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THE PRIME SPECTRUM AND DIMENSION OF IDEAL TRANSFORM ALGEBRAS

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ABSTRACT. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring of dimension $d \geq 1$, and let I be a non-nilpotent ideal of R such that the ideal transform functor $D_I(-)$ is exact. In this paper, it is shown that the finitely generated flat R-algebra $D_I(R)$ is a Noetherian ring of dimension $n = \dim R/\Gamma_I(R) - 1$. Also, it is shown that, under Zariski topologies on the sets $\operatorname{Spec} D_I(R)$ and $\operatorname{Spec} R/\Gamma_I(R)$, there is a homeomorphism of topological spaces:

$$\widetilde{\eta^*} : \operatorname{Spec} D_I(R) \longrightarrow \operatorname{Spec} R/\Gamma_I(R) \setminus V((I+\Gamma_I(R))/\Gamma_I(R)).$$

1. Introduction. Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. The local cohomology modules $H_I^i(M)$, $i = 0, 1, 2, \ldots$, of an R-module M with respect to I were introduced by Grothendieck **[7]**. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e.,

$$\bigcup_{n=1}^{\infty} (0 :_M I^n).$$

There is a natural isomorphism:

$$H_I^i(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

The reader is referred to **[3, 7]** for more details about local cohomology.

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Recall that, for an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \sup\{i \in \mathbb{Z} : H_I^i(M) \neq 0\}.$$

The cohomological dimension has been studied by several authors, see, for example, Faltings [**5**], Hartshorne [**8**], Huneke-Lyubeznik [**12**], Divaani-Aazar, et al. [**4**], Hellus [**9**], Hellus-Stückrad [**10**], Mehrvarz, et al. [**14**], and Ghasemi, et al. [**6**].

Recall that, for any proper ideal I of R, the *arithmetic rank* of I, denoted by $\operatorname{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I. Also, recall that, for any ideal I of an arbitrary Noetherian ring R, the *I-transform functor*, denoted by $D_I(-)$, is defined as:

$$D_I(-) = \varinjlim_{n \geq 1} \operatorname{Hom}_R(I^n, -).$$

It is well known that the R-module $D_I(R)$ has a finitely generated R-algebra structure whenever the functor $D_I(-)$ is exact.

In this paper, as our main result, we shall prove that if the I-transform functor $D_I(-)$ is exact and non-zero, then the finitely generated flat R-algebra $D_I(R)$ is a Noetherian ring of dimensiom $n = \dim R/\Gamma_I(R) - 1$. In addition, it is shown that, under the Zariski topologies on the sets $\operatorname{Spec} D_I(R)$ and $\operatorname{Spec} R/\Gamma_I(R)$, there is a homeomorphism of topological spaces:

$$\widetilde{\eta}^* : \operatorname{Spec} D_I(R) \longrightarrow \operatorname{Spec} R/\Gamma_I(R) \setminus V((I + \Gamma_I(R))/\Gamma_I(R)).$$

For each R-module L, we denote by $\operatorname{Assh}_R L$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. For any ideal $\mathfrak{a}$ of R, we denote by $V(\mathfrak{a})$ the set $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$. Also, for any ideal $\mathfrak{b}$ of R, the *radical* of $\mathfrak{b}$, denoted $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Finally, for each ring T, we denote the set of all maximal ideals of T by $\operatorname{Max}(T)$. For any undefined notation and terminology the reader is referred to [**3, 13**].

2. Preliminaries. In this section, we prove some technical results, which will be used later. We begin this section with the following well-known result, which is needed in the proof of Lemma 2.3.

Lemma 2.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$*, and let* $x_1, \ldots, x_d$ *be a system of parameters of* R*. Then, for each* $1 \leq i \leq d$*, there exists a minimal prime ideal* $\mathfrak{p}$ *over* $(x_1, \ldots, x_i)$ *such that* $\operatorname{height} \mathfrak{p} = i$ *and* $\dim R/\mathfrak{p} = d - i$*.*

Proof. See [**2,** Lemma 3.1]. $\square$

The next definition will be quite useful in this section.

Definition 2.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then, we define

$$\Upsilon_i(R) := \{\mathfrak{p} \in \operatorname{Spec} R : \operatorname{height} \mathfrak{p} = i \text{ and } \dim R/\mathfrak{p} = d - i\},$$

for every integer $0 \leq i \leq d$.

Note that, if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, then

$$\Upsilon_0(R) = \operatorname{Assh}_R R \quad \text{and} \quad \Upsilon_d(R) = \{\mathfrak{m}\}.$$

The following result is needed in the proofs of Lemma 2.4 and Proposition 2.7.

Lemma 2.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$*. Let* $x \in \mathfrak{m}$*, and let* n *be an integer such that* $0 \leq n \leq d-1$*. If* $x \notin \mathfrak{q}$ *for some* $\mathfrak{q} \in \Upsilon_0(R)$*, then there exists a prime ideal* $\mathfrak{p} \in \Upsilon_n(R)$ *such that* $\mathfrak{q} \subseteq \mathfrak{p}$ *and* $x \notin \mathfrak{p}$*.*

Proof. As the assertion is clear for $n = 0$, we may assume that $1 \leq n \leq d - 1$. Since, by hypothesis, we have $x \notin \mathfrak{q}$ and $\mathfrak{q} \in \Upsilon_0(R)$, it follows that

$$\dim R/(\mathfrak{q} + Rx) = \dim R/\mathfrak{q} - 1 = d - 1,$$

and thus, $y_1 := x + \mathfrak{q}$ is a subset of a system of parameters for the local ring $R/\mathfrak{q}$. Then, there are elements $y_2 = x_2 + \mathfrak{q}, \ldots, y_d = x_d + \mathfrak{q} \in \mathfrak{m}/\mathfrak{q}$ such that $y_1, y_2, \ldots, y_d$ is a system of parameters for the local ring $R/\mathfrak{q}$. Then, by Lemma 2.1, there exists a minimal prime ideal $\mathfrak{p}/\mathfrak{q}$ over $(y_2, \ldots, y_{n+1})$ such that $\operatorname{height} \mathfrak{p}/\mathfrak{q} = n$ and $\dim(R/\mathfrak{q})/(\mathfrak{p}/\mathfrak{q}) = d - n$. Now, it is straightforward to see $\mathfrak{p} \in [\Upsilon_n(R) \bigcap V(\mathfrak{q})]$ and $y_1 \notin \mathfrak{p}/\mathfrak{q}$, which implies that $x \notin \mathfrak{p}$. $\square$

The following result plays a key role in the proof of Theorem 3.1.

Lemma 2.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$, *and let* n *be an integer such that* $0 \leq n \leq d-1$. *Then, we have*

$$\bigcap_{\mathfrak{p} \in \Upsilon_n(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p}.$$

In particular, we have

$$\bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \Upsilon_1(R)} \mathfrak{p} = \cdots = \bigcap_{\mathfrak{p} \in \Upsilon_{d-2}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \Upsilon_{d-1}(R)} \mathfrak{p}.$$

Proof. As the assertion is clear for $n = 0$, we may assume that $1 \leq n \leq d-1$. Now, let $\mathfrak{p} \in \Upsilon_n(R)$. Since, by hypothesis, we have $\operatorname{height} \mathfrak{p} = n$, it follows that a chain of distinct prime ideals of R exists as

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{n-1} \subset \mathfrak{p}.$$

Also, by the hypothesis, $\dim R/\mathfrak{p} = d - n$. Thus, a chain of distinct prime ideals of R exists as

$$\mathfrak{p} \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{d-n-1} \subset \mathfrak{p}_{d-n} = \mathfrak{m}.$$

Now, from the chain

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{n-1} \subset \mathfrak{p} \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{d-n-1} \subset \mathfrak{p}_{d-n} = \mathfrak{m},$$

we conclude that $\mathfrak{q}_0 \in \Upsilon_0(R)$, and thus, we have

$$\bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p} \subseteq \mathfrak{q}_0 \subseteq \mathfrak{p}.$$

This yields

$$\bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in \Upsilon_n(R)} \mathfrak{p}.$$

Therefore, it is enough to prove that

$$\bigcap_{\mathfrak{p} \in \Upsilon_n(R)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p}.$$

In order to do this, assume the opposite. There is an element

$$x \in \Big[\Big(\bigcap_{\mathfrak{p}\in\Upsilon_n(R)} \mathfrak{p}\Big)\Big] \setminus \Big(\bigcap_{\mathfrak{p}\in\Upsilon_0(R)} \mathfrak{p}\Big)\Big].$$

Then, there exists an element $\mathfrak{q} \in \Upsilon_0(R)$ such that $x \notin \mathfrak{q}$. Now, it follows from Lemma 2.3 that $x \notin P$ for some $P \in [\Upsilon_n(R) \bigcap V(\mathfrak{q})]$. However, this is a contradiction since, by hypothesis, we have $x \in \bigcap_{\mathfrak{p}\in\Upsilon_n(R)} \mathfrak{p}$. $\square$

The following results are some consequences of Lemma 2.4.

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension $d \geq 1$. Then, we have*

$$0 = \bigcap_{\mathfrak{p}\in\Upsilon_1(R)} \mathfrak{p} = \cdots = \bigcap_{\mathfrak{p}\in\Upsilon_{d-2}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\Upsilon_{d-1}(R)} \mathfrak{p}.$$

Proof. This follows from Lemma 2.4. $\square$

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring of dimension $d \geq 1$, and let n be an integer such that $0 \leq n \leq d-1$. Then, we have*

$$\bigcap_{\mathfrak{p}\in\Upsilon_n(R)} \mathfrak{p} = \mathrm{Rad}(0).$$

In particular, this yields

$$\mathrm{Rad}(0) = \bigcap_{\mathfrak{p}\in\Upsilon_0(R)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\Upsilon_1(R)} \mathfrak{p} = \cdots = \bigcap_{\mathfrak{p}\in\Upsilon_{d-2}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\Upsilon_{d-1}(R)} \mathfrak{p}.$$

Proof. The assertion is clear by [**13,** Theorems 17.3, 17.4] and Lemma 2.4. $\square$

We close this section with the next result.

Proposition 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$, and let n be an integer such that $1 \leq n \leq d-1$. Then, for each $\mathfrak{q}_1 \in \Upsilon_0$, the following set is infinite*:

$$\Phi_n(\mathfrak{q}_1) := \{\mathfrak{p} \in \Upsilon_n(R) : \{\mathfrak{q} \in \Upsilon_0(R) : \mathfrak{p} \in V(\mathfrak{q})\} = \{\mathfrak{q}_1\}\}.$$

Proof. If we have $\Upsilon_0(R) = \{\mathfrak{q}_1\}$, then the assertion is clear by Lemma 2.4. Thus, we may assume $\Upsilon_0(R) \neq \{\mathfrak{q}_1\}$. Then, there is an element

$$x \in \bigg[\bigg(\bigcap_{\mathfrak{q}\in(\Upsilon_0(R)\setminus\{\mathfrak{q}_1\})} \mathfrak{q}\bigg) \setminus \mathfrak{q}_1\bigg].$$

By Lemma 2.3, there exists a prime ideal $P \in [V(\mathfrak{q}_1) \cap \Upsilon_n(R)]$ such that $x \notin P$. In particular, we have $P \in \Phi_n(\mathfrak{q}_1)$, and thus, $\Phi_n(\mathfrak{q}_1) \neq \emptyset$.

Now, we claim that $\Phi_n(\mathfrak{q}_1)$ is an infinite set. Assume the opposite. Then the set $\Phi_n(\mathfrak{q}_1)$ is a nonempty finite set. Let $\Phi_n(\mathfrak{q}_1) = \{P_1, \ldots, P_k\}$. Thus, by definition, we have

$$\bigg[\bigg(\bigcap_{\mathfrak{q}\in(\Upsilon_0(R)\setminus\{\mathfrak{q}_1\})} \mathfrak{q}\bigg) \bigcap \bigg(\bigcap_{i=1}^{k} P_i\bigg)\bigg] \not\subseteq \mathfrak{q}_1.$$

Hence, there exists an element

$$x_1 \in \bigg[\bigg(\bigcap_{\mathfrak{q}\in(\Upsilon_0(R)\setminus\{\mathfrak{q}_1\})} \mathfrak{q}\bigg) \bigcap \bigg(\bigcap_{i=1}^{k} P_i\bigg)\bigg]$$

such that $x_1 \notin \mathfrak{q}_1$. Then, by Lemma 2.3, a prime ideal $Q \in [V(\mathfrak{q}_1) \cap \Upsilon_n(R)]$ exists such that $x_1 \notin Q$. In particular, we have $Q \in \Phi_n(\mathfrak{q}_1)$ and $x_1 \notin Q$, which is a contradiction. $\square$

3. Main results. In this section we shall prove our main results, Theorems 3.1 and 3.2.

Theorem 3.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$, *and let* I *be an ideal of* R *with* $\operatorname{cd}(I, R) = 1$. *Then,* $D_I(R)$ *is a Noetherian flat* R*-algebra of dimension* $n = \dim R/\Gamma_I(R) - 1$.

Proof. Let $\dim R/\Gamma_I(R) = d'$. First, note that, since we have $\operatorname{cd}(I, R) = 1$, it follows from [**3,** Lemma 6.3.1] that the functor $D_I(-)$ is exact. Thus, by [**3,** Proposition 6.3.5], we have $D_I(R) = ID_I(R)$. Hence, it follows from [**3,** Proposition 6.3.4] that $D_I(R)$ is a finitely generated R-algebra. Therefore, $D_I(R)$ is a Noetherian ring. Moreover, it follows from [**1,** Theorem 3.11] that the ring $D_I(R)$ is a Noetherian flat R-algebra.

Now, in order to prove $\dim D_I(R) = \dim R/\Gamma_I(R) - 1$ (as a ring), first we use the isomorphism $D_I(R) \simeq D_I(R/\Gamma_I(R))$, given in [**3,** Corollary 2.2.8 (ii)]. By this isomorphism, and using [**3,** Corollary 2.1.7 (iii)], replacing $R/\Gamma_I(R)$ with R, without loss of generality, we may assume that $\Gamma_I(R) = 0$, $\mathrm{cd}(I, R) = 1$ and $\dim R = d'$. Then, we must prove $\dim D_I(R) = \dim R - 1 = d' - 1$. However, under this hypothesis, it follows from [**3,** Theorem 6.2.7] that $\mathrm{grade}(I, R) = 1$. Now, using this fact, it follows from [**3,** Corollary 6.3.6] that $\mathrm{height}\, I = 1$. Therefore, we have

$$I \nsubseteq \bigcup_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p},$$

where $\Upsilon_0(R) = \mathrm{Assh}_R\, R$. In particular,

$$I \nsubseteq \bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p}.$$

Then, by Lemma 2.4, we have

$$I \nsubseteq \bigcap_{\mathfrak{p} \in \Upsilon_{d'-1}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \Upsilon_0(R)} \mathfrak{p}.$$

Therefore, by definition, there exists an element $\mathfrak{Q} \in \Upsilon_{d'-1}(R)$ such that $I \nsubseteq \mathfrak{Q}$. Then, we have $\mathrm{Rad}(I + \mathfrak{Q}) = \mathfrak{m}$ and $\mathrm{height}\, \mathfrak{Q} = d' - 1$. Since the R-module $H^1_I(R)$ is I-torsion, in view of [**3,** Exercise 2.1.9], we have

$$H^i_{\mathfrak{Q}}(H^1_I(R)) \simeq H^i_{I+\mathfrak{Q}}(H^1_I(R)) = H^i_{\mathfrak{m}}(H^1_I(R)),$$

for each integer $i \geq 0$. In particular, we have $\mathrm{Supp}\, H^i_{\mathfrak{Q}}(H^1_I(R)) \subseteq \{\mathfrak{m}\}$ for each integer $i \geq 0$. On the other hand, since $\mathrm{height}\, \mathfrak{Q} = d' - 1$, it follows from Grothendieck's non-vanishing theorem that $\mathfrak{Q} \in \mathrm{Supp}\, H^{d'-1}_{\mathfrak{Q}}(R)$. Next, by [**3,** Theorem 2.2.4], there exists an exact sequence

$$0 \longrightarrow R \longrightarrow D_I(R) \longrightarrow H^1_I(R) \longrightarrow 0,$$

which induces the following exact sequence

$$H^{d'-2}_{\mathfrak{Q}}(H^1_I(R)) \longrightarrow H^{d'-1}_{\mathfrak{Q}}(R) \longrightarrow H^{d'-1}_{\mathfrak{Q}}(D_I(R)) \longrightarrow H^{d'-1}_{\mathfrak{Q}}(H^1_I(R)).$$

Now, the last exact sequence implies that $\mathfrak{Q} \in \mathrm{Supp}\, H^{d'-1}_{\mathfrak{Q}}(D_I(R))$. In particular, using the Independence theorem, we have

$$H^{d'-1}_{\mathfrak{Q}D_I(R)}(D_I(R)) \simeq H^{d'-1}_{\mathfrak{Q}}(D_I(R)) \neq 0.$$

Thus, it follows from Grothendieck's vanishing theorem that $\dim D_I(R) \geq d' - 1$.

At this point, it is enough to prove $\dim D_I(R) \leq d' - 1$.

By [**3,** Theorem 2.2.4] there exists an exact sequence

$$0 \longrightarrow R \xrightarrow{\eta_R} D_I(R) \xrightarrow{\zeta^0_R} H^1_I(R) \longrightarrow 0,$$

of R-modules and R-homomorphisms. Moreover, by [**3,** Exercise 2.2.10], the map η_R is a ring homomorphism. Therefore, without loss of generality, we may assume that R is a subring of the ring $D_I(R)$ and $H^1_I(R) = D_I(R)/R$. Now, let $\mathfrak{n}$ be an arbitrary maximal ideal of $D_I(R)$. Since we have $ID_I(R) = D_I(R)$, $I \subseteq R \subseteq D_I(R)$ and $\mathfrak{n} \neq D_I(R)$, it follows that there exists an element $a \in I$ such that $a \notin \mathfrak{n}$. Since the R-module $H^1_I(R) = D_I(R)/R$ is I-torsion, from the hypothesis $a \in I$, it follows that $(D_I(R)/R)_a = 0$, where $(D_I(R)/R)_a$ is the localization of the R-module $D_I(R)/R$ to the multiplicatively closed set $\{1_R, a, a^2, a^3, \ldots\}$. In particular, we have $R_a = (D_I(R))_a$. Now, since $\{1_R, a, a^2, a^3, \ldots\} \subseteq (D_I(R) \setminus \mathfrak{n})$, we have the following isomorphism of $D_I(R)$-modules: $(D_I(R))_{\mathfrak{n}} \simeq ((D_I(R))_a)_{\mathfrak{n}}$. Hence, we have:

$$\begin{aligned} \operatorname{height} \mathfrak{n} = \dim(D_I(R))_{\mathfrak{n}} &= \dim((D_I(R))_a)_{\mathfrak{n}} \leq \dim(D_I(R))_a \\ &= \dim R_a \leq \dim R - 1 = d' - 1, \end{aligned}$$

whence,

$$\dim D_I(R) = \sup\{\operatorname{height} \mathfrak{n} : \mathfrak{n} \in \operatorname{Max}(D_I(R))\} \leq d' - 1,$$

as required. $\square$

Note that, if A and B are two commutative rings with identities and $\varphi : A \to B$ is a ring homomorphism, then, for each prime ideal $\mathfrak{q}$ of B, the ideal $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of A is a prime ideal, and hence, φ induces a mapping $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$. Also, it is well known that φ^* is a continuous map, under the Zariski topologies on both $\operatorname{Spec} B$ and $\operatorname{Spec} A$. Now, let $(R, \mathfrak{m})$ be an arbitrary Noetherian local ring and I a non-nilpotent ideal of R. Then, by [**3,** Exercise 2.2.10], there is a ring homomorphism $\eta : R/\Gamma_I(R) \to D_I(R/\Gamma_I(R))$ which is a monomorphism. Also, there is a ring isomorphism $D_I(R/\Gamma_I(R)) \simeq D_I(R)$. Therefore, there is a natural mapping $\eta^* : \operatorname{Spec} D_I(R) \to \operatorname{Spec} R/\Gamma_I(R)$ which is a continuous map.

Theorem 3.2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$*, and let* I *be an ideal of* R *with* $\operatorname{cd}(I, R) = 1$*. Consider the sets* $\operatorname{Spec} D_I(R)$ *and subspace* $\operatorname{Spec} R/\Gamma_I(R) \setminus V((I + \Gamma_I(R))/\Gamma_I(R))$ *of* $\operatorname{Spec} R/\Gamma_I(R)$ *with the usual Zariski topologies. Then, there is a homeomorphism*

$$\widetilde{\eta^*} : \operatorname{Spec} D_I(R) \longrightarrow \operatorname{Spec} R/\Gamma_I(R) \setminus V((I + \Gamma_I(R))/\Gamma_I(R)),$$

which is induced by η^**.*

Proof. Without loss of generality, we may assume that $\Gamma_I(R) = 0$. Then, we must find a homeomorphism

$$\widetilde{\eta^*} : \operatorname{Spec} D_I(R) \longrightarrow \operatorname{Spec} R \setminus V(I).$$

In order to do this, by the notation of [**3,** Theorem 2.2.4], there is a ring monomorphism $\eta_R : R \to D_I(R)$ which induces a mapping $\eta_R^* : \operatorname{Spec} D_I(R) \to \operatorname{Spec} R$. We claim that $\operatorname{im} \eta_R^* = \operatorname{Spec} R \setminus V(I)$. In order to do so, as by hypothesis we have $\operatorname{cd}(I, R) = 1$, it follows from [**3,** Lemma 6.3.1] that the ideal transform functor $D_I(-)$ is exact. Therefore, by [**3,** Proposition 6.3.5], we have $D_I(R) = ID_I(R)$. Now, it is clear that $\operatorname{im} \eta_R^* \subseteq \operatorname{Spec} R \setminus V(I)$. On the other hand, for every $\mathfrak{p} \in \operatorname{Spec} R \setminus V(I)$, there is an element $a \in I$ such that $a \notin \mathfrak{p}$. Then, by the same argument as in the proof of Theorem 3.1, we may assume that R is a subring of $D_I(R)$ and $(D_I(R))_a = R_a$. Thus, there is an ideal Q of $D_I(R)$ such that $Q(D_I(R))_a = \mathfrak{p}R_a$. Now it is straightforward to see that $\eta_R^*(Q) = \mathfrak{p}$. Hence, we have $\operatorname{im} \eta_R^* = \operatorname{Spec} R \setminus V(I)$.

Also, we claim that the map η_R^* is injective. In order to see this, let Q_1 and Q_2 be distinct elements of $\operatorname{Spec} D_I(R)$ such that $\eta_R^*(Q_1) = \eta_R^*(Q_2)$. Then, we have $\eta_R^{-1}(Q_1) = \eta_R^{-1}(Q_2)$. Now, again, by the same argument as in the proof of Theorem 3.1, we may assume that R is a subring of $D_I(R)$. Since we have $I \nsubseteq \eta_R^{-1}(Q_1) = \eta_R^{-1}(Q_2)$, it follows that there is an element $a \in I$ such that $a \notin \eta_R^{-1}(Q_1) = \eta_R^{-1}(Q_2)$.

Next, we have $a \notin Q_1$ and $a \notin Q_2$. Then, as R is a subring of $D_I(R)$ and $(D_I(R))_a = R_a$, it follows that $Q_1(D_I(R))_a \neq Q_2(D_I(R))_a$, and thus, $Q_1R_a \neq Q_2R_a$; however, $(Q_1 \cap R)R_a = (Q_2 \cap R)R_a$, which is a contradiction. Therefore, the map η_R^* is injective. Furthermore, the map η_R^* induces a mapping

$$\widetilde{\eta^*} : \operatorname{Spec} D_I(R) \longrightarrow \operatorname{Spec} R \setminus V(I),$$

which is injective and surjective.

Let $\varepsilon := \widetilde{\eta^*}$. We claim that ε is a homeomorphism. In order to prove this assertion, we must show that both of the maps ε and ε^{-1} are continuous. First, we show that ε is continuous. In order to see this, let Y be an open subset of $\operatorname{Spec} R \setminus V(I)$. Then, there is an ideal J of R such that

$$Y = [\operatorname{Spec} R \setminus V(I)] \cap [\operatorname{Spec} R \setminus V(J)] = \operatorname{Spec} R \setminus V(I \cap J).$$

Thus, Y is an open subset of $\operatorname{Spec} R$. Therefore, the set $X = (\eta_R^*)^{-1}(Y) = \varepsilon^{-1}(Y)$ is an open subset of $\operatorname{Spec} D_I(R)$. (Note that the map η_R^* is continuous.) Hence, by the definition, the map ε is continuous. Now, in order to prove that ε^{-1} is continuous, let X be a closed subset of $\operatorname{Spec} D_I(R)$. If $X = \emptyset$, then it is clear that $\varepsilon(X) = \emptyset$ is a closed subset of $\operatorname{Spec} R \setminus V(I)$. Thus, we may assume $X \neq \emptyset$. Then, there is a proper ideal $\mathfrak{J}$ of $D_I(R)$ such that $X = V(\mathfrak{J})$. Since $D_I(R)$ is a Noetherian ring, it follows that there are prime ideals $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_n$ such that

$$V(\mathfrak{J}) = \bigcup_{i=1}^{n} V(\mathfrak{Q}_i).$$

Let $\mathfrak{p}_i = \eta_R^*(\mathfrak{Q}_i)$ for $i = 1, 2, \ldots, n$. Then, it is straightforward to see that

$$\varepsilon(V(\mathfrak{J})) = V\Big(\bigcap_{i=1}^{n} \mathfrak{p}_i\Big) \setminus V(I) = [\operatorname{Spec}(R) \setminus V(I)] \bigcap V\Big(\bigcap_{i=1}^{n} \mathfrak{p}_i\Big),$$

which implies that $\varepsilon(V(\mathfrak{J}))$ is a closed subset of $\operatorname{Spec}(R) \setminus V(I)$. Hence, the map ε^{-1} is continuous. Therefore, the map ε is a homeomorphism, as required. $\square$

The next corollary of Theorem 3.2 shows that the algebra $D_I(R)$ is rarely semilocal, whenever I is an ideal of a Noetherian local ring $(R, \mathfrak{m})$ with $\operatorname{cd}(I, R) = 1$.

Corollary 3.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring, and let* I *be an ideal of* R *with* $\mathrm{cd}(I, R) = 1$*. Then, the Noetherian ring* $D_I(R)$ *is not semilocal if and only if* $\dim R/\Gamma_I(R) \geq 2$*.*

Proof. Since we have $D_I(R) \simeq D_I(R/\Gamma_I(R))$, replacing R with $R/\Gamma_I(R)$, without loss of generality, we may assume that $\Gamma_I(R) = 0$. Then, it is easy to see that the set of maximal elements of the set $\mathrm{Spec}(R)\setminus V(I)$ is not finite if, and only if, $\dim R \geq 2$. Now the assertion follows from Theorem 3.2. $\square$

Remark 3.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. In the case where $I = Rx$ for some nilpotent element $x \in \mathfrak{m}$, we have $\mathrm{ara}(I) = \mathrm{cd}(I, R) = 1$ and, by [**3,** Theorem 2.2.16], we have $D_I(R) \simeq R_x$. Therefore, the veracity of Theorems 3.1 and 3.2 can easily be seen in this case. However, there are examples of Noetherian local rings $(R, \mathfrak{m})$ with proper ideals I for which $\mathrm{cd}(I, R) = 1$ and $\mathrm{ara}(I) \geq 2$. For instance, the following example is given by Hellus and Stückrad in [**11**].

Example 3.5. Let k be a field, and let $S = k[[x, y, z, w]]$, where x, y, z and w are independent indeterminates over k. Let $f = xw - yz$, $g = y^3 - x^2z$ and $h = z^3 - w^2y$. Let $R = S/fS$ and $I = (f, g, h)S/fS$. Then, R is a Noetherian local ring of dimension 3 with maximal ideal $\mathfrak{m} = (x, y, z, w)S/fS$. Also, for the ideal I of R, we have $\mathrm{cd}(I, R) = 1$ and $\mathrm{ara}(I) = 2$. (See [**11,** Remark 2.1 (ii)].)

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