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MULTIGRADED HILBERT SCHEMES PARAMETRIZING IDEALS IN THE WEYL ALGEBRA

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ABSTRACT. Results of Haiman and Sturmfels **[2]** on multigraded Hilbert schemes are used to establish a quasi-projective scheme which parametrizes all left homogeneous ideals in the Weyl algebra having a fixed Hilbert function with respect to a given grading by an abelian group.

1. Introduction. Let $S = k[x_1, \ldots, x_n]$ be the polynomial algebra over a commutative ring k. The monomials x^u in S are identified with their exponents $u \in \mathbb{N}^n$. A grading of S by an abelian group A is a semigroup homomorphism

$$\deg : \mathbb{N}^n \longrightarrow A.$$

We may assume that A is generated by $\deg(x_i)$ for $i = 1, \ldots, n$. For $a \in A$, let S_a be the k-span of the monomial x^u with $\deg(u) = a$. We have the decomposition

$$S = \bigoplus_{a \in A} S_a$$

which satisfies $S_a \cdot S_b \subseteq S_{a+b}$. An admissible ideal in S is a homogeneous ideal I with the property that $(S/I)_a = S_a/I_a$ is a locally free k-module of finite rank (constant on $\operatorname{Spec} k$) for all $a \in A$. The Hilbert function of an admissible ideal I is a map

$$h_I : A \longrightarrow \mathbb{N}$$

defined by $h_I(a) := \operatorname{rank}_k(S_a/I_a)$.

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By fixing any function

$$h : A \longrightarrow \mathbb{N},$$

Haiman and Sturmfels constructed **[2]** a scheme H_S^h over k (called the multigraded Hilbert scheme) which parametrizes all admissible ideals I in S with Hilbert function $h_I = h$. As discussed in **[2]**, their results recover many special cases, including Hilbert schemes of points in affine space, toric Hilbert schemes, Hilbert schemes of abelian groups orbits and Grothendieck Hilbert schemes. It is also mentioned in **[2,** subsection 6.2] that their results can be applied to the universal enveloping algebra of an A-graded Lie algebra. The purpose of this note is to verify this claim for the special case of the Weyl algebra $W = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$.

In order to have a well-defined degree function on the set

$$\mathcal{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$$

of all monomials in W, we assume that our ground ring k is an integral domain of characteristic 0. By Proposition 2.1 of **[1,** Chapter 1] (the proof works for any integral domain k), the set $\mathcal{B}$ forms a k-basis for W. In general, this does not hold in the non-domain case. For example, if $k = \mathbb{Z}[t]/\langle 2t\rangle$, then $t\partial^2 \in k\langle x, \partial\rangle$ acts as the zero operator on $k[x]$. On the other hand, in view of the relations $\partial_i x_i - x_i \partial_i = 1$ in W, it may be quickly noticed that we must have $\deg(x_i) = -\deg(\partial_i)$. Therefore, any A-grading

$$\deg : \mathbb{N}^n \longrightarrow A$$

on S extends to an A-grading $\deg : \mathcal{B} \to A$ on W by $\deg(x^\alpha \partial^\beta) = \deg(\alpha) - \deg(\beta)$. We have the decomposition

$$W = \bigoplus_{a \in A} W_a$$

satisfying $W_a \cdot W_b \subseteq W_{a+b}$, where W_a is the k-span of the monomials in $\mathcal{B}$ with degree a.

Similarly to the case of polynomial algebras, we call a homogeneous left ideal I of W *admissible* if $(W/I)_a = W_a/I_a$ is a locally free k-module of finite rank (constant on $\operatorname{Spec} k$) for all $a \in A$. Note that the Hilbert function $h_I : A \to \mathbb{N}$ of an admissible ideal I in W defined by $h_I(a) = \operatorname{rank}_k(W_a/I_a)$ cannot have finite support. This follows from the fact that there is no left ideal of W with finite co-rank over k.

Indeed, if $\mathrm{rank}(W/I)$ is finite, then the two k-linear maps ϕ_x and ϕ_∂ on W/I induced by multiplications of x and ∂, respectively, would satisfy the equality $\phi_\partial\phi_x - \phi_x\phi_\partial = \mathrm{id}_{W/I}$, which is not possible by comparing the traces of the linear maps from both sides.

Our goal is to prove the following analog of [**2,** Theorem 1.1].

Theorem 1.1. *Given a Hilbert function $h : A \to \mathbb{N}$, there exists a quasi-projective scheme over k that represents the Hilbert functor*

$$H_W^h : \underline{k\text{-Alg}} \longrightarrow \underline{\mathrm{Set}}$$

where, for a k-algebra R, the set $H_W^h(R)$ consists of homogeneous ideals $I \subseteq R \otimes_k W$ such that

$$(R \otimes_k W_a)/I_a$$

is a locally free R-module of $\mathrm{rank}\, h(a)$ *for every $a \in A$.*

In Section 4, we will recall the techniques from [**2**] that are needed in the proof of Theorem 1.1. Roughly speaking, we first show that, for any finite set of degrees D, the Hilbert functor $H_{W_D}^h$ is represented by a quasi-projective scheme that is a closed subscheme of a certain relative Grassmann scheme. Here, the k-module

$$W_D = \bigoplus_{a \in D} W_a,$$

and, by abusing notation, the restriction of the Hilbert function $h : A \to \mathbb{N}$ to D is also denoted by h. Then, we specify a special finite set D such that H_W^h is a subfunctor of $H_{W_D}^h$ represented by a closed subscheme of $H_{W_D}^h$.

Although the strategy of proving Theorem 1.1 is very similar to the polynomial algebra case, there are still several issues that require some modifications. For example, the key feature that makes these mechanisms work for the multigraded Hilbert scheme H_S^h is the nice behaviors of monomial ideals in S, e.g., the fact that antichains of monomial ideals in S are finite [**3**] is essential in the construction of H_S^h. In Section 2, we will see that monomial ideals in W do not have the expected behaviors in general. In particular, the naive generalization of Gröbner basis theory to Weyl algebra does not work very well. For example, the ideal $\langle \partial^2, x\partial - 1\rangle$ and its naive initial ideal $\langle \partial^2, x\partial\rangle = \langle \partial \rangle$

in $W = k\langle x, \partial\rangle$ do not have the same Hilbert function. In order to get around this, we consider the initial ideal of a left ideal in W as a monomial ideal in the associated graded algebra $\operatorname{gr} W$ (which is a polynomial algebra) and utilize the Gröbner basis theory for the Weyl algebra developed in [**4**]. Basic facts about the Gröbner basis theory for W will be reviewed in Section 3. Finally, the proof of Theorem 1.1 will be elaborated in Section 5.

The same results of this paper applied to right ideals may be achieved similarly.

2. Monomial ideals in the Weyl algebra. Let k be an integral domain of characteristic 0, and let $W = k\langle x, \partial\rangle = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the nth Weyl algebra. Many classical facts regarding the Weyl algebra are proved under the assumption that k is a field, see e.g., [**1**]. This has an advantage in that W is left and right Noetherian. For our purposes, we will not make this assumption. Many classical properties of the Weyl algebra extend to this more general setting. For example, by [**1,** Chapter 1, Proposition 2.1] (the proof of which works for integral domain k) the set

$$\mathcal{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$$

forms a k-basis for W, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. The unique expression of an element δ of W in terms of this k-basis $\mathcal{B}$ is called the *canonical form* of δ. In this paper, elements in $\mathcal{B}$ are the only monomials of the Weyl algebra W, and a product of monomials in W may not be a monomial. Also, the total degree of the monomial $x^\alpha\partial^\beta$ is $|\alpha| + |\beta|$, and the total degree of an element in W is defined as the total degree of its leading monomials. The total degrees of elements of W induce the Bernstein filtration on W, whose associated graded ring $\operatorname{gr} W = k[x, \xi] = k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ is the polynomial algebra of $2n$ variables over k. Moreover, by considering the isomorphism of free k-modules

$$\Psi : k[x, \xi] \longrightarrow W = k\langle x, \partial\rangle$$

that sends $x^\alpha\xi^\beta$ to $x^\alpha\partial^\beta$, we have the following Leibniz formula which is helpful for the multiplication of elements in W. Note that, in the formula of Proposition 2.1, the denominator $k_1! \cdots k_n!$ is used only to obtain a nice expression. We will never need to find the inverse of elements in the domain k.

Proposition 2.1. [**4,** Theorem 1.1.1]. *For any two polynomials f and g in $k[x,\xi]$, we have*

$$\Psi(f)\cdot\Psi(g)=\sum_{k_1,\dots,k_n\geq 0}\frac{1}{k_1!\cdots k_n!}\,\Psi\left(\frac{\partial^k f}{\partial\xi^k}\cdot\frac{\partial^k g}{\partial x^k}\right).$$

In particular, we have a convenient formula for multiplying two monomials.

Corollary 2.2. *Let $x^\alpha\partial^\beta$, $x^{\alpha'}\partial^{\beta'}$ be monomials in W. We have*

$$(x^\alpha\partial^\beta)\cdot(x^{\alpha'}\partial^{\beta'})=\sum_{k_1,\dots,k_n\geq 0}\left(\prod_{i=1}^{n}k_i!\binom{\beta_i}{k_i}\binom{\alpha'_i}{k_i}x^{\alpha_i+\alpha'_i-k_i}\partial^{\beta_i+\beta'_i-k_i}\right).$$

A left ideal in W is called a *left monomial ideal* if it is generated by monomials. Unlike the monomial ideals in a polynomial algebra, it may occur that an element in a left monomial ideal I is a sum of monomials which are not in I. For example, in the first Weyl algebra $W=k\langle x,\partial\rangle$, the element $\partial x=x\partial+1$ is in the principal ideal $I=Wx$ generated by x, but the identity 1 (hence, $x\partial$) is not in I by considering the total degrees of elements in W. Moreover, there exists an infinite antichain of monomial ideals in W, see Example 2.6. Thus, the direct generalization of the main theorem in [**3**] does not hold for the Weyl algebra. Nonetheless, we still have the following analog of Dickson's lemma for monomial ideals in polynomial algebras.

Proposition 2.3. *Every left monomial ideal of W is generated by finitely many monomials.*

Proof. Let I be a left monomial ideal of W. By passing to the associated graded algebra with respect to the Bernstein filtration, it may be observed that the ideal $\operatorname{gr} I$ is a monomial ideal of $\operatorname{gr} W$. By Dickson's lemma, $\operatorname{gr} I$ is finitely generated by monomials of degrees $\leq m$ for some m. Standard arguments regarding filtered algebras, see, for example, the proof of [**1,** Theorem 8.2.3], show that I is generated by elements with total degree $\leq m$, say $f_1,\dots,f_t$. We only need finitely many monomials in I to generate $f_1,\dots,f_t$; thus, I is, in fact, generated by finitely many monomials. □

Example 2.4. Every left monomial ideal in the first Weyl algebra $W = k\langle x, \partial\rangle$ is principally generated by one monomial. In order to see this, suppose that I is a left monomial ideal of W and assume that $x^\alpha\partial^\beta \in I$. Observe that, by Corollary 2.2, we have

(i) $(x\partial)(x^\alpha\partial^\beta) = x^{\alpha+1}\partial^{\beta+1} + \alpha x^\alpha\partial^\beta$, so $x^{\alpha+1}\partial^{\beta+1} \in I$, and hence, $x^{\alpha+s}\partial^{\beta+t} \in I$ for all $0 \le t \le s$;

(ii) for $\alpha \ge 1$, $\partial(x^\alpha\partial^\beta) = x^\alpha\partial^{\beta+1} + ax^{\alpha-1}\partial^\beta$, $x^\alpha\partial^{\beta+1} \in I$ if and only if $x^{\alpha-1}\partial^\beta \in I$.

It suffices to show that, for any two monomials $x^\alpha\partial^\beta$ and $x^{\alpha'}\partial^{\beta'}$ in I, there exists a monomial $x^{\alpha''}\partial^{\beta''} \in I$ such that $x^\alpha\partial^\beta, x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha''}\partial^{\beta''}$. We may assume by symmetry that $\beta' \ge \beta$ and consider only the following cases.

(a) $[0 \le \beta'-\beta \le \alpha'-\alpha]$. In this case, $x^{\alpha'}\partial^{\beta'} = x^{\alpha+(\alpha'-\alpha)}\partial^{\beta+(\beta'-\beta)} \in Wx^\alpha\partial^\beta$ by (i); thus, we simply take $(\alpha, \beta) = (\alpha'', \beta'')$.

(b) $[\alpha' - \alpha < \beta' - \beta]$. In this case, take $\beta'' = \beta$ and

$$\alpha'' = \begin{cases} 0 & \text{if } \alpha' - \beta' + \beta \le 0; \\ \alpha' - \beta' + \beta & \text{otherwise.} \end{cases}$$

It follows from (i) that $x^\alpha\partial^\beta, x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha''}\partial^{\beta''}$. On the other hand, the Leibniz formula in (ii) also shows that, if $x^{\alpha''+1}\partial^{\beta''}$ and $x^{\alpha''+1}\partial^{\beta''+1}$ are both in I, then $x^{\alpha''}\partial^{\beta''} \in I$. Hence, if there exists an $m \in \mathbb{N}$ such that

$$\{x^{\alpha''+m}\partial^{\beta''+i} \mid 0 \le i \le m\} \subset I,$$

then $x^{\alpha''}\partial^{\beta''} \in I$. The proof of this example is completed by taking $m = \alpha + \beta' - \beta - \alpha''$. Indeed, for $\beta' - \beta \le i \le m$,

$$0 \le (\beta'' + i) - \beta' \le m + (\beta - \beta') \le m + (\alpha'' - \alpha') = (\alpha'' + m) - \alpha';$$

thus, by (i),

$$\{x^{\alpha''+m}\partial^{\beta''+i} \mid \beta' - \beta \le i \le m\} \subset Wx^{\alpha'}\partial^{\beta'} \subset I.$$

Also, for $0 \le i < \beta' - \beta$,

$$0 \le i = (\beta'' + i) - \beta < \beta' - \beta = (\alpha'' + m) - \alpha;$$

thus, again by (i),

$$\{x^{\alpha''+m}\partial^{\beta''+i} \mid 0 \le i < \beta' - \beta\} \subset Wx^\alpha\partial^\beta \subset I.$$

The observations in Example 2.4 also imply the following lemma.

Lemma 2.5. *In the first Weyl algebra W, if $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$ with $\alpha \geq 1$, then*

$$[(\alpha', \beta') - (\alpha, \beta)] \in \Sigma = \{(s,t) \in \mathbb{N}^2 \mid 0 \leq t \leq s\}.$$

Proof. Suppose otherwise that $\alpha' - \alpha < \beta' - \beta$. Applying Example 2.4 (i) to $x^{\alpha}\partial^{\beta} \in Wx^{\alpha}\partial^{\beta}$, we obtain

$$\{x^{\beta'-\beta+\alpha-1}\partial^{\beta+i} \mid 0 \leq i \leq \beta' - \beta - 1\} \subset Wx^{\alpha}\partial^{\beta}.$$

Since $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$, we have

$$x^{\beta'-\beta+\alpha-1}\partial^{\beta'} = x^{(\beta'-\beta)-(\alpha'-\alpha)-1}(x^{\alpha'}\partial^{\beta'}) \in Wx^{\alpha}\partial^{\beta}.$$

Therefore,

$$\{x^{\beta'-\beta+\alpha-1}\partial^{\beta+i} \mid 0 \leq i \leq \beta' - \beta\} \subset Wx^{\alpha}\partial^{\beta}.$$

Through repeated use of Example 2.4 (ii), we obtain $x^{\alpha-1}\partial^{\beta} \in Wx^{\alpha}\partial^{\beta}$, which is impossible in view of the Bernstein filtration on W. □

We remark that the same argument in the proof of Lemma 2.5 generalizes to the nth Weyl algebra W, namely, if $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$ with $\alpha_i \geq 1$ for some $i \in \{1, \dots, n\}$, then

$$[(\alpha_i', \beta_i') - (\alpha_i, \beta_i)] \in \Sigma = \{(s,t) \in \mathbb{N}^2 \mid 0 \leq t \leq s\}.$$

Example 2.6. Using Lemma 2.5, it may readily be verified that

$$\{Wx\partial^{\beta} \mid \beta \in \mathbb{N}\}$$

is an infinite antichain of monomial ideals in the first Weyl algebra W.

3. Gröbner bases in the Weyl algebra. In this section, the ground ring k is a field of characteristic 0. We recall the Gröbner bases theory for Weyl algebra over k developed in [**4,** subsection 1.1].

A total order $\prec$ on the set $\mathcal{B}$ of monomials in W is called a *term order* for W if the following two conditions hold:

(1) $1 = x^0\partial^0$ is the $\prec$-smallest element;
(2) $x^{\alpha}\partial^{\beta} \prec x^a\partial^b$ implies $x^{\alpha+s}\partial^{\beta+t} \prec x^{a+s}\partial^{b+t}$ for all $(s,t) \in \mathbb{N}^{2n}$.

The initial monomial $\mathrm{in}_{\prec}(\delta)$ of an element $\delta \in W$ is the commutative monomial $x^\alpha \xi^\beta \in k[x,\xi]$ such that $x^\alpha \partial^\beta$ is the $\prec$-largest monomial appearing in the canonical form of δ. For a W-ideal I, its initial ideal is the ideal in $k[x,\xi]$, generated by $\{\mathrm{in}_{\prec} \delta \mid \delta \in I\}$. A finite set G of W is said to be a *Gröbner basis* for a W-left ideal I with respect to $\prec$ if I is generated by G and the initial ideal $\mathrm{in}_{\prec} I$ is generated by $\{\mathrm{in}_{\prec} g \mid g \in G\}$. From [**4,** Theorem 1.1.10], every left ideal I of W admits a Gröbner basis G with respect to any given term order $\prec$. Note that not every finite monomial generating set of a monomial ideal forms a Gröbner basis. For example, the initial ideal of $I = Wx + W\partial = W$ is $k[x,\xi]$, which is not generated by x and ξ. Nonetheless, we have the following analog of the normal form algorithm: every element $\delta \in W$ has a unique normal form $\overline{\delta}^G \in W$ with respect to G such that $\delta \equiv \overline{\delta}^G$ modulo I and that every monomial appearing in the canonical form of $\overline{\delta}^G$ is not divisible by $\Psi(\mathrm{in}_{\prec} g)$ for any $g \in G$. Here, a monomial $x^\alpha \partial^\beta$ is said to be divisible by $x^a \partial^b$ in W if $\alpha_i \geq a_i$ and $\beta_i \geq b_i$ for all i. A monomial of W is called a *standard monomial* of I with respect to $\prec$ if it is not divisible by $\Psi(\mathrm{in}_{\prec} g)$ for any g in a Gröbner basis G for I.

The next lemma is an immediate consequence of the normal form algorithm.

Lemma 3.1. *Let $\prec$ be a term order on W, and let I be a left ideal of W.*

(1) *The images of the standard monomials of I in W/I form a k-basis.*

(2) *The map*

$$\Psi : \mathrm{gr}\, W = k[x,\xi] \longrightarrow W$$

induces an isomorphism between the k-vector spaces $\mathrm{gr}\, W/\mathrm{in}_{\prec} I$ and W/I, which sends the standard monomials of $\mathrm{in}_{\prec} I$ in $\mathrm{gr}\, W$ to the standard monomials of I in W.

(3) *If I is homogeneous with respect to an A-grading of W, then $\mathrm{in}_{\prec} I$ is homogeneous with respect to the induced A-grading on $\mathrm{gr}\, W$, and the map Ψ restricts to an isomorphism between $(\mathrm{gr}\, W/\mathrm{in}_{\prec} I)_a$ and $(W/I)_a$ for each $a \in A$. In particular, the Hilbert functions of I and $\mathrm{in}_{\prec} I$ are identical.*

4. Main techniques. For the reader's convenience, we recall the general framework for the construction of the multigraded Hilbert scheme in [**2**].

Fix a commutative ring k and an arbitrary index set A. Consider the pair (T, F) of graded k-modules

$$T = \bigoplus_{a \in A} T_a$$

with a collection of operators

$$F = \bigcup_{a,b \in A} F_{a,b},$$

where $F_{a,b} \subseteq \mathrm{Hom}_k(T_a, T_b)$ satisfies $F_{b,c} \circ F_{a,b} \subseteq F_{a,c}$ and $\mathrm{id}_{T_a} \in F_{a,a}$. In fact, (T, F) is a small category of k-modules with objects T_a and arrows, the elements of which are in F.

For a commutative k-algebra R, we denote by $R \otimes T$ the graded R-module

$$\bigoplus_a (R \otimes T_a)$$

with operators $\widehat{F}_{a,b} = (1_R \otimes -)(F_{a,b})$. A homogeneous submodule

$$L = \bigoplus_a L_a \subseteq R \otimes T$$

is an F-submodule if $\widehat{F}_{a,b}(L_a) \subseteq L_b$ for all $a, b \in A$. Fix a function

$$h : A \longrightarrow \mathbb{N}.$$

Let $H_T^h(R)$ be the set of F-submodules $L \subseteq R \otimes T$ such that $(R \otimes T_a)/L_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$. We have the Hilbert functor

$$H_T^h : \underline{k\text{-Alg}} \longrightarrow \underline{\mathrm{Set}}.$$

For any subset D of A, denote by (T_D, F_D) the full subcategory of (T, F) with objects T_a and the set of arrows $F_{D,a,b} = F_{a,b}$ for $a, b \in A$. We have a natural transformation of Hilbert functors

$$H_T^h \longrightarrow H_{T_D}^h$$

given by restriction of degrees.

Theorem 4.1. [**2,** Theorem 2.2]. *Let (T, F) be a graded k-module with operators as above. Suppose that there exist homogeneous k-submodules $M \subseteq N \subseteq T$ and a subset $F' \subseteq F$ satisfying the following conditions*:

(i) *N is a finitely generated k-module*;
(ii) *N generates T as an F'-module*;
(iii) *for every field $K \in \underline{k\text{-Alg}}$ and every $L \in H_T^h(K)$, M spans $(K \otimes T)/L$*;
(iv) *there is a subset $G \subseteq F'$, generating F' as a category, such that $GM \subseteq N$.*

Then, H_T^h is represented by a quasi-projective scheme over k.

We remark that the statement of Theorem 4.1 is slightly stronger than that of [**2,** Theorem 2.2]. However, the same proof works in this setting. Indeed, in the proof of [**2,** Theorem 2.2], only Step 6 applies to conditions (ii) and (iv) where any element in T needs to be produced using elements in N and operators in F. However, this does not require the full set of F. Any subset $F' \subseteq F$ satisfying conditions (ii) and (iv) will suffice.

Moreover, hypothesis (iii) implies that

$$\dim_K (K \otimes T)/L = \sum_{a \in A} h(a)$$

is finite; thus, Theorem 4.1 works only for h having finite support. For the general case, we need the following theorem.

Theorem 4.2. [**2,** Theorem 2.3]. *Let (T, F) be graded k-modules with operators, and let $D \subseteq A$ be such that $H_{T_D}^h$ is represented by a scheme over k. Assume that, for each degree $a \in A$*:

(v) *there is a finite subset*

$$E \subseteq \bigcup_{b \in D} F_{b,a}$$

such that

$$T_a / \sum_{b \in D} E_{b,a}(T_b)$$

is a finitely generated k-module;

(vi) *for every field* $K \in \underline{k\text{-Alg}}$ *and every* $L_D \in H^h_{T_D}(K)$*, if* L' *denotes the* F*-submodule of* $K \otimes T$ *generated by* L_D*, then* $\dim(K \otimes T_a)/L'_a \leq h(a)$.

Then, the natural transformation

$$H^h_T \longrightarrow H^h_{T_D}$$

makes H^h_T *a subfunctor of* $H^h_{T_D}$*, represented by a closed subscheme of the Hilbert scheme* $H^h_{T_D}$.

In order to find a suitable finite set D of degrees satisfying hypotheses (v) and (vi) in Theorem 4.2, we also need the following facts.

Proposition 4.3. [**2,** Proposition 3.2]. *Let* S *be an* A*-graded polynomial ring. Given a degree function*

$$\deg : \mathbb{N}^n \longrightarrow A$$

and a Hilbert function

$$h : A \longrightarrow \mathbb{N},$$

there is a finite set of degrees $D \subseteq A$ *that satisfies the following two properties*:

(g) *every monomial ideal of* S *with Hilbert function* h *is generated by monomials of degrees in* D*, and*

(h′) *every monomial ideal* I *of* S *generated in degrees* D *satisfies*: *if* $h_I(a) = h(a)$ *for all* $a \in D$*, then* $h_I(a) \leq h(a)$ *for all* $a \in A$.

Such a set D in Proposition 4.3 is called a supportive set of degrees in [**2**]. There is also a so-called very supportive set E which is used to define equations of H^h_S in the positive grading case. Since the A-grading on W is never positive, we will not pursue here the analogous results on very supportive sets.

5. Proof of Theorem 1.1. Fix any Hilbert function $h : A \to \mathbb{N}$, and let D be a finite subset of degrees in A. Our first task is to construct k-submodules M, N and a subset $F'_D \subseteq F_D$ satisfying the hypotheses in Theorem 4.1 for the graded k-modules

$$W_D = \bigoplus_{a \in D} W_a$$

with the set of operators F_D to be defined later.

Let F be the monoid of operators on W generated by multiplication of monomials in W. Note that this is slightly different from the polynomial case due to the non-commutativity of W. Denote the set of all operators in F that send W_a into W_b by $F_{a,b}$. Then,

$$F = \bigcup_{a,b \in A} F_{a,b}.$$

Moreover, we have $F_{b,c} \circ F_{a,b} \subseteq F_{a,c}$ for all $a, b, c \in A$, and $F_{a,a}$ contains the identity map on W_a for all $a \in A$. Thus, (W, F) is a small category of k-modules with the components W_a of W as objects and elements of F as arrows. Note also that, for a k-algebra R, an admissible left ideal in $R \otimes_k W$ is equivalent to a left F-submodule L of $R \otimes_k W$ such that $(R \otimes_k W_a)/L_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$.

Define

$$F_D := \bigcup_{a,b \in D} F_{a,b}.$$

Then, (W_D, F_D) is a full subcategory of (W, F). Consider, for each k-algebra R, the set $H^h_{W_D}(R)$ of all admissible F_D-submodules of $R \otimes_k W_D$ and, for each k-algebra homomorphism

$$\phi : R \longrightarrow S,$$

the map

$$H^h_{W_D}(\phi) : H^h_{W_D}(R) \longrightarrow H^h_{W_D}(S).$$

There is a natural transformation of Hilbert functors

$$H^h_W \longrightarrow H^h_{W_D},$$

given by sending $L \in H^h_W(R)$ to

$$L_D := \bigoplus_{a \in D} L_a \in H^h_{W_D}(R).$$

For each $a \in A$, let $\mathcal{B}_a$ be the set of monomials (excluding $1 = x^0\partial^0$) with degree a. Denote the set of minimal elements in $\mathcal{B}_a$ with respect to the partial ordering

$$x^\alpha\partial^\beta \leq x^{\alpha'}\partial^{\beta'} \iff (\alpha, \beta) \leq (\alpha', \beta')$$

by G'_a. Recall that $x^{\alpha'}\partial^{\beta'}$ is said to be divisible by $x^\alpha\partial^\beta$ if $(\alpha, \beta) \leq (\alpha', \beta')$. By Dickson's lemma, we have G'_a is finite for each $a \in A$. For $a, b \in A$, let $G_{a,b}$ be the set of operators on W consisting of left multiplication by elements in G'_{b-a}. Denote by F'_D the monoid (category) generated by

$$G_D := \bigcup_{a,b \in D} G_{a,b}.$$

For $a, b \in D$, denote the set of all operators in F'_D that send W_a into W_b by $F'_{a,b}$. The following example shows that strict inequality $F'_D \subsetneq F_D$ can occur.

Example 5.1. Consider the $\mathbb{Z}$-grading on the first Weyl algebra $W = k\langle x, \partial\rangle$ with $\deg(x) = -\deg(\partial) = 1$. Let $D = \{0, 2\} \subseteq \mathbb{Z}$. Then, $G_D = \{x\partial, x^2, \partial^2\}$. Observe that the element $\partial x^3 \in F_{0,2} \subseteq F_D$ does not lie in the monoid F'_D generated by elements in G_D.

The A-grading on W induces an A-grading on $\operatorname{gr} W$ by setting $\deg \xi := \deg \partial$. The Hilbert function $h : A \to \mathbb{N}$ can be viewed as a Hilbert function for ideals in the polynomial algebra $\operatorname{gr} W$ with this induced A-grading. Let $\mathcal{C}_D$ be the set of ideals of $\operatorname{gr} W$ generated by monomials in degrees D with Hilbert functions agreeing with h on D. Denote the union over all $I \in \mathcal{C}_D$ of the Ψ-images of the standard monomials of I in $(\operatorname{gr} W)_D$ by M', i.e.,

$$M' = \{\Psi(x^\alpha\xi^\beta) \mid x^\alpha\xi^\beta \in (\operatorname{gr} W)_D \setminus I, \text{ for some } I \in \mathcal{C}_D\}.$$

Since $\mathcal{C}_D$ is finite by **[3]**, the set M' is also finite.

Let

$$N' = G_D M' \cup \Big(\bigcup_{a \in D} G'_a\Big),$$
$$M = kM' \quad \text{and} \quad N = kN'.$$

We verify that (W_D, F'_D, F_D), N, M and G_D satisfy the hypotheses of Theorem 4.1 which is rewritten below.

(i) N is a finitely generated k-module;
(ii) N generates W_D as an F'_D-module;
(iii) for every field $K \in \underline{k\text{-Alg}}$ and every $L_D \in H^h_{W_D}(K)$, M spans $(K \otimes W_D)/L_D$;
(iv) there is a subset $G_D \subseteq F'_D$, generating F'_D as a category, such that $G_D M \subseteq N$.

Conditions (i) and (iv) obviously hold by our construction. For condition (ii), given $a \in D$ and $x^\alpha \partial^\beta \in W_a$, we want to show that $x^\alpha \partial^\beta$ is generated by N over F'_D by induction on the total degree $|\alpha| + |\beta|$ of $x^\alpha \partial^\beta$. If $x^\alpha \partial^\beta \in G'_a \subseteq N$, the statement is automatically true. For

$$x^\alpha \partial^\beta \in W_a \setminus G'_a,$$

there exists an $x^{\alpha'} \partial^{\beta'} \in G'_a \subseteq N$ such that $x^\alpha \partial^\beta$ is divisible by $x^{\alpha'} \partial^{\beta'}$. Note that the total degree of the element

$$(x^\alpha \partial^\beta - x^{\alpha-\alpha'} \partial^{\beta-\beta'} \cdot x^{\alpha'} \partial^{\beta'}) \in W_a$$

is strictly less than that of $x^\alpha \partial^\beta$; thus, by inductive hypothesis, it is generated by N over F'_D. Therefore, it suffices to show that $x^{\alpha-\alpha'} \partial^{\beta-\beta'} \in W_0$ acts on W_a as a sum of operators in F'_D. In fact, we will show that every element $x^{\bar\alpha} \partial^{\bar\beta} \in W_0$ acts as a sum of operators in F'_D by induction on the total degree of $x^{\bar\alpha} \partial^{\bar\beta}$. Recall that F'_D is the monoid generated by

$$\bigcup_{a,b \in D} G'_{a,b}.$$

In particular, if $x^{\bar\alpha} \partial^{\bar\beta} \in G'_0$, it acts as an operator in $G_{a,a} \subseteq F'_D$ for any $a \in D$. For

$$x^{\bar\alpha} \partial^{\bar\beta} \in W_0 \setminus G'_0,$$

there exists an $x^{\bar\alpha'} \partial^{\bar\beta'} \in G'_0$ such that $x^{\bar\alpha} \partial^{\bar\beta}$ is divisible by $x^{\bar\alpha'} \partial^{\bar\beta'}$. Since $x^{\bar\alpha-\bar\alpha'} \partial^{\bar\beta-\bar\beta'}$ and $(x^{\bar\alpha} \partial^{\bar\beta} - x^{\bar\alpha-\bar\alpha'} \partial^{\bar\beta-\bar\beta'} \cdot x^{\bar\alpha'} \partial^{\bar\beta'})$ are in W_0 and have total degree strictly less than $x^{\bar\alpha} \partial^{\bar\beta}$, they both act as a sum of operators in F'_D. We conclude that

$$x^{\bar\alpha}\partial^{\bar\beta} = (x^{\bar\alpha}\partial^{\bar\beta} - x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'}\cdot x^{\bar\alpha'}\partial^{\bar\beta'}) + x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'}\cdot x^{\bar\alpha'}\partial^{\bar\beta'}$$

also acts as a sum of operators in F'_D. This establishes condition (ii).

We need the next lemma to verify condition (iii).

Lemma 5.2. *Let $R \in \underline{k\text{-Alg}}$, $L_D \in H^h_{W_D}(R)$ and $L \subseteq R \otimes_k W$ be the left ideal generated by L_D. Then, $L_a = L_{D,a}$ for all $a \in D$.*

Proof. Observe that, for $a \in D$,

$$L_a = \sum_{b\in D} F_{b,a}(L_{D,b}) \supseteq F_{a,a}(L_{D,a}) \supseteq L_{D,a}.$$

Conversely, we have $F_{b,a}(L_{D,b}) \subseteq L_{D,a}$ for any $a, b \in D$, since $L_D \in H^h_{W_D}(R)$ is an F_D-submodule of $R \otimes_k W_D$. □

For condition (iii), fix a field $K \in \underline{k\text{-Alg}}$ and an F_D-submodule $L_D \in H^h_{W_D}(K)$. Let $L \subseteq K \otimes_k W$ be the ideal generated by L_D. Fix any term order $\prec$ on W, and let $I = \operatorname{in}_\prec L \subset \operatorname{gr} W$ be the initial ideal of L with respect to $\prec$. By Lemma 3.1 (3), the Hilbert functions h_I and h_L coincide. Hence, $h_I(a) = h_{L_D}(a)$ for all $a \in D$ by Lemma 5.2. In particular, the ideal $I \in \mathcal{C}_D$. From Lemma 3.1 (2), M' spans $(K \otimes_k W_D)/L_D$, which is exactly the statement of condition (iii).

At this point, we have shown, by using Theorem 4.1, that $H^h_{W_D}$ is represented by a quasi-projective scheme for any finite set D of degrees in A. In order to complete the proof of Theorem 1.1, it remains to verify conditions (v) and (vi) (for each degree $a \in A$) in Theorem 4.2 for some suitable finite subset D of A.

(v) There is a finite subset $E \subseteq \bigcup_{b\in D} F_{b,a}$ such that

$$W_a/\sum_{b\in D} E_{b,a}(W_b)$$

is a finitely generated k-module;

(vi) for every field $K \in \underline{k\text{-Alg}}$ and every $L_D \in H^h_{W_D}(K)$, if L denotes the F-submodule of $K \otimes W$ generated by L_D, then $\dim(K \otimes W_a)/L_a \le h(a)$.

Applying Proposition 4.3 to the case where $S = \operatorname{gr} W$ with the induced degree function

$$\deg : \mathbb{N}^{2n} \longrightarrow A$$

given by $\deg(\xi) = \deg(\partial)$, we can find a finite subset D of A that satisfies conditions (g) and (h′) for $\operatorname{gr} W$ with respect to the same Hilbert function h. From now on, fix such a finite set D. The goal is to use Theorem 4.2 to show that the natural transformation

$$H^h_W \longrightarrow H^h_{W_D}$$

makes H^h_W a subfunctor of $H^h_{W_D}$, represented by a closed subscheme of the Hilbert scheme $H^h_{W_D}$. We may assume that there exists an admissible function L of W, whose Hilbert function is $h_L = h$, for otherwise, the statement of Theorem 4.2 is null. Choose any term order $\prec$ on W. By Lemma 3.1 (3), the Hilbert function $h_{\mathrm{in}_\prec L}$ of the initial ideal $\mathrm{in}_\prec L$ of L in $\operatorname{gr} W$ coincides with h, and hence, $\mathrm{in}_\prec L$ is generated in degrees D by condition (g) in Proposition 4.3. Since the s-pair of two homogeneous elements in the Weyl algebra is still homogeneous, there exists a Gröbner basis for L consisting of homogeneous elements in degrees D. In particular, the ideal L of W is also generated in degrees D. Therefore, for each $a \in A$, the component

$$L_a = \sum_{b \in D} F_{b,a}(L_b),$$

and it has finite k-codimension $h(a)$ in W_a.

In order to verify condition (v), it suffices to find a finite subset

$$E \subseteq \bigcup_{b \in D} F_{b,a}$$

such that, for any $b \in D$, $a \in A$,

$$F_{b,a}(L_b) \subseteq \sum_{b' \in D} E_{b',a}(W_b).$$

Take $E_{b,a} = G_{b,a}$, and let

$$E = \bigcup_{b \in D} E_{b,a}.$$

We claim that, in fact,

$$F_{b,a}(W_b) \subseteq E_{b,a}(W_b) = G_{b,a}(W_b).$$

Since each operator in $F_{b,a}$, which is a product of monomials, can be written as a sum of monomials in degree $a-b$ by Corollary 2.2, we verify only that, if $\deg(x^\alpha\partial^\beta) = a-b$, then $x^\alpha\partial^\beta(W_b) \subseteq G_{b,a}(W_b)$. It is certainly true that

$$x^\alpha\partial^\beta(W_b) \subseteq G_{b,a}(W_b)$$

when $x^\alpha\partial^\beta \in G'_{a-b}$. In general, we have $x^\alpha\partial^\beta$ is divisible by some element $x^{\alpha'}\partial^{\beta'} \in G'_{a-b}$ and, by inductive hypothesis,

$$[x^\alpha\partial^\beta(W_b) - x^{\alpha'}\partial^{\beta'} \cdot x^{\alpha-\alpha'}\partial^{\beta-\beta'}(W_b)] \subseteq G_{b,a}(W_b).$$

Since $\deg(x^{\alpha-\alpha'}\partial^{\beta-\beta'}) = 0$, we have $x^{\alpha-\alpha'}\partial^{\beta-\beta'}(W_b) \subseteq W_b$, and hence,

$$x^\alpha\partial^\beta(W_b) \subseteq G_{b,a}(W_b),$$

as desired.

For condition (vi), fix a field $K \in \underline{k}\text{-Alg}$, an element $L_D \in H^h_{W_D}(K)$, and let $L \subseteq K \otimes_k W$ be the ideal generated by L_D. From Lemma 5.2, $L_a = L_{D,a}$, and hence, $h_L(a) = h(a)$ for all $a \in D$. Also, we have $h_L(a) = h_{\mathrm{in}_\prec L}(a)$ for all $a \in A$ by Lemma 3.1 (3). Let I be the monomial ideal in $\operatorname{gr} W$ generated by $(\mathrm{in}_\prec L)_D$. Then, $I_a = (\mathrm{in}_\prec L)_a$ for all $a \in D$ by the same argument of Lemma 5.2, and hence,

$$h_I(a) = h_{\mathrm{in}_\prec L}(a) = h_L(a) = h(a) \quad \text{for all } a \in D.$$

Therefore, by condition (h$'$) of Proposition 4.3,

$$h_L(a) = h_{\mathrm{in}_\prec L}(a) \le h_I(a) \le h(a) \quad \text{for all } a \in A.$$

This establishes condition (vi).

Example 5.3. Let k be a field. Consider the finest possible A-grading on W, where $A = \mathbb{Z}^n$ and

$$\deg(x_i) = -\deg(\partial_i) = e_i.$$

Under this A-grading, any homogeneous ideals are generated by elements in W of the form $x^a p(\theta)\partial^b$, $a, b \in \mathbb{N}^n$. From [**4,** Lemma 2.3.1], such ideals are the torus-fixed ideals of W, which are used in the algorithms for solving systems of linear partial differential equations.

Fixing a Hilbert function $h : A \to \mathbb{N}$, we remark that, if $I, J \in H_W^h(k)$ and if I is holonomic, then J is also holonomic. Indeed, using the notation in [**4**], the Hilbert functions of $\mathrm{in}_{\prec_{(0,e)}} I$ and $\mathrm{in}_{\prec_{(0,e)}} J$ coincide by Lemma 3.1. Therefore, from [**4,** Theorem 1.1.6], the ideals $\mathrm{in}_{(0,e)} I$ and $\mathrm{in}_{(0,e)} J$ in $\mathrm{gr}\, W$ also have the same Hilbert functions under the A-grading inherited from that of W. In particular, $\dim \mathrm{in}_{(0,e)} I = \dim \mathrm{in}_{(0,e)} J$, and the holonomicity of J follows.

REFERENCES

1. S.C. Coutinho, *A primer of algebraic D-modules*, Lond. Math. Soc. **33**, Cambridge University Press, Cambridge, 1995.

2. Mark Haiman and Bernd Sturmfels, *Multigraded Hilbert schemes*, J. Alg. Geom. **13** (2004), 725–769.

3. Diane Maclagan, *Antichains of monomial ideals are finite*, Proc. Amer. Math. Soc. **129** (2001), 1609–1615 (electronic).

4. Mutsumi Saito, Bernd Sturmfels and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, in *Algorithms and computation in mathematics*, Volume 6, Springer-Verlag, Berlin, 2000.

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