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MULTIPLE SOLUTIONS FOR A KIRCHHOFF-TYPE PROBLEM INVOLVING NONLOCAL FRACTIONAL p-LAPLACIAN AND CONCAVE-CONVEX NONLINEARITIES

CHANG-MU CHU, JIAO-JIAO SUN AND ZHI-PENG CAI

ABSTRACT. This paper examines a class of Kirchhoff nonlocal operators involving concave-convex nonlinearities and sign-changing weight functions. With the aid of the Nehari manifold, the existence of multiple nontrivial nonnegative solutions is obtained.

1. Introduction. In this paper, we study the multiplicity of nontrivial nonnegative solutions to the Dirichlet boundary value problem by the nonlocal operator:

$$\begin{cases} -M\bigg(\displaystyle\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p dx\,dy\bigg)\mathcal{L}_K^p u & \\ \qquad = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $M(t) = a + bt^m$ and the parameters $a, b, \lambda > 0$, $0 < s < 1 < q < p < \infty$, $N > ps$,

$$0 \le m < \frac{ps}{N-ps}, \qquad (m+1)p < r < p_s^* = \frac{Np}{N-ps}.$$

The weight functions $f(x),\ g(x) \in C(\overline{\Omega})$ satisfy $f^+ = \max\{f, 0\} \neq 0$ and $g^+ = \max\{g, 0\} \neq 0$. The nonlocal operator $\mathcal{L}_K^p$ is defined as:

$$\mathcal{L}_K^p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x)-u(y)|^{p-2}(u(x)-u(y))K(x-y)\,dy,$$

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$x \in \mathbb{R}^N$, and $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a measurable function with the property:

$$
\begin{cases}
\gamma K \in L^1(\mathbb{R}^N) & \text{where } \gamma(x) = \min\{|x|^p, 1\}; \\
\text{there exists a } k_0 > 0 \text{ such that} & \\
K(x) \geq k_0 |x|^{-(N+ps)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}; \\
K(x) = K(-x) & \text{for any } x \in \mathbb{R}^N \setminus \{0\}.
\end{cases} \tag{1.2}
$$

A typical example for K is given by the singular kernel $K(x) = |x|^{-(N+ps)}$. In this case, problem (1.1) becomes

$$
\begin{cases}
M\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right) (-\Delta)_p^s u & \\
\qquad = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.3}
$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator which, up to normalization factors, may be defined as

$$
(-\Delta)_p^s u(x) = -2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy.
$$

Setting $m = 0$ and $a + b = 1$, problems (1.1) and (1.3) reduce to

$$
\begin{cases}
-\mathcal{L}_K^p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.4}
$$

and

$$
\begin{cases}
(-\Delta)_p^s u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases} \tag{1.5}
$$

Recently, much attention has been given to the study of Kirchhoff-type problems and nonlocal elliptic operators. Kirchhoff-type equations arise in the description of nonlinear vibrations of an elastic string, see [**16**]. Solvability of the Kirchhoff-type problem with the Dirichlet boundary using variational methods is studied in [**1, 3, 7, 17**], and the references therein. Fractional and nonlocal operators arise in many different contexts, such as the thin obstacle problem, finance, optimization, quasi-geostrophic flow, etc. There is much literature on fractional and nonlocal operators; the interested reader is referred to **[2, 4, 8, 9, 13]**–**[15, 18, 20, 22]**–**[26, 28]**, and the references therein. However, there is little research concerning the existence of solutions for Kirchhoff-type problems in the fractional setting. A detailed discussion regarding the physical meaning underlying fractional Kirchhoff models and their applications was first provided in [**12,** Appendix A]. More precisely, Fiscella and Valdinoci proposed a stationary Kirchhoff variational model, which takes into account the nonlocal aspects of tension arising from nonlocal measurements of the fractional length of the string. Some results of Kirchhoff-type problems involving nonlocal operators may also be found in **[19, 21, 27]**, and the references therein. In these papers, multiple results with concave-convex nonlinearities and sign changing weight functions were obtained using the Nehari manifold and fibering map analysis. However, as far as is known, there has been no research on Kirchhoff-type problems and nonlocal operators with concave-convex nonlinearities and sign changing weight functions.

Here, we use the variational approach on the Nehari manifold to solve problem (1.1). Motivated by **[7, 13]**, the aim of this paper is to investigate multiple solutions of problem (1.1) and extend the results of **[8, 13]**.

Problem (1.1) has variational structure, and solutions may be constructed as critical points of an associated energy functional on some appropriate space. In the norm $\|u\|_{H^s(\mathbb{R}^N)}$, the interaction between Ω and $\mathbb{R}^N \setminus \Omega$ provides a positive contribution, which should be considered when encoding the boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the weak formulation.

Now, we introduce the linear space

$$X = \Big\{ u \mid u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega) \\ \text{and } (u(x) - u(y)) \sqrt[p]{K(x-y)} \in L^p(Q) \Big\},$$

where $Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$. The space X is endowed with the norm, defined as

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \bigg(\int_Q |u(x) - u(y)|^p K(x-y)\, dx\, dy \bigg)^{1/p}.$$

Moreover, we shall work in the closed linear subspace

$$X_0 = \{u \in X : u = 0 \text{ almost everywhere in } \mathbb{R}^N \setminus \Omega\},$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\, dx\, dy \right)^{1/p}.$$

Let

$$K = \mathbb{R}^N \setminus \{0\} \longrightarrow (0, +\infty)$$

satisfy assumption (1.2). We have that $C_0^\infty(\Omega) \subset X_0$, and $(X_0, \|\cdot\|_{X_0})$ is a reflexive Banach space, see **[10, 27]**. Moreover,

$$X \subset W^{s,p}(\Omega)$$

and

$$X_0 \subset W^{s,p}(\mathbb{R}^N),$$

where $W^{s,p}(\Omega)$ is the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dx\, dy \right)^{1/p}.$$

In addition, the embedding

$$X_0 \hookrightarrow L^{p_s^*}(\Omega)$$

is continuous, and there exists a positive constant $C_0 = C_0(N, p, s)$ such that, for any $v \in X_0$, $1 < k < p_s^*$,

$$\|v\|_{L^k(\Omega)} \le C_0 \|v\|_{X_0}. \tag{1.6}$$

Now, the definition of weak solutions for problem (1.1) is given.

Definition 1.1. We say that $u \in X_0$ is a weak solution of problem (1.1) if u satisfies

$$M(\|u\|_{X_0}^p)\int_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy$$
$$= \lambda\int_\Omega f(x)|u|^{q-2}u\varphi\,dx + \int_\Omega g(x)|u|^{r-2}u\varphi\,dx,$$

for any $\varphi \in X_0$.

The main results of this paper are as follows.

Theorem 1.2. *Let*

$$K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0, +\infty)$$

be a function satisfying (1.2). *Suppose that* $M(t) = a+bt^m$, $f(x), g(x) \in C(\overline{\Omega})$ *satisfy*

$$f^+ = \max\{f, 0\} \neq 0 \quad \textit{and} \quad g^+ = \max\{g, 0\} \neq 0.$$

If $0 < s < 1 < q < p < \infty$, $N > ps$, $a > 0$, $b > 0$, $0 \leq m < ps/(N-ps)$, $(m+1)p < r < p_s^*$, *then there exists a* $\lambda^* > 0$ *such that problem* (1.1) *for any* $\lambda \in (0, \lambda^*)$ *has at least two nontrivial nonnegative solutions.*

Remark 1.3. The multiple solutions of problem (1.4) with $p = 2$, $f(x) = g(x) \equiv 1$, are obtained in [**8**], and the multiple solutions of problem (1.5) are obtained in [**13**]. Obviously, the results there are the special case of our Theorem 1.2.

This paper is organized as follows. In Section 2, we give some notation and preliminaries regarding the Nehari manifold. In Section 3, we give the proof of Theorem 1.2.

2. Notation and preliminaries. The energy functional

$$J_\lambda : X_0 \longrightarrow \mathbb{R}$$

associated to problem (1.1) is defined as

$$J_\lambda(u) = \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{r}\int_\Omega g(x)|u|^r dx,$$

where

$$\widehat{M}(t) = \int_0^t M(\tau)\, d\tau = at + \frac{b}{m+1}\, t^{m+1}.$$

It is well known that J_λ is of class C^1 in X_0, and the solutions of problem (1.1) are critical points of the energy functional J_λ in X_0. In fact,

$$\begin{aligned}\langle J_\lambda'(u), \varphi\rangle = M(\|u\|_{X_0}^p) &\int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y)) \\ &\times (\varphi(x) - \varphi(y))K(x-y)\, dx\, dy \\ &- \lambda \int_\Omega f(x)|u|^{q-2}u\varphi\, dx - \int_\Omega g(x)|u|^{r-2}u\varphi\, dx.\end{aligned}$$

Since $r > (m+1)p$, it is easy to see that J_λ is unbounded from below on X_0. In order to obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in X_0 \setminus \{0\} : \langle J_\lambda'(u), u\rangle = 0\},$$

where $\langle\ ,\ \rangle$ denotes the duality between X_0 and its dual space. Thus, $u \in \mathcal{N}_\lambda$ if and only if

$$M(\|u\|_{X_0}^p)\|u\|_{X_0}^p - \lambda \int_\Omega f(x)|u|^q dx - \int_\Omega g(x)|u|^r dx = 0. \tag{2.1}$$

It is clear that all nonzero solutions of problem (1.1) must lie on $\mathcal{N}_\lambda$, and $\mathcal{N}_\lambda$ is a much smaller set than X_0. Thus, it is easier to study J_λ on $\mathcal{N}_\lambda$. The Nehari manifold $\mathcal{N}_\lambda$ is closely linked to the behavior of functions of the form

$$\psi_u : t \longrightarrow J_\lambda(tu) \quad \text{for } t > 0.$$

Such maps are called *fiber maps*, introduced by Drabek and Pohozaev in **[11]** and discussed by Brown and Zhang in **[6]**. For $u \in X_0$, we have

$$\psi_u(t) = \frac{1}{p}\widehat{M}(t^p\|u\|_{X_0}^p) - \frac{\lambda t^q}{q}\int_\Omega f(x)|u|^q dx - \frac{t^r}{r}\int_\Omega g(x)|u|^r dx,$$

$$\psi_u'(t) = t^{p-1}M(t^p\|u\|_{X_0}^p)\|u\|_{X_0}^p - \lambda t^{q-1}\int_\Omega f(x)|u|^q dx - t^{r-1}\int_\Omega g(x)|u|^r dx,$$

$$\begin{aligned}\psi_u''(t) = (p-1)&t^{p-2}M(t^p\|u\|_{X_0}^p)\|u\|_{X_0}^p + pt^{2p-2}M'(t^p\|u\|_{X_0}^p)\|u\|_{X_0}^{2p} \\ &- \lambda(q-1)t^{q-2}\int_\Omega f(x)|u|^q dx - (r-1)t^{r-2}\int_\Omega g(x)|u|^r dx.\end{aligned}$$

Clearly,

$$t\psi_u'(t) = M(t^p\|u\|_{X_0}^p)\|tu\|_{X_0}^p - \lambda\int_\Omega f(x)|tu|^q dx - \int_\Omega g(x)|tu|^r dx,$$

and thus, for $u \in X_0$ and $t > 0$, $\psi_u'(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$. In particular, $\psi_u'(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. Therefore, it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection. For this, we set

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \psi_u''(1) > 0\},\\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \psi_u''(1) = 0\},\\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \psi_u''(1) < 0\}.\end{aligned}$$

Thus, for each $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned}\psi_u''(1) &= pM'(\|u\|_{X_0}^p)\|u\|_{X_0}^{2p}\\ &\quad + (p-q)M(\|u\|_{X_0}^p)\|u\|_{X_0}^p - (r-q)\int_\Omega g(x)|u|^r dx,\end{aligned}\tag{2.2}$$

or

$$\begin{aligned}\psi_u''(1) &= pM'(\|u\|_{X_0}^p)\|u\|_{X_0}^{2p}\\ &\quad - (r-p)M(\|u\|_{X_0}^p)\|u\|_{X_0}^p + \lambda(r-q)\int_\Omega f(x)|u|^q dx.\end{aligned}\tag{2.3}$$

Define

$$\begin{aligned}\phi_\lambda(u) = \langle J_\lambda'(u), u\rangle &= M(\|u\|_{X_0}^p)\|u\|_{X_0}^p\\ &\quad - \lambda\int_\Omega f(x)|u|^q dx - \int_\Omega g(x)|u|^r dx.\end{aligned}\tag{2.4}$$

Then, for $u \in \mathcal{N}_\lambda$,

$$\begin{aligned}\langle\phi_\lambda'(u), u\rangle &= (p-1)M(\|u\|_{X_0}^p)\|u\|_{X_0}^p + pM'(\|u\|_{X_0}^p)\|u\|_{X_0}^{2p}\\ &\quad - (q-1)\lambda\int_\Omega f(x)|u|^q dx - (r-1)\int_\Omega g(x)|u|^r dx\\ &= \psi_u''(1).\end{aligned}\tag{2.5}$$

Similar to the argument of [**6,** Theorem 2.3], we conclude the following result.

Lemma 2.1. *Suppose that u_0 is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$. Then, $J'_\lambda(u_0) = 0$ in X_0^*.*

Proof. Since u_0 is a local minimizer on $\mathcal{N}_\lambda$ to J_λ, by the theory of Lagrange multipliers and (2.4), there exists a $\sigma \in \mathbb{R}$ such that

$$J'_\lambda(u_0) = \sigma \phi'_\lambda(u_0).$$

Thus,

$$\langle J'_\lambda(u_0), u_0\rangle = \sigma\langle \phi'_\lambda(u_0), u_0\rangle = \sigma \psi''_{u_0}(1).$$

Since $u_0 \notin \mathcal{N}_\lambda^0$, we have $\psi''_{u_0}(1) \neq 0$. Hence, $\sigma = 0$, that is, $J'_\lambda(u_0) = 0$. □

Lemma 2.2. *J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. For $u \in \mathcal{N}_\lambda$, by (1.6) and (2.1),

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{r}\int_\Omega g(x)|u|^r dx \\ &= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{1}{r}M(\|u\|_{X_0}^p)\|u\|_{X_0}^p - \frac{(r-q)\lambda}{rq}\int_\Omega f(x)|u|^q dx \\ &\geq \frac{(r-p)a}{pr}\|u\|_{X_0}^p + \frac{(r-(m+1)p)b}{(m+1)pr}\|u\|_{X_0}^{(m+1)p} \\ &\quad - \frac{(r-q)\lambda}{rq}\|f^+\|_\infty C_0^q \|u\|_{X_0}^q. \end{aligned}$$

Since $0 < q < 1 < p$, $(m+1)p < r < p_s^*$, we obtain that J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

Lemma 2.3. *There exists a $\lambda_1 > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$ for any $\lambda \in (0, \lambda_1)$.*

Proof. We argue by contradiction. Assume that $\mathcal{N}_\lambda^0 \neq \emptyset$ for any $\lambda > 0$. Then, for $u \in \mathcal{N}_\lambda^0$, we have $\langle J'_\lambda(u), u\rangle = 0$ and $\psi''_u(1) = 0$. In addition, from (2.2) and (2.3),

$$(p-q)M(\|u\|_{X_0}^p)\|u\|_{X_0}^p + pM'(\|u\|_{X_0}^p)\|u\|_{X_0}^{2p} = (r-q)\int_\Omega g(x)|u|^r dx,$$

and

$$(r-p)M(\|u\|_{X_0}^p)\|u\|_{X_0}^p - pM'(\|u\|_{X_0}^p)\|u\|_{X_0}^{2p} = \lambda(r-q)\int_\Omega f(x)|u|^q dx.$$

From (1.6),

$$((m+1)p-q)b\|u\|_{X_0}^{(m+1)p} \le C_0^r(r-q)\|g^+\|_\infty\|u\|_{X_0}^r,$$

and

$$(r-(m+1)p)b\|u\|_{X_0}^{(m+1)p} \le C_0^q(r-q)\lambda\|f^+\|_\infty\|u\|_{X_0}^q.$$

This yields that

$$\left(\frac{((m+1)p-q)b}{C_0^r(r-q)\|g^+\|_\infty}\right)^{1/(r-(m+1)p)} \le \|u\|_{X_0} \le \left(\frac{(r-q)\|f^+\|_\infty C_0^q\lambda}{(r-(m+1)p)b}\right)^{1/((m+1)p-q)}.$$

This is impossible if λ is sufficiently small. Thus, we obtain that a $\lambda_1 > 0$ exists such that $\mathcal{N}_\lambda^0 = \emptyset$ for any $\lambda \in (0, \lambda_1)$. $\square$

By Lemma 2.3, for $\lambda \in (0, \lambda_1)$, we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$a_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u), \qquad a_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

Let

$$\lambda_2 = \frac{a(r-p)}{(r-q)C_0^q\|f^+\|_\infty}\left(\frac{a(p-q)}{(r-q)C_0^r\|g^+\|_\infty}\right)^{(p-q)/(r-p)}.$$

Then, we have following results.

Lemma 2.4. *Suppose that* $0 < \lambda < \lambda_2$. *Then, for each* $u \in X_0$ *with*

$$\int_\Omega g(x)|u|^r dx > 0,$$

there exists a $t_{b,\max} > 0$ *such that,*

(i) *if*

$$\int_\Omega f(x)\, dx \le 0,$$

then there is a unique $t^- > t_{b,\max}$ *such that* $t^-u \in \mathcal{N}_\lambda^-$ *and*

$$J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu);$$

(ii) *if*

$$\int_\Omega f(x)\,dx > 0,$$

then there are unique t^+ *and* t^- *with* $0 < t^+ < t_{b,\max} < t^-$ *such that* $t^\pm u \in \mathcal{N}_\lambda^\pm$ *and*

$$J_\lambda(t^+u) = \inf_{0\le t\le t_{b,\max}} J_\lambda(tu), \qquad J_\lambda(t^-u) = \sup_{t\ge t_{b,\max}} J_\lambda(tu).$$

Proof. Fix $u \in X_0$ with

$$\int_\Omega g(x)|u|^r dx > 0.$$

Let

$$h_b(t) = at^{p-q}\|u\|_{X_0}^p + bt^{(m+1)p-q}\|u\|_{X_0}^{(m+1)p} - t^{r-q}\int_\Omega g(x)|u|^r dx,$$

for $a, t \ge 0$. Clearly, $tu \in \mathcal{N}_\lambda$ if and only if

$$h_b(t) = \lambda \int_\Omega f(x)|u|^q dx.$$

Since $r > (m+1)p$ and

$$\int_\Omega g(x)|u|^r dx > 0,$$

we have $h_b(0) = 0$,

$$h(t) \longrightarrow -\infty \qquad \text{as } t \to \infty.$$

Moreover, there is a unique $t_{b,\max} > 0$ such that $h_b(t)$ achieves its maximum at $t_{b,\max}$, increasing for $t \in [0, t_{b,\max})$ and decreasing for $t \in (t_{b,\max}, +\infty)$. In addition,

$$t_{0,\max} = \left(\frac{a(p-q)\|u\|_{X_0}^p}{(r-q)\int_\Omega g(x)|u|^r dx}\right)^{1/(r-p)},$$

and

$$h_0(t_{0,\max}) = \left(\frac{a(p-q)\|u\|_{X_0}^p}{(r-q)\int_\Omega g(x)|u|^r dx}\right)^{(p-q)/(r-p)} a\|u\|_{X_0}^p$$

$$
\begin{aligned}
(2.6) \qquad & - \left(\frac{a(p-q)\|u\|_{X_0}^p}{(r-q)\int_\Omega g(x)|u|^r dx} \right)^{(r-q)/(r-p)} \int_\Omega g(x)|u|^r dx \\
&= \|u\|_{X_0}^p \frac{a(r-p)}{r-q} \left(\frac{a(p-q)\|u\|_{X_0}^r}{(r-q)\int_\Omega g(x)|u|^r dx} \right)^{(p-q)/(r-p)} \\
&\geq \|u\|_{X_0}^p \frac{a(r-p)}{r-q} \left(\frac{a(p-q)}{(r-q)C_0^r\|g^+\|_\infty} \right)^{(p-q)/(r-p)}.
\end{aligned}
$$

(i)

$$\int_\Omega f(x)\, dx \leq 0.$$

There is a unique $t^- > t_{b,\max}$ such that

$$h_b(t^-) = \lambda \int_\Omega f(x)|u|^q dx$$

and $h_b'(t^-) < 0$. Now,

$$
\begin{aligned}
\psi_{t^-u}'(1) &= t^-\psi_u'(t^-) \\
(2.7) \qquad &= M(\|t^-u\|_{X_0}^p)\|t^-u\|_{X_0}^p - \lambda \int_\Omega f(x)|t^-u|^q dx - \int_\Omega g(x)|t^-u|^r dx \\
&= (t^-)^q \left(h_b(t^-) - \lambda \int_\Omega f(x)|u|^q dx \right) \\
&= 0,
\end{aligned}
$$

and

$$(2.8) \qquad \psi_{t^-u}''(1) = (t^-)^2\psi_u''(t^-) = (t^-)^{q+1}h_b'(t^-) < 0.$$

Thus, $t^-u \in \mathcal{N}_\lambda^-$. Since $t > t_{b,\max}$, we have $h_b'(t) < 0$. It follows from (2.7) and (2.8) that

$$J_\lambda(t^-u) = \psi_u(t^-) = \sup_{t\geq 0} \psi_u(t) = \sup_{t\geq 0} J_\lambda(tu).$$

(ii)

$$\int_\Omega f(x)\, dx > 0.$$

From (1.6) and (2.6), we have

$$
\begin{aligned}
h_b(0) = 0 < \lambda \int_\Omega f(x)|u|^q dx &\le \lambda \|f^+\|_\infty C_0^q \|u\|_{X_0}^q \\
&\le \|u\|_{X_0}^q \frac{a(r-p)}{r-q} \left(\frac{a(p-q)}{(r-q)C_0^r \|g^+\|_\infty} \right)^{(p-q)/(r-p)} \\
&\le h_0(t_0, \max) < h_b(t_b, \max),
\end{aligned}
$$

for $\lambda \in (0, \lambda_2)$. Therefore, there are unique t^+ and t^- such that $0 < t^+ < t_{b,\max} < t^-$,

$$
h_b(t^+) = \lambda \int_\Omega f(x)|u|^q dx = h_b(t^-) \quad \text{and} \quad h_b'(t^+) > 0 > h_b'(t^-).
$$

Similar to the argument in part (i), we conclude that $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$. Moreover,

$$
J_\lambda(t^-u) \ge J_\lambda(tu) \ge J_\lambda(t^+u) \quad \text{for each } t \in [t^+, t^-]
$$

and

$$
J_\lambda(t^+u) \le J_\lambda(tu) \quad \text{for each } t \in [0, t^+].
$$

Thus,

$$
J_\lambda(t^+u) = \inf_{0 \le t \le t_{b,\max}} J_\lambda(tu), \qquad J_\lambda(t^-u) = \sup_{t \ge t_{b,\max}} J_\lambda(tu). \qquad \square
$$

Lemma 2.5. *Suppose that* $0 < \lambda < \lambda_2$. *Then, for each* $u \in X_0$ *with*

$$
\int_\Omega f(x)|u|^q dx > 0,
$$

there exists a $\bar{t}_{b,\max} > 0$ *such that,*

(i) *if*

$$
\int_\Omega g(x)|u|^r dx \le 0,
$$

then there is a unique $0 < t^+ < \bar{t}_{b,\max}$ *such that* $t^+u \in \mathcal{N}_\lambda^+$ *and*

$$
J_\lambda(t^+u) = \inf_{t \ge 0} J_\lambda(tu);
$$

(ii) *if*

$$\int_\Omega g(x)|u|^r dx > 0,$$

then there are unique t^+ *and* t^- *with* $0 < t^+ < t_{b,\max} < t^-$ *such that* $t^\pm u \in \mathcal{N}_\lambda^\pm$ *and*

$$J_\lambda(t^+u) = \inf_{0\le t\le t_{b,\max}} J_\lambda(tu), \qquad J_\lambda(t^-u) = \sup_{t\ge t_{b,\max}} J_\lambda(tu).$$

Proof. Fix $u \in X_0$ with

$$\int_\Omega f(x)|u|^q dx > 0.$$

Let

$$\overline{h}_b(t) = at^{p-r}\|u\|_{X_0}^p + bt^{(m+1)p-r}\|u\|_{X_0}^{(m+1)p} - \lambda t^{q-r}\int_\Omega f(x)|u|^q dx,$$

for $a, t \ge 0$. Clearly,

$$\overline{h}_b(t) \longrightarrow -\infty \quad \text{as } t \to 0^+$$

and

$$\overline{h}_b(t) \longrightarrow 0 \qquad \text{as } t \to +\infty.$$

Moreover, there is a unique $\overline{t}_{b,\max} > 0$ such that $\overline{h}_b(t)$ achieves its maximum at $\overline{t}_{b,\max}$, increasing for $t \in [0, \overline{t}_{b,\max})$ and decreasing for $t \in (\overline{t}_{b,\max}, +\infty)$. In addition,

$$\overline{t}_{0,\max} = \left(\frac{(r-q)\lambda\int_\Omega f(x)|u|^q dx}{a(r-p)\|u\|_{X_0}^p}\right)^{1/(p-q)},$$

and

$$\begin{aligned}\overline{h}_0(\overline{t}_{0,\max}) &= \left(\frac{a(r-p)\|u\|_{X_0}^p}{(r-q)\lambda\int_\Omega f(x)|u|^q dx}\right)^{(r-p)/(p-q)} a\|u\|_{X_0}^p \\ &\quad - \left(\frac{a(r-p)\|u\|_{X_0}^p}{(r-q)\lambda\int_\Omega f(x)|u|^q dx}\right)^{(r-q)/(p-q)} \lambda\int_\Omega f(x)|u|^q dx \\ &= \|u\|_{X_0}^r \frac{a(p-q)}{r-q}\left(\frac{a(r-p)\|u\|_{X_0}^q}{(r-q)\lambda\int_\Omega f(x)|u|^q dx}\right)^{(r-q)/(p-q)}\end{aligned}$$

$$\geq \|u\|_{X_0}^r \frac{a(p-q)}{r-q}\left(\frac{a(r-p)}{(r-q)\lambda C_0^q\|f^+\|_\infty}\right)^{(r-q)/(p-q)}.$$

The results of Lemma 2.5 are obtained by repeating the same argument of Lemma 2.4. □

Lemma 2.6. *Suppose that* $0<\lambda<(q\lambda_2)/(2p)$. *Then, we have*

(i) $\alpha_\lambda^+<0$,

(ii) $\alpha_\lambda^->d_0$ *for some* $d_0>0$.

Proof.

(i) Let $u\in\mathcal{N}_\lambda^+$. We have $\psi_u''(1)>0$. By (2.2),

$$(r-q)\int_\Omega g(x)|u|^r dx \leq a(p-q)\|u\|_{X_0}^p + ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p}.$$

Therefore,

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{r}\int_\Omega g(x)|u|^r dx\\
&= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{1}{q}M(\|u\|_{X_0}^p)\|u\|_{X_0}^p + \frac{r-q}{rq}\int_\Omega g(x)|u|^r dx\\
&\leq \frac{a(q-p)}{pq}\|u\|_{X_0}^p + \frac{(q-(m+1)p)b}{(m+1)pq}\|u\|_{X_0}^{(m+1)p}\\
&\quad + \frac{r-q}{rq}(a(p-q)\|u\|_{X_0}^p + ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p})\\
&\leq -\frac{a(p-q)(r-p)}{pqr}\|u\|_{X_0}^p\\
&\quad - \frac{((m+1)p-q)(r-(m+1)p)b}{(m+1)prq}\|u\|_{X_0}^{(m+1)p}\\
&< 0.
\end{aligned}$$

(ii) Let $u\in\mathcal{N}_\lambda^-$. We have $\psi_u''(1)<0$. From (1.6) and (2.2),

(2.9)

$$\begin{aligned}
a(p-q)\|u\|_{X_0}^p &\leq (r-q)\int_\Omega g(x)|u|^r dx - ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p}\\
&\leq (r-q)\int_\Omega g(x)|u|^r dx
\end{aligned}$$

$$\leq (r-q)C_0^r\|g^+\|_\infty\|u(x)\|_{X_0}^r.$$

It follows from (2.9) that

$$\|u\|_{X_0} > \left(\frac{a(p-q)}{C_0^r\|g^+\|_\infty(r-q)}\right)^{1/(r-p)}.$$

According to (1.6) and (2.1), we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{r}\int_\Omega g(x)|u|^r dx \\
&= \frac{1}{p}\widehat{M}(\|u\|_{X_0}^p) - \frac{1}{r}M(\|u\|_{X_0}^p)\|u\|_{X_0}^p \\
&\quad - \frac{(r-q)\lambda}{rq}\int_\Omega f(x)|u|^q dx \\
&= \frac{(r-p)a}{pr}\|u\|_{X_0}^p + \frac{(r-(m+1)p)b}{(m+1)pr}\|u\|_{X_0}^{(m+1)p} \\
&\quad - \frac{(r-q)\lambda}{rq}\int_\Omega f(x)|u|^q dx \\
&\geq \frac{(r-p)a}{pr}\|u\|_{X_0}^p + \frac{(r-(m+1)p)b}{(m+1)pr}\|u\|_{X_0}^{(m+1)p} \\
&\quad - \frac{(r-q)\lambda}{rq}\|f^+\|_\infty C_0^q\|u\|_{X_0}^q \\
&\geq \|u\|_{X_0}^q\left(\frac{a(p-q)}{rp}\|u\|_{X_0}^{p-q} - \frac{(r-q)\lambda}{rq}\|f^+\|_\infty C_0^q\right) \\
&\geq \left(\frac{a(p-q)}{C_0^r\|g^+\|_\infty(r-q)}\right)^{q/(r-p)}\left(\frac{a(r-p)}{rp}\left(\frac{a(p-q)}{C_0^r\|g^+\|_\infty(r-q)}\right)^{(p-q)/(r-p)}\right. \\
&\qquad\qquad \left. - \frac{(r-q)\lambda}{rq}\|f^+\|_\infty C_0^q\right).
\end{aligned}$$

Thus, if $\lambda < (q\lambda_2)/(2p)$, then $\alpha_\lambda^- > d_0$ for some $d_0 > 0$. □

3. Proof of the main result. In this section, we show the existence of minimizers in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ for $\lambda > 0$ small enough. Let

$$\lambda^* = \min\left\{\lambda_1, \frac{q}{2p}\lambda_2\right\}.$$

We have the following results.

Proposition 3.1. *If $0 < \lambda < \lambda^*$, then the functional J_λ has a minimizer u_0^+ in $\mathcal{N}_\lambda^+$ and satisfies*:

(i) $J_\lambda(u_0^+) = \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$,

(ii) u_0^+ *is a nonzero solution of problem* (1.1).

Proof. Since J_λ is bounded from below on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^+. \tag{3.1}$$

Thus, by Lemma (2.2), the sequence $\{u_n\}$ is bounded in X_0. Then, there exists a u_0^+, up to a sequence, such that

$$u_n \rightharpoonup u_0^+ \text{ weakly in } X_0 \text{ as } n \to \infty.$$

Moreover, by [**23,** Lemma 8],

$$\begin{aligned} u_n &\longrightarrow u_0^+ \quad \text{in } L^k(\mathbb{R}^N),\\ u_n &\longrightarrow u_0^+ \quad \text{almost everywhere in } \mathbb{R}^N \end{aligned}$$

as $n \to \infty$. For any $1 \le k < p_s^*$, by [**5,** Theorem IV-9], there exists an $l(x) \in L^k(\mathbb{R}^N)$ such that

$$|u_n(x)| \le l(x) \text{ almost everywhere in } \mathbb{R}^N.$$

Therefore, by the dominated convergence theorem, we have that

$$\begin{aligned} \lambda\int_\Omega f(x)|u_n|^q dx &\longrightarrow \lambda\int_\Omega f(x)|u_0^+|^q dx,\\ \int_\Omega g(x)|u_n|^r dx &\longrightarrow \int_\Omega g(x)|u_0^+|^r dx, \end{aligned} \tag{3.2}$$

as $n \to \infty$. Now, on $\mathcal{N}_\lambda$, we have

$$\begin{aligned} \frac{\lambda(r-q)}{rq}\int_\Omega f(x)|u_n|^q dx = \frac{a(r-p)}{pr}&\|u_n\|_{X_0}^p\\ &+ \frac{[r-(m+1)p]b}{(m+1)pr}\|u_n\|_{X_0}^{(m+1)p} - J_\lambda(u_n). \end{aligned} \tag{3.3}$$

Letting $n \to \infty$, from Lemma 2.6, (3.1) and (3.2), we obtain

$$\int_\Omega f(x)|u_0^+|^q dx > 0.$$

By Lemma 2.5, there exists a $t_1 > 0$ such that

$$t_1 u_0^+ \in \mathcal{N}_\lambda^+.$$

Next, we show that

$$u_n \longrightarrow u_0^+ \text{ strongly in } X_0.$$

Suppose that this is not true. Then,

$$\|u_0^+\|_{X_0} < \liminf_{n\to\infty} \|u_n\|_{X_0}.$$

Thus, for $u_n \in \mathcal{N}_\lambda^+$,

$$\begin{aligned}\lim_{n\to\infty} \psi'_{u_n}(t_1) &= \lim_{n\to\infty} \bigg(t_1^{p-1} M(t_1^p \|u_n\|_{X_0}^p) \|u_n\|_{X_0}^p \\ &\qquad - \lambda t_1^{q-1} \int_\Omega f(x)|u_n|^q dx - t_1^{r-1} \int_\Omega g(x)|u_n|^r dx \bigg) \\ &> t_1^{p-1} M(t_1^p \|u_0^+\|_{X_0}^p) \|u_0^+\|_{X_0}^p \\ &\quad - \lambda t_1^{q-1} \int_\Omega f(x)|u_0^+|^q dx - t_1^{r-1} \int_\Omega g(x)|u_0^+|^r dx \\ &= \psi'_{u_0^+}(t_1) \\ &= 0,\end{aligned}$$

that is, $\psi'_{u_n}(t_1) > 0$ for n large enough. Since $u_n = 1 \cdot u_n \in \mathcal{N}_\lambda^+$, we see that $\psi'_{u_n}(t) < 0$ for $t \in (0,1)$ and $\psi'_{u_n}(1) = 0$ for all n. Therefore, we must have $t_1 > 0$. On the other hand, $\psi_{u_n}(t)$ is decreasing on $(0, t_1)$; thus,

$$J_\lambda(t_1 u_0^+) \le J_\lambda(u_0^+) < \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^+,$$

a contradiction. Hence,

$$u_n \longrightarrow u_0^+ \text{ strongly in } X_0.$$

This implies

$$J_\lambda(u_n) \longrightarrow J_\lambda(u_0^+) = \alpha_\lambda^+ \quad \text{as } n \to \infty,$$

namely, u_0^+ is a minimizer of J_λ on $\mathcal{N}_\lambda^+$. Using Lemma 2.1, u_0^+ is a nonzero solution to problem (1.1). $\square$

Proposition 3.2. *If $0 < \lambda < \lambda^*$, then the functional J_λ has a minimizer u_0^- in $\mathcal{N}_\lambda^-$ and satisfies:*

(i) $J_\lambda(u_0^-) = \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) > 0$,

(ii) u_0^- *is a nonzero solution of problem* (1.1).

Proof. J_λ is bounded from below such that

$$\lim_{J \to \infty} J_\lambda(\widetilde{u}_n) = \alpha_\lambda^-. \tag{3.4}$$

By the same argument given in the proof of Proposition 3.1, there exists a u_0^-, up to a sequence, such that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup u_0^- \text{ weakly in } X_0, \\ &\widetilde{u}_n \longrightarrow u_0^- \quad \text{in } L^k(\mathbb{R}^N) \end{aligned}$$

and

$$\widetilde{u}_n \longrightarrow u_0^- \text{ almost everywhere in } \mathbb{R}^N,$$

as $n \to \infty$, for any $1 \le k \le p_s^*$. Moreover,

$$\begin{aligned} \lambda \int_\Omega f(x)|\widetilde{u}_n|^q dx &\longrightarrow \lambda \int_\Omega f(x)|u_0^-|^q dx, \\ \int_\Omega g(x)|\widetilde{u}_n|^r dx &\longrightarrow \int_\Omega g(x)|u_0^-|^r dx, \end{aligned} \tag{3.5}$$

as $n \to \infty$. Similarly, we have

$$\frac{r-q}{rq}\int_\Omega g(x)|\widetilde{u}_n|^r dx = J_\lambda(\widetilde{u}_n) + \frac{a(p-q)}{pq}\|\widetilde{u}_n\|_{X_0}^p + \frac{((m+1)p-q)b}{(m+1)pq}\|\widetilde{u}_n\|_{X_0}^{(m+1)p}.$$

Letting $n \to \infty$, from Lemma 2.6, (3.4) and (3.5), we obtain

$$\int_\Omega g(x)|u_0^-|^r dx > 0.$$

From Lemma 2.4, there exists a $t_2 > 0$ such that $t_2 u_0^- \in \mathcal{N}_\lambda^-$.

Next, we show that

$$\widetilde{u}_n \longrightarrow u_0^- \text{ strongly in } X_0.$$

Suppose that this is not true. Then,

$$\|u_0^-\|_{X_0} < \liminf_{n\to\infty} \|\widetilde{u}_n\|_{X_0}.$$

Thus, for $\widetilde{u}_n \in \mathcal{N}_\lambda^-$, similar to the argument in the proof of Proposition 3.1, we have

$$J_\lambda(t_2 u_0^-) < \liminf_{n\to\infty} J_\lambda(t_2\widetilde{u}_n) \leq \lim_{n\to\infty} J_\lambda(t_2\widetilde{u}_n) \leq \lim_{n\to\infty} J_\lambda(\widetilde{u}_n) = \alpha_\lambda^-,$$

a contradiction. Hence, $\widetilde{u}_n \to u_0^-$ strongly in X_0. This implies

$$J_\lambda(\widetilde{u}_n) \longrightarrow J_\lambda(u_0^-) = \alpha_\lambda^- \quad \text{as } n \to \infty,$$

namely, u_0^- is a minimizer of J_λ on $\mathcal{N}_\lambda^-$. Using Lemma 2.1, u_0^- is a nonzero solution to problem (1.1). □

Proof of Theorem 1.1. By Propositions 3.1 and 3.2, we obtain that problem (1.1) has two nonzero solutions $u_0^+ \in \mathcal{N}_\lambda^+$ and $u_0^- \in \mathcal{N}_\lambda^-$ in X_0. Since

$$J_\lambda(u_0^\pm) = J_\lambda(|u_0^\pm|) \quad \text{and} \quad |u_0^\pm| \in \mathcal{N}_\lambda^\pm,$$

we may assume that $u_0^\pm$ are nonnegative solutions of problem (1.1). The proof of Theorem 1.1 is complete. □

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Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China
Email address: **gzmychuchangmu@sina.com**

Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China
Email address: **295284483@qq.com**

Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China
Email address: **136577802@qq.com**