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A CONSTRUCTION OF A BSE-ALGEBRA OF TYPE I WHICH IS ISOMORPHIC TO NO C*-ALGEBRAS

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ABSTRACT. In this note, we construct a BSE-algebra of type I which is isomorphic to no C*-algebras. This affirmatively solves the problem posed by Takahasi and Hatori [**5**], "is there a BSE-algebra of type I isomorphic to no C*-algebras?"

1. Introduction and main result. Let A be a semisimple commutative Banach algebra with Gelfand space Φ_A. Let $C^b(\Phi_A)$ be the Banach algebra of all bounded continuous complex-valued functions on Φ_A with sup-norm $\| \ \|_\infty$. A function $\sigma \in C^b(\Phi_A)$ is said to be a BSE-function if there is a constant $\beta > 0$ such that, for any finite number of elements $\varphi_1, \ldots, \varphi_n$ in Φ_A, and for the same number of $c_1, \ldots, c_n \in \mathbb{C}$, the inequality

$$\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| \leq \beta \left\| \sum_{i=1}^{n} c_i \varphi_i \right\|_{A^*}$$

holds, where A^* is the dual space of A. The BSE-norm of σ, $\|\sigma\|_{\mathrm{BSE}}$, is the infimum of all such β. Let $C_{\mathrm{BSE}}(\Phi_A)$ be the set of all BSE-functions. Then, it becomes a semisimple commutative Banach algebra under the BSE-norm. The algebra A is said to be a BSE-*algebra* if the BSE-functions on Φ_A are precisely the Gelfand transforms of the elements of the multiplier algebra $M(A)$ of A. In addition, a BSE-algebra A is said to be of *type* I if the functions in $C^b(\Phi_A)$ are precisely the Gelfand transforms of the elements of $M(A)$ (see [**5,** pages 151, 152]).

Hatori and the second author showed that any BSE-algebra of type I with a bounded approximate identity is isomorphic to a commutative

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C*-algebra, and vice versa (see [**5,** Theorem 3]), and they proposed the following problem: is there a BSE-algebra of type I which is isomorphic to no C*-algebras? (see [**5,** Problem 1]). We solve this problem affirmatively in this note by constructing a concrete example of one.

In the following, G denotes a noncompact non-discrete LCA group with Harr measure dx, $C_0(G)$ the commutative C*-algebra of all continuous complex-valued functions on G vanishing at infinity and $L^p(G)$ the L^p-space on G, where $1 \leq p < \infty$. Then $C_{0,p}(G) := C_0(G) \cap L^p(G)$ becomes a semisimple commutative Banach algebra with l^1-norm:

$$\|f\|_{\infty,p} = \|f\|_\infty + \|f\|_p, \quad f \in C_{0,p}(G).$$

In [**2**], the authors extended the notion of Reiter's Segal algebra in commutative group algebras to the notion of Segal algebra in commutative semisimple Banach algebras $(A, \| \ \|_A)$, which satisfy the following conditions:

(α) A is regular.

(β) There exists a bounded approximate identity of A composed of elements in A_c consisting of all $x \in A$ such that $\widehat{x}$ has compact support, where $\widehat{x}$ denotes the Gelfand transform of $x \in A$.

When A is equal to a group algebra $L^1(G)$ of an LCA group G, Segal algebras in A coincide with Reiter's Segal algebras in $L^1(G)$. For Reiter's Segal algebras, the reader is referred to [**3, 4**].

Definition 1.1. A subspace S in A is called a *Segal algebra* in A if the following conditions are satisfied:

(a) S is a dense ideal in A.

(b) S itself constitutes a Banach space under a norm $\| \ \|_S$ satisfying $\|x\|_A \leq \|x\|_S$, $x \in S$.

(c) $\|ax\|_S \leq \|a\|_A \|x\|_S$, $a \in A$, $x \in S$.

(d) S has approximate units.

Note that $(C_0(G), \| \ \|_\infty)$ is a commutative Banach algebra satisfying conditions (α) and (β). The main result of this note is the following.

Theorem 1.2. *There exists a* BSE-*algebra of type* I *which is isomorphic to no* C*-*algebras.*

2. Lemmas.

Lemma 2.1. *Let*

$$C_{0,p}(G) = C_0(G) \cap L^p(G)$$

be a subspace of $C_0(G)$ *with norm*

$$\|f\|_{\infty,p} := \|f\|_\infty + \|f\|_p, \quad f \in C_0(G).$$

Then, $(C_{0,p}(G), \|\ \|_{\infty,p})$ *is a proper* (*meaning strictly contained*) *Segal algebra in* $C_0(G)$.

Proof. We need only prove that $C_{0,p}(G)$ satisfies conditions (a)–(d) in Definition 1.1. We can easily see that $C_{0,p}(G)$ satisfies conditions (a) and (b). Since

$$\begin{aligned}\|fg\|_{\infty,p} &= \|fg\|_\infty + \|fg\|_p \le \|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g\|_p \\ &= \|f\|_\infty(\|g\|_\infty + \|g\|_p) = \|f\|_\infty\|g\|_{\infty,p}\end{aligned}$$

for all $f \in C_0(G)$ and $g \in C_{0,p}(G)$, it follows that $C_{0,p}(G)$ satisfies condition (c).

If $g \in C_{0,p}(G)$ and $\varepsilon > 0$, we can choose a continuous complex-valued function f on G with compact support such that $\|g - fg\|_\infty < \varepsilon/2$ and $\|g - fg\|_p < \varepsilon/2$. Then,

$$\|g - fg\|_{\infty,p} = \|g - fg\|_\infty + \|g - fg\|_p < \varepsilon.$$

This shows that $C_{0,p}(G)$ satisfies condition (d). It is also easy to see that $C_{0,p}(G)$ is proper in $C_0(G)$ since $C_0(G)$ is not contained in $L^p(G)$ by non-compactness of G. $\square$

Lemma 2.2. *The Segal algebra* $C_{0,p}(G)$ *is a* BSE-*algebra of type* I.

Proof. It is well known that the Gelfand space of $C_0(G)$ can be identified with G, and the Gelfand transform of elements in $C_0(G)$ is the identity mapping. Since $C_{0,p}(G)$ is a Segal algebra in $C_0(G)$ from Lemma 2.1, it follows from [2, Theorem B$'$-(ii)] that the Gelfand space of $C_{0,p}(G)$ can also be identified with G, and the Gelfand transform of elements of $C_{0,p}(G)$ is again the identity mapping. Therefore, it is trivial that the multiplier algebra $M(C_{0,p}(G))$ is contained in $C^b(G)$. On the other hand, it is easy to see that $fg \in C_0(G) \cap L^p(G) = C_{0,p}(G)$ whenever $f \in C^b(G)$ and $g \in C_{0,p}(G)$, that is, $M(C_{0,p}(G)) = C^b(G)$ holds.

Next, we observe that $C_{0,p}(G)$ has a bounded weak approximate identity in the sense of Jones-Lahr. In fact, let Λ be the net consisting of all finite subsets of G. For any $\lambda = \{x_1, \dots, x_n\} \in \Lambda$, choose $e_\lambda \in C_{0,p}(G)$ such that $e_\lambda(x_i) = 1$ $(i = 1, \dots, n)$ and $\|e_\lambda\|_\infty = \|e_\lambda\|_p = 1$. Then, $\{e_\lambda : \lambda \in \Lambda\}$ is a weak approximate identity bounded by two in the sense of Jones-Lahr. Therefore, from [2, Theorem 9.10], the Segal algebra $C_{0,p}(G)$ is BSE, that is, $M(C_{0,p}(G)) = C_{\mathrm{BSE}}(\Phi_{C_{0,p}(G)})$ holds. Thus, we have proved that $C_{0,p}(G)$ is a BSE-algebra of type I. □

Remark 2.3. We can show that $C^b(G) = C_{\mathrm{BSE}}(\Phi_{C_{0,p}(G)})$ by a concrete calculation. In fact, take $\sigma \in C^b(G)$ arbitrarily, and let $\{e_\lambda : \lambda \in \Lambda\}$ be as in the proof of Lemma 2.2, which is a weak approximate identity of $C_{0,p}(G)$ bounded by two in the sense of Jones-Lahr. For each $t \in G$ and $f \in C_0(G)$, set $\varphi_t(f) = f(t)$. Then, we have

$$\varphi_t|_{C_{0,p}(G)} \in \Phi_{C_{0,p}(G)} \quad \text{for each } t \in G$$

since $C_{0,p}(G)$ is dense in $C_0(G)$. Let $c_1, \dots, c_n \in \mathbb{C}, t_1, \dots, t_n \in G$ and $\lambda_0 = \{t_1, \dots, t_n\} \in \Lambda$. Then, we have

$$\begin{aligned}
\left|\sum_{i=1}^n c_i \sigma(t_i)\right| &\le \left|\sum_{i=1}^n c_i (\sigma e_{\lambda_0} - \sigma)(t_i)\right| + \left|\sum_{i=1}^n c_i \sigma e_{\lambda_0}(t_i)\right| \\
&= \left|\sum_{i=1}^n c_i \sigma e_{\lambda_0}(t_i)\right| \\
&\le \|\sigma e_{\lambda_0}\|_{\infty,p} \left\|\sum_{i=1}^n c_i \varphi_{t_i}|_{C_{0,p}(G)}\right\|_{C_{0,p}(G)^*} \\
&\le \|\sigma\|_\infty \|e_{\lambda_0}\|_{\infty,p} \left\|\sum_{i=1}^n c_i \varphi_{t_i}|_{C_{0,p}(G)}\right\|_{C_{0,p}(G)^*} \\
&\le 2\|\sigma\|_\infty \left\|\sum_{i=1}^n c_i \varphi_{t_i}|_{C_{0,p}(G)}\right\|_{C_{0,p}(G)^*};
\end{aligned}$$

hence, σ is a BSE-function on $\Phi_{C_{0,p}(G)}$ ($\cong G$) with $\|\sigma\|_{\mathrm{BSE}} \leq 2\|\sigma\|_\infty$. Thus, we obtain that $C^b(G) = C_{\mathrm{BSE}}(\Phi_{C_{0,p}(G)})$ since σ is arbitrary.

3. Proof of Theorem 1.2. From Lemma 2.2, $C_{0,p}(G)$, $1 \leq p < \infty$, is a BSE-algebra of type I. Also, since $C_{0,p}(G)$ is a proper Segal algebra in $C_0(G)$ by Lemma 2.1, it does not contain a bounded approximate identity by [2, Theorem C'-(ii)]. Therefore, $C_{0,p}(G)$ is isomorphic to no commutative C*-algebras. $\square$

For more details on BSE-algebras, the reader is referred to [**1**].

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