\| \le \Big\| \sum_{j=1}^{n} \beta_{t_j}(a_j) \Big\|.$$

Take $r \in G$ and $b \in B$ as before, and note that $pa_j\beta_{t_j^{-1}r}(b) \in A_{t_j^{-1}r}$. Then,

$$\begin{aligned}
\Big\|\sum_{j=1}^n \beta_{t_j}(pa_j)\beta_r(b)\Big\| &= \Big\|\sum_{j=1}^n \beta_{t_j}(pa_j\beta_{t_j^{-1}r}(b))\Big\| \cdots \\
&= \Big\|\sum_{j=1}^n \beta_{r^{-1}t_j}(pa_j\beta_{t_j^{-1}r}(b))\Big\| = \Big\|\sum_{j=1}^n \alpha_{r^{-1}t_j}(pa_j\beta_{t_j^{-1}r}(b))\Big\| \cdots \\
&= \Big\|\sum_{j=1}^n p\alpha_{r^{-1}t_j}(a_j\beta_{t_j^{-1}r}(b))\Big\| \leq \Big\|\sum_{j=1}^n \alpha_{r^{-1}t_j}(a_j\beta_{t_j^{-1}r}(b))\Big\| \cdots \\
&= \Big\|\sum_{j=1}^n \beta_{r^{-1}t_j}(a_j\beta_{t_j^{-1}r}(b))\Big\| \leq \Big\|\sum_{j=1}^n \beta_{t_j}(a_j)\Big\|.
\end{aligned}$$

Set $\mathcal{Y} := (1-\overline{p})A\overline{p} \subset B$, note that $\mathcal{X} \subset \mathcal{Y}$ and define $\iota\colon \mathcal{X} \to \mathcal{Y}$ as the canonical inclusion. Remark 2.1 implies that, if $\beta|_{\mathcal{Y}}|_{\mathcal{X}} = \beta|_{\mathcal{X}} = \beta|_A|_{\mathcal{X}} = \alpha|_{\mathcal{X}} = \gamma$, then $(\mathcal{Y}, \beta|_{\mathcal{Y}}, \mathcal{X}, \iota)$ is an Hb-globalization of γ. $\square$

Corollary 5.7. *An* Hb-*partial action has an* Hb-*globalization if and only if its linking partial action has a* C*-*globalization.*

Proof. Let γ be a partial action of G on the A-B-equivalence bimodule $\mathcal{X}$. The thesis directly follows from the previous theorem with $\alpha = \mathbb{L}(\gamma)$ and $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ since $\mathbb{L}(\gamma)|_{p\mathbb{L}(\mathcal{X})(1-p)}$ is isomorphic to γ. $\square$

As a consequence of Corollary 5.7, we obtain that the group's topology does not affect the existence of Hb-globalizations since it does not affect the existence of C*-globalizations. This conclusion may also be derived from our final result, which also implies that a C*-partial action has a C*-globalization if and only if it has an Hb-globalization.

Theorem 5.8. *An* Hb-*partial action has an* Hb-*globalization if and only if its left and right sides have* C*-*globalizations.*

Proof. The direct implication follows from Proposition 5.5. In order to prove the converse, assume that γ is an Hb-partial action of the topological group G on ${}_A\mathcal{X}_B$ and that both sides of γ, $\alpha := {}^l\gamma$ and $\beta := \gamma^r$, have C*-globalizations. Given $t \in G$, set $A_t := {}_A\mathcal{X}_t$ and $B_t := {}_{\mathcal{X}_t}B$.

It suffices to show that $\mathbb{L}(\gamma)$ has a C*-globalization. To do this, we use Proposition 4.6 with

$$U = V = \left\{ \begin{pmatrix} 0 & x \\ \widetilde{y} & 0 \end{pmatrix} : x, y \in \mathcal{X} \right\}.$$

Fix $(t, \xi, \eta) \in G \times U \times V$. Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of $A_{t^{-1}}$ and $\{w_\mu\}_{\mu\in M}$ one of $B_{t^{-1}}$. Consider $K = \Lambda \times M$ with the order

$$(\lambda, \mu) \leq (\lambda', \mu') \iff \lambda \leq \lambda' \quad \text{and} \quad \mu \leq \mu',$$

and define, for every $\kappa = (\lambda, \mu) \in K$,

$$d_\kappa := \begin{pmatrix} v_\lambda & 0 \\ 0 & w_\mu \end{pmatrix}.$$

Then, $\{d_\kappa\}_{\kappa\in K}$ is an approximate unit of $\mathbb{L}(\mathcal{X})_t$.

Suppose that

$$\xi = \begin{pmatrix} 0 & x \\ \widetilde{y} & 0 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 0 & u \\ \widetilde{v} & 0 \end{pmatrix}.$$

Then, for all $\kappa = (\lambda, \mu) \in K$,

$$\mathbb{L}(\gamma)_t(d_\kappa\xi)\eta = \begin{pmatrix} \langle \gamma_t(v_\lambda x), v\rangle_l & 0 \\ 0 & \langle \gamma_t(y w_\mu), u\rangle_r \end{pmatrix}.$$

By the Cohen-Hewitt theorem, there exist $b, c \in B$ and $z, w \in \mathcal{X}$ such that $x = zb$ and $v = wc$. In addition, $\{\beta_t(w_\mu b)c^*\}_{\mu\in M}$ converges to an element $p \in B_{t^{-1}}$. The upper left corner of $\mathbb{L}(\gamma)_t(d_\kappa\xi)\eta$ is

$$\begin{aligned} \langle \gamma_t(v_\lambda x), v\rangle_l &= \langle \gamma_t(v_\lambda a z b), wc\rangle_l = \lim_\mu \lim_\nu \langle \gamma_t(v_\lambda z w_\mu f_\nu b)c^*, w\rangle_l \\ &= \lim_\mu \lim_\nu \langle \gamma_t(v_\lambda z w_\mu)\beta_t(f_\nu b)c^*, w\rangle_l = \lim_\mu \langle \gamma_t(v_\lambda z w_\mu)p, w\rangle_l \\ &= \lim_\mu \langle \gamma_t(v_\lambda z w_\mu \beta_{t^{-1}}(p)), w\rangle_l = \langle \gamma_t(v_\lambda z \beta_{t^{-1}}(p)), w\rangle_l. \end{aligned}$$

Note that $z\beta_{t^{-1}}(p) \in \mathcal{X}_{t^{-1}}$; thus, $\lim_\lambda \langle \gamma_t(v_\lambda x), v\rangle_l = \langle \gamma_t(z\beta_{t^{-1}}(p)), w\rangle_l$.

Using symmetry, we conclude that $\{\langle \gamma_t(y w_\mu), u\rangle_r\}_{\mu\in M}$ converges. Hence, $\{\mathbb{L}(\gamma)_t(d_\kappa\xi)\eta\}_{\kappa\in K}$ converges. $\square$

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