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OSCILLATORY CRITERIA FOR THE SYSTEMS OF TWO FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. A definition of strict oscillation of the system of two first-order linear ordinary differential equations is given. It is shown that oscillation follows from strict oscillation of its system, but strict oscillation does not follow. Sturm-type theorems are proven. Oscillatory and strict oscillatory criteria in terms of coefficients of the system are obtained.

1. Introduction. Let $a_{jk}(t)$, $j,k=1,2$, $p(t)$, $q(t)$ and $r(t)$ be real-valued continuous functions on $R \equiv (-\infty;\infty)$, and let $p(t)>0$, $t\in R$. Consider the system

$$\begin{cases}\phi'(t)=a_{11}(t)\phi(t)+a_{12}(t)\psi(t),\\ \psi'(t)=a_{21}(t)\phi(t)+a_{22}(t)\psi(t),\end{cases} \tag{1.1}$$

and the equation

$$(p(t)\phi'(t))'+q(t)\phi'(t)+r(t)\phi(t)=0, \tag{1.2}$$

on R. The substitution $\psi(t)\equiv p(t)\phi'(t)$ in (1.2) leads to the system

$$\begin{cases}\phi'(t)=(1/p(t))\psi(t),\\ \psi'(t)=-r(t)\phi(t)-(q(t)/p(t))\psi(t),\end{cases} \tag{1.3}$$

which is a particular case of (1.1). It is not difficult to show that, when $a_{12}(t)\neq 0$, $at \in R$ or $a_{21}(t)\neq 0$, $t\in R$, the system (1.1) can be reduced to equation (1.2). Other conditions also exist on the functions $a_{jk}(t)$, $j,k=1,2$, for which system (1.1) is reducible to equation (1.2). In spite of this and the fact that the study of equation (1.2) has been the subject

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of much research, see [**1, 2, 5, 7, 9, 10, 12, 13, 14, 15, 16**], and the references cited therein, taking into account the fact that there is no known way of reducing system (1.1) to equation (1.2) in the general case, gives our premise for the study of system (1.1).

An important problem in qualitative theory regarding differential equations is the study of the question of oscillation of system (1.1). Although the study of the question of oscillation of equation (1.2) has been the subject of much research, see [**1, 10, 14, 15**] and references cited therein, as well as [**5, 7, 9, 12, 13, 16**], a result in this direction, related to system (1.1), has heretofore not been found. The set of zeros of the components of solutions of system (1.1), in contrast to the set of zeros of solutions of equation (1.2), may have a very complex structure. However, as is shown in this work, see Section 3, these sets can be partitioned into separate classes (null-classes), for which Sturm-type theorems are valid. In Section 3, we give a definition of strict oscillation of system (1.1) in terms of null-classes. We show that its oscillation follows from strict oscillation of system (1.1). It turns out that the oscillation of equation (1.2) is equivalent to the strict oscillation of system (1.3). As an example, we show that, from the oscillation of system (1.1), its strict oscillation does not follow. In Section 4, we prove oscillatory and strict oscillatory theorems for system (1.1). From Theorem 4.3, Ph. Hartman's oscillatory criterion follows (see Corollary 4.1 ff. and Theorem 4.4 ff.). We give an example, showing that Corollary 4.1 does not follow from Ph. Hartman's criterion.

2. Auxiliary propositions. Let $a(t)$, $b(t)$ and $c(t)$ be real-valued continuous functions on R. Consider the Riccati equation

$$z'(t) + a(t)z^2(t) + b(t)z(t) + c(t) = 0. \tag{2.1}$$

The solutions $z(t)$ of equation (2.1) exist on an interval $I(\subset R)$ and are connected with solutions $(\phi(t), \psi(t))$ of the system

$$\begin{cases} \phi'(t) = a(t)\psi(t), \\ \psi'(t) = -c(t)\phi(t) - b(t)\psi(t), \end{cases} \tag{2.2}$$

by relations (see [**4,** pages 153, 154])

$$\phi(t) = \phi(t_0) \exp\left\{ \int_{t_0}^{t} a(\tau)z(\tau)\, d\tau \right\}, \tag{2.3}$$

$$\psi(t) = z(t)\phi(t), \quad t, t_0 \in I,\ \phi(t_0) \neq 0$$

(t_0 is fixed).

Lemma 2.1. *For every $t_0 \in R$ and for each complex number $z_{(0)} = x_{(0)} + iy_{(0)}$ with $y_{(0)} \neq 0$, equation* (2.1) *has a solution $z_0(t)$ on R, satisfying initial value condition $z_0(t_0) = z_{(0)}$.*

The proof of this lemma is elementary, and we omit it, see **[8]**.

Let $z(t)$ be a solution of equation (2.1) on R. Denote

$$x(t) \equiv \operatorname{Re} z(t), \qquad y(t) \equiv \operatorname{Im} z(t).$$

We substitute $z(t) = x(t) + iy(t)$ in (2.1) and separate real and imaginary parts. For $y(t)$, we obtain

$$y'(t) + [2a(t)x(t) + b(t)]y(t) = 0,$$

from which

$$y(t) = y(t_0)\exp\left\{-\int_{t_0}^{t}\Big[2a(\tau)x(\tau) + b(\tau)\Big]d\tau\right\} \tag{2.4}$$

follows. Consider the Riccati equation

$$z'(t) + a_{12}(t)z^2(t) + B(t)z(t) - a_{21}(t) = 0, \tag{2.5}$$

where $B(t) \equiv a_{11}(t) - a_{22}(t)$. Let $z_0(t)$ be the solution of this equation, satisfying the initial value condition $z_0(t_0) = i$. By virtue of Lemma 2.1, $z_0(t)$ exists on R. Let $(\phi_0(t), \psi_0(t))$ be the solution of (1.1) with $\phi_0(t_0) = 1$ and $\psi_0(t_0) = i$. Then, it is easy to show that

$$\begin{aligned}\phi_0(t) &= \exp\left\{\int_{t_0}^{t}[a_{12}(\tau)z_0(\tau) + a_{11}(\tau)]\,d\tau\right\},\\ \psi_0(t) &= z_0(t)\phi_0(t).\end{aligned} \tag{2.6}$$

Since $a_{jk}(t)$, $j, k = 1, 2$, are real valued, $(\overline{\phi_0(t)}, \overline{\psi_0(t)})$ are also solutions of system (1.1). Let

$$\phi_\pm(t) \equiv \frac{\phi_0(t) \pm \overline{\phi_0(t)}}{2i^{(1\mp1)/2}}, \qquad \psi_\pm(t) \equiv \frac{\psi_0(t) \pm \overline{\psi_0(t)}}{2i^{(1\mp1)/2}}.$$

Then $(\phi_\pm(t), \psi_\pm(t))$ are real-valued solutions of system (1.1). It is evident that

$$\begin{cases} \phi_+(t) = 1, & \psi_+(t) = 0, \\ \phi_-(t) = 0, & \psi_-(t) = 1. \end{cases} \tag{2.7}$$

Denote $x_0(t) \equiv \operatorname{Re} z_0(t)$, $y_0(t) \equiv \operatorname{Im} z_0(t)$. Taking into account (2.5), from (2.6), it is easy to derive the equalities

$$\phi_+(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \cos\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right), \tag{2.8}$$

$$\psi_+(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \left[x_0(t) \cos\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right) - y_0(t) \sin\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right)\right], \tag{2.9}$$

$$\phi_-(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \sin\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right), \tag{2.10}$$

$$\psi_-(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \left[x_0(t) \sin\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right) + y_0(t) \cos\left(\int_{t_0}^t a_{12}(\tau) y_0(\tau)\, d\tau\right)\right], \tag{2.11}$$

where

$$S(t) \equiv a_{11}(t) + a_{22}(t),$$
$$J_u(t) \equiv \exp\left\{\int_{t_0}^t u(\tau)\, d\tau\right\},$$

$u(t)$ is an arbitrary continuous function on R. Denote

$$\alpha_0(t) \equiv \arcsin \frac{x_0(t)}{\sqrt{x_0^2(t) + y_0^2(t)}}.$$

Then,

$$\sin \alpha_0(t) = \frac{x_0(t)}{\sqrt{x_0^2(t) + y_0^2(t)}}, \qquad \cos \alpha_0(t) = \frac{y_0(t)}{\sqrt{x_0^2(t) + y_0^2(t)}}.$$

Now, from (2.9) and (2.11), it follows that

(2.12)
$$\psi_+(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}}\sqrt{x_0^2(t)+y_0^2(t)}\sin\left(\alpha_0(t) - \int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau\right),$$

(2.13)
$$\psi_-(t) = \frac{J_{S/2}(t)}{\sqrt{y_0(t)}}\sqrt{x_0^2(t)+y_0^2(t)}\cos\left(\alpha_0(t) - \int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau\right),$$

Since $y_0(t_0) = 1$, by virtue of (2.4), $y_0(t) > 0$, $t \in R$. Therefore,

$$-\frac{\pi}{2} < \alpha_0(t) < \frac{\pi}{2}, \quad t \in R. \tag{2.14}$$

Let $(\phi(t), \psi(t))$ be an arbitrary real-valued solution of system (1.1). Since, by (2.7), the solutions $(\phi_\pm(t), \psi_\pm(t))$ are linearly independent,

$$\phi(t) = c_+\phi_+(t) + c_-\phi_-(t), \qquad \psi(t) = c_+\psi_+(t) + c_-\psi_-(t),$$

where $c_\pm$ are some real constants. Then, from (2.8), (2.10), (2.12) and (2.13), we get

$$\phi(t) = \mu\frac{J_{S/2}(t)}{\sqrt{y_0(t)}}\sin\left(\int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau + \widetilde{\theta}\right), \tag{2.15}$$

(2.16)
$$\psi(t) = \mu\sqrt{x_0^2(t)+y_0^2(t)}\frac{J_{S/2}(t)}{\sqrt{y_0(t)}}\cos\left(\int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau + \widetilde{\theta} - \alpha_0(t)\right),$$

where

$$\mu = \mu(c_-; c_+) \equiv \sqrt{c_-^2 + c_+^2},$$
$$\widetilde{\theta} = \widetilde{\theta}(c_-; c_+) \equiv \begin{cases} \arctan(c_+/c_-), & \text{for } c_- \neq 0, \\ \arctan(c_-/c_+), & \text{for } c_+ \neq 0. \end{cases}$$

By analogy, the following equalities

(2.17)
$$\phi(t) = \mu\sqrt{x_1^2(t)+y_1^2(t)}\frac{J_{S/2}(t)}{\sqrt{y_1(t)}}\cos\left(\int_{t_0}^t a_{21}(\tau)y_1(\tau)\,d\tau + \widetilde{\theta} - \beta_0(t)\right),$$

$$\psi(t) = \mu \frac{J_{S/2}(t)}{\sqrt{y_1(t)}} \sin\bigg(\int_{t_0}^{t} a_{21}(\tau) y_1(\tau)\, d\tau + \widetilde{\theta}\bigg), \tag{2.18}$$

are derived, where $z_1(t) \equiv x_1(t) + i y_1(t)$ is the solution of

$$z'(t) + a_{21}(t) z^2(t) - B(t) z(t) - a_{12}(t) = 0, \tag{2.19}$$

with $z_1(t_0) = i$, $\beta_0(t) \equiv x_1(t)/\sqrt{x_1^2(t) + y_1^2(t)}$. Note that an equality for the general solution of equation (1.2) with $q(t) \equiv 0$, which is an analogue of (2.15), is given in [**11,** pages 152–154].

Definition 2.2. A real-valued solution of equation (2.1) is said to be t_1-*regular* if it exists on $[t_1; +\infty)$.

Definition 2.3. A t_1-regular solution $x(t)$ of equation (2.1) is said to be t_1-*normal* if there exists a $\delta > 0$ such that every solution $x_1(t)$ of equation (2.1) with $x_1(t_1) \in (x(t_1) - \delta; x(t_1) + \delta)$ is t_1-regular. Otherwise, it is said to be t_1-*marginal*.

Definition 2.4. A t_1-marginal solution $x(t)$ of equation (2.1) is said to be a *lower* (*upper*) t_1-marginal solution if every solution $x_1(t)$ of equation (2.1) with $x_1(t_1) < x(t_1)$, $(x_1(t_1) > x(t_1))$ is not t_1-regular.

Denote by $\operatorname{reg}(t_1)$ the set of $x_{(0)} \in R$, for which the solution $x(t)$ of equation (2.1) with $x(t_1) = x_{(0)}$ is t_1-regular. In the sequel, we suppose that the function $a(t)$ has unbounded support on $[t_0; +\infty)$.

Lemma 2.5. *Let* $a(t) \geq 0$, $t \geq t_1$, *and suppose that equation* (2.1) *has a* t_1*-regular solution. Then, it has a lower* t_1*-marginal solution* $x_*(t)$, *and*

$$\operatorname{reg}(t_1) = [x_*(t_1); +\infty).$$

Proof. See [**6**]. □

Let $x(t)$ be a t_1-regular solution of equation (2.1). Consider the integral

$$\nu_x(t) \equiv \int_t^{+\infty} a(\tau) \exp\bigg\{ -\int_t^{\tau} \Big[2a(s)x(s) + b(s) \Big] ds \bigg\}\, d\tau, \quad t \geq t_1.$$

Theorem 2.6 ([**6,** Theorem 2.A])**.** *Let $a(t) \geq 0$, $t \geq t_1$. Then the integral $\nu_x(t)$ converges for all $t \geq t_1$ if and only if $x(t)$ is t_1-normal.*

Denote

$$\left(\frac{b(t)}{2a(t)}\right)_0 \equiv \begin{cases} b(t)/(2a(t)) & a(t) \neq 0, \\ 0 & a(t) = 0. \end{cases}$$

The next lemma is a modification of the mandatory condition of Ph. Hartman's lemma, see [**10,** page 431, Lemma 7.1].

Lemma 2.7. *Let the following conditions hold*:

(i) $a(t) \geq 0$, $t \geq t_0$;
(ii) $\int_{t_0}^{+\infty} a(\tau)\, d\tau = +\infty$;
(iii) supp $b(t)\backslash$supp $a(t)$ *has null measure and* $(b(t)/(2a(t)))_0 \in \mathcal{L}_1^{\mathrm{loc}}(t_0; +\infty)$;
(iv) *there exists a t_0-regular solution $x(t)$ of equation* (2.1) *such that*

$$\int_{t_0}^{+\infty} a(\tau)\left[x(\tau) - \left(\frac{b(t)}{2a(t)}\right)_0\right]^2 d\tau < +\infty.$$

Then, there exists a finite limit

$$P \equiv \lim_{T\to+\infty}\left(\int_{t_0}^{T} a(\tau)\, d\tau\right)^{-1} \cdot \int_{t_0}^{T}\left[a(\tau)\int_{t_0}^{\tau}\left\{\left(\frac{b(s)}{2a(s)}\right)_0 b(s) - 2c(s)\right\} ds - \frac{b(\tau)}{2}\right] d\tau.$$

Proof. From conditions (iii) and (iv) it follows that

$$x(t) - \int_{t}^{+\infty} a(\tau)\left[x(\tau) - \left(\frac{b(\tau)}{2a(\tau)}\right)_0\right]^2 d\tau = k + \int_{t_0}^{t}\left[\left(\frac{b(\tau)}{2a(\tau)}\right)_0^2 - c(\tau)\right] d\tau, \quad t \geq t_0,$$

where $k \equiv x(t_0) - \int_{t_0}^{+\infty} a(\tau)[x(\tau) - (b(\tau)/(2a(\tau)))_0]^2 d\tau$. Then,

(2.20)
$$\left(\int_{t_0}^{T} a(\tau)\,d\tau\right)^{-1}\left\{\int_{t_0}^{T} a(\tau)\left[x(\tau) - \left(\frac{b(\tau)}{2a(\tau)}\right)_0\right] d\tau \right.$$
$$\left. - \int_{t_0}^{T} a(\tau)\,d\tau \int_{\tau}^{+\infty} a(s)\left[x(s) - \left(\frac{b(s)}{2a(s)}\right)_0\right]^2 ds\right\} = k$$
$$+ \left(\int_{t_0}^{T} a(\tau)\,d\tau\right)^{-1}\int_{t_0}^{T}\left[a(\tau)\int_{t_0}^{\tau}\left\{\left(\frac{b(s)}{2a(s)}\right)_0^2 - c(s)\right\} ds - \frac{b(\tau)}{2}\right] d\tau,$$
$$T \geq t_0.$$

By virtue of Schwartz's inequality

(2.21)
$$\left(\int_{t_0}^{T} a(\tau)\,d\tau\right)^{-1}\int_{t_0}^{T} a(\tau)\left[x(\tau) - \left(\frac{b(\tau)}{2a(\tau)}\right)_0\right] d\tau$$
$$\leq \left\{\left(\int_{t_0}^{T} a(\tau)\,d\tau\right)^{-1}\int_{t_0}^{T} a(\tau)\left[x(\tau) - \left(\frac{b(\tau)}{2a(\tau)}\right)_0\right]^2 d\tau\right\}^{1/2}, \quad T \geq t_0.$$

Let $\varepsilon > 0$. We choose $N(> t_0)$ large enough such that

$$\int_{\tau}^{+\infty} a(s)\left[x(s) - \left(\frac{b(s)}{2a(s)}\right)_0\right]^2 ds \leq \varepsilon \quad \text{for } \tau \geq N.$$

Then, by virtue of (i) for $T \geq N$, the following inequality holds:

$$\int_{t_0}^{T} a(\tau)\,d\tau \int_{\tau}^{+\infty} a(s)\left[x(s) - \left(\frac{b(s)}{2a(s)}\right)_0\right]^2 ds$$
$$\leq \int_{t_0}^{N} a(\tau)\,d\tau \int_{t_0}^{+\infty} a(s)\left[x(s) - \left(\frac{b(s)}{2a(s)}\right)_0\right]^2 ds + \varepsilon \int_{N}^{T} a(\tau)\,d\tau.$$

By (ii), it follows that, for large enough values of T,

$$f(T) \equiv \left(\int_{t_0}^{T} a(\tau)\,d\tau\right)^{-1}\int_{t_0}^{T} a(\tau)\,d\tau$$
$$\cdot \int_{\tau}^{+\infty} a(s)\left[x(s) - \left(\frac{b(s)}{2a(s)}\right)_0\right]^2 ds \leq 2\varepsilon.$$

Therefore, $\lim_{T\to+\infty} f(T) = 0$. From here, from (i), (ii), (iv), (2.20) and (2.21), existence of the finite limit P follows. The lemma is proven. $\square$

3. Sturm-type theorems. In the sequel, we assume the solutions of all equations and systems of equations to be real valued. On the set 2^R of subsets of R define an order relation $\prec$, assuming $x \prec y$ if and only if, for every $t_x \in x \in 2^R$ and $t_y \in y \in 2^R$, the inequality $t_x < t_y$ holds. Obviously, $\prec$ is a partial ordering.

Let $(\phi(t), \psi(t))$ be a nontrivial solution of system (1.1). Since the functions $\phi(t)$ and $\psi(t)$ are continuous, their zeros form closed sets.

Definition 3.1. A connected component of zeros of the function $\phi(t)$ $(\psi(t))$ will be called a null-element of the function $\phi(t)$ $(\psi(t))$.

Let $N(\phi)$ and $N(\psi)$ be null-elements of $\phi(t)$ and $\psi(t)$, correspondingly. By virtue of (2.15) and (2.18) we have:

$$\int_{t_0}^{t} a_{12}(\tau)y_0(\tau)\,d\tau + \widetilde{\theta} = \pi k_0, \quad t \in N(\phi),\ k_0 \in Z; \tag{3.1}$$

$$\int_{t_0}^{t} a_{21}(\tau)y_1(\tau)\,d\tau + \widetilde{\theta} = \pi k_1, \quad t \in N(\psi),\ k_1 \in Z; \tag{3.2}$$

$\widetilde{\theta} = \text{const}$. In the sequel, for arbitrary $t_1, t_2 \in R$ under the symbol $[t_1; t_2]$, we mean the set of points of R inclusive, lying between t_1 and t_2.

Definition 3.2. Null-elements $N_1(\phi)$ and $N_2(\phi)$ $(N_1(\psi)$ and $N_2(\psi))$ of the function $\phi(t)$ $(\psi(t))$ of the solution $(\phi(t), \psi(t))$ of system (1.1) are called *congenerous* if, for every $t_j \in N_j(\phi)$ $(\in N_j(\psi))$, $j = 1, 2$, the following inequality holds

$$\left| \int_{t_1}^{t} a_{12}(\tau)y_0(\tau)\,d\tau \right| < \pi \left(\left| \int_{t_1}^{t} a_{21}(\tau)y_1(\tau)\,d\tau \right| < \pi \right), \quad t \in [t_1; t_2].$$

It follows from this definition that, for every congenerous element $N_j(\phi)$ $(N_j(\psi))$, $j = 1, 2$, and for every $t_j \in N_j(\phi)$ $(\in N_j(\psi))$, $j = 1, 2$, the inequality

$$\Big|\int_{t_1}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau\Big| < \pi\bigg(\Big|\int_{t_1}^{t_2} a_{21}(\tau)y_1(\tau)\,d\tau\Big| < \pi\bigg), \quad t\in[t_1;t_2].$$

holds. At the same time, by virtue of (3.1) (by virtue of (3.2))

$$\Big|\int_{t_1}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau\Big| = \Big|\int_{t_0}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau - \int_{t_0}^{t_1} a_{12}(\tau)y_0(\tau)\,d\tau\Big| = \pi\widetilde{k_0},$$
$$\widetilde{k_0}\in Z,$$

$$\bigg(\Big|\int_{t_1}^{t_2} a_{21}(\tau)y_1(\tau)\,d\tau\Big| = \pi\widetilde{k_1},\ \widetilde{k_1}\in Z\bigg).$$

Therefore,

$$\int_{t_1}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau = 0\bigg(\int_{t_1}^{t_2} a_{21}(\tau)y_1(\tau)\,d\tau = 0\bigg). \tag{3.3}$$

It follows that, for every $t\in[t_1;t_2]$, the inequality

$$\begin{aligned}\Big|\int_{t_2}^{t} a_{12}(\tau)y_0(\tau)\Big| &= \Big|\int_{t_1}^{t} a_{12}(\tau)\,d\tau y_0(\tau)\,d\tau - \int_{t_1}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau\Big|\\ &= \Big|\int_{t_1}^{t} a_{12}(\tau)y_0(\tau)\,d\tau\Big| < \pi\end{aligned}$$

holds (analogously,

$$\Big|\int_{t_2}^{t} a_{21}(\tau)y_1(\tau)\Big| = \Big|\int_{t_1}^{t} a_{21}(\tau)y_1(\tau)\Big| < \pi).$$

This means that the congeniality relation is symmetric. Let $N_j(\phi)$, $j=\overline{1,3}$ be null-elements of $\phi(t)$, and suppose that $N_1(\phi)$ is congenerous with $N_2(\phi)$, and $N_2(\phi)$ is congenerous with $N_3(\phi)$. We show that $N_1(\phi)$ and $N_3(\phi)$ are congenerous. Let $t_j\in N_j(\phi)$, $j=\overline{1,3}$, and let $t\in[t_1;t_3]$. Then, $t\in[t_1;t_2]$ or $t\in[t_2;t_3]$. If $t\in[t_1;t_2]$, then

$$\Big|\int_{t_1}^{t} a_{12}(\tau)y_0(\tau)\,d\tau\Big| < \pi,$$

by virtue of congeniality of $N_1(\phi)$ and $N_2(\phi)$. However, if $t \in [t_2; t_3]$, then, taking (3.3) into account, we have

$$\left| \int_{t_1}^{t} a_{12}(\tau) y_0(\tau)\, d\tau \right| = \left| \int_{t_1}^{t_2} a_{12}(\tau) y_0(\tau)\, d\tau + \int_{t_2}^{t} a_{12}(\tau) y_0(\tau)\, d\tau \right|$$
$$= \left| \int_{t_2}^{t} a_{12}(\tau) y_0(\tau)\, d\tau \right| < \pi,$$

by virtue of the congeniality of $N_2(\phi)$ and $N_3(\phi)$. Therefore, $N_1(\phi)$ and $N_3(\phi)$ are congenerous. This means that the relation of congeniality between null-elements of the function $\phi(t)$ is transitive. From (3.1), it immediately follows reflexivity of the relation of congeniality among null-elements of function $\phi(t)$. Thus, the congeniality relations among null-elements of function $\phi(t)$ is an equivalence relation. Similarly, it is shown that the congeniality relations among null-elements of function $\psi(t)$ are equivalence relations.

Definition 3.3. The equivalence class, generated by the relation of congeniality among null-elements of function $\phi(t)$ ($\psi(t)$) of the solution $(\phi(t), \psi(t))$ of system (1.1) is called the *null-class* of function $\phi(t)$ ($\psi(t)$).

Theorem 3.4. *Null-classes of function $\phi(t)$ ($\psi(t)$) of the solution $(\phi(t), \psi(t))$ of system* (1.1) *are linearly ordered by $\prec$.*

Proof. Let $n_1(\phi)$ and $n_2(\phi)$ be distinct null classes of function $\phi(t)$. We show that $n_1(\phi) \prec n_2(\phi)$ or $n_2(\phi) \prec n_1(\phi)$. Suppose that $t_j^0 \in n_j(\phi)$, $j = 1, 2$. Then, $t_1^0 < t_2^0$ or $t_2^0 < t_1^0$. Suppose that $t_1^0 < t_2^0$. This yields

$$t_1^0 < t, \quad t \in n_2(\phi). \tag{3.4}$$

Suppose the converse, i.e., there exists a $\widetilde{t} \in n_2(\phi)$ such that $\widetilde{t} < t_1^0$ (it is evident that the equality $\widetilde{t} = t_1$ is impossible). Since, by hypothesis, $t_1^0 < t_2^0$, it follows that $t_1^0 \in [\widetilde{t}; t_2^0]$. Then,

$$\left| \int_{\widetilde{t}}^{t_1^0} a_{12}(\tau) y_0(\tau)\, d\tau \right| < \pi,$$

since $\widetilde{t}, t_2^0 \in n_2(\phi)$. This means that $\widetilde{t}$ and t_1^0 belong to the same null-class. The contradiction just obtained proves (3.4). Similarly, it can be shown that

$$t < t_2^0, \quad t \in n_1(\phi). \tag{3.5}$$

From (3.4), it follows that the set $n_2(\phi)$ is upper bounded and, from (3.5), it follows that the set $n_1(\phi)$ is lower bounded. Then, let $\overline{t_1} \equiv \sup\{t : t \in n_1(\phi)\}$ and $\underline{t_2} \equiv \inf\{t : t \in n_2(\phi)\}$. It is easy to see that $n_j(\phi)$, $j = 1, 2$, are closed sets. Then, $\overline{t_1} \in n_1(\phi)$ and $\underline{t_2} \in n_2(\phi)$. Therefore, by virtue of (3.4), we have $t_1^0 < \underline{t_2}$. Furthermore, by virtue of (3.5), $\overline{t_1} < \underline{t_2}$. It follows that, for every $t_j \in n_j(\phi)$, the inequality $t_1 \le \overline{t_1} < \underline{t_2} \le t_2$ holds. Thus, if $t_1^0 < t_2^0$, then $n_1(\phi) \prec n_2(\phi)$.

Likewise, it can be shown that, if $t_2^0 < t_1^0$, then $n_2(\phi) \prec n_1(\phi)$ or else $n_1(\phi) \prec n_2(\phi)$ or $n_2(\phi) \prec n_1(\phi)$. Hence, the set of null-classes of the function $\phi(t)$ is linearly ordered by $\prec$. Furthermore, it can be shown that the set of null-classes of function $\psi(t)$ is linearly ordered by $\prec$. The theorem is proven. □

Theorem 3.5. *There exists at least one null-element of function $\psi(t)$ ($\phi(t)$) which lies between two null-classes of function $\phi(t)$ ($\psi(t)$) of the nontrivial solution $(\phi(t), \psi(t))$ of system* (1.1).

Proof. Let $n_1(\phi) \prec n_2(\phi)$ be two null-classes of function $\phi(t)$. Denote $\overline{t_1} \equiv \sup\{t : t \in n_1(\phi)\}$ and $\underline{t_2} \equiv \inf\{t : t \in n_2(\phi)\}$. Since $\overline{t_1} \in n_1(\phi)$, then, by virtue of (2.15), the equality

$$\int_{t_0}^{\overline{t_2}} a_{12}(\tau) y_0(\tau)\, d\tau + \theta_\phi = \pi k, \quad k \in Z, \tag{3.6}$$

holds, where θ_ϕ is defined from representation (2.15) for $\phi(t)$ on $[\overline{t_1}; \underline{t_2}]$.

We consider the function

$$f_0(t) \equiv \int_{\overline{t_1}}^{t} a_{12}(\tau) y_0(\tau)\, d\tau.$$

Since $\overline{t_1}$ and $\underline{t_2}$ belong to different null-classes, there exists a $t_1 \in [\overline{t_1}; \underline{t_2}]$ such that $|f_0(t_1)| \ge \pi$. Since $f_0(\overline{t_1}) = 0$ and $f_0(t)$ is continuous, from the last inequality, $f_0(t_2) = \pm\pi$ follows for some $t_2 \in [\overline{t_1}; \underline{t_2}]$. From this and (3.6), it follows that

$$\int_{t_0}^{t_2} a_{12}(\tau) y_0(\tau)\, d\tau + \theta_\phi = \pi k_1, \tag{3.7}$$

where $k_1 = k \pm 1$. We prove the theorem in the case where $k_1 = k + 1$ (the proof in the case $k_1 = k - 1$ is similar). From (2.14) and (3.6), the inequality

$$\int_{t_0}^{\overline{t_1}} a_{12}(\tau)y_0(\tau)\,d\tau + \theta_\phi - \alpha_0(\overline{t_1}) < \pi k + \frac{\pi}{2}$$

follows and, from (2.14) and (3.17), the inequality

$$\int_{t_0}^{t_2} a_{12}(\tau)y_0(\tau)\,d\tau + \theta_\phi - \alpha_0(t_2) > \pi(k+1) - \frac{\pi}{2} = \pi k + \frac{\pi}{2}$$

follows. Therefore, there exists a $t_3 \in [\overline{t_1}; t_2]$ such that

$$\int_{t_0}^{t_3} a_{12}(\tau)y_0(\tau)\,d\tau + \theta_\phi - \alpha_0(t_3) = \pi k + \frac{\pi}{2}.$$

By virtue of (2.16), it follows that $\psi(t_3) = 0$. Let $N(\psi)$ be the null-element of the function $\psi(t)$ containing the point t_3. Since, by virtue of the uniqueness theorem, the null-elements of functions $\phi(t)$ and $\psi(t)$ are disjoint, and $\overline{t_1} < t_3 < t_2$, we have $\{\overline{t_1}\} \prec N(\psi) \prec \{\underline{t_2}\}$. It follows from here that $n_1(\phi) \prec N(\psi) \prec n_2(\phi)$. Likewise, it can be proven (using equalities (2.17) and (2.18) in place of (2.15) and (2.16)), that there exists at least one null-element of the function $\phi(t)$, which lies between two null-classes of the function $\psi(t)$. The theorem is proven. □

Corollary 3.6. *Every segment of the real line R intersects with at most a finite number of null-classes of the functions $\phi(t)$ and $\psi(t)$ of every solution $(\phi(t), \psi(t))$ of system* (1.1).

Proof. Let $(\phi(t), \psi(t))$ be a nontrivial solution of system (1.1). Suppose that some segment $[\overline{t_1}; \overline{t_2}]$ intersects with an infinite number of null-classes of the function $\phi(t)$. Then, by virtue of Theorem 3.1, there exists an infinite-ordered sequence

$$n_1(\phi) \prec n_2(\phi) \prec \cdots \prec n_m(\phi) \prec \cdots \tag{3.8}$$

or a sequence

$$\cdots \prec n_m(\phi) \prec \cdots \prec n_2(\phi) \prec n_1(\phi), \tag{3.9}$$

of null-classes of the function $\phi(t)$, each element of which intersects with $[\overline{t_1}; \overline{t_2}]$. Suppose that sequence (3.8) exists with the above-mentioned property. By virtue of Theorem 3.2, there exist null-elements $N_j(\psi)$, $j = 1, 2, \ldots$, of function $\psi(t)$ such that

$$n_1(\phi) \prec N_1(\psi) \prec n_2(\phi) \prec N_3(\psi) \prec n_3(\phi) \prec \cdots . \tag{3.10}$$

Suppose that $t_j \in n_j(\phi) \cap [\overline{t_1}; \overline{t_2}]$, $j = 1, 2, \ldots$. It follows from (3.10) that $\{t_j\} \prec N_j(\psi) \prec \{t_{j+1}\}$, $j = 1, 2, \ldots$. Therefore, the intersections $N_j(\psi) \cap [\overline{t_1}; \overline{t_2}]$, $j = 1, 2, \ldots$, are nonempty. Let $t'_j \in N_j(\psi) \cap [\overline{t_1}; \overline{t_2}]$, $j = 1, 2, \ldots$. It follows from (3.10) that

$$\overline{t_1} \le t_1 < t'_1 < t_2 < t'_2 < \cdots < t_j < t'_j < t_{j+1} < \cdots < \overline{t_2}.$$

Therefore, limits exist which are equal to each other.

$$\lim_{j\to+\infty} t_j = \lim_{j\to+\infty} t'_j \equiv t_0 \in [\overline{t_1}; \overline{t_2}]. \tag{3.11}$$

Since $\phi(t_j) = \psi(t'_j) = 0$, $j = 1, 2, \ldots$, by virtue of (3.11) and continuity of the functions $\phi(t)$ and $\psi(t)$, equality takes place at $\phi(t_0) = \psi(t_0) = 0$, which is impossible. The contradiction thus obtained shows that every segment of R intersects with at most a finite number of null-classes of function $\phi(t)$.

Likewise, it can be proven that every segment of R intersects with at most a finite number of null-classes of function $\psi(t)$. The corollary is proven. □

The proof of sequence (3.9), with the above-mentioned assumptions, is similar.

If $a_{12}(t)$ does not change sign, then the function

$$g_0(t) \equiv \int_{t_0}^{t} a_{12}(\tau) y_0(\tau)\, d\tau$$

is monotone. In this case, by virtue of (2.15), every null-class of function $\phi(t)$ of every solution $(\phi(t), \psi(t))$ of system (1.1) consists of only one null-element. Similarly, if $a_{21}(t)$ does not change sign, then every null-class of function $\psi(t)$ of the above-mentioned solution consists of only one null-element. Therefore, from Theorem 3.5, we immediately obtain the next corollary.

Corollary 3.7. *If $a_{12}(t)$ and $a_{21}(t)$ do not change signs, then, for every nontrivial solution $(\phi(t), \psi(t))$ of system* (1.1)*, the null-elements* *of function $\phi(t)$ are separate null-elements of function $\psi(t)$ and are separated by them.*

The next theorem is an analog of Sturm's separation theorem, see, [**10,** page 396], [**14,** pages 167, 168].

Theorem 3.8. *Let $(\phi_j(t), \psi_j(t))$, $j = 1, 2$, be linearly independent solutions of system* (1.1). *Then, between two null-classes of function $\phi_1(t)$ $(\psi_1(t))$ lies at least one null-element of function $\phi_2(t)$ $(\psi_2(t))$.*

Proof. By virtue of (2.15), we have

$$\phi_j(t) = \mu_j \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \sin\bigg(\int_{t_0}^{t} a_{12}(\tau) y_0(\tau)\, d\tau + \theta_j\bigg), \tag{3.12_j}$$

where μ_j and θ_j are some real constants, $j = 1, 2$. Let $n_1(\phi_1) \prec n_2(\phi_1)$ be two neighbor null-classes of function $\phi_1(t)$. Denote $\overline{t_1} \equiv \sup\{t : t \in n_1(\phi)\}$ and $\underline{t_2} \equiv \inf\{t : t \in n_2(\phi)\}$. Consider the functions

$$f_j(t) \equiv \int_{t_0}^{t} a_{12}(\tau) y_0(\tau)\, d\tau + \theta_j, \quad j = 1, 2.$$

Since $\overline{t_1} \in n_1(\phi_1)$ and $\underline{t_2} \in n_2(\phi_1)$, from (3.12_j), $j = 1, 2$, it follows that $f_2(\underline{t_2}) - f_2(\overline{t_1}) = f_1(\underline{t_2}) - f_1(\overline{t_1}) = \pm\pi$. From this and from (3.12_2), by virtue of the continuity of $f_2(t)$, it follows that $\phi_2(t_3) = 0$ for some $t_3 \in [\overline{t_1}; \underline{t_2}]$. Let $N(\phi_2)$ be a null-element of function $\phi_2(t)$, containing the point t_3. Since $(\phi_j(t), \psi_j(t))$ $j = 1, 2$, are linearly independent, $\overline{t_1}, \underline{t_2} \notin N(\phi_2)$. Therefore, $\{\overline{t_1}\} \prec N(\phi_2) \prec \{\underline{t_2}\}$. It follows that $n_1(\phi_1) \prec N(\phi_2) \prec n_2(\phi_1)$. Similarly, it can be shown that between two null-classes of function $\psi_1(t)$ lies at least one null-element of function $\psi_2(t)$. The theorem is proven. □

If $a_{12}(t)$ $(a_{21}(t))$ does not change sign, then, as shown above, every null-class of function $\phi(t)$ $(\psi(t))$ of every solution $(\phi(t), \psi(t))$ of system (1.1) consists of only one null-element. Therefore, from Theorem 3.3, we immediately obtain:

Corollary 3.9. *Let $(\phi_j(t), \psi_j(t))$ $j = 1, 2$, be linearly independent solutions of system* (1.1), *and suppose that $a_{12}(t)$ $(a_{21}(t))$ does not* *change sign. Then the null-elements of function $\phi_1(t)$ ($\psi_1(t)$) separate null-elements of function $\phi_2(t)$ ($\psi_2(t)$) and are separated by them.*

4. Oscillation. Hereinafter, we consider equations and systems of equations on the half axis $[t_0; +\infty)$ (t_0 fixed).

Definition 4.1. System (1.1) is said to be *oscillatory* if, for every nontrivial solution $(\phi(t), \psi(t))$, functions $\phi(t)$ and $\psi(t)$ have arbitrary large zeros.

Definition 4.2. System (1.1) is said to be *strictly oscillatory* if, for every nontrivial solution $(\phi(t), \psi(t))$, functions $\phi(t)$ and $\psi(t)$ have an infinite number of null-classes.

By virtue of Corollary 3.1, if (1.1) is strictly oscillatory, then it is oscillatory. The next example shows that the converse assertion is false.

Example 4.3. Suppose that $t_0 < \xi_0 < \eta_0 < t_1 < \xi_1 < \eta_1 < t_0 + T$, $\eta_0 - \xi_0 = \eta_1 - \xi_1 = 2\pi$, and let $a(t)$ be a continuous and periodic function on $[t_0; +\infty)$ with period T satisfying the conditions: $a(t) > 0$, $t \in (t_0; t_1)$, $a(t) < 0$, $t \in (t_1; t_0 + T)$, $a(t) = 1$, $t \in [\xi_0; \eta_0]$, $a(t) = -1$, $t \in [\xi_1; \eta_1]$.

Define

$$\Lambda(t) \equiv \begin{cases} \lambda_k & t \in [t_0 + kT; t_1 + kT], \qquad k = 0, 1, 2, \ldots, \\ \mu_k & t \in [t_1 + kT; t_0 + (k+1)T], \quad k = 0, 1, 2, \ldots. \end{cases}$$

We set $a_{12}(t) \equiv \Lambda(t)a(t)$, $a_{21}(t) \equiv -a_{12}(t)$, $t \geq t_0$. Consider the system

$$\begin{cases} \phi'(t) = a_{12}(t)\psi(t), \\ \psi'(t) = a_{21}(t)\phi(t). \end{cases} \tag{4.1}$$

Let $(\phi_j(t), \psi_j(t))$, $j = 1, 2$, be the solutions of this system such that

$$\begin{cases} \phi_1(t_0) = 0 \quad \psi_1(t_0) = 1, \\ \psi_2(t_0) = 1 \quad \psi_2(t_0) = 0. \end{cases} \tag{4.2}$$

It is not difficult to see that

$$\phi_2(t) = \psi_1(t), \qquad \psi_2(t) = -\psi_1(t), \quad t \geq t_0. \tag{4.3}$$

Obviously, on segment $[\xi_0 + 2kT; \eta_0 + 2kT]$, system (4.1) is equivalent to the equation $u''(t) + \lambda_k^2 u(t) = 0$, and on segment $[\xi_0 + (2k+1)T; \eta_0 + (2k+1)T]$ system (4.1) is equivalent to the equation $u''(t) + \mu_k^2 u(t) = 0$, $k = 1, 2, \dots$. Then, since $\eta_0 - \xi_0 = \eta_1 - \xi_1 = 2\pi$ for $\lambda_k = 1$ ($\mu_k = 1$), each of the functions $\phi_j(t)$ ($\psi_j(t)$), $j = 1, 2$, on each segment $[t_0 + kT; t_1 + kT]$ ($[t_1 + kT; t_0 + (k+1)T]$) has at least one zero. By virtue of (2.15), it follows that, for $\lambda_k = \mu_k = 1$, the following inequalities hold:

$$\int_{t_0+kT}^{t_1+kT} a_{12}(\tau) y_0(\tau)\, d\tau > \pi,$$

$$\int_{t_1+kT}^{t_0+(k+1)T} a_{12}(\tau) y_0(\tau)\, d\tau < -\pi, \quad k = 0, 1, 2, \dots.$$

Taking these inequalities and continuous dependence $y_0(t)$ on λ_k and μ_k, $k = 0, 1, \dots$, into account, we choose $\lambda_0 > 0$ such that

$$\int_{t_0}^{t_1} a_{12}(\tau) y_0(\tau)\, d\tau = \pi.$$

Next, choose $\mu_0 > 0$ such that

$$\int_{t_1}^{t_0+T} a_{12}(\tau) y_0(\tau)\, d\tau = -\pi.$$

Finally, choose $\lambda_1 > 0$ and $\mu_1 > 0$ such that

$$\int_{t_0+T}^{t_1+T} a_{12}(\tau) y_0(\tau)\, d\tau = \pi, \qquad \int_{t_1}^{t_0+2T} a_{12}(\tau) y_0(\tau)\, d\tau = -\pi,$$

and so on. After a countable number of such operations of successive determinations of $\lambda_k > 0$ and $\mu_k > 0$, $k = 0, 1, 2, \dots$, we obtain a function $a_{12}(t)$ such that

$$\int_{t_0+kT}^{t_1+kT} a_{12}(\tau) y_0(\tau)\, d\tau = \pi, \tag{4.4}$$

$$\int_{t_1+kT}^{t_0+(k+1)T} a_{12}(\tau) y_0(\tau)\, d\tau = -\pi, \quad k = 0, 1, 2, \dots.$$

Then,

$$\phi_1(t_0 + kT) = \phi_2(t_1 + kT) = 0, \quad k = 0, 1, 2, \ldots. \tag{4.5}$$

Note that

$$0 \le \int_{t_0}^{t} a_{12}(\tau) y_0(\tau)\, d\tau \le \pi, \quad t \ge t_0.$$

Then, by virtue of (2.15) and (4.2), $\phi_1(t) \ge 0$, $t \ge t_0$. It follows that all zeros of function $\psi_1(t)$ are contained in one (unique) null-class. Thus, system (4.1) is not strictly oscillatory. We show that it is oscillatory. From (4.5), it follows that $\phi_1(t)$ has arbitrary large zeros. Let $(\phi_3(t), \psi_3(t))$ be a solution of (4.1), which is linearly independent of $(\phi_1(t), \psi_1(t))$. From (4.4), it follows that $t_0, t_1 + T, t_0 + 2T, \ldots$, pairwise belong to different null-classes of function $\phi_1(t)$. By virtue of Theorem 3.3, and (4.5), it follows that $\phi_3(t)$ has arbitrary large zeros. Similarly, using (4.3)–(4.5), it can be shown that, for every solution $(\phi(t), \psi(t))$ of system (4.1), function $\psi(t)$ has arbitrary large zeros. Therefore, system (4.1) is oscillatory.

In equation (2.5), we make the change $z(t) \equiv ctg\theta(t)$, which leads to the equation

$$\theta'(t) = a_{12}(t)\cos^2\theta(t) + B(t)\sin\theta(t)\cos\theta(t) - a_{21}(t)\sin^2\theta(t). \tag{4.6}$$

Since the right hand side of (4.6) for every continuous function $\theta(t)$ on $[t_0; t_1)$, $t_1 < +\infty$ is bounded on $[t_0; t_1)$ (see [**3,** pages 274, 275], [**10,** page 19], Picard's theorem 1.1), for every $\theta_{(0)} \in R$, it has a unique solution $\theta(t)$ on $[t_0; +\infty)$ satisfying the initial value condition $\theta(t_0) = \theta_{(0)}$. Rewrite equation (4.6) in the form

$$2\theta'(t) = a_{12}(t) - a_{21}(t) + \sqrt{B^2(t) + (a_{12}(t) + a_{21}(t))^2}\,\sin(2\theta(t) + \gamma(t)), \tag{4.7}$$

where

$$\gamma(t) \equiv \begin{cases} \arcsin(a_{12}(t) + a_{21}(t))/\sqrt{B^2(t) + (a_{12}(t) + a_{21}(t))^2} & \\ & a_{12}(t) + a_{21}(t) \ne 0, \\ 0 & a_{12}(t) + a_{21}(t) = 0. \end{cases}$$

Denote $R_\pm(t) \equiv a_{12}(t) - a_{21}(t) \pm \sqrt{B^2(t) + (a_{12}(t) + a_{21}(t))^2}$. Let $\theta(t)$ be a solution of equation (4.7). Then, obviously,

$$R_-(t) \le 2\theta'(t) \le R_+(t), \quad t \ge t_0.$$

It follows that

$$\int_{t_0}^{t} R_-(\tau)\,d\tau \le 2(\theta(t)-\theta(t_0)) \le \int_{t_0}^{t} R_+(\tau)\,d\tau, \quad t \ge t_0. \tag{4.8}$$

Lemma 4.4. *Let* $(\phi(t),\psi(t))$ *be a solution of system* (1.1) *with* $\phi(\xi) \neq 0$, *and let* $\theta_0(t)$ *be the solution of equation* (4.7), *satisfying the initial value condition* $\cot\theta_0(\xi) = \psi(\xi)/\phi(\xi)$. *Then, in all of the points of null-class* $n(\phi)$ *of function* $\phi(t)$, *the solution* $\theta_0(\xi)$ *takes the same value* πk_0, $k_0 \in Z$.

Proof. By the uniqueness theorem, $\cot\theta_0(t)$ coincides with the solution $x(t) \equiv \psi(t)/\phi(t)$ of equation (2.5) on I, where $I = (p;q)$ is a maximum interval of existence for $x(t)$, or $I = [t_0;q)$, $x(t)$ exists on $[t_0;q)$ and cannot be continued beyond q. Therefore, $\theta_0(q) = \pi k_0$ for some $k_0 \in Z$. Let $t_1 \in n(\phi)$. We show that $\theta_0(t_1) = \pi k_0$. Consider the case $t_1 > q$ (the proof in the case $t_1 < q$ is similar). Suppose that $\theta_0(t_1) \neq \pi k_0$. Then, $\theta(t_1) = \pi k_1$, $k_1 \in Z$, $k_1 \neq k_0$. This means that $|\theta_0(t_1) - \theta_0(q)| \ge \pi$. By continuity of $\theta_0(t)$, it follows that $\theta_0(t_2) = \theta_0(q) \pm \pi$ for some $t_2 \in (q;t_1]$.

Suppose that $\theta_0(t_2) = \theta_0(q) + \pi$ (the proof in the case $\theta_0(t_2) = \theta_0(q) - \pi$ is similar). Denote $\overline{t_1} \equiv \max\{t \in [q;t_1] \cap n(\phi) : \theta_0(t) = \pi k_0\}$ and $\underline{t_2} \equiv \min\{t \in [\overline{t_1};t_1] \cap n(\phi) : \theta_0(t) = \pi k_0 + \pi\}$. It is evident that $\overline{t_1} < \underline{t_2}$,

$$\theta_0(\overline{t_1}) = \pi k_0, \qquad \theta_0(\underline{t_2}) = \pi k_0 + \pi, \tag{4.9}$$

and

$$\pi k_0 < \theta_0(t) < \pi k_0 + \pi, \quad t \in (\overline{t_1};\underline{t_2}). \tag{4.10}$$

This means that $\widetilde{x}(t) \equiv \cot\theta_0(t)$ exists on $(\overline{t_1};\underline{t_2})$ and defines a solution of equation (2.5) there. Since, by (2.6), $\widetilde{x}(t) = \widetilde{\psi(t)}/\widetilde{\phi(t)}$, where $(\widetilde{\phi(t)},\widetilde{\psi(t)})$ is a solution of system (1.1), and, by (4.9), $\widetilde{x}(\overline{t_1}) = \pm\infty$, $\widetilde{x}(\underline{t_2}) = \pm\infty$, we have $\widetilde{\phi}(\overline{t_1}) = \widetilde{\phi}(\underline{t_2}) = 0$. It follows that $(\widetilde{\phi(t)},\widetilde{\psi(t)}) = \lambda(\phi(t),\psi(t))$, $\lambda = \text{const} \neq 0$. Therefore, $\widetilde{x}(t) = \psi(t)/\phi(t)$, $t \in (\overline{t_1};\underline{t_2})$. Furthermore, it follows that $\phi(t)$ does not change sign on $(\overline{t_1};\underline{t_2})$.

By (2.16), the following equality occurs:
(4.11)
$$\psi(t)=\mu_\phi\sqrt{x_0^2(t)+y_0^2(t)}\,\frac{J_{S/2}(t)}{\sqrt{y_0(t)}}\cos\bigg(\int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau+\theta_\phi-\alpha_0(t)\bigg),$$
where μ_ϕ and θ_ϕ are some constants. Since $\overline{t_1}, \underline{t_2} \in n(\phi)$, by (3.3),
$$\int_{t_0}^{\overline{t_1}} a_{12}(\tau)y_0(\tau)\,d\tau+\theta_\phi=\int_{t_0}^{\underline{t_2}} a_{12}(\tau)y_0(\tau)\,d\tau+\theta_\phi=\pi\widetilde{k}, \quad \widetilde{k}\in Z.$$
Now, from (2.14) and (4.11), it follows that $\operatorname{sign}\psi(\overline{t_1})=\operatorname{sign}\psi(\underline{t_2})\neq 0$. Then, since $\phi(t)$ does not change sign on $(\overline{t_1};\underline{t_2})$, we have $\widetilde{x}(\overline{t_1}+0)=x(\overline{t_1}+0)=x(\underline{t_2}-0)=\widetilde{x}(\underline{t_2}-0)=\pm\infty$. However, on the other hand, from (4.9) and (4.10), it follows that $\widetilde{x}(\overline{t_1}+0)=\cot\theta_0(\overline{t_1}+0)\neq\cot\theta_0(\underline{t_2}-0)=\widetilde{x}(\underline{t_2}-0)$. The contradiction just obtained shows that $\theta_0(t)$ takes the same value πk_0 on $n(\phi)$. The lemma is proven. □

In equation (2.19), we change $z(t)=\cot\theta(t)$, which yields
$$2\theta'(t)=a_{21}(t)-a_{12}(t)+\sqrt{B^2(t)+(a_{12}(t)+a_{21}(t))^2}\sin(2\theta(t)+\gamma(t)).$$
It is evident that, for every solution $\theta(t)$ of this equation,
$$-R_+(t)\le 2\theta'(t)\le -R_-(t), \quad t\ge t_0.$$
Therefore,
$$\int_{t_0}^t -R_+(\tau)\,d\tau\le 2(\theta(t)-\theta(t_0))\le-\int_{t_0}^t R_-(\tau)\,d\tau, \quad t\ge t_0. \tag{4.12}$$

Theorem 4.5. *If*
$$\limsup_{t\to+\infty}\int_{t_0}^t R_-(\tau)\,d\tau=+\infty \quad \textit{or} \quad \liminf_{t\to+\infty}\int_{t_0}^t R_+(\tau)\,d\tau=-\infty, \tag{4.13}$$
then system (1.1) *is oscillatory, and, if*
$$\int_{t_0}^{+\infty} R_-(\tau)\,d\tau=+\infty \quad \textit{or} \quad \int_{t_0}^{+\infty} R_+(\tau)\,d\tau=-\infty, \tag{4.14}$$
then system (1.1) *is strictly oscillatory.*

Proof. Suppose that system (1.1) is not oscillatory. Then, there exists a solution $(\phi(t),\psi(t))$ of system (1.1) such that $\phi(t)\neq 0$, $t\ge T$, or $\psi(t) \neq 0$, $t \geq T$, for some $T \geq t_0$. Suppose that $\phi(t) \neq 0$, $t \geq T$. Then, by (3.6), $x(t) \equiv \psi(t)/\phi(t)$ is a solution of equation (2.5) on $[T; +\infty)$. Therefore, $\theta(t) \equiv \operatorname{arccot} x(t)$ is a solution of equation (4.7) on $[T; +\infty)$. By (4.8), we have:

$$\int_T^t R_-(\tau)\, d\tau \leq 2[\theta(t) - \theta(T)] \leq \int_T^t R_+(\tau)\, d\tau, \quad t \geq T.$$

It follows that

$$\int_T^t R_-(\tau)\, d\tau \leq 2\pi - 2\theta(T), \quad t \geq T, \tag{4.15}$$

$$\int_T^t R_+(\tau)\, d\tau \geq -2\pi - 2\theta(T), \quad t \geq T. \tag{4.16}$$

Suppose that the first of equalities (4.13) occurs. Then,

$$\limsup_{t\to+\infty} \int_T^t R_-(\tau)\, d\tau = +\infty,$$

which contradicts (4.15). Suppose that the second of equalities (4.13) occurs. Then,

$$\liminf_{t\to+\infty} \int_T^t R_+(\tau)\, d\tau = -\infty,$$

which contradicts (4.16). Thus, the assumption that $\phi(t) \neq 0$, $t \geq T$, leads to the contradiction.

Similarly, using (4.12) in place of (4.8), it can be shown that the assumption $\psi(t) \neq 0$, $t \geq T$, for some $T \geq t_0$, leads to a contradiction. Therefore, system (1.1) is oscillatory. Suppose that (4.14) takes place. Then (4.13) holds. Therefore, as already proven, system (1.1) is oscillatory. Thus, for every solution $(\phi(t), \psi(t))$ of system (1.1), the function $\phi(t)$ ($\psi(t)$) has at least one null-class.

Suppose that (1.1) is not strictly oscillatory. Then, for its solution $(\phi(t), \psi(t))$, function $\phi(t)$ or $\psi(t)$ has a finite number of null-classes. Suppose that $\phi(t)$ has a finite number of null-classes $n_1(\phi) \prec n_2(\phi) \prec \cdots \prec n_m(\phi)$ (the proof which assumes that $\psi(t)$ has a finite number of null-classes, is similar). Since $\phi(t)$ has arbitrary large zeros, there exists an infinitely large sequence $\{t_n\} \subset n_m(\phi)$.

Suppose that $\phi(\xi) \neq 0$, and let $\theta_0(t)$ be the solution of equation (4.7) satisfying the condition $\cot \theta_0(\xi) = \psi(\xi)/\phi(\xi)$. Then, by virtue of Lemma 4.1, $\theta_0(t_n) = \pi k_0$, $k_0 \in Z$, $n = 1, 2, \ldots$. By virtue of (4.8), it follows that

$$\int_{t_1}^{t_n} R_-(\tau)\, d\tau \leq 0, \quad n = 1, 2, \ldots, \tag{4.17}$$

$$\int_{t_1}^{t_n} R_+(\tau)\, d\tau \geq 0, \quad n = 1, 2, \ldots. \tag{4.18}$$

Then, the first of equalities (4.14) contradicts (4.17), and the second contradicts (4.18). The above-obtained contradiction proves the theorem. □

Remark 4.6. It is not difficult to check that $-R_-(t)R_+(t)$ coincides with the characteristic discriminant of system (1.1). Therefore, in the case of constant coefficients $a_{jk}(t)$, $j, k = 1, 2$, the conditions of the theorem are necessary for oscillation of system (1.1) as well.

Theorem 4.7. *Suppose that the integrals* $\int_{t_0}^{+\infty} R_\pm(\tau)\, d\tau$ *converge. Then, system* (1.1) *has a solution* $(\phi(t), \psi(t))$ *such that* $\phi(t) \neq 0$, $\psi(t) \neq 0$, $t \geq T$, *for some* $T \geq t_0$.

Proof. We choose $T \geq t_0$ large enough such that

$$\left| \int_T^{+\infty} R_\pm(\tau)\, d\tau \right| < \frac{\pi}{8}. \tag{4.19}$$

Let $\theta(t)$ be a solution of equation (4.7) with $\theta(T) = \pi/8$. Then, by (4.8), we have:

$$\int_T^t R_-(\tau)\, d\tau \leq 2\theta(t) - \frac{\pi}{4} \leq \int_T^t R_+(\tau)\, d\tau, \quad t \geq T.$$

From this and (4.19), it follows that

$$\frac{\pi}{16} \leq \theta(t) \leq \frac{3\pi}{16}, \quad t \geq T.$$

Then, $x(t) \equiv \cot\theta(t)$ is a solution of equation (2.5) on $[T;+\infty)$, and $x(t) \neq 0$, $t \geq T$. By (2.6), it follows that the functions

$$\phi(t) \equiv \exp\left\{\int_T^t [a_{12}(\tau)x(\tau) + a_{11}(\tau)]\,d\tau\right\}, \qquad \psi(t) \equiv x(t)\phi(t),$$

form a solution $(\phi(t), \psi(t))$ of system (1.1) on $[T;+\infty)$, and $\phi(t) \neq 0$, $\psi(t) \neq 0$, $t \geq T$. Evidently, the continuation of this solution on $[t_0;+\infty)$ is the required solution of system (1.1). The theorem is proven. □

Lemma 4.8. *Suppose that $a_{12}(t)$ does not change sign. Then, if, for some solution $(\phi(t), \psi(t))$ of system* (1.1), *function $\phi(t)$ has an infinite number of null-classes, then system* (1.1) *is strictly oscillatory.*

Proof. Since $a_{12}(t)$ does not change sign, the function

$$f_0(t) \equiv \int_{t_0}^t a_{12}(\tau)y_0(\tau)\,d\tau$$

is monotone. Then, by (2.15), every null-class of function $\phi(t)$ consists only of one null-element. Therefore, the number of null-classes of function $\phi(t)$ is infinite. Let

$$n_1(\phi) \prec n_2(\phi) \prec \cdots \prec n_m(\phi) \prec \cdots$$

all be null-classes of function $\phi(t)$. By Theorem 3.2, between $n_j(\phi)$ and $n_{j+1}(\phi)$ lies some null-element $N_j(\psi)$ of function $\psi(t)$. Thus,

$$N_j(\psi) \prec n_{j+1}(\phi) \prec N_{j+1}(\psi), \quad j = 1, 2, \ldots, \tag{4.20}$$

and there is no other null-class of function $\phi(t)$ between $N_j(\psi)$ and $N_{j+1}(\psi)$. Let $t_j \in N_j(\psi)$, $\widetilde{t_j} \in n_j(\phi)$, $j = 1, 2, \ldots$. By (4.20), we have:

$$t_j < \widetilde{t_{j+1}} < t_{j+1}, \quad j = 1, 2, \ldots. \tag{4.21}$$

Since $f_0(t)$ is monotone and $\phi(\widetilde{t_j}) = \psi(t_j) = 0$, by (2.15) and (4.21), it follows that function $\phi(t)$ takes the values of different signs in the points t_j and t_{j+1} (due to only one null-element of function $\phi(t)$ which lies between $N_j(\psi)$ and $N_{j+1}(\psi)$). By (2.18), it follows that

$$\left|\int_{t_j}^{t_{j+1}} a_{21}(\tau)y_1(\tau)\,d\tau\right| = \pi.$$

Then, from (4.20) and (4.21), it follows that $N_j(\psi)$ and $N_m(\psi)$ are contained in different null-classes of function $\psi(t)$ for all $j, m = 1, 2, \ldots, j \neq m$. Therefore, $\psi(t)$ has infinitely many null-classes. Let $(\phi_1(t), \psi_1(t))$ be a nontrivial solution of system (1.1). Since $a_{12}(t)$ does not change sign, and $\phi(t)$ has infinitely many null-classes, by (2.15),

$$\int_{t_0}^{+\infty} |a_{12}(\tau)| y_0(\tau)\, d\tau = +\infty.$$

By (2.15), it follows that $\phi_1(t)$ has infinitely many null-classes. Then, that the function $\psi_1(t)$ has infinitely many null-classes follows from the above observations. The lemma is proven. $\square$

Denote

$$\left(\frac{B(t)}{a_{12}(t)}\right)_0 \equiv \begin{cases} B(t)/a_{12}(t) & a_{12}(t) \neq 0, \\ 0 & a_{12}(t) = 0, \end{cases}$$

$$F(t) \equiv \left(\int_{t_0}^{T} a_{12}(\tau)\, d\tau\right)^{-1} \cdot \int_{t_0}^{T} \left[a_{12}(\tau) \int_{t_0}^{\tau} \left\{\left(\frac{B(s)}{a_{12}(s)}\right)_0 \frac{B(s)}{2} + 2a_{21}(s)\right\} ds - B(\tau)\right] d\tau,$$

$$T \geq t_0.$$

Theorem 4.9. *Suppose that the following conditions hold*:

(1°) $a_{12}(t) \geq 0$, $t \geq t_0$;

(2°) $\int_{t_0}^{+\infty} a_{12}(\tau)\, d\tau = +\infty$;

(3°) supp $B(t) \setminus$ supp $a_{12}(t)$ *has zero measure, and* $(B(t)/a_{12}(t))_0 \in \mathcal{L}_1^{\text{loc}}(t_0; +\infty)$;

(4°) $\liminf_{T\to+\infty} F(T) < \limsup_{T\to+\infty} F(T)$;

(5°) *there exists a* $k_0 \geq 0$ *such that*

$$\int_{t_0}^{+\infty} a_{12}(T) \exp\left\{\int_{t_0}^{T} \left\{a_{12}(\tau)\left[k_0 - \int_{t_0}^{\tau} \left\{2a_{21}(s) + \frac{B(s)}{2}\left(\frac{B(s)}{a_{12}(s)}\right)_0\right\} ds\right] - B(\tau)\right\} d\tau\right\} dT = +\infty.$$

Then, system (1.1) *is strictly oscillatory.*

Proof. Let $(\phi(t), \psi(t))$ be a nontrivial solution of system (1.1) with $\phi(t_0) = 0$. By condition (1°) and Lemma 4.2, it is enough to show that $\phi(t)$ has infinitely many null-elements. Suppose that $\phi(t)$ has a finite number of null-elements $N_1(\phi) \prec \cdots \prec N_m(\phi)$ (since $\phi(t_0) = 0$, then $\phi(t)$ has at least one null-element). We show that $N_m(\phi)$ is upper bounded.

For the converse, suppose that it is not so. Then, by (2.15), from (1°), it follows that $N_m(\phi) = [t_1; +\infty)$ for some $t_1 \geq t_0$. From this and from the first equation of system (1.1), it follows that $a_{12}(t) \equiv 0$ on $[t_1; +\infty)$. However, then

$$\int_{t_0}^{+\infty} a_{12}(\tau)\, d\tau = \int_{t_0}^{t_1} a_{12}(\tau)\, d\tau < +\infty,$$

which contradicts condition (2°). The above-obtained contradiction proves the upper boundedness of $N_m(\phi)$. Therefore, there exists a $t_2 \geq t_0$ such that $\phi(t) \neq 0$, $t \geq t_2$. By (2.6), it follows that equation (2.5) has a t_2-regular solution. Then, by Lemma 2.2 and (1°), it follows that (2.5) has a t_2-normal solution $x_N(t)$. Taking into account (3°), we have:

$$x_N(t) = x_N(t_2) - \int_{t_2}^{t} a_{12}(\tau)\left[x_N(\tau) + \frac{1}{2}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right]^2 d\tau + \int_{t_2}^{t}\left[a_{12}(\tau) + \frac{B(\tau)}{4}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right] d\tau, \quad t \geq t_0. \tag{4.22}$$

By virtue of Lemma 2.3, and (1°)–(4°), it follows that

$$\int_{t_2}^{+\infty} a_{12}(\tau)\left[x_N(\tau) - \frac{1}{2}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right]^2 d\tau = +\infty.$$

In view of this, we choose $t_3 \geq t_2$ such that

$$x_N(t_2) - \int_{t_2}^{t} a_{12}(\tau)\left[x_N(\tau) - \frac{1}{2}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right]^2 d\tau \leq -\frac{k_0}{2} + \int_{t_0}^{t_2}\left[a_{21}(\tau) + \frac{B(\tau)}{4}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right] d\tau,$$

for $t \geq t_3$. Then, from (4.22), we obtain

(4.23)
$$\int_{t_3}^{+\infty} a_{12}(T) \exp\left\{-\int_{t_3}^{T}\left[2a_{12}(t)x_N(t) + B(t)\right] dt\right\} dT$$
$$\geq M\int_{t_3}^{+\infty} a_{12}(T) \exp\left\{\int_{t_0}^{T}\left\{a_{12}(t)\left[k_0 - \int_{t_0}^{t}\left\{2a_{21}(\tau) + \frac{B(\tau)}{2}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right\}d\tau\right] - B(t)\right\}dt\right\}dT,$$

where

$$M \equiv \exp\left\{ - \int_{t_9}^{t_3} \left\{a_{12}(t)\left[k_0 - \int_{t_0}^{t} \left\{2a_{21}(\tau) + \frac{B(\tau)}{2}\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0\right\} d\tau\right] - B(t)\right\} dt\right\}.$$

Since $x_N(t)$ is t_3-normal (because it is t_2-normal), by Theorem 2.1, the left hand side of (4.23) is finite, while from (5°), it follows that its right hand side is equal to $+\infty$. The above-obtained contradiction shows that $\phi(t)$ has infinitely many null-classes. The theorem is proven. □

By virtue of the connection between (1.2) and (1.3) from Theorem 4.3, we immediately obtain:

Corollary 4.10. *Suppose that the following conditions hold*:

(1) $\liminf_{T\to+\infty}(1/T)\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau < \limsup_{T\to+\infty}(1/T)\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau$;

(2) *there exists a* $k_0 \geq 0$ *such that*

$$\int_{t_0}^{+\infty} \exp\left\{k_0 T + 2\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau\right\} dT = +\infty.$$

Then, the equation

(4.24)
$$\phi''(t) + r(t)\phi(t) = 0,$$

is oscillatory.

It is not difficult to see that, from Corollary 4.1, Ph. Hartman's theorem follows, see [**5,** page 958], [**10,** page 433].

Theorem 4.11 ([**10**])**.** *If*

$$-\infty < \liminf_{T\to+\infty} \frac{1}{T}\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau < \limsup_{T\to+\infty} \frac{1}{T}\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau,$$

then equation (4.24) *is oscillatory.*

Example 4.12. Let $r(t)$ define the equality

$$\frac{1}{T}\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau = \begin{cases} \sin^3 T & t\in[2\pi k;(2k+1)\pi), \quad k=0,1,2,\ldots, \\ e^T \sin^3 T & t\in[(2k+1)\pi; 2(k+1)\pi), \quad k=0,1,2,\ldots. \end{cases}$$

Then,

$$-\infty = \liminf_{T\to+\infty} \frac{1}{T}\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau < \limsup_{T\to+\infty} \frac{1}{T}\int_{t_0}^{T} dt \int_{t_0}^{t} r(\tau)\, d\tau.$$

Therefore, Theorem 4.4 is not applicable to equation (4.24) with such an $r(t)$. It is easy to see that Corollary 4.1 is applicable to equation (4.26) with the above-mentioned $r(t)$.

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