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THE PRIMITIVE IDEAL SPACE OF THE PARTIAL-ISOMETRIC CROSSED PRODUCT OF A SYSTEM BY A SINGLE AUTOMORPHISM

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ABSTRACT. Let (A, α) be a system consisting of a C^*-algebra A and an automorphism α of A. We describe the primitive ideal space of the partial-isometric crossed product $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ of the system by using its realization as a full corner of a classical crossed product and applying some results of Williams and Echterhoff.

1. Introduction. Lindiarni and Raeburn [**8**] introduced the partial-isometric crossed product of a dynamical system (A, Γ^+, α) in which Γ^+ is the positive cone of a totally ordered abelian group Γ and α is an action of Γ^+ by endomorphisms of A. Note that, since the C^*-algebra A is not necessarily unital, we require that each endomorphism α_s must extend to a strictly continuous endomorphism $\overline{\alpha}_s$ of the multiplier algebra $M(A)$. This occurs for an endomorphism α of A if and only if there exists an approximate identity (a_λ) in A and a projection $p \in M(A)$ such that $\alpha(a_\lambda)$ strictly converges to p in $M(A)$. It should be stressed that, if α is extendible, then we may not have $\overline{\alpha}(1_{M(A)}) = 1_{M(A)}$. A covariant representation of the system (A, Γ^+, α) is defined for which the endomorphisms α_s are implemented by partial isometries, and the associated partial-isometric crossed product $A \times_{\alpha}^{\mathrm{piso}} \Gamma^+$ of the system is a C^*-algebra generated by a universal covariant representation such that there is a bijection between covariant representations of the system and nondegenerate representations of $A \times_{\alpha}^{\mathrm{piso}} \Gamma^+$. This generalizes the covariant isometric representation theory that uses isometries to represent the semigroup of endomorphisms in a covariant representation

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of the system, see [**3**]. The authors of [**8**], in particular, studied the structure of the partial-isometric crossed product of the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$, where the action τ of Γ^+ on the subalgebra B_{Γ^+} of $\ell^\infty(\Gamma^+)$ is given by right translation. Later, in [**4**], the authors showed that $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is a full corner in a subalgebra of the C^*-algebra $\mathcal{L}(\ell^2(\Gamma^+) \otimes A)$ of adjointable operators on the Hilbert A-module

$$\ell^2(\Gamma^+) \otimes A \simeq \ell^2(\Gamma^+, A).$$

This realization led them to identify the kernel of the natural homomorphism

$$q : A \times_\alpha^{\mathrm{piso}} \Gamma^+ \longrightarrow A \times_\alpha^{\mathrm{iso}} \Gamma^+$$

as a full corner of the compact operators $\mathcal{K}(\ell^2(\mathbb{N}) \otimes A)$, when Γ^+ is $\mathbb{N} := \mathbb{Z}^+$. Thus, as an application, they recovered the Pimsner-Voiculescu exact sequence in [**10**]. Then, in their subsequent work [**5**], they proved that, for an extendible α-invariant ideal I of A (see the definition in [**1**]), the partial-isometric crossed product $I \times_\alpha^{\mathrm{piso}} \Gamma^+$ naturally sits as an ideal in $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ such that

$$\frac{A \times_\alpha^{\mathrm{piso}} \Gamma^+}{I \times_\alpha^{\mathrm{piso}} \Gamma^+} \simeq \frac{A}{I} \times_{\tilde{\alpha}}^{\mathrm{piso}} \Gamma^+.$$

This is actually a generalization of [**2,** Theorem 2.2]. They then combined these results to show that the large commutative diagram of [**8,** Theorem 5.6] associated to the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ is valid for any totally ordered abelian group, not only for subgroups of $\mathbb{R}$. In particular, they used this large commutative diagram for $\Gamma^+ = \mathbb{N}$ to explicitly describe the ideal structure of the algebra $B_{\mathbb{N}} \times_\tau^{\mathrm{piso}} \mathbb{N}$.

Here, we now consider a system (A, α) consisting of a C^*-algebra A and an automorphism α of A. Thus, we actually have an action of the positive cone $\mathbb{N} = \mathbb{Z}^+$ of integers $\mathbb{Z}$ by automorphisms of A. In the present work, we want to study $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$, the primitive ideal space of the partial-isometric crossed product $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ of the system. Since $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is in fact a full corner of the classical crossed product $(B_{\mathbb{Z}} \otimes A) \times \mathbb{Z}$, see [**4,** Section 5], $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$ is homeomorphic to $\mathrm{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$. Therefore, it is sufficient to describe $\mathrm{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$. In order to do so, we apply the results on describing the primitive ideal space (ideal structure) of the classical crossed products from [**7, 12**]. Therefore, we consider the following two conditions:

(1) when A is separable and abelian;
(2) when A is separable and $\mathbb{Z}$ acts on $\operatorname{Prim} A$ freely, see Section 2.

For the first condition, by applying a theorem of Williams,

$$\operatorname{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$$

is homeomorphic to a quotient space of

$$\Omega(B_{\mathbb{Z}}) \times \Omega(A) \times \mathbb{T},$$

where $\Omega(B_{\mathbb{Z}})$ and $\Omega(A)$ are the spectrums of the C^*-algebras $B_{\mathbb{Z}}$ and A, respectively (recall that the dual $\widehat{\mathbb{Z}}$ is identified with $\mathbb{T}$ via the map $z \mapsto (\gamma_z : n \mapsto z^n))$. By computing $\Omega(B_{\mathbb{Z}})$, we parameterize the quotient space as a disjoint union, and then we precisely identify the open sets. For the second condition, we apply a result of Echterhoff which shows that $\operatorname{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$ is homeomorphic to the quasi-orbit space of

$$\operatorname{Prim}(B_{\mathbb{Z}} \otimes A) = \operatorname{Prim} B_{\mathbb{Z}} \times \operatorname{Prim} A,$$

(see in Section 2 that this is a quotient space of $\operatorname{Prim}(B_{\mathbb{Z}} \otimes A)$). Again by a similar argument to the first condition, we precisely describe the quotient space and its topology.

We begin with a preliminary section in which the theory of the partial-isometric crossed products is recalled, as well as some brief discussions on the primitive ideal space of the classical crossed products. In Section 3, for a system (A, α) consisting of a C^*-algebra A and an automorphism α of A, we apply the works of Williams and Echterhoff to describe $\operatorname{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N})$ using the realization of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ as a full corner of the classical crossed product $(B_{\mathbb{Z}} \otimes A) \times \mathbb{Z}$. As some examples, we compute the primitive ideal space of $C(\mathbb{T}) \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$, where the action α is given by rotation through the angle $2\pi\theta$ with θ rational and irrational. Moreover, the description of the primitive ideal space of the Pimsner-Voiculescu Toeplitz algebra associated to the system (A, α) is completely obtained, as it is isomorphic to $A \times_{\alpha^{-1}}^{\mathrm{piso}} \mathbb{N}$. Also, we discuss necessary and sufficient conditions under which $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is GCR (postliminal or type I). Finally, in Section 4, we discuss the primitivity and simplicity of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$.

2. Preliminaries.

2.1. The partial-isometric crossed product. A *partial-isometric representation* of $\mathbb{N}$ on a Hilbert space H is a map

$$V : \mathbb{N} \longrightarrow B(H)$$

such that each $V_n := V(n)$ is a partial isometry, and $V_{n+m} = V_n V_m$ for all $n, m \in \mathbb{N}$.

A *covariant partial-isometric representation* of (A, α) on a Hilbert space H is a pair (π, V) consisting of a nondegenerate representation

$$\pi : A \longrightarrow B(H)$$

and a partial-isometric representation $V : \mathbb{N} \to B(H)$ such that

$$\pi(\alpha_n(a)) = V_n \pi(a) V_n^* \quad \text{and} \quad V_n^* V_n \pi(a) = \pi(a) V_n^* V_n \tag{2.1}$$

for all $a \in A$ and $n \in \mathbb{N}$.

Note that every system (A, α) admits a nontrivial covariant partial-isometric representation [**8,** Example 4.6]: let π be a nondegenerate representation of A on H. Define

$$\Pi : A \longrightarrow B(\ell^2(\mathbb{N}, H))$$

by $(\Pi(a)\xi)(n) = \pi(\alpha_n(a))\xi(n)$. If

$$\mathcal{H} := \overline{\text{span}}\{\xi \in \ell^2(\mathbb{N}, H) : \xi(n) \in \overline{\pi}(\overline{\alpha}_n(1))H \text{ for all } n\},$$

then the representation Π is nondegenerate on $\mathcal{H}$. Now, for every $m \in \mathbb{N}$, define V_m on $\mathcal{H}$ by $(V_m\xi)(n) = \xi(n + m)$. Then, the pair $(\Pi|_{\mathcal{H}}, V)$ is a partial-isometric covariant representation of (A, α) on $\mathcal{H}$. It is easily seen that, if we take π faithful, then Π will be faithful as well, and $\mathcal{H} = \ell^2(\mathbb{N}, H)$ whenever $\overline{\alpha}(1) = 1$ (e.g., when α is an automorphism).

Definition 2.1. A partial-isometric crossed product of (A, α) is a triple $(B, j_A, j_{\mathbb{N}})$ consisting of a C^*-algebra B, a nondegenerate homomorphism $i_A : A \to B$, and a partial-isometric representation $i_{\mathbb{N}} : \mathbb{N} \to M(B)$ such that:

(i) the pair $(j_A, j_{\mathbb{N}})$ is a covariant representation of (A, α) in B;

(ii) for every covariant partial-isometric representation (π, V) of (A, α) on a Hilbert space H, there exists a nondegenerate representation

$$\pi \times V : B \longrightarrow B(H)$$

such that $(\pi \times V) \circ i_A = \pi$ and $(\overline{\pi \times V}) \circ i_{\mathbb{N}} = V$; and

(iii) the C^*-algebra B is spanned by $\{i_{\mathbb{N}}(n)^* i_A(a) i_{\mathbb{N}}(m) : n, m \in \mathbb{N},\ a \in A\}$.

From [**8,** Proposition 4.7], the partial-isometric crossed product of (A, α) always exists, and it is unique up to isomorphism. Thus, we write the partial-isometric crossed product B as $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$.

We recall that, by [**8,** Theorem 4.8], a covariant representation (π, V) of (A, α) on H induces a faithful representation $\pi \times V$ of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ if and only if π is faithful on the range of $(1 - V_n^* V_n)$ for every $n > 0$ (it can actually be seen that it is sufficient to verify that π is faithful on the range of $(1 - V^*V)$, where $V := V_1$).

2.2. The primitive ideal space of crossed products associated to second countable locally compact transformation groups. Let Γ be a discrete group which acts on a topological space X. For every $x \in X$, the set

$$\Gamma \cdot x := \{s \cdot x : s \in \Gamma\}$$

is called the Γ-*orbit* of x. The set $\Gamma_x := \{s \in \Gamma : s \cdot x = x\}$, which is a subgroup of Γ, is called the *stability group* of x. We say the Γ-action is *free* or Γ acts on X *freely* if $\Gamma_x = \{e\}$ for all $x \in X$. Consider a relation $\sim$ on X such that, for $x, y \in X$, $x \sim y$ if and only if $\overline{\Gamma \cdot x} = \overline{\Gamma \cdot y}$. It may be observed that this is an equivalence relation on X. The set of all equivalence classes equipped with the quotient topology is denoted by $\mathcal{O}(X)$ and called the *quasi-orbit space*, which is always a T_0-topological space. The equivalence class of each $x \in X$ is denoted by $\mathcal{O}(x)$ and called the *quasi-orbit* of x.

Now, let Γ be an abelian countable discrete group which acts on a second countable locally compact Hausdorff space X. So (Γ, X) is a second countable locally compact transformation group with Γ abelian. Then, the associated dynamical system $(C_0(X), \Gamma, \tau)$ is separable with Γ abelian, and thus, the primitive ideals of $C_0(X) \times_\tau \Gamma$ are known, see [**12,** Theorem 8.21]. Furthermore, the topology of $\mathrm{Prim}(C_0(X) \times_\tau \Gamma)$

has been beautifully described [**12,** Theorem 8.39]. Therefore, here, we want to briefly recall the discussion on $\mathrm{Prim}(C_0(X) \times_\tau \Gamma)$. The interested reader may consult [**12**] to find that this is indeed a huge and deep discussion.

Let N be a subgroup of Γ. If we restrict the action τ to N, then we obtain a dynamical system $(C_0(X), N, \tau|_N)$ with the associated crossed product $C_0(X) \times_{\tau|_N} N$. Suppose that X_N^Γ is the Green's $((C_0(X) \otimes C_0(\Gamma/N)) \times_{\tau\otimes \mathrm{lt}} \Gamma) - (C_0(X) \times_{\tau|_N} N)$-imprimitivity bimodule, the structure of which can be found in [**12,** Theorem 4.22]. If (π, V) is a covariant representation of $(C_0(X), N, \tau|_N)$, then $\mathrm{Ind}_N^\Gamma(\pi \times V)$ denotes the representation of $C_0(X) \times_\tau \Gamma$ induced from the representation $\pi \times V$ of $C_0(X) \times_{\tau|_N} N$ via X_N^Γ. Now, for $x \in X$, let

$$\varepsilon_x : C_0(X) \longrightarrow \mathbb{C} \simeq B(\mathbb{C})$$

be the evaluation map at x and w a character of Γ_x. Then, the pair (ε_x, w) is a covariant representation of $(C_0(X), \Gamma_x, \tau|_{\Gamma_x})$ such that the associated representation $\varepsilon_x \times w$ of $C_0(X) \times \Gamma_x$ is irreducible, and hence, from [**12,** Proposition 8.27], $\mathrm{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w)$ is an irreducible representation of $C_0(X) \times_\tau \Gamma$. Thus, $\ker(\mathrm{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w))$ is a primitive ideal of $C_0(X) \times_\tau \Gamma$. Note that, if a primitive ideal is obtained in this way, then we say it is *induced from a stability group*. In fact, by [**12,** Theorem 8.21], all primitive ideals of $C_0(X) \times_\tau \Gamma$ are induced from stability groups. Moreover, since, for every $w \in \widehat{\Gamma_x}$, there is a $\gamma \in \widehat{\Gamma}$ such that $w = \gamma|_{\Gamma_x}$, every primitive ideal of $C_0(X) \times_\tau \Gamma$ is actually given by the kernel of an induced irreducible representation $\mathrm{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})$ corresponding to a pair (x, γ) in $X \times \widehat{\Gamma}$. In order to see the description of the topology of $\mathrm{Prim}(C_0(X) \times_\tau \Gamma)$, first note that, if (x, γ) and (y, μ) belong to $X \times \widehat{\Gamma}$ such that $\overline{\Gamma \cdot x} = \overline{\Gamma \cdot y}$ (which implies that $\Gamma_x = \Gamma_y$) and $\gamma|_{\Gamma_x} = \mu|_{\Gamma_x}$, then by [**12,** Lemma 8.34],

$$\ker\big(\mathrm{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})\big) = \ker\big(\mathrm{Ind}_{\Gamma_y}^\Gamma(\varepsilon_y \times \mu|_{\Gamma_y})\big).$$

Thus, define a relation on $X \times \widehat{\Gamma}$ such that $(x, \gamma) \sim (y, \mu)$ if

$$\overline{\Gamma \cdot x} = \overline{\Gamma \cdot y} \quad \text{and} \quad \gamma|_{\Gamma_x} = \mu|_{\Gamma_x}. \tag{2.2}$$

It may easily be seen that $\sim$ is an equivalence relation on $X \times \widehat{\Gamma}$. Now, consider the quotient space $X \times \widehat{\Gamma}/\sim$ equipped with the quotient topology. Then we have:

Theorem 2.2 ([**12,** Theorem 8.39]). *Let* (Γ, X) *be a second countable locally compact transformation group with* Γ *abelian. Then, the map*

$$\Phi : X \times \widehat{\Gamma} \longrightarrow \mathrm{Prim}(C_0(X) \times_\tau \Gamma)$$

defined by

$$\Phi(x,\gamma) := \ker\left(\mathrm{Ind}_{\Gamma_x}^{\Gamma}(\varepsilon_x \times \gamma|_{\Gamma_x})\right)$$

is a continuous and open surjection and factors through a homeomorphism of $X \times \widehat{\Gamma}/\sim$ *onto* $\mathrm{Prim}(C_0(X) \times_\tau \Gamma)$.

Remark 2.3. In Theorem 2.2, note that $\mathrm{Prim}(C_0(X) \times_\tau \Gamma)$ is then a second countable space. This is due to the fact that it is mentioned in [**12,** Remark 8.40], the quotient map

$$\mathsf{q} : X \times \widehat{\Gamma} \longrightarrow X \times \widehat{\Gamma}/\sim$$

is open. Moreover, X and $\widehat{\Gamma}$ both are second countable.

Theorem 2.2 can be applied to see that the primitive ideal space of the rational rotation algebra is homeomorphic to $\mathbb{T}^2$. The interested reader is referred to [**12,** Example 8.45] for the proof.

2.3. The primitive ideal space of crossed products by free actions. Let (A, Γ, α) be a classical dynamical system with Γ discrete. Then, the system gives an action of Γ on the spectrum $\widehat{A}$ of A by $s \cdot [\pi] := [\pi \circ \alpha_s^{-1}]$ for every $s \in \Gamma$ and $[\pi] \in \widehat{A}$, see [**11,** Lemma 7.1] and [**12,** Lemma 2.8]. This also induces an action of Γ on $\mathrm{Prim}\, A$ such that $s \cdot P := \alpha_s(P)$ for each $s \in \Gamma$ and $P \in \mathrm{Prim}\, A$.

Recall that, if π is a (nondegenerate) representation of A on H with $\ker \pi = J$, then $\mathrm{Ind}\, \pi$ denotes the induced representation $\widetilde{\pi} \times U$ of $A \times_\alpha \Gamma$ on $\ell^2(\Gamma, H)$ associated to the covariant pair $(\widetilde{\pi}, U)$ of (A, Γ, α) defined by

$$(\widetilde{\pi}(a)\xi)(s) = \pi(\alpha_s^{-1}(a))\xi(s) \quad \text{and} \quad (U_t\xi)(s) = \xi(t^{-1}s)$$

for every $a \in A$, $\xi \in \ell^2(\Gamma, H)$ and $s, t \in \Gamma$. Note that, by $\mathrm{Ind}\, J$, we mean $\ker(\mathrm{Ind}\, \pi)$.

Now, let (A, Γ, α) be a classical dynamical system in which A is separable and Γ is an abelian discrete countable group. If Γ acts on $\mathrm{Prim}\, A$ freely, then each primitive ideal $\ker \pi = P$ of A induces a primitive ideal of $A \times_\alpha \Gamma$, namely, $\operatorname{Ind} P = \ker(\operatorname{Ind}\pi)$, and the description of $\operatorname{Prim}(A \times_\alpha \Gamma)$ is completely available:

Theorem 2.4 ([**7,** Corollary 10.16])**.** *Suppose in the system (A, Γ, α) that A is separable and Γ is an amenable discrete countable group. If Γ acts on $\operatorname{Prim} A$ freely, then the map*

$$\begin{aligned}\mathcal{O}(\operatorname{Prim} A) &\longrightarrow \operatorname{Prim}(A \times_\alpha \Gamma)\\ \mathcal{O}(P) &\longmapsto \operatorname{Ind} P = \ker(\operatorname{Ind}\pi)\end{aligned}$$

is a homeomorphism, where π is an irreducible representation of A with $\ker \pi = P$. In particular, $A \times_\alpha \Gamma$ is simple if and only if every Γ-orbit is dense in $\operatorname{Prim} A$.

The above theorem may be applied to see that the irrational rotation algebras are simple. The interested reader may refer to [**7,** Example 10.18] or [**12,** Example 8.46] for more details.

3. The primitive ideal space of $A \times_\alpha^{\text{piso}} \mathbb{N}$ by automorphic action.

First, recall that, if T is the isometry in $B(\ell^2(\mathbb{N}))$ such that $T(e_n) = e_{n+1}$ on the usual orthonormal basis $\{e_n\}_{n=0}^\infty$ of $\ell^2(\mathbb{N})$, then we have

$$\mathcal{K}(\ell^2(\mathbb{N})) = \overline{\operatorname{span}}\{T_n(1 - TT^*)T_m^* : n, m \in \mathbb{N}\}.$$

Now, consider a system (A, α) consisting of a C^*-algebra A and an automorphism α of A. Let the triples $(A \times_\alpha^{\text{piso}} \mathbb{N}, j_A, v)$ and $(A \times_\alpha \mathbb{Z}, i_A, u)$ be the partial-isometric crossed product and the classical crossed product of the system, respectively. Here, our goal is to completely describe the primitive ideal space of $A \times_\alpha^{\text{piso}} \mathbb{N}$ and its topology. Observe [**4**] that the kernel of the natural homomorphism

$$q : (A \times_\alpha^{\text{piso}} \mathbb{N}, j_A, v) \longrightarrow (A \times_\alpha \mathbb{Z}, i_A, u),$$

given by $q(v_n^* j_A(a) v_m) = u_n^* i_A(a) u_m$, is isomorphic to the algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$. Therefore, we have a short exact sequence

$$0 \longrightarrow (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) \xrightarrow{\mu} A \times_\alpha^{\text{piso}} \mathbb{N} \xrightarrow{q} A \times_\alpha \mathbb{Z} \longrightarrow 0, \tag{3.1}$$

where $\mu(T_n(1 - TT^*)T_m^* \otimes a) = v_n^* j_A(a)(1 - v^*v)v_m$ for all $a \in A$ and $n, m \in \mathbb{N}$. Thus, $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$, as a set, is given by the sets $\mathrm{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ and $\mathrm{Prim}\left(A \times_\alpha \mathbb{Z}\right)$. With no conditions on the system, we do not have much information regarding $\mathrm{Prim}(A \times_\alpha \mathbb{Z})$ in general. However, from [**4,** Proposition 2.5], we do know that $\ker q \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is an essential ideal of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$. Therefore, $\mathrm{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$, which is homeomorphic to $\mathrm{Prim}\, A$, sits in $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$ as an open dense subset. We will identify this open dense subset, namely, the primitive ideals $\{\mathcal{I}_P : P \in \mathrm{Prim}\, A\}$ of $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$, derived from $\mathrm{Prim}\, A$, shortly. Moreover, see in [**4,** Section 5] that $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is a full corner of the classical crossed product $(B_\mathbb{Z} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z}$, where

$$B_\mathbb{Z} := \overline{\mathrm{span}}\{1_n : n \in \mathbb{Z}\} \subset \ell^\infty(\mathbb{Z}),$$

and the action β of $\mathbb{Z}$ on $B_\mathbb{Z}$ is given by translation such that $\beta_m(1_n) = 1_{n+m}$ for all $m, n \in \mathbb{Z}$. Thus, $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$ is homeomorphic to $\mathrm{Prim}((B_\mathbb{Z} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$, and hence, it suffices to describe $\mathrm{Prim}((B_\mathbb{Z} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$ and its topology. In order to do this, we consider two conditions on the system that enable us to apply a theorem of Williams and a result by Echterhoff. We shall also identify those primitive ideals of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ derived from $\mathrm{Prim}(A \times_\alpha \mathbb{Z})$, which form a closed subset of $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$. However, first, let us identify the primitive ideals $\mathcal{I}_P$.

Proposition 3.1. *Let $\pi : A \to B(H)$ be a nonzero irreducible representation of A with $P := \ker \pi$. If the pair (Π, V) is defined as in* [**8,** Example 4.6]*, see Section 2, then the associated representation of $(A \times_\alpha^{\mathrm{piso}} \mathbb{N}, j_A, v)$, denoted by $(\Pi \times V)_P$, is irreducible on $\ell^2(\mathbb{N}, H)$, and does not vanish on $\ker q \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$.*

Proof. In order to see that $(\Pi \times V)_P$ is irreducible, we show that every $\xi \in \ell^2(\mathbb{N}, H)\backslash\{0\}$ is a cyclic vector for $(\Pi \times V)_P$, that is,

$$\ell^2(\mathbb{N}, H) = \overline{\mathrm{span}}\{(\Pi \times V)_P(x)(\xi) : x \in (A \times_\alpha^{\mathrm{piso}} \mathbb{N})\}.$$

We show that

$$\mathcal{H} := \overline{\mathrm{span}}\{(\Pi \times V)_P(v_n^* j_A(a)(1 - v^*v)v_m)(\xi) : a \in A, n, m \in \mathbb{N}\} \tag{3.2}$$

equals $\ell^2(\mathbb{N}, H)$ which is enough. By viewing $\ell^2(\mathbb{N}, H)$ as the Hilbert space $\ell^2(\mathbb{N}) \otimes H$, it suffices to see that each $e_n \otimes h$ belongs to $\mathcal{H}$, where $\{e_n\}_{n=0}^\infty$ is the usual orthonormal basis of $\ell^2(\mathbb{N})$ and $h \in H$. Since $\xi \neq 0$

in $\ell^2(\mathbb{N}, H)$, there is an $m \in \mathbb{N}$ such that $\xi(m) \neq 0$ in H. However, $\xi(m)$ is a cyclic vector for the representation

$$\pi : A \longrightarrow B(H)$$

as π is irreducible. Thus, we have

$$\overline{\text{span}}\{\pi(a)(\xi(m)) : a \in A\} = H,$$

and hence,

$$\text{span}\{e_n \otimes \big(\pi(a)\xi(m)\big) : n \in \mathbb{N}, a \in A\}$$

is dense in

$$\ell^2(\mathbb{N}) \otimes H \simeq \ell^2(\mathbb{N}, H).$$

Therefore, we must only show that $\mathcal{H}$ contains each element $e_n \otimes (\pi(a)\xi(m))$. Straightforward calculation shows

$$\begin{aligned} e_n \otimes \big(\pi(a)\xi(m)\big) &= (V_n^*\Pi(a)(1 - V^*V)V_m)(\xi) \\ &= (\Pi \times V)_P(v_n^* j_A(a)(1 - v^*v)v_m)(\xi), \end{aligned}$$

and therefore, $e_n \otimes (\pi(a)\xi(m)) \in \mathcal{H}$ for every $a \in A$ and $n \in \mathbb{N}$. Thus, we have $\mathcal{H} = \ell^2(\mathbb{N}, H)$.

In order to show that $(\Pi \times V)_P$ does not vanish on $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, first note that, since π is nonzero, $\pi(a)h \neq 0$ for some $a \in A$, $h \in H$. Now, if we take

$$(1 - TT^*) \otimes a \in \mathcal{K}(\ell^2(\mathbb{N})) \otimes A,$$

then

$$(\Pi \times V)_P(\mu((1 - TT^*) \otimes a)) = (\Pi \times V)_P(j(a)(1 - v^*v)) \neq 0.$$

This is due to the fact that, for $(e_0 \otimes h) \in \ell^2(\mathbb{N}, H)$, we have

$$(\Pi \times V)_P(j_A(a)(1 - v^*v))(e_0 \otimes h) = \Pi(a)(1 - V^*V)(e_0 \otimes h) = e_0 \otimes \pi(a)h,$$

which is not zero in $\ell^2(\mathbb{N}, H)$ as $\pi(a)h \neq 0$. $\square$

Remark 3.2. The primitive ideals $\mathcal{I}_P$ are actually kernels of the irreducible representations $(\Pi \times V)_P$ which form the open dense subset

$$\mathcal{U} := \{\mathcal{I} \in \text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \not\subset \mathcal{I}\}$$

of $\text{Prim}\big(A \times_\alpha^{\text{piso}} \mathbb{N}\big)$ homeomorphic to $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$. Now, $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ itself is homeomorphic to $\text{Prim}\, A$ via the (Rieffel)

homeomorphism
$$P \longmapsto \mathcal{K}(\ell^2(\mathbb{N})) \otimes P.$$

However, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes P$ is the kernel of the irreducible representation $(\mathrm{id} \otimes \pi)$ of $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, where $(\mathrm{id} \otimes \pi)$ indeed equals the restriction $(\Pi \times V)_P|_{\mathcal{K}(\ell^2(\mathbb{N}))\otimes A}$. Therefore, we have

$$\begin{aligned}\mathcal{I}_P \cap (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) &= \ker((\Pi \times V)_P|_{\mathcal{K}(\ell^2(\mathbb{N}))\otimes A}) \\ &= \ker(\mathrm{id} \otimes \pi) = \mathcal{K}(\ell^2(\mathbb{N})) \otimes P.\end{aligned}$$

Consequently, the map $P \mapsto \mathcal{I}_P$ is a homeomorphism of $\mathrm{Prim}\, A$ onto the open dense subset $\mathcal{U}$ of $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$.

Now, we want to describe the topology of

$$\mathrm{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z}) \simeq \mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N}) \tag{3.3}$$

and identify the primitive ideals of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ derived from $A \times_\alpha \mathbb{Z}$ under the following two conditions:

(1) when A is separable and abelian, by applying a theorem of Williams, namely, Theorem 2.2;
(2) when A is separable and $\mathbb{Z}$ acts on $\mathrm{Prim}\, A$ freely, by applying Theorem 2.4.

3.1. The topology of $\mathrm{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$ **when** A **is separable and abelian.** Suppose that A is separable and abelian. Then, $(B_{\mathbb{Z}} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z}$ is isomorphic to the crossed product $C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_\tau \mathbb{Z}$ associated to the second countable locally compact transformation group $(\mathbb{Z}, \Omega(B_{\mathbb{Z}} \otimes A))$. Therefore, by Theorem 2.2, $\mathrm{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$ is homeomorphic to $\Omega(B_{\mathbb{Z}} \otimes A) \times \mathbb{T}/\sim$. However, we want to describe $\Omega(B_{\mathbb{Z}} \otimes A) \times \mathbb{T}/\sim$ precisely. In order to do so, we need to analyze $\Omega(B_{\mathbb{Z}} \otimes A)$, and, since $\Omega(B_{\mathbb{Z}} \otimes A) \simeq \Omega(B_{\mathbb{Z}}) \times \Omega(A)$, see [**11,** Theorem B.37] or [**11,** Theorem B.45], we must first compute $\Omega(B_{\mathbb{Z}})$.

Lemma 3.3. *Let*
$$\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$$
be the two-point compactification of $\mathbb{Z}$. *Then,* $\Omega(B_{\mathbb{Z}})$ *is homeomorphic to the open dense subset* $\mathbb{Z} \cup \{\infty\}$.

Proof. First, note that $B_{\mathbb{Z}}$ exactly consists of those functions

$$f : \mathbb{Z} \longrightarrow \mathbb{C}$$

such that $\lim_{n\to-\infty} f(n) = 0$ and $\lim_{n\to\infty} f(n)$ exists. Thus, the complex homomorphisms (irreducible representations) of $B_{\mathbb{Z}}$ are given by the evaluation maps $\{\varepsilon_n : n \in \mathbb{Z}\}$, and the map

$$\varepsilon_\infty : B_{\mathbb{Z}} \to \mathbb{C}$$

defined by $\varepsilon_\infty(f) := \lim_{n\to\infty} f(n)$ for all $f \in B_{\mathbb{Z}}$. Hence, we have $\Omega(B_{\mathbb{Z}}) = \{\varepsilon_n : n \in \mathbb{Z}\} \cup \{\varepsilon_\infty\}$. Note that the kernel of ε_∞ is the ideal

$$C_0(\mathbb{Z}) = \overline{\operatorname{span}}\{1_n - 1_m : n < m \in \mathbb{Z}\}$$

of $B_{\mathbb{Z}}$. Now, let $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ be the two-point compactification of $\mathbb{Z}$, which is homeomorphic to the subspace

$$X := \{-1\} \cup \{-1 + 1/(1-n) : n \in \mathbb{Z}, n < 0\} \\ \cup \{1 - 1/(1+n) : n \in \mathbb{Z}, n \geq 0\} \cup \{1\}$$

of $\mathbb{R}$. Then, the map

$$f \in B_{\mathbb{Z}} \longmapsto \widetilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}),$$

where

$$\widetilde{f}(r) := \begin{cases} \lim_{n\to\infty} f(n) & \text{if } r = \infty, \\ f(r) & \text{if } r \in \mathbb{Z}, \text{ and} \\ 0 & \text{if } r = -\infty, \end{cases}$$

embeds $B_{\mathbb{Z}}$ in $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})$ as the maximal ideal

$$I := \{\widetilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}) : \widetilde{f}(-\infty) = 0\}.$$

Thus, it follows that $\Omega(B_{\mathbb{Z}})$ is homeomorphic to $\widehat{I}$, and $\widehat{I}$ itself is homeomorphic to the open subset

$$\{\pi \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge : \pi|_I \neq 0\} = \{\widetilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\}$$

of $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge$ in which each $\widetilde{\varepsilon}_r$ is an evaluation map. Thus, by the homeomorphism between $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge$ and $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, the open subset $\{\widetilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\}$ is homeomorphic to the open (dense) subset $\mathbb{Z} \cup \{\infty\}$ of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ equipped with the relative topology. Therefore, $\Omega(B_{\mathbb{Z}})$ is in fact homeomorphic to $\mathbb{Z} \cup \{\infty\}$. It can easily be seen that $\mathbb{Z} \cup \{\infty\}$ is indeed a second countable locally compact Hausdorff space with

$$\mathcal{B} := \{\{n\} : n \in \mathbb{Z}\} \cup \{J_n : n \in \mathbb{Z}\}$$

as a countable basis for its topology, where $J_n := \{n, n+1, n+2, \ldots\} \cup \{\infty\}$ for every $n \in \mathbb{Z}$. $\square$

Remark 3.4. Before continuing, it needs to be mentioned that, if A is a separable C^*-algebra (not necessarily abelian), then, by **[11,** Theorem B.45] and using Lemma 3.3, $(C_0(\mathbb{Z}) \otimes A)\widehat{}$ and $(B_\mathbb{Z} \otimes A)\widehat{}$ are homeomorphic to $\mathbb{Z} \times \widehat{A}$ and $(\mathbb{Z} \cup \{\infty\}) \times \widehat{A}$, respectively. Also, $\mathrm{Prim}(C_0(\mathbb{Z}) \otimes A)$ and $\mathrm{Prim}(B_\mathbb{Z} \otimes A)$ are homeomorphic to $\mathbb{Z} \times \mathrm{Prim}\, A$ and $(\mathbb{Z} \cup \{\infty\}) \times \mathrm{Prim}\, A$, respectively (note that these homeomorphisms are $\mathbb{Z}$-equivariant for the action of $\mathbb{Z}$). Since $C_0(\mathbb{Z}) \otimes A$ is an (essential) ideal of $B_\mathbb{Z} \otimes A$, we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} \times \hat{A} & \longrightarrow & (C_0(\mathbb{Z}) \otimes A)\widehat{} & \xrightarrow{\Theta} & \mathrm{Prim}(C_0(\mathbb{Z}) \otimes A) & \longrightarrow & \mathbb{Z} \times \mathrm{Prim}\, A \\
\downarrow \mathrm{id} & & \downarrow \iota & & \downarrow \tilde{\iota} & & \downarrow \mathrm{id} \\
(\mathbb{Z} \cup \{\infty\}) \times \hat{A} & \longrightarrow & (B_\mathbb{Z} \otimes A)\widehat{} & \xrightarrow{\tilde{\Theta}} & \mathrm{Prim}(B_\mathbb{Z} \otimes A) & \longrightarrow & (\mathbb{Z} \cup \{\infty\}) \times \mathrm{Prim}\, A,
\end{array}$$

where Θ and $\widetilde{\Theta}$ are the canonical continuous, open surjections, and ι and $\widetilde{\iota}$ are the canonical embedding maps. Now, to see in what manner $\mathbb{Z}$ acts on $(\mathbb{Z} \cup \{\infty\}) \times \widehat{A}$ (and accordingly on $(\mathbb{Z} \cup \{\infty\}) \times \mathrm{Prim}\, A$), note that, since the crossed products $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z}$ and $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \mathrm{id}} \mathbb{Z}$ are isomorphic, see **[12,** Lemma 7.4], we have

$$n \cdot (m, [\pi]) = (m + n, [\pi])$$

and

$$n \cdot (\infty, [\pi]) = (n + \infty, n \cdot [\pi]) = (\infty, [\pi \circ \alpha_n])$$

for all $n, m \in \mathbb{Z}$ and $[\pi] \in \widehat{A}$. Accordingly,

$$n \cdot (m, P) = (m + n, P) \quad \text{and} \quad n \cdot (\infty, P) = (\infty, \alpha_n^{-1}(P))$$

for all $n, m \in \mathbb{Z}$ and $P \in \mathrm{Prim}\, A$.

Thus, when A is separable and abelian, using Lemma 3.3,

$$\Omega(B_{\mathbb{Z}} \otimes A) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A).$$

Now, in order to describe

$$((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T}/\sim,$$

note that, by Remark 3.4, $\mathbb{Z}$ acts on $(\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$ as follows:

$$n \cdot (m, \phi) = (m + n, \phi) \quad \text{and} \quad n \cdot (\infty, \phi) = (\infty, \phi \circ \alpha_n)$$

for all $n, m \in \mathbb{Z}$ and $\phi \in \Omega(A)$. Therefore, the stability group of each (m, ϕ) is $\{0\}$, and the stability group of each (∞, ϕ) equals the stability group $\mathbb{Z}_\phi$ of ϕ. Accordingly, the $\mathbb{Z}$-orbit of each (m, ϕ) is $\mathbb{Z} \times \{\phi\}$, and the $\mathbb{Z}$-orbit of (∞, ϕ) is $\{\infty\} \times \mathbb{Z} \cdot \phi$, where $\mathbb{Z} \cdot \phi$ is the $\mathbb{Z}$-orbit of ϕ. Thus, for the pairs (or triples) $((m, \phi), z)$ and $((n, \psi), w)$ of $(\mathbb{Z} \times \Omega(A)) \times \mathbb{T}$, we have

$$\begin{aligned}
((m, \phi), z) \sim ((n, \psi), w) &\iff \overline{\mathbb{Z} \cdot (m, \phi)} = \overline{\mathbb{Z} \cdot (n, \psi)} \\
&\iff \overline{\mathbb{Z} \times \{\phi\}} = \overline{\mathbb{Z} \times \{\psi\}} \\
&\iff \overline{\mathbb{Z}} \times \overline{\{\phi\}} = \overline{\mathbb{Z}} \times \overline{\{\psi\}} \\
&\iff (\mathbb{Z} \cup \{\infty\}) \times \overline{\{\phi\}} = (\mathbb{Z} \cup \{\infty\}) \times \overline{\{\psi\}} \\
&\iff (\mathbb{Z} \cup \{\infty\}) \times \{\phi\} = (\mathbb{Z} \cup \{\infty\}) \times \{\psi\}.
\end{aligned}$$

The last equivalence follows from the fact that $\Omega(A)$ is Hausdorff. Therefore, $((m, \phi), z)$ and $((n, \psi), w)$ are in the same equivalence class in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T}/\sim$ if and only if $\phi = \psi$, while $((m, \phi), z) \nsim ((\infty, \psi), w)$ for every $\psi \in \Omega(A)$ and $w \in \mathbb{T}$, since

$$\overline{\mathbb{Z} \cdot (\infty, \psi)} = \overline{\{\infty\} \times \mathbb{Z} \cdot \psi} = \overline{\{\infty\}} \times \overline{\mathbb{Z} \cdot \psi} = \{\infty\} \times \overline{\mathbb{Z} \cdot \psi}.$$

Thus, if $\phi \in \Omega(A)$, then all pairs $((m, \phi), z)$ for every $m \in \mathbb{Z}$ and $z \in \mathbb{T}$ are in the same equivalence class, which can be parameterized by $\phi \in \Omega(A)$. On the other hand, for the pairs $((\infty, \phi), z)$ and $((\infty, \psi), w)$, we have

$$((\infty, \phi), z) \sim ((\infty, \psi), w) \iff \overline{\mathbb{Z} \cdot (\infty, \phi)} = \overline{\mathbb{Z} \cdot (\infty, \psi)}$$

and

$$\gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\phi} \iff \{\infty\} \times \overline{\mathbb{Z} \cdot \phi} = \{\infty\} \times \overline{\mathbb{Z} \cdot \psi}$$

and

$$\gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\phi}.$$

Therefore,

$$((\infty,\phi),z) \sim ((\infty,\psi),w) \Longleftrightarrow \overline{\mathbb{Z}\cdot\phi} = \overline{\mathbb{Z}\cdot\psi} \quad \text{and} \quad \gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\phi},$$

which means that if and only if the pairs (ϕ, z) and (ψ, w) are in the same equivalence class in the quotient space is $\Omega(A) \times \mathbb{T}/\sim$ homeomorphic to $\mathrm{Prim}\,(A \times_\alpha \mathbb{Z})$. Therefore, $((\infty,\phi),z) \sim ((\infty,\psi),w)$ in $((\mathbb{Z}\cup\{\infty\}) \times \Omega(A)) \times \mathbb{T}/\sim$ precisely when $(\phi,z) \sim (\psi,w)$ in $\Omega(A)\times\mathbb{T}/\sim$, and hence, the class of each $((\infty,\phi),z)$ in $((\mathbb{Z}\cup\{\infty\}) \times \Omega(A))\times\mathbb{T}/\sim$ can be parameterized by the class of (ϕ,z) in $\Omega(A)\times\mathbb{T}/\sim$. Thus, we can identify $((\mathbb{Z}\cup\{\infty\})\times\Omega(A))\times\mathbb{T}/\sim$ with the disjoint union

$$\Omega(A) \sqcup (\Omega(A)\times\mathbb{T}/\sim).$$

Now, we have:

Theorem 3.5. *Let (A,α) be a system consisting of a separable abelian C^*-algebra A and an automorphism α of A. Then,* $\mathrm{Prim}(A \times_\alpha^{\mathrm{piso}} \mathbb{N})$ *is homeomorphic to $\Omega(A)\sqcup(\Omega(A)\times\mathbb{T}/\sim)$, equipped with the (quotient) topology in which the open sets are of the form*

$$\{U \subset \Omega(A) : U \text{ is open in } \Omega(A)\}$$
$$\cup\,\{U\cup W : U \text{ is a nonempty open subset of } \Omega(A), \text{ and } W \text{ is open in } (\Omega(A)\times\mathbb{T}/\sim)\}.$$

Proof. Since the quotient map

$$\mathsf{q} : ((\mathbb{Z}\cup\{\infty\})\times\Omega(A))\times\mathbb{T} \longrightarrow \Omega(A)\sqcup(\Omega(A)\times\mathbb{T}/\sim)$$

is open, as well as $\widetilde{\mathsf{q}} : \Omega(A)\times\mathbb{T} \to \Omega(A)\times\mathbb{T}/\sim$, for every $n\in\mathbb{Z}$, every open subset O of $\Omega(A)$, and every open subset V of $\mathbb{T}$, the forward image of open subsets $\{n\}\times O\times V$ and $J_n\times O\times V$ by q, forms a basis for the topology of $\Omega(A)\sqcup(\Omega(A)\times\mathbb{T}/\sim)$, which is

$$\{O \subset \Omega(A) : O \text{ is open in } \Omega(A)\}$$
$$\cup\,\{O\cup\widetilde{\mathsf{q}}(O\times V) : O \text{ is a nonempty open subset of } \Omega(A), \text{ and } V \text{ is open in } \mathbb{T}\}.$$

As the open subsets $\widetilde{\mathsf{q}}(O \times V)$ also form a basis for the quotient topology of $\Omega(A) \times \mathbb{T}/\sim$, we can see that each open subset of

$$\Omega(A) \sqcup (\Omega(A) \times \mathbb{T}/\sim)$$

is either an open subset U of $\Omega(A)$ or of the form $U \cup W$ for some nonempty open subset U in $\Omega(A)$ and some open subset W in $\Omega(A) \times \mathbb{T}/\sim$. $\square$

Remark 3.6. Under the condition of Theorem 3.5, the primitive ideals of $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N})$ derived from $\mathrm{Prim}(A \times_{\alpha} \mathbb{Z})$, which form the closed subset

$$\mathcal{F} := \{\mathcal{J} \in \mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subset \mathcal{J}\},$$

are the kernels of the irreducible representations $(\mathrm{Ind}_{\mathbb{Z}_\phi}^{\mathbb{Z}}(\phi \times \gamma_z|_{\mathbb{Z}_\phi})) \circ q$ corresponding to the equivalence classes of the pairs (ϕ, z) in $\Omega(A) \times \mathbb{T}/\sim$ (again, by using Theorem 2.2). Therefore, if $\mathcal{J}_{[(\phi,z)]}$ denotes $\ker(\mathrm{Ind}_{\mathbb{Z}_\phi}^{\mathbb{Z}}(\phi \times \gamma_z|_{\mathbb{Z}_\phi}) \circ q)$, then

$$\mathcal{F} = \{\mathcal{J}_{[(\phi,z)]} : \phi \in \Omega(A), z \in \mathbb{T}\},$$

and the map

$$[(\phi, z)] \longmapsto \mathcal{J}_{[(\phi,z)]}$$

is a homeomorphism of $\mathrm{Prim}(A \times_{\alpha} \mathbb{Z}) \simeq \Omega(A) \times \mathbb{T}/\sim$ onto $\mathcal{F}$.

Proposition 3.7. *Let (A, α) be a system consisting of a separable abelian C^*-algebra A and an automorphism α of A. Then, $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is* GCR *if and only if $\mathbb{Z}\backslash\Omega(A)$ is a T_0 space.*

Proof. From [**9,** Theorem 5.6.2], $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is GCR if and only if

$$\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q$$

and

$$A \times_{\alpha} \mathbb{Z} \simeq C_0(\Omega(A)) \times_{\tau} \mathbb{Z}$$

are GCR. However, since A is abelian, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is automatically CCR, and hence, it is GCR. Therefore $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is GCR precisely when $A \times_{\alpha} \mathbb{Z}$ is GCR. From [**12,** Theorem 8.43], $A \times_{\alpha} \mathbb{Z}$ is GCR if and only if $\mathbb{Z}\backslash\Omega(A)$ is T_0. $\square$

Proposition 3.8. *Let* (A, α) *be a system consisting of a separable abelian* C^**-algebra* A *and an automorphism* α *of* A*. Then,* $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ *is not* CCR.

Proof. Note that $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is CCR if and only if

$$(B_{\mathbb{Z}} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z} \simeq C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_{\tau} \mathbb{Z}$$

is CCR since they are Morita equivalent (see [**12,** Proposition I.43]). Since, for the $\mathbb{Z}$-orbit of a pair (m, ϕ), we have

$$\overline{\mathbb{Z} \cdot (m, \phi)} = \overline{\mathbb{Z} \times \{\phi\}} = \overline{\mathbb{Z}} \times \overline{\{\phi\}} = (\mathbb{Z} \cup \{\infty\}) \times \{\phi\},$$

it follows that the $\mathbb{Z}$-orbit of (m, ϕ) is not closed in $\Omega(B_{\mathbb{Z}} \otimes A) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$. Therefore, by [**12,** Theorem 8.44], $C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_{\tau} \mathbb{Z}$ is not CCR, and hence, $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ is not CCR. □

Example 3.9 (Pimsner-Voiculescu Toeplitz algebra)**.** Suppose that $\mathcal{T}(A, \alpha)$ is the Pimsner-Voiculescu Toeplitz algebra associated to the system (A, α) (see [**10**]). It was shown [**4,** Section 5] that $\mathcal{T}(A, \alpha)$ is isomorphic to the partial-isometric crossed product $A \times_{\alpha^{-1}}^{\mathrm{piso}} \mathbb{N}$ associated to the system (A, α^{-1}). Therefore, when A is abelian and separable, the description of $\mathrm{Prim}(\mathcal{T}(A, \alpha))$ completely follows from Theorem 3.5. In particular, for the trivial system $(\mathbb{C}, \mathrm{id})$, $\mathcal{T}(\mathbb{C}, \mathrm{id})$ is the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$ of integers isomorphic to $\mathbb{C} \times_{\mathrm{id}}^{\mathrm{piso}} \mathbb{N}$. Thus, again from Theorem 3.5, $\mathrm{Prim}(\mathcal{T}(\mathbb{Z}))$ corresponds to the disjoint union $\{0\} \sqcup \mathbb{T}$ in which every (nonempty) open set is of the form $\{0\} \cup W$ for some open subset W of $\mathbb{T}$. This description is known to coincide with the description of $\mathrm{Prim}(\mathcal{T}(\mathbb{Z}))$ obtained from the well-known short exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T}(\mathbb{Z}) \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

Example 3.10. Consider the system $(C(\mathbb{T}), \alpha)$ in which the action α is given by rotation through the angle $2\pi\theta$ with θ rational. By using the discussion in [**12,** Example 8.46], $\mathrm{Prim}(C(\mathbb{T}) \times_{\alpha}^{\mathrm{piso}} \mathbb{N})$ can be identified with the disjoint union

$$\mathbb{T} \sqcup \mathbb{T}^2,$$

in which, by Theorem 3.5, each open set is given by

$$\{U \subset \mathbb{T} : U \text{ is open in } \mathbb{T}\} \\ \cup \{U \cup W : U \text{ is a nonempty open subset of } \mathbb{T}, \\ \text{and } W \text{ is open in } \mathbb{T}^2\}.$$

Moreover, the orbit space $\mathbb{Z}\backslash\mathbb{T}$ is homeomorphic to $\mathbb{T}$, which is obviously T_0 (in fact, Hausdorff). Thus, it follows from Proposition 3.7 that $C(\mathbb{T}) \times_\alpha^{\text{piso}} \mathbb{N}$ is GCR.

3.2. The topology of $\operatorname{Prim}((B_\mathbb{Z} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$ **when** A **is separable and** $\mathbb{Z}$ **acts on** $\operatorname{Prim} A$ **freely.** Consider a system (A, α) in which A is separable and $\mathbb{Z}$ acts freely on $\operatorname{Prim} A$. It follows that $\mathbb{Z}$ acts freely on $\operatorname{Prim}(B_\mathbb{Z} \otimes A)$, too. This is due to the facts that, firstly, by **[11,** Theorem B.45], $\operatorname{Prim}(B_\mathbb{Z} \otimes A)$ is homeomorphic to $\operatorname{Prim} B_\mathbb{Z} \times \operatorname{Prim} A$, and hence, it is homeomorphic to $(\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A$. Then, $\mathbb{Z}$ acts on

$$(\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A$$

such that

$$n \cdot (m, P) = (m + n, P) \quad \text{and} \quad n \cdot (\infty, P) = (\infty, \alpha_n^{-1}(P))$$

for all $n, m \in \mathbb{Z}$ and $P \in \operatorname{Prim} A$. Therefore, the stability group of each (∞, P) equals the stability group $\mathbb{Z}_P$ of P, which is $\{0\}$ as $\mathbb{Z}$ acts freely on $\operatorname{Prim} A$, and the stability group of each (m, P) is clearly $\{0\}$. Thus, in the separable system $(B_\mathbb{Z} \otimes A, \mathbb{Z}, \beta \otimes \alpha^{-1})$ (with $\mathbb{Z}$ abelian), $\mathbb{Z}$ freely acts on

$$\operatorname{Prim}(B_\mathbb{Z} \otimes A) \simeq (\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A.$$

Therefore, by Theorem 2.4, $\operatorname{Prim}((B_\mathbb{Z} \otimes A) \times_{\beta\otimes\alpha^{-1}} \mathbb{Z})$ is homeomorphic to the quasi-orbit space

$$\mathcal{O}(\operatorname{Prim}(B_\mathbb{Z} \otimes A)) = \mathcal{O}((\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A),$$

which describes $\operatorname{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ as well. We want to precisely describe the quotient topology of $\mathcal{O}((\mathbb{Z} \cup \{\infty\}) \times \operatorname{Prim} A)$ and to identify the primitive ideals of $A \times_\alpha^{\text{piso}} \mathbb{N}$ derived from $\operatorname{Prim}(A \times_\alpha \mathbb{Z})$. We have

$$\begin{aligned} \mathcal{O}(m, P) = \mathcal{O}(n, Q) &\Longleftrightarrow \overline{\mathbb{Z} \cdot (m, P)} = \overline{\mathbb{Z} \cdot (n, Q)} \\ &\Longleftrightarrow \overline{\mathbb{Z} \times \{P\}} = \overline{\mathbb{Z} \times \{Q\}} \\ &\Longleftrightarrow \overline{\mathbb{Z}} \times \overline{\{P\}} = \overline{\mathbb{Z}} \times \overline{\{Q\}} \end{aligned}$$

$$\begin{aligned}\iff (\mathbb{Z}\cup\{\infty\})\times\overline{\{P\}}\\ = (\mathbb{Z}\cup\{\infty\})\times\overline{\{Q\}}.\end{aligned}$$

Therefore, $\mathcal{O}(m,P)=\mathcal{O}(n,Q)$ if and only if $\overline{\{P\}}=\overline{\{Q\}}$, and this occurs precisely when $P=Q$ by the definition of the hull-kernel (Jacobson) topology on $\operatorname{Prim} A$ (which is why the primitive ideal space of any C^*-algebra is always T_0 [**9,** Theorem 5.4.7]). Hence, all pairs (m,P) for every $m\in\mathbb{Z}$ have the same quasi-orbit which can be parameterized by $P\in\operatorname{Prim} A$, and, since

$$\overline{\mathbb{Z}\cdot(\infty,Q)}=\overline{\{\infty\}\times\mathbb{Z}\cdot Q}=\overline{\{\infty\}}\times\overline{\mathbb{Z}\cdot Q}=\{\infty\}\times\overline{\mathbb{Z}\cdot Q},$$

$\mathcal{O}(m,P)\neq\mathcal{O}(\infty,Q)$ for all $m\in\mathbb{Z}$ and $P,Q\in\operatorname{Prim} A$. Moreover,

$$\begin{aligned}\mathcal{O}(\infty,P)=\mathcal{O}(\infty,Q) &\iff \overline{\mathbb{Z}\cdot(\infty,P)}=\overline{\mathbb{Z}\cdot(\infty,Q)}\\ &\iff \{\infty\}\times\overline{\mathbb{Z}\cdot P}=\{\infty\}\times\overline{\mathbb{Z}\cdot Q}.\end{aligned}$$

Thus, $\mathcal{O}(\infty,P)=\mathcal{O}(\infty,Q)$ if and only if $\overline{\mathbb{Z}\cdot P}=\overline{\mathbb{Z}\cdot Q}$, which means if and only if P and Q have the same quasi-orbit ($\mathcal{O}(P)=\mathcal{O}(Q)$) in

$$\mathcal{O}(\operatorname{Prim} A)\simeq\operatorname{Prim}(A\times_\alpha\mathbb{Z}).$$

Hence, each quasi-orbit $\mathcal{O}(\infty,P)$ can be parameterized by the quasi-orbit $\mathcal{O}(P)$ in $\mathcal{O}(\operatorname{Prim} A)$, and we can therefore identify $\mathcal{O}((\mathbb{Z}\cup\{\infty\})\times\operatorname{Prim} A)$ by the disjoint union

$$\operatorname{Prim} A\sqcup\mathcal{O}(\operatorname{Prim} A).$$

Then, we have:

Theorem 3.11. *Let (A,α) be a system consisting of a separable C^*-algebra A and an automorphism α of A. Suppose that $\mathbb{Z}$ freely acts on* $\operatorname{Prim} A$. *Then,* $\operatorname{Prim}(A\times_\alpha^{\mathrm{piso}}\mathbb{N})$ *is homeomorphic to* $\operatorname{Prim} A\sqcup\mathcal{O}(\operatorname{Prim} A)$, *equipped with the (quotient) topology in which the open sets are of the form*

$$\begin{aligned}&\{U\subset\operatorname{Prim} A : U \text{ is open in } \operatorname{Prim} A\}\\ &\quad\cup\{U\cup W : U \text{ is a nonempty open subset of } \operatorname{Prim} A,\\ &\qquad\qquad \text{and } W \text{ is open in } \mathcal{O}(\operatorname{Prim} A)\}.\end{aligned}$$

Proof. Note that since, from [**12,** Lemma 6.12], the quasi-orbit map

$$\mathsf{q} : \mathrm{Prim}(B_{\mathbb{Z}} \otimes A) \longrightarrow \mathcal{O}(\mathrm{Prim}(B_{\mathbb{Z}} \otimes A))$$

is continuous and open, the proof follows from a similar argument to the proof of Theorem 3.5. Thus, we skip it here. □

Remark 3.12. Under the condition of Theorem 3.11, we want to identify the primitive ideals of $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N})$ derived from $\mathrm{Prim}(A \times_{\alpha} \mathbb{Z})$, which form the closed subset

$$\mathcal{F} := \{\mathcal{J} \in \mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subset \mathcal{J}\}$$

homeomorphic to $\mathrm{Prim}(A \times_{\alpha} \mathbb{Z}) \simeq \mathcal{O}(\mathrm{Prim}\, A)$ (see Theorem 2.4). These ideals are actually the kernels of the irreducible representations

$$(\mathrm{Ind}\, \pi) \circ q = (\widetilde{\pi} \times U) \circ q$$

of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$, where π is an irreducible representation of A with $\ker \pi = P$. However, since the pair $(\widetilde{\pi}, U)$ is clearly a covariant partial-isometric representation of (A, α), we can see that, in fact, $(\mathrm{Ind}\, \pi) \circ q = \widetilde{\pi} \times^{\mathrm{piso}} U$, where $\widetilde{\pi} \times^{\mathrm{piso}} U$ is the associated representation of $A \times_{\alpha}^{\mathrm{piso}} \mathbb{N}$ corresponding to the pair $(\widetilde{\pi}, U)$. Thus, each element of $\mathcal{F}$ is of the form $\ker(\widetilde{\pi} \times^{\mathrm{piso}} U)$ corresponding to the quasi-orbit $\mathcal{O}(P)$, and therefore, we denote $\ker(\widetilde{\pi} \times^{\mathrm{piso}} U)$ by $\mathcal{J}_{\mathcal{O}(P)}$. Thus, the map

$$\mathcal{O}(P) \longmapsto \mathcal{J}_{\mathcal{O}(P)}$$

is a homeomorphism of $\mathcal{O}(\mathrm{Prim}\, A)$ onto the closed subspace $\mathcal{F}$ of $\mathrm{Prim}(A \times_{\alpha}^{\mathrm{piso}} \mathbb{N})$.

For the next remark, we need to recall that the primitive ideal space of any C^*-algebra A is locally compact [**6,** Corollary 3.3.8]. A locally compact space X (not necessarily Hausdorff) is called *almost Hausdorff* if each locally compact subspace U contains a relatively open nonempty Hausdorff subset (see [**12,** Definition 6.1.]). If a C^*-algebra is GCR, then it is almost Hausdorff (see the discussion in [**12,** pages 171, 172]). Finally if A is separable, then, by applying [**11,** Theorem A.38 and Proposition A.46], it follows that $\mathrm{Prim}\, A$ is second countable.

Remark 3.13. It follows from [**13**] that, if $(A, \mathbb{Z}, \alpha)$ is a separable system in which $\mathbb{Z}$ acts on $\widehat{A}$ freely, then $A \times_{\alpha} \mathbb{Z}$ is GCR if and only if

A is GCR and every $\mathbb{Z}$-orbit in $\widehat{A}$ is discrete. However, every $\mathbb{Z}$-orbit in $\widehat{A}$ is discrete if and only if, for each $[\pi] \in \widehat{A}$, the map $\mathbb{Z} \to \mathbb{Z} \cdot [\pi]$ defined by

$$n \longmapsto n \cdot [\pi] = [\pi \circ \alpha_n^{-1}]$$

is a homeomorphism, and this statement itself, by **[12,** Theorem 6.2 (Mackey-Glimm Dichotomy)], is equivalent to stating that the orbit space $\mathbb{Z}\backslash\widehat{A}$ is T_0. Therefore, we can rephrase the statement of **[13]** to state that, if $(A, \mathbb{Z}, \alpha)$ is a separable system in which $\mathbb{Z}$ acts on $\widehat{A}$ freely, then $A \times_\alpha \mathbb{Z}$ is GCR if and only if A is GCR and the orbit space $\mathbb{Z}\backslash\widehat{A}$ is T_0.

Proposition 3.14. *Let (A, α) be a system consisting of a separable C^*-algebra A and an automorphism α of A. Suppose that $\mathbb{Z}$ freely acts on $\widehat{A}$. Then, $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is* GCR *if and only if A is* GCR *and the orbit space $\mathbb{Z}\backslash\widehat{A}$ is T_0.*

Proof. The proof follows from a similar argument to the proof of Proposition 3.7 and Remark 3.13. □

Example 3.15. Consider the system $(C(\mathbb{T}), \alpha)$ in which the action α is given by rotation through the angle $2\pi\theta$ with θ irrational. Then, $\mathbb{Z}$ freely acts on $\mathrm{Prim}(C(\mathbb{T})) = C(\mathbb{T})\widehat{\ } = \mathbb{T}$ (see **[7,** Example 10.18] or **[12,** Example 8.45]). Therefore, from Theorem 3.11, $\mathrm{Prim}(C(\mathbb{T}) \times_\alpha^{\mathrm{piso}} \mathbb{N})$ can be identified with the disjoint union $\mathbb{T} \sqcup \mathcal{O}(\mathbb{T})$. However, the quasi-orbit space $\mathcal{O}(\mathbb{T})$ contains only one point as each $\mathbb{Z}$-orbit is dense in $\mathbb{T}$ (see **[12,** Lemma 3.29]). We parameterize this one point by 0 (note that $\mathcal{O}(\mathbb{T})$ is homeomorphic to the primitive ideal space of the irrational rotation algebra $A_\theta := C(\mathbb{T}) \times_\alpha \mathbb{Z}$, which is simple). Thus, $\mathrm{Prim}(C(\mathbb{T}) \times_\alpha^{\mathrm{piso}} \mathbb{N})$ is actually identified with

$$\mathbb{T} \sqcup \{0\},$$

where each open set is given by

$$\{U \subset \mathbb{T} : U \text{ is open in } \mathbb{T}\} \cup \{U \cup \{0\} : U \text{ is a nonempty open subset of } \mathbb{T}\}.$$

Here, we would like to mention that 0 in $\mathbb{T} \sqcup \{0\}$ corresponds to the primitive ideal $\mathcal{K}(\ell^2(\mathbb{N})) \otimes C(\mathbb{T})$ of $C(\mathbb{T}) \times_\alpha^{\mathrm{piso}} \mathbb{N}$. Finally, although $C(\mathbb{T})$ is GCR (in fact CCR), the orbit space $\mathbb{Z}\backslash\mathbb{T}$ is not T_0 as each

$\mathbb{Z}$-orbit is dense in $\mathbb{T}$. Therefore, it follows from Proposition 3.14 that $C(\mathbb{T}) \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is not GCR.

Remark 3.16. Recall that, since the Pimsner-Voiculescu Toeplitz algebra $\mathcal{T}(A, \alpha)$ is isomorphic to $A \times_{\alpha^{-1}}^{\mathrm{piso}} \mathbb{N}$ (see Example 3.9), if A is separable and $\mathbb{Z}$ freely acts on $\operatorname{Prim} A$, then the description of $\operatorname{Prim}(\mathcal{T}(A, \alpha))$ is obtained completely from Theorem 3.11.

4. Primitivity and simplicity of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$. In this section, we discuss the primitivity and simplicity of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$. Recall that a C^*-algebra is called *primitive* if it has a faithful nonzero irreducible representation, and it is called *simple* if it has no nontrivial ideal.

Theorem 4.1. *Let (A, α) be a system consisting of a C^*-algebra A and an automorphism α of A. Then, $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is primitive if and only A is primitive.*

Proof. If $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ is primitive, it has a faithful nonzero irreducible representation

$$\rho : A \times_\alpha^{\mathrm{piso}} \mathbb{N} \longrightarrow B(\mathcal{H}).$$

Then, since the restriction of ρ to the ideal $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q$ is nonzero, it gives an irreducible representation of $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ which is clearly faithful. Thus, it follows that $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is primitive, and therefore, A must be primitive as well.

Conversely, if A is primitive, then it has a faithful nonzero irreducible representation π on some Hilbert space H ($P = \ker \pi = \{0\}$). We show that the associated irreducible representation $(\Pi \times V)_P$ of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$ on $\ell^2(\mathbb{N}, H)$ is faithful. From [**8,** Theorem 4.8], it is sufficient to see that, if $\Pi(a)(1 - V^*V) = 0$, then $a = 0$. If $\Pi(a)(1 - V^*V) = 0$, then

$$\Pi(a)(1 - V^*V)(e_0 \otimes h) = (e_0 \otimes \pi(a)h) = 0 \quad \text{for all } h \in H.$$

It follows that $\pi(a)h = 0$ for all $h \in H$, and therefore, $\pi(a) = 0$. Since π is faithful, we must have $a = 0$. This completes the proof. $\square$

Remark 4.2. Note that Theorem 4.1 simply means that, in the homeomorphism $P \mapsto \mathcal{I}_P$ mentioned in Remark 3.2, P is the zero ideal if and only if $\mathcal{I}_P$ is the zero ideal. This is due to the fact that, if

$A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is primitive, then its zero ideal as one of its primitive ideals is of the form $\mathcal{I}_P$ (derived from $\operatorname{Prim} A$), as $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \neq 0$.

Finally it is easy to see that $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is not simple. This is due to the fact that, as we see, it contains $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ as a nonzero ideal. Moreover, if

$$\mathcal{K}(\ell^2(\mathbb{N})) \otimes A = A \times_{\alpha}^{\text{piso}} \mathbb{N},$$

then

$$A \times_{\alpha} \mathbb{Z} \simeq (A \times_{\alpha}^{\text{piso}} \mathbb{N})/(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$$

must be the zero algebra. Thus, it follows that $A = 0$, which is a contradiction as we have $A \neq 0$. Therefore, $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ contains $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ as a proper nonzero ideal, and hence, we have proved the following:

Theorem 4.3. *Let (A, α) be a system consisting of a C^*-algebra A and an automorphism α of A. Then $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is not simple.*

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