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A NOTE ON THE RANK OF $B\mathrm{ext}^1_A(G,A)$

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ABSTRACT. The goal of this paper is to compare ranks of the divisible groups $\mathrm{Ext}(G,A)$ and $B\mathrm{ext}^1_A(G,A)$ whenever G is a countable torsion-free A-solvable group and A has a right hereditary endomorphism ring.

Every Abelian group A induces functors $H_A(\cdot) = \mathrm{Hom}(A,\cdot)$ and $T_A(\cdot) = \cdot \otimes_E A$ between the category of Abelian groups and the category of right E-modules where $E = E(A)$ denotes the endomorphism ring of A. These functors induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$ for all $\alpha \in H_A(G)$, $x \in M$ and $a \in A$. A group G is A-*generated* if $S_A(G) = G$ where $S_A(G) = \mathrm{im}(\theta_G)$, and it is A-*solvable* if θ_G is an isomorphism. A group P is (*finitely*) A-*projective* if it is a direct summand of $\oplus_I A$ for some (finite) index-set I. If A is a torsion-free group of finite rank, then all A-projective groups are A-solvable. Finally, $G^* = \mathrm{Hom}(G,A)$ for all Abelian groups G.

One of the central problems in the theory of Abelian groups is to determine the structure of the divisible group $\mathrm{Ext}(A,B)$ in the case where A and B are torsion-free. However, properties of an Abelian group A are often better described by considering subgroups of Ext, as can be seen in the discussion of Butler groups. Similarly, the group $\mathrm{Ext}(G,A)$ is not the right tool for studying homological properties of A-solvable groups. These are better described by the subgroup $B\mathrm{ext}^1_A(G,H)$ of $\mathrm{Ext}(G,H)$. It consists of the equivalence classes of sequences

$$0 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 0$$

with G and H A-solvable, with respect to which A is projective **[3]**.

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Since $B\mathrm{ext}^1_A(G,H) \cong \mathrm{Ext}^1_E(H_A(G),H_A(H))$, we obtain that $B\mathrm{ext}^1_A$ is a right exact functor if E is right hereditary. Standard homological arguments show that in this case $B\mathrm{ext}^1_A(G,A)$ is a divisible group for a torsion-free A-solvable group G whose structure is determined by its torsion-free and its p-ranks, where the *p-rank of a divisible group* D is defined as $r_p(D) = \dim_{\mathbb{Z}/p\mathbb{Z}} D[p]$. On the other hand, $r_p(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} G/pG$ is the *p-rank* of a torsion-free group G [**4**]. Naturally, the question arises of how the ranks of $\mathrm{Ext}(G,A)$ and $B\mathrm{ext}^1_A(G,A)$ are related if G is A-solvable. Since

$$2^{\aleph_0} = r_0(\mathrm{Ext}(P,A)) > r_0(B\mathrm{ext}^1_A(P,A)) = 0,$$

for a countable A-projective group P unless $\mathrm{Ext}(A,A) = 0$, these two ranks need not coincide in general.

Theorem 0.1. *Let A be a countable reduced torsion-free Abelian group whose endomorphism ring is a hereditary subring of a finite-dimensional $\mathbb{Q}$-algebra. If G is a countable torsion-free A-solvable group which is not A-projective, then*

$$r_0(B\mathrm{ext}^1_A(G,A)) = 2^{\aleph_0} = r_0(\mathrm{Ext}(G,A)).$$

Proof. Since E is a hereditary subring of a finite-dimensional $\mathbb{Q}$-algbera, E is a semi-prime right and left Noetherian ring and $\mathbb{Q}E$ is semi-simple Artinian by [**8**]. In particular, a right E-module M is non-singular, see [**7**], if and only if its additive group is torsion-free. Moreover, A is faithfully flat as an E-module by [**1**]. Consequently, A-generated subgroups of A-solvable groups are A-solvable by [**3**]; and every exact sequence $G \to P \to 0$ such that G is A-generated and P is A-projective splits [**1**].

Let F be a finitely A-projective subgroup of a torsion-free A-solvable group G. We consider the induced sequence

$$0 \longrightarrow H_A(F) \longrightarrow H_A(G),$$

and select a submodule U of $H_A(G)$ such that $H_A(G)/U$ is torsion-free and $U/H_A(F)$ is torsion. Then, $T_A(U)/T_AH_A(F)$ is torsion. Since A is flat as an E-module,

$$T_AH_A(G)/T_A(U) \cong T_A(H_A(G)/U)$$

is torsion-free. Therefore, the purification F_* of F in G is isomorphic to $T_A(U)$ since G is A-solvable. In particular, F_* is A-solvable by the remarks in the last paragraph.

Suppose that F_* is A-projective for all finitely A-projective subgroups F of a countable A-solvable group G. Then, $H_A(F_*)$ is a projective right E-module which contains the finitely generated projective module $H_A(F)$ as an essential submodule. By Sandomierski's theorem **[5]**, $H_A(F_*)$ is finitely generated, and $F_* \cong T_AH_A(F_*)$ is finitely A-projective. Thus, G is the union of an ascending chain $\{U_n \mid n < \omega\}$ of pure finitely A-projective subgroups. For each $n < \omega$, we obtain the exact sequence

$$0 \longrightarrow H_A(U_n) \longrightarrow H_A(U_{n+1}) \longrightarrow H_A(U_{n+1}/U_n)$$

which yields that $H_A(U_{n+1})/H_A(U_n)$ is a finitely generated nonsingular right E-module. Since $\mathbb{Q}E$ is semi-simple Artinian, this module is projective by **[7]**, and $H_A(U_n)$ is a direct summand of $H_A(U_{n+1})$. Hence, we obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_AH_A(U_n) & \longrightarrow & T_AH_A(U_{n+1}) \\ & & \wr\Big\downarrow \theta_{U_1} & & \wr\Big\downarrow \theta_{U_2} \\ 0 & \longrightarrow & U_n & \longrightarrow & U_{n+1} \end{array}$$

whose top row splits. Thus, the same holds for the bottom row. Consequently, G is A-projective, a contradiction.

Therefore, G contains a finitely A-projective subgroup F and a pure subgroup U containing F such that U/F is torsion, but U is not A-projective. Since E is right hereditary, $\mathrm{Ext}_E^1(H_A(U),E) \cong B\mathrm{ext}_A^1(U,A)$ is an epimorphic image of

$$\mathrm{Ext}_E^1(H_A(G),E) \cong B\mathrm{ext}_A^1(G,A).$$

Because $|\mathrm{Ext}(G,A)| \leq 2^{\aleph_0}$, it suffices to show that $B\mathrm{ext}_A^1(U,A)$ has torsion-free rank at least $2^{\aleph_0}$.

Hence, we may assume that G contains a finitely A-projective subgroup F such that G/F is torsion and consider the induced sequence

$$0 \longrightarrow H_A(F) \longrightarrow H_A(G) \overset{H_A(\pi)}{\longrightarrow} M \longrightarrow 0$$

where $\pi : G \to G/F$ is the projection and $M = \operatorname{im} H_A(\pi)$. The last sequence induces

$$\mathrm{Hom}_E(H_A(F), E) \longrightarrow \mathrm{Ext}^1_E(M, E) \longrightarrow \mathrm{Ext}^1_E(H_A(G), E) \longrightarrow 0.$$

Since $\mathrm{Hom}_E(H_A(F), E)$ is countable and

$$B\mathrm{ext}^1_A(G, A) \cong \mathrm{Ext}^1_E(H_A(G), E)$$

by **[3]**, it remains to show that $\mathrm{Ext}^1_E(M, E)$ has torsion-free rank at least $2^{\aleph_0}$.

Observe that $T_A(M) \cong G/F$ is torsion due to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A H_A(F) & \longrightarrow & T_A H_A(G) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\ & & \wr\downarrow \theta_F & & \wr\downarrow \theta_G & & & & \\ 0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & G/F & \longrightarrow & 0. \end{array}$$

On the other hand, $T_A(M/tM)$ is torsion-free since A is E-flat. Consequently, $T_A(M/tM) = 0$, from which we obtain

$$M = tM = \oplus_p M_p$$

because A is faithfully flat as a left E-module by **[2]**, and $\mathrm{Ext}^1_E(M, E) \cong \Pi_p \mathrm{Ext}^1_E(M_p, E)$. Moreover, $(G/F)_p \neq 0$ if and only if $M_p \neq 0$, since multiplication by an integer relatively prime to p is an automorphism of M_p.

In order to see that $\mathrm{Ext}^1_E(N, E) \neq 0$ whenever N is a non-zero right E-module whose additive group is torsion, consider an essential right ideal I of E such that $E/I \neq 0$ is isomorphic to a submodule of N. Since E is hereditary, $\mathrm{Ext}^1_E(E/I, E) = 0$ if $\mathrm{Ext}^1_E(N, E) = 0$. Thus, we obtain the exact sequence

$$0 = E/I^* \longrightarrow E^* \longrightarrow I^* \longrightarrow \mathrm{Ext}^1_E(E/I, E) = 0,$$

in which $(E/I)^* = 0$ because E/I is singular and E is non-singular. Since I is finitely generated and projective, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{**} & \longrightarrow & E^{**} & \longrightarrow & 0 & & \\
 & & \wr\uparrow\psi_I & & \wr\uparrow\psi_E & & & & \\
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & E/I & \longrightarrow & 0,
\end{array}$$

where $M^* = \mathrm{Hom}_E(M,E)$ for all right (left) E-modules M, and the natural map $\psi_M : M \to M^{**}$ is an isomorphism whenever M is a finitely generated projective module **[9]**. Therefore, $E/I = 0$, a contradiction.

If $(G/F)_p \neq 0$ for infinitely many primes, then $\mathrm{Ext}(M_p, E) \neq 0$ for infinitely many primes by the last paragraph, and $\mathrm{Ext}^1_E(M,E) \cong \Pi_p \mathrm{Ext}^1_E(M_p,E)$ has torsion-free rank $2^{\aleph_0}$.

Hence, it remains to consider the case where $(G/F)_p \neq 0$ for only finitely many primes p. In this case, M_p cannot be bounded as an Abelian group for all primes p since this would yield $kH_A(G) \subseteq H_A(F)$ for some non-zero integer k. Therefore, $G \cong T_A H_A(G)$ would be A-projective since E is right hereditary.

In order to see that $\mathrm{Ext}(M_p, E)$ has torsion-free rank at least $2^{\aleph_0}$ whenever M_p is unbounded as an Abelian group, choose submodules V_n of $H_A(G)$ containing $H_A(F)$ such that $V_n/H_A(F) = M[p^n] \subseteq M_p$. Since $V_{n+1}/V_n \cong M[p^{n+1}]/M[p^n] \neq 0$ by the last paragraph and $pV_{n+1} \subseteq V_n$, we have $\mathrm{Ext}^1_E(V_{n+1}/V_n, E) \neq 0$ by what has already been shown. By **[8]**, there is an epimorphism

$$\mathrm{Ext}^1_E(M_p,E) \longrightarrow \varprojlim \mathrm{Ext}^1_E(V_n/H_A(F),E)$$

since

$$M = \bigcup_{n<\omega} V_n/H_A(F).$$

However, the latter Ext-group is torsion-free of at least rank $2^{\aleph_0}$ as in **[8]**, since the sequences

$$0 \longrightarrow \mathrm{Ext}^1_E(V_{n+1}/V_n,E) \longrightarrow \mathrm{Ext}^1_E(V_{n+1}/H_A(F),E) \longrightarrow \mathrm{Ext}^1_E(V_n/H_A(F),A) \longrightarrow 0$$

are exact since $\mathrm{Hom}((V_n/H_A(F)),E) = 0$ for all $n < \omega$. $\square$

We now turn to the p-rank of $B\text{ext}^1_A(G,A)$.

Theorem 0.2. *Let A be a torsion-free Abelian group A of finite rank.*

(a) *The following conditions are equivalent for a prime p:*
  (i) $r_p(E) = [r_p(A)]^2$.
  (ii) $t_p\text{Ext}(G,A) = t_pB\text{ext}^1_A(G,A)$ *for all torsion-free A-solvable groups G.*
  (iii) $[\text{Ext}(G,A)/B\text{ext}^1_A(G,A)][p] = 0$ *for all torsion-free A-solvable groups G.*

(b) *If $r_p(E) = [r_p(A)]^2$ for some prime p, then $r_p(B\text{ext}^1_A(G,A))$ is either finite or $2^{\aleph_0}$ whenever G is an A-solvable group.*

Proof.

(a) (i) $\Rightarrow$ (iii). By Warfield's result **[10]**, (i) yields

$$r_p(\text{Ext}(A,A)) = [r_p(A)]^2 - r_p(E) = 0$$

and $\text{Ext}(A,A)[p] = 0$. For an A-solvable group G, there is an exact sequence

$$0 \longrightarrow U \longrightarrow F \longrightarrow G \longrightarrow 0$$

such that F is a direct sum of copies of A with respect to which A is projective. Since $B\text{ext}^1_A(F,A) \cong \text{Ext}^1_E(H_B(F),E) = 0$ by **[3]**, it induces

$$0 \longrightarrow G^* \longrightarrow F^* \longrightarrow U^* \xrightarrow{\delta} B\text{ext}^1_A(G,A) \longrightarrow 0.$$

Combining this with the regular Cartan-Eilenberg sequence yields the exact sequence

$$0 \longrightarrow \frac{\text{Ext}(G,A)}{B\text{ext}^1_A(G,A)} \longrightarrow \text{Ext}(F,A) \longrightarrow \text{Ext}(U,A) \longrightarrow 0$$

which splits since $\text{Ext}(G,A)$ is divisible. Since $\text{Ext}(A,A)[p] = 0$, the same holds for $\text{Ext}(G,A)/B\text{ext}^1_A(G,A)[p]$.

(iii) $\Rightarrow$ (ii). Due to the fact that $[\text{Ext}(G,A)/B\text{ext}^1_A(G,A)][p] = 0$, we obtain an exact sequence

$$0 \longrightarrow B\text{ext}^1_A(G,A)[p] \longrightarrow \text{Ext}(G,A)[p] \longrightarrow 0,$$

which yields that $t_pB\text{ext}^1_A(G,A)$ is an essential subgroup of $t_p\text{Ext}(G,A)$. Since $B\text{ext}^1_A(G,A) \cong \text{Ext}^1_E(H_A(G),E)$ is divisible, $t_pB\text{ext}^1_A(G,A)$ is a direct summand of $t_p\mathrm{Ext}(G,A)$, which is only possible if $t_p\mathrm{Ext}(G,A) = t_pB\mathrm{ext}^1_A(G,A)$.

(ii) ⇒ (i). If (i) fails, then $r_p(\mathrm{Ext}(A,A)) = [r_p(A)]^2 - r_p(E) > 0$ by **[10]**, and there exists an exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow A \longrightarrow 0,$$

which represents an element of order p in $\mathrm{Ext}(A,A)$. By (ii), this sequence belongs to $B\mathrm{ext}^1_A(A,A) \cong \mathrm{Ext}^1_E(E,E) = 0$, a contradiction.

(b) Let

$$R_A(G) = \cap\{\ker\alpha \mid \alpha \in G^*\}.$$

Then $G/R_A(G)$ is isomorphic to an A-generated subgroup of A^I for some index-set I. However, all countable A-generated subgroups of A^I are A-projective by [**1**]. Hence,

$$G = R_A(G) \oplus P$$

for some A-projective group P by the remarks of the first paragraph of the proof of Theorem 0.1. Since $B\mathrm{ext}^1_A(P,A) = 0$ and $R_A(G)^* = 0$, we may assume that $G^* = 0$. Then,

$$\mathrm{Hom}_E(H_A(G),E) \cong G^* = 0$$

by the adjoint functor theorem. Observe that

$$B\mathrm{ext}^1_A(G,A) \cong \mathrm{Ext}^1_E(H_A(G),E),$$

and consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}_E(H_A(G),E/pE) \longrightarrow \mathrm{Ext}^1_E(H_A(G),E) \xrightarrow{\cdot p} \mathrm{Ext}^1_E(H_A(G),E),$$

because of which we need to show that $r_p(\mathrm{Hom}_E(H_A(G),E/pE)$ is either finite or $2^{\aleph_0}$. However,

$$\mathrm{Hom}_E(H_A(G),E/pE) \cong \mathrm{Hom}_E\left(\frac{H_A(G)}{pH_A(G)}, E/pE\right).$$

Since $r_p(E) = [r_p(A)]^2$, the ring E/pE is simple, and $E/pE \cong S^n$ for some simple E-module S and some $n < \omega$. Therefore,

$$\frac{H_A(G)}{pH_A(G)} \cong \oplus_I S$$

for some index-set I, and the conclusion immediately follows. □

A torsion-free Abelian group A of finite rank is a *finitely faithful S-group* if $r_p(E) = [r_p(B)]^2$ for all primes p.

Corollary 0.3. *The following are equivalent for a torsion-free Abelian group A of finite rank with a hereditary endomorphism ring*:

(a) *A is a finitely faithful S-group.*
(b) $t\mathrm{Ext}(G,A) = tB\mathrm{ext}^1_A(G,A)$ *whenever G is a torsion-free A-solvable group.*
(c) $\mathrm{Ext}(G,A)/B\mathrm{ext}^1_A(G,A)$ *is torsion-free divisible whenever G is a torsion-free A-solvable group.*

In particular, $\mathrm{Ext}(G,A)$ *and* $B\mathrm{ext}^1_A(G,A)$ *are isomorphic whenever A is a finitely faithful S-group and G is a countable torsion-free A-solvable group which is not A-projective.* □

Finally, we remark that the results of this paper can be extended to uncountable A-solvable groups by combining the arguments used in the proofs of Theorems 0.1 and 0.2 with the set-theoretic tools of [**6**]. However, we will not discuss this extension here since the algebraic, rather than the set-theoretic, tools needed for the investigation of $B\mathrm{ext}^1_A$ are the focus of this paper.

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