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REMARKS ON REGULARITY CRITERIA FOR 2D GENERALIZED MHD EQUATIONS

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ABSTRACT. In this paper, we establish two regularity criteria for the two-dimensional (2D) incompressible generalized magnetohydrodynamic (GMHD) equations in terms of only one quantity, namely, the current density $j = \nabla \times b$ or the vorticity $\omega = \nabla \times u$. More precisely, it is proved that, if one of the following holds true:

$$\int_0^T \|j(t)\|_{\dot{B}^0_{\infty,\,\infty}(\mathbb{R}^2)}\, dt < \infty,$$

$$\int_0^T \|\omega(t)\|_{\dot{B}^0_{\infty,\,\infty}(\mathbb{R}^2)}\, dt < \infty,$$

then the solution $(u,\, b)$ actually remains regular on $[0,T]$.

1. Introduction.

In this paper, we are interested in studying the following 2D incompressible GMHD equations in the entire space $\mathbb{R}^2$:

$$(1.1)\qquad \begin{cases} \partial_t u + (u\cdot\nabla)u + \nu\Lambda^{2\alpha}u = -\nabla p + (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b + \eta\Lambda^{2\beta}b = (b\cdot\nabla)u, \\ \nabla\cdot u = 0, \qquad\qquad \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x), \end{cases}$$

where $u = u(x,t) = (u_1(x,\, t),\, u_2(x,\, t))$, $b = b(x,t) = (b_1(x,\, t),\, b_2(x,\, t))$ and $p = p(x,t)$ denote the velocity vector, magnetic vector and pressure scalar fields respectively. Here, $\alpha \in [0,\, 2]$ and $\beta \in [0,\, 2]$ are real parameters, while $\nu > 0$, $\eta > 0$ are the kinematic viscosity and magnetic diffusivity, respectively; for simplicity, we set $\nu = \eta = 1$. The fractional Laplacian operator $\Lambda^{2\alpha} \triangleq (-\Delta)^\alpha$ is defined through the Fourier

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transform, namely,

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi)$$

$\widehat{f}$ the Fourier transform of f given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

We recall the convention that, by $\alpha = 0$, it is meant that there is no dissipation in $(1.1)_1$, and similarly, $\beta = 0$ represents that there is no diffusion in $(1.1)_2$.

When $\alpha = \beta = 1$, system (1.1) reduces to the standard magnetohydrodynamic (MHD) equations which govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas and reflect basic physics conservation laws. Due to their physical applications and mathematical significance, GMHD and MHD equations have been extensively studied, and important progress has been made.

Let us first briefly review some existence theories of the 2D case. Global regularity of system (1.1) with both Laplacian dissipation and magnetic diffusion, namely, $\alpha = \beta = 1$, was proven, see, e.g., [**9, 26, 30**], while the question of whether a solution to completely inviscid MHD equations ($\alpha = \beta = 0$) can develop a finite-time singularity from smooth initial data with finite energy remains a challenging open problem. Thus, examination of the intermediate cases has been an attractive direction of research. Recently, Cao and Wu [**3**] showed that smooth solutions are global for 2D MHD equations with mixed partial and magnetic diffusion, see [**8**] for more general cases. Very recently, Wu [**33**] proved global-in-time regularity as long as the powers α and β satisfy

$$\alpha \geq \frac{1}{2} + \frac{n}{4}, \quad \beta > 0, \qquad \alpha + \beta \geq 1 + \frac{n}{2},$$

where n is the spatial dimension. It is a remarkable fact that, when spatial dimension $n = 2$, the above conditions can be greatly weakened. Actually, recent efforts have been devoted to the global regularity of (1.1) with the smallest possible $\alpha \in [0, 2]$ and $\beta \in [0, 2]$ (see [**5, 13, 20, 21, 27, 30, 35, 36, 41, 42**] for more details). To the best of our knowledge, global regularity or finite time singularity for system (1.1) with $(\alpha,\ \beta) \in \mathfrak{M}$ currently is also an unsettled issue, where

$$\mathfrak{M} \triangleq \Big\{(\alpha,\ \beta) \mid \alpha \geq 0,\ 0 \leq \beta < 1,\ \alpha + \beta < 2\Big\} \cup \{(0,\ 1)\}. \tag{1.2}$$

Considerable work has been devoted to studying regularity criteria for the case $(\alpha, \beta) \in \mathfrak{M}$. When $\alpha = 1$, $\beta = 0$, we refer to **[14, 19, 40, 48]** for some interesting regularity criteria results. In addition, several regularity criteria for $\alpha > 1$, $\beta = 0$ may be found in **[39]**.

Now, we mention some results concerning the 3D case. Similarly to generalized Navier-Stokes equations, when α and β belong to a suitable range, system (1.1) with large initial data clearly admits a unique global smooth solution, see, e.g., **[28, 30, 33, 37, 38]**). Due to the presence of the Navier-Stokes equations in (1.1), it remains unknown whether or not 3D MHD equations with large initial data have a unique globally smooth solution. For this reason, a large amount of literature is devoted to addressing sufficient conditions with which to guarantee global regularity of the weak solution. Various regularity criteria in terms of the velocity and magnetic fields, pressure and their derivatives have been proposed. We list only a few, with no intention of comprehensiveness, see **[1, 2, 4, 6, 7, 11, 12, 16, 17, 18, 23, 24, 29, 31, 32, 34, 43, 45, 46, 47, 49]** and the references therein. It is noteworthy to point out that both velocity vector field and magnetic field conditions are needed to characterize the regularity criterion to completely inviscid MHD equations. In particular, the following are results from **[1, 2, 44]**, respectively:

$$\int_0^T (\|\omega(t)\|_{L^\infty(\mathbb{R}^n)} + \|j(t)\|_{L^\infty(\mathbb{R}^n)})\, dt < \infty,$$

$$\int_0^T (\|\omega(t)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)} + \|j(t)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)})\, dt < \infty,$$

$$\lim_{\varepsilon\to 0}\sup_{k\in\mathbb{Z}} \int_{T-\varepsilon}^T (\|\Delta_k\omega(t)\|_{L^\infty(\mathbb{R}^n)} + \|\Delta_k j(t)\|_{L^\infty(\mathbb{R}^n)})\, dt = \delta < M,$$

for some positive constant M, and Δ_k is a frequency localization on $|\xi| \approx 2^k$. Of course, these results hold true for system (1.1) with ν, $\eta > 0$.

Since there is no global well-posedness result for system (1.1) with $(\alpha, \beta) \in \mathfrak{M}$, it is natural to examine regularity criteria. In this paper, we establish the regularity criteria in terms of only one quantity, namely, the velocity or magnetic vector field, when the fractional powers of the Laplacian for system (1.1) belong to some certain range.

These results will be useful for further investigations on the global regularity problem of 2D MHD equations.

We now state our main results as follows. The first result concerns the $\dot{B}^0_{\infty,\infty}$ norm of current density j.

Theorem 1.1. *Suppose that* $\alpha+\beta>1$ *with* $\alpha>1/2$*,* $(\alpha,\beta)\in\mathfrak{M}$ *and* $(u_0,b_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ *for any* $s>2$*. Let* (u,b) *be a locally smooth solution to system* (1.1)*. Then,* (u,b) *can be extended beyond time* T*, provided that*

$$\int_0^T \|j(t)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^2)}\,dt<\infty. \tag{1.3}$$

The last regularity criterion is expressed in terms of the $\dot{B}^0_{\infty,\infty}$ norm of vorticity w. More precisely, we have the next theorem.

Theorem 1.2. *Suppose that* $\alpha+\beta>1$ *with* $(\alpha,\beta)\in\mathfrak{M}$ *and* $(u_0,b_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ *for any* $s>2$*. Let* (u,b) *be a locally smooth solution to system* (1.1)*. Then,* (u,b) *can be extended beyond time* T*, provided that*

$$\int_0^T \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^2)}\,dt<\infty. \tag{1.4}$$

Remark 1.3. In this paper, for simplicity of presentation, we merely prove Theorems 1.1 and 1.2 for $\alpha<1$ and $\beta<1$. When $\alpha\geq 1$ or $\beta\geq 1$, the proofs are much easier. Moreover, the global-in-time regularity solution in cases $\alpha=0$, $\beta>1$ and $\alpha>0$, $\beta=1$ has been proved by **[5, 13, 21]**, respectively.

Remark 1.4. At present, we are not able to show that Theorem 1.1 holds true under the condition $\alpha+\beta>1$ for $\alpha\leq 1/2$. The estimation of (3.10) in Section 3 prevents this possibility.

The general outline of the paper is as follows. In the next section, we first present some notation we shall use throughout this study, as well as preliminary inequalities. Section 3 is devoted to proving Theorem 1.1. In Section 4, we aim at the proof of Theorem 1.2 by using the same arguments adopted in proving Theorem 1.1.

2. Preliminaries. In this section, before we state the main results, we shall present some notation used throughout this study, as well as preliminary inequalities. Throughout the paper, C stands for some real positive constants which may differ in each occurrence. We shall sometimes use the notation $A \lesssim B$, which stands for $A \leq CB$. For brevity, we ∂_{x_i} by ∂_i for $i = 1, 2$.

Now, we briefly recall the Calderón-Zygmund estimate which will be frequently used throughout this paper.

Lemma 2.1. *For any smooth divergence-free vector field u with vorticity $\omega \in L^p$ and $p \in (1, \infty)$, an absolute constant $C > 0$ exists satisfying the following property:*

$$\|\nabla u\|_{L^p} \leq C \frac{p^2}{p-1} \|\omega\|_{L^p}.$$

Next, we present the following well-known fractional version of the Gagliardo-Nirenberg inequality, see [**15**].

Lemma 2.2 (Gagliardo-Nirenberg inequality)**.** *Let $1 < p, q, r < \infty$, $0 \leq \theta \leq 1$ and $s, s_1, s_2 \in \mathbb{R}$. Assume that $u \in C_c^\infty(\mathbb{R}^2)$. Then,*

$$\|\Lambda^s u\|_{L^p} \leq C \|\Lambda^{s_1} u\|_{L^q}^{1-\theta} \|\Lambda^{s_2} u\|_{L^r}^{\theta}, \tag{2.1}$$

where

$$\frac{1}{p} - \frac{s}{2} = (1-\theta)\left(\frac{1}{q} - \frac{s_1}{2}\right) + \theta\left(\frac{1}{r} - \frac{s_2}{2}\right), \quad s \leq (1-\theta)s_1 + \theta s_2.$$

In particular, we have the following lemma, see [**10, 25**].

Lemma 2.3. *If the spatial dimension is 2, then an absolute positive constant C exists such that the following interpolation inequalities hold true*:

$$\begin{aligned} \|u\|_{L^\infty} &\leq C \|u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2}, \\ \|\nabla u\|_{L^4} &\leq C \|u\|_{L^\infty}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2}, \\ \|u\|_{L^4} &\leq C \|u\|_{L^2}^{(2\varrho-1)/(2\varrho)} \|\Lambda^\varrho u\|_{L^2}^{1/(2\varrho)}, \qquad \varrho \geq \frac{1}{2}, \end{aligned}$$

$$\|\nabla f\|_{L^4} \leq C\|\Lambda^\delta f\|_{L^2}^{(2\delta+1)/4}\|\Lambda^\delta \nabla^2 f\|_{L^2}^{(3-2\delta)/4}, \quad 0 \leq \delta \leq 1,$$
$$\|\nabla^2 g\|_{L^4} \leq C\|\Lambda^\gamma g\|_{L^2}^{(2\gamma-1)/4}\|\Lambda^\gamma \nabla^2 g\|_{L^2}^{(5-2\gamma)/4}, \quad \frac{1}{2} \leq \gamma \leq \frac{5}{2}.$$

Now, we introduce the differential form Gronwall-type inequality to conclude this section. The proof is quite straightforward and is omitted.

Lemma 2.4. *Let $f(t)$ be a nonnegative, absolutely continuous function on $[0,T]$ which satisfies, for almost every t the differential inequality,*

$$f'(t) \leq g(t)F(f(t)),$$

where $g(t)$ is a nonnegative, integrable function on $[0,T]$ and nonnegative function $F(s)$ satisfies the following conditions, for any $0 < a \leq b < \infty$,

$$\int_a^b \frac{1}{F(s)}\, ds \leq C < \infty \quad \text{and} \quad \int_a^\infty \frac{1}{F(s)}\, ds = \infty.$$

Then, $f(t)$ is bounded for any $t \in [0,T]$.

3. Proof of Theorem 1.1. The existence of locally smooth solutions can easily be obtained, see, for example, **[26]**. Thus, in order to complete the proof of Theorem 1.1, it is sufficient to establish a priori uniformly strong estimates in $t \in [0, T)$. Therefore, in the following, we assume that solution (u, b) is sufficiently smooth on $[0, T)$. Keep in mind that we only consider $\alpha < 1$ and $\beta < 1$.

Proof of Theorem 1.1. The following logarithmic-type Sobolev inequality is needed before beginning the proof of Theorem 1.1, see, for example, **[22]**.

$$\begin{aligned} &\|\nabla f\|_{L^\infty(\mathbb{R}^n)} \\ &\quad \leq C\Big(1 + \|f\|_{L^2(\mathbb{R}^n)} + \|\nabla \times f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)} \log(1 + \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)})\Big), \end{aligned} \tag{3.1}$$

with $s > 1 + n/p$, $f \in L^2(\mathbb{R}^n) \cap \dot{W}^{s,p}(\mathbb{R}^n)$ and $\nabla \cdot f = 0$.

In order to prove Theorem 1.1, we begin with the following basic energy estimate.

Lemma 3.1. *Let $\alpha \geq 0$ and $\beta \geq 0$. For any corresponding solution (u, b) of* (1.1)*, some constants C exist such that, for any $t \in [0, T]$,*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^T (\|\Lambda^\alpha u(t)\|_{L^2}^2 + \|\Lambda^\beta b(t)\|_{L^2}^2)\, dt \leq C. \tag{3.2}$$

In order to obtain the H^1 estimate on (u, b), we first take curls on the GMHD equation (1.1) to obtain the equation of vorticity ω and the current density j:

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda^{2\alpha}\omega - (b \cdot \nabla) j = 0, \tag{3.3}$$

$$\partial_t j + (u \cdot \nabla) j + \Lambda^{2\beta} j - (b \cdot \nabla) w - T(\nabla u, \nabla b) = 0, \tag{3.4}$$

where $T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$.

Taking the inner product of equations (3.3) and (3.4) with ω and j, respectively, and adding, we deduce:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + \|\Lambda^\alpha \omega(t)\|_{L^2}^2 + \|\Lambda^\beta j(t)\|_{L^2}^2 \\ &\quad \leq \int_{\mathbb{R}^2} |T(\nabla u, \nabla b)||j|\, dx \\ &\quad \leq C\|\nabla b\|_{L^\infty}\|\nabla u\|_{L^2}\|j\|_{L^2} \leq C\|\nabla b\|_{L^\infty}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2). \end{aligned} \tag{3.5}$$

We denote

$$M(t) \triangleq \max_{\mu \in [T_0,\, t]} (\|\nabla^2 \omega(\mu)\|_{L^2}^2 + \|\nabla^2 j(\mu)\|_{L^2}^2).$$

It is an obvious observation that $M(t)$ is a monotonically increasing function. This observation will be useful later. Let $T_0 \in (0,\, T)$, which is to be fixed hereafter such that

$$T - T_0 \leq 1 \quad \text{and} \quad \log(1 + M(t)) \geq 1 \text{ for all } t \in [T_0,\, T].$$

The goal of this section is to show that, if assumption (1.3) holds, then the following holds:

$$\lim_{t \to T^-} M(t) \leq C < \infty,$$

for some positive constant C that depends only upon u_0, b_0, T and $M(T_0)$. The above estimate is enough to extend the smooth solution $(u,\ b)$ beyond T.

The logarithmic Sobolev (3.1) and Gronwall inequalities enable us to deduce that

$$
\begin{aligned}
(3.6)\quad & \|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2 \\
&\quad +\int_{T_0}^{t}(\|\Lambda^{\alpha}\omega(s)\|_{L^2}^2+\|\Lambda^{\beta}j(s)\|_{L^2}^2)\,ds \\
&\le C(\|\omega(T_0)\|_{L^2}^2+\|j(T_0)\|_{L^2}^2)\exp\left[\int_{T_0}^{t}\|\nabla b(s)\|_{L^\infty}\,ds\right] \\
&\le C(\|\omega(T_0)\|_{L^2}^2+\|j(T_0)\|_{L^2}^2) \\
&\cdot\exp\left[C\int_{T_0}^{t}\Big(1+\|b\|_{L^2}+\|j\|_{\dot{B}^0_{\infty,\infty}}\log(1+\|b\|_{\dot{H}^3}\Big)(s)\,ds\right] \\
&\le C\exp\left[\int_{T_0}^{t}C(1+\|b_0\|_{L^2})\,ds\right] \\
&\cdot\exp\left[C\Big(\int_{T_0}^{t}\|j(s)\|_{\dot{B}^0_{\infty,\infty}}\,ds\Big)\log(1+M(t))\right] \\
&\le C\exp\left[C\Big(\int_{T_0}^{t}\|j(s)\|_{\dot{B}^0_{\infty,\infty}}\,ds\Big)\log(1+M(t))\right],
\end{aligned}
$$

for all $T_0\le t<T$. Due to

$$\int_0^T\|j(s)\|_{\dot{B}^0_{\infty,\infty}}ds<\infty,$$

we can choose T_0 close enough to T such that

$$C\int_{T_0}^T\|j(s)\|_{\dot{B}^0_{\infty,\infty}}ds\le\epsilon$$

for a sufficiently small $\epsilon>0$, to be chosen hereafter. Therefore, it is easy to conclude that, for any $T_0\le t<T$,

$$
\begin{aligned}
(3.7)\quad \|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\int_{T_0}^{t}(\|\Lambda^{\alpha}\omega(s)\|_{L^2}^2+\|\Lambda^{\beta}j(s)\|_{L^2}^2)\,ds& \\
\le C(1+M(t))^{\epsilon}.&
\end{aligned}
$$

Next, we prove a global a priori bound for H^2-estimates on (ω,j). Applying ∇^2 to equations (3.3) and (3.4), taking the L^2 inner product of the so-obtained equations with $\nabla^2\omega$ and $\nabla^2 j$, respectively, and adding, we arrive at:

(3.8)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^2\omega(t)\|_{L^2}^2+\|\nabla^2 j(t)\|_{L^2}^2)+\|\Lambda^\alpha\nabla^2\omega(t)\|_{L^2}^2+\|\Lambda^\beta\nabla^2 j(t)\|_{L^2}^2\\
&\quad=\int_{\mathbb{R}^2}\big[u\cdot\nabla\nabla^2\omega-\nabla^2(u\cdot\nabla\omega)\big]\nabla^2\omega\,dx\\
&\qquad+\int_{\mathbb{R}^2}\big[\nabla^2(b\cdot\nabla j)-b\cdot\nabla\nabla^2 j\big]\nabla^2\omega\,dx\\
&\qquad+\int_{\mathbb{R}^2}\big[u\cdot\nabla\nabla^2 j-\nabla^2(u\cdot\nabla j)\big]\nabla^2 j\,dx\\
&\qquad+\int_{\mathbb{R}^2}\big[\nabla^2(b\cdot\nabla\omega)-b\cdot\nabla\nabla^2\omega\big]\nabla^2 j\,dx\\
&\qquad+\int_{\mathbb{R}^2}\nabla^2 T(\nabla u,\,\nabla b)\nabla^2 j\,dx\\
&\quad\lesssim\int_{\mathbb{R}^2}|\nabla^2 u|\,|\nabla\omega|\,|\nabla^2\omega|\,dx+\int_{\mathbb{R}^2}|\nabla u|\,|\nabla^2\omega|^2\,dx\\
&\qquad+\int_{\mathbb{R}^2}|\nabla^2 b|\,|\nabla j|\,|\nabla^2\omega|\,dx\\
&\qquad+\bigg(\int_{\mathbb{R}^2}|\nabla^2 b|\,|\nabla\omega|\,|\nabla^2 j|\,dx+\int_{\mathbb{R}^2}|\nabla^2 u|\,|\nabla j|\,|\nabla^2 j|\,dx\\
&\qquad\qquad+\int_{\mathbb{R}^2}|\nabla^2 u|\,|\nabla^2 b|\,|\nabla^2 j|\,dx\bigg)\\
&\qquad+\bigg(\int_{\mathbb{R}^2}|\nabla b|\,|\nabla^2 j|\,|\nabla^2\omega|\,dx+\int_{\mathbb{R}^2}|\nabla^3 u|\,|\nabla b|\,|\nabla^2 j|\,dx\bigg)\\
&\qquad+\bigg(\int_{\mathbb{R}^2}|\nabla u|\,|\nabla^2 j|^2\,dx+\int_{\mathbb{R}^2}|\nabla u|\,|\nabla^3 b|\,|\nabla^2 j|\,dx\bigg)\\
&\quad\triangleq J_1+J_2+\cdots+J_6,
\end{aligned}
$$

where we used the facts

$$\int_{\mathbb{R}^2}u\cdot\nabla\nabla^2\omega\cdot\nabla^2\omega\,dx=\int_{\mathbb{R}^2}u\cdot\nabla\nabla^2 j\cdot\nabla^2 j\,dx=0$$

and

$$\int_{\mathbb{R}^2}b\cdot\nabla\nabla^2 j\cdot\nabla^2\omega\,dx+\int_{\mathbb{R}^2}b\cdot\nabla\nabla^2\omega\cdot\nabla^2 j\,dx=0.$$

Now, we estimate the terms on the right-hand side of (3.8) one-by-one as follows. We begin with an estimate of the term J_1. Applying the Hölder and Gagliardo-Nirenberg inequalities to J_1 yields:

$$\begin{aligned} J_1 &\lesssim \|\nabla^2 u\|_{L^4}\|\nabla w\|_{L^4}\|\nabla^2 w\|_{L^2} \\ &\lesssim \|\nabla\omega\|_{L^4}^2\|\nabla^2\omega\|_{L^2} \\ &\lesssim \|\Lambda^\alpha\omega\|_{L^2}^{(2\alpha+1)/2}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^{(3-2\alpha)/2}\|\nabla^2\omega\|_{L^2} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C\|\Lambda^\alpha\omega\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^{4/(2\alpha+1)}. \end{aligned} \tag{3.9}$$

Note that here and in what follows the Gagliardo-Nirenberg inequality holds true for $\alpha \le 1$. However, if $\alpha > 1$, the case can be handled easily by a similar argument.

By the Hölder and Gagliardo-Nirenberg inequalities, we estimate J_2 as:

$$\begin{aligned} J_2 &\lesssim \|\nabla u\|_{L^2}\|\nabla^2\omega\|_{L^4}^2 \\ &\lesssim \|\omega\|_{L^2}\|\Lambda^\alpha\omega\|_{L^2}^{(2\alpha-1)/2}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^{(5-2\alpha)/2} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C\|\omega\|_{L^2}^{4/(2\alpha-1)}\|\Lambda^\alpha\omega\|_{L^2}^2, \end{aligned} \tag{3.10}$$

where we need the restriction $\alpha > 1/2$. According to the Hölder and Gagliardo-Nirenberg inequalities, J_3 can be bounded as follows:

$$\begin{aligned} J_3 &\lesssim \|\nabla^2 b\|_{L^4}\|\nabla j\|_{L^4}\|\nabla^2\omega\|_{L^2} \lesssim \|\nabla j\|_{L^4}^2\|\nabla^2 w\|_{L^2} \\ &\lesssim \|j\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2} \lesssim \|\nabla b\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2}. \end{aligned} \tag{3.11}$$

Due to Lemma 2.1 and $\alpha + \beta > 1$, the following is obtained for J_4:

$$\begin{aligned} J_4 &\lesssim \|\nabla^2 b\|_{L^4}\|\nabla\omega\|_{L^4}\|\nabla^2 j\|_{L^2} \lesssim \|\nabla j\|_{L^4}\|\nabla\omega\|_{L^4}\|\nabla^2 j\|_{L^2} \\ &\lesssim \|\Lambda^\beta j\|_{L^2}^{(2\beta+1)/4}\|\Lambda^\beta\nabla^2 j\|_{L^2}^{(3-2\beta)/4}\|\Lambda^\alpha\omega\|_{L^2}^{(2\alpha+1)/4} \\ &\quad \cdot \|\Lambda^\alpha\nabla^2\omega\|_{L^2}^{(3-2\alpha)/4}\|\nabla^2 j\|_{L^2} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + \frac{1}{16}\|\Lambda^\beta\nabla^2 j\|_{L^2}^2 \\ &\quad + C(\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2 j\|_{L^2}^{4/(1+\alpha+\beta)}. \end{aligned} \tag{3.12}$$

For J_5, we obtain

$$J_5 \le C\|\nabla b\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2}.$$

For J_6, we directly achieve, for $\alpha+\beta>1$,

(3.13)

$$\begin{aligned} J_6 &\lesssim \|\nabla u\|_{L^{2/(1-\alpha)}}\|\nabla^2 j\nabla^2 j\|_{L^{2/(1+\alpha)}} + \|\nabla u\|_{L^{2/1-\alpha}}\|\nabla^3 b\nabla^2 j\|_{L^{2/1+\alpha}} \\ &\lesssim \|\Lambda^\alpha \omega\|_{L^2}\|\nabla^2 j\|_{L^2}\|\nabla^2 j\|_{L^{2/\alpha}} \\ &\lesssim \|\Lambda^\alpha \omega\|_{L^2}\|\nabla^2 j\|_{L^2}\|\Lambda^\beta j\|_{L^2}^{1-(3-\alpha-\beta)/2}\|\Lambda^\beta\nabla^2 j\|_{L^2}^{(3-\alpha-\beta)/2} \\ &\le \frac{1}{16}\|\Lambda^\beta\nabla^2 j\|_{L^2}^2 + C(\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2 j\|_{L^2}^{4/(\alpha+\beta+1)}. \end{aligned}$$

Combining all of the estimates $J_1, J_2, \ldots, J_6$, we get:

$$\begin{aligned} &\frac{d}{dt}(\|\nabla^2\omega(t)\|_{L^2}^2 + \|\nabla^2 j(t)\|_{L^2}^2) + \|\Lambda^\alpha\nabla^2\omega(t)\|_{L^2}^2 + \|\Lambda^\beta\nabla^2 j(t)\|_{L^2}^2 \\ &\quad\le C\|\Lambda^\alpha\omega\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^{4/(2\alpha+1)} + C\|\omega\|_{L^2}^{4/(2\alpha-1)}\|\Lambda^\alpha\omega\|_{L^2}^2 \\ &\qquad + C(\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2 j\|_{L^2}^{4/(1+\alpha+\beta)} \\ &\qquad + C\|\nabla b\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2}. \end{aligned}$$

Using the logarithmic Sobolev inequality (3.1) and ignoring the dissipative term, we can thus deduce from the above inequality that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla^2\omega(t)\|_{L^2}^2 + \|\nabla^2 j(t)\|_{L^2}^2) \\ &\quad\le C\big(1 + \|j\|_{\dot B^0_{\infty,\infty}}\log(1+\|\nabla^2 j\|_{L^2}^2)\big)\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2} \\ &\qquad + C\|\Lambda^\alpha\omega\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^{4/(2\alpha+1)} + C\|\omega\|_{L^2}^{4/(2\alpha-1)}\|\Lambda^\alpha\omega\|_{L^2}^2 \\ &\qquad + C(\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2 j\|_{L^2}^{4/(1+\alpha+\beta)}. \end{aligned} \tag{3.14}$$

Integrating over interval (T_0, t) and observing that $M(t)$ is a monotonically increasing function, it follows that

(3.15)

$$\begin{aligned} &1 + M(t) - M(T_0) \\ &\quad\le C(T_0)\int_{T_0}^t \big(1 + \|j\|_{\dot B^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds \\ &\qquad + C(T_0)\int_{T_0}^t M(s)^{2/(2\alpha+1)}\|\Lambda^\alpha\omega\|_{L^2}^2\,ds \end{aligned}$$

$$
\begin{aligned}
&+ C(T_0)\int_{T_0}^{t} \|\omega\|_{L^2}^{4/(2\alpha-1)}\|\Lambda^\alpha\omega\|_{L^2}^2\,ds \\
&+ C(T_0)\int_{T_0}^{t} \big(1+M(s)^{2/(1+\alpha+\beta)}\big)(\|\Lambda^\alpha\omega\|_{L^2}^2+\|\Lambda^\beta j\|_{L^2}^2)\,ds \\
\le\ & C(T_0)\int_{T_0}^{t} \big(1+\|j\|_{\dot B^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds \\
&+ C(T_0)M(t)^{2/(2\alpha+1)}\int_{T_0}^{t}\|\Lambda^\alpha\omega\|_{L^2}^2\,ds + C(T_0) \\
&\cdot \int_{T_0}^{t} \|\omega\|_{L^2}^{4/(2\alpha-1)}\|\Lambda^\alpha\omega\|_{L^2}^2\,ds \\
&+ C(T_0)M(t)^{2/(1+\alpha+\beta)}\int_{T_0}^{t}(\|\Lambda^\alpha\omega\|_{L^2}^2+\|\Lambda^\beta j\|_{L^2}^2)\,ds \\
\le\ & C(T_0)\int_{T_0}^{t} \big(1+\|j\|_{\dot B^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds \\
&+ C(T_0)(1+M(t))^{2/(2\alpha+1)+\epsilon} + C(T_0)(1+M(t))^{[2\epsilon/(2\alpha-1)]+\epsilon} \\
&+ C(T_0)\big(1+M(t)\big)^{2/(1+\alpha+\beta)}(1+M(t))^{\epsilon} + C(T_0).
\end{aligned}
$$

We remark that estimate (3.7) has been used several times.

Taking

$$
\epsilon = \frac{1}{2}\min\left\{\frac{2\alpha-1}{2\alpha+1},\quad \frac{2\alpha-1}{4\alpha-1},\quad \frac{\alpha+\beta-1}{\alpha+\beta+1}\right\} > 0,
$$

a simple calculation shows that

(3.16)

$$
\begin{aligned}
1+M(t) \le\ & C(T_0)\int_{T_0}^{t} \big(1+\|j(s)\|_{\dot B^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds \\
&+ C(T_0)(1+M(t))^{\gamma} + C(T_0) \\
\le\ & C(T_0)\int_{T_0}^{t} \big(1+\|j(s)\|_{\dot B^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds \\
&+ \frac{1}{2}(1+M(t)) + C(T_0),
\end{aligned}
$$

with some $\gamma \in (0, 1)$. Thus, we have

$$1 + M(t) \le C + C \int_{T_0}^{t} \big(1 + \|j(s)\|_{\dot{B}^0_{\infty,\infty}} \log\big(1 + M(s)\big)\big)(1 + M(s))\, ds.$$

For simplicity of exposition, we denote

$$V(t) \triangleq C + C \int_{T_0}^{t} \big(1 + \|j(s)\|_{\dot{B}^0_{\infty,\infty}} \log(1 + M(s))\big)(1 + M(s))\, ds.$$

Thus, we have

$$1 + M(t) \le V(t).$$

Therefore,

$$\begin{aligned} \frac{d}{dt} V(t) &= C\big(1 + \|j(t)\|_{\dot{B}^0_{\infty,\infty}} \log(1 + M(t))\big)(1 + M(t)) \\ &\le C\big(1 + \|j(t)\|_{\dot{B}^0_{\infty,\infty}} \log V(t)\big)V(t). \end{aligned} \tag{3.17}$$

Applying the standard Log-Gronwall type inequality, see Lemma 2.4, now tells us that $M(t)$ remains bounded for any $t \in [0, T]$, which implies that

$$\max_{0\le t\le T} (\|\nabla^2 w(t)\|_{L^2}^2 + \|\nabla^2 j(t)\|_{L^2}^2) \le C.$$

Thus, we have completed the proof of Theorem 1.1. □

4. Proof of Theorem 1.2. This section aims at proving Theorem 1.2 which follows the approach used in the proof of Theorem 1.1. For the sake of completeness, detailed proofs are given as follows.

Proof of Theorem 1.2. To begin, we obtain the following L^2 bounds for (ω, j)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + \|\Lambda^\alpha \omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2 \\ &\quad \le \int_{\mathbb{R}^2} |T(\nabla u, \nabla b)||j|\, dx \\ &\quad \le C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}\|j\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|j\|_{L^2}^2 \\ &\quad \le C\|\nabla u\|_{L^\infty}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2). \end{aligned} \tag{4.1}$$

The definition of

$$M(t) \triangleq \max_{\mu \in [T_0,\, t]} (\|\nabla^2 \omega(\mu)\|_{L^2}^2 + \|\nabla^2 j(\mu)\|_{L^2}^2)$$

remains the same. We deduce from the logarithmic Sobolev (3.1) and Gronwall inequalities that

(4.2)

$$\begin{aligned}
&\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_{T_0}^{t} (\|\Lambda^{\alpha}\omega(s)\|_{L^2}^2 + \|\Lambda^{\beta} j(s)\|_{L^2}^2)\, ds \\
&\quad \le C(\|\omega(T_0)\|_{L^2}^2 + \|j(T_0)\|_{L^2}^2) \\
&\quad \cdot \exp\left[\int_{T_0}^{t} \|\nabla u(s)\|_{L^\infty}\, ds\right] \\
&\quad \le C(\|\omega(T_0)\|_{L^2}^2 + \|j(T_0)\|_{L^2}^2) \\
&\quad \cdot \exp\left[C \int_{T_0}^{t} \big(1 + \|u\|_{L^2} + \|\omega\|_{\dot{B}^0_{\infty,\,\infty}} \log(1 + \|u\|_{\dot{H}^3})\big)\, ds\right] \\
&\quad \le C \exp\left[\int_{T_0}^{t} C(1 + \|u_0\|_{L^2})\, ds\right] \\
&\quad \cdot \exp\left[C\bigg(\int_{T_0}^{t} \|\omega\|_{\dot{B}^0_{\infty,\,\infty}}\, ds\bigg) \log(1 + M(t))\right] \\
&\quad \le C \exp\left[C\bigg(\int_{T_0}^{t} \|\omega\|_{\dot{B}^0_{\infty,\,\infty}}\, ds\bigg) \log(1 + M(t))\right], \quad \text{for all } T_0 \le t < T.
\end{aligned}$$

Due to

$$\int_0^T \|\omega(s)\|_{\dot{B}^0_{\infty,\,\infty}}\, ds < \infty,$$

we can choose T_0 close enough to T such that

$$C \int_{T_0}^{T} \|\omega(s)\|_{\dot{B}^0_{\infty,\,\infty}}\, ds \le \epsilon$$

for sufficiently small number $\epsilon > 0$ to be chosen later. Thus, we get

$$\text{(4.3)} \quad \|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_{T_0}^{t} (\|\Lambda^{\alpha}\omega(s)\|_{L^2}^2 + \|\Lambda^{\beta} j(s)\|_{L^2}^2)\, ds \le C(1 + M(t))^{\epsilon}.$$

Noting (3.8), it is sufficient to estimate the terms $J_1 - J_6$. The Gagliardo-Nirenberg inequality tells us that

$$J_1 \lesssim \|\nabla^2 u\|_{L^4}\|\nabla w\|_{L^4}\|\nabla^2 w\|_{L^2} \lesssim \|\nabla\omega\|_{L^4}^2\|\nabla^2\omega\|_{L^2} \lesssim (\|\omega\|_{L^\infty}\|\nabla^2\omega\|_{L^2})\|\nabla^2\omega\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\|\nabla^2\omega\|_{L^2}^2, \tag{4.4}$$

and

$$J_2 \le \|\nabla u\|_{L^\infty}\|\nabla^2\omega\|_{L^2}^2. \tag{4.5}$$

Due to $\alpha+\beta>1$, we may choose p_1 satisfying

$$\max\left\{1,\ \frac{2}{1+\alpha},\ \frac{2(2+\alpha-\beta)}{(2\alpha+1)(2-\beta)}\right\} < p_1 < \min\left\{2,\ \frac{2}{2-\beta}\right\}; \tag{4.6}$$

therefore, by using the Gagliardo-Nirenberg and the Young inequalities, we obtain:

(4.7)

$$\begin{aligned} J_3 &\lesssim \|\nabla^2 b\|_{L^{2p_1}}\|\nabla j\|_{L^{2p_1}}\|\nabla^2\omega\|_{L^{p_1/(p_1-1)}} \\ &\lesssim \|\nabla j\|_{L^{2p_1}}^2\|\nabla^2\omega\|_{L^{p_1/(p_1-1)}} \\ &\lesssim \|\Lambda^\beta j\|_{L^2}^{2(1-\lambda_1)}\|\nabla^2 j\|_{L^2}^{2\lambda_1}\|\nabla^2\omega\|_{L^2}^{1-\lambda_2}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^{\lambda_2} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C\|\Lambda^\beta j\|_{L^2}^{[4(1-\lambda_1)]/[2-\lambda_2]}(\|\nabla^2 j\|_{L^2}^{2\lambda_1}\|\nabla^2\omega\|_{L^2}^{1-\lambda_2})^{2/(2-\lambda_2)} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C(1+\|\Lambda^\beta j\|_{L^2}^2)(\|\nabla^2 j\|_{L^2}^{2\lambda_1}\|\nabla^2\omega\|_{L^2}^{1-\lambda_2})^{2/(2-\lambda_2)} \\ &\le \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C(1+\|\Lambda^\beta j\|_{L^2}^2)(\|\nabla^2 j\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2)^{\nu_1}, \end{aligned}$$

where

$$\lambda_1 = 1 - \frac{1}{(2-\beta)p_1}, \qquad \lambda_2 = \frac{2-p_1}{\alpha p_1}, \qquad \nu_1 = \left(\lambda_1 + \frac{1-\lambda_2}{2}\right)\frac{2}{2-\lambda_2}.$$

It should be noted that, when p satisfies (4.10) it is then easy to check

$$\nu_1 < 1. \tag{4.8}$$

By the Hölder, Gagliardo-Nirenberg and Young inequalities, we obtain

$$J_4 \lesssim \|\nabla^2 b\|_{L^{2(2+\beta)}}\|\nabla\omega\|_{L^{[2(2+\beta)]/(1+\beta)}}\|\nabla^2 j\|_{L^2} \tag{4.9}$$

$$
\begin{aligned}
&\lesssim \|\nabla j\|_{L^{2(2+\beta)}} \|\nabla\omega\|_{L^{[2(2+\beta)]/(1+\beta)}} \|\nabla^2 j\|_{L^2} \\
&\lesssim \|j\|_{L^2}^{1-[1/(2+\beta)]} \|\Lambda^\beta \nabla^2 j\|_{L^2}^{1/(2+\beta)} \\
&\quad \cdot \|\Lambda^\alpha \omega\|_{L^2}^{1-[((1-\alpha)(2+\beta)+1)/((2-\alpha)(2+\beta))]} \\
&\quad \cdot \|\nabla^2 \omega\|_{L^2}^{[(1-\alpha)(2+\beta)+1]/[(2-\alpha)(2+\beta)]} \|\nabla^2 j\|_{L^2} \\
&\le \frac{1}{16}\|\Lambda^\beta \nabla^2 j\|_{L^2}^2 + C\|j\|_{L^2}^{[2(1+\beta)]/(3+2\beta)} \|\Lambda^\alpha \omega\|_{L^2}^{1/(2-\alpha)} \\
&\quad \cdot \left(\|\nabla^2 \omega\|_{L^2}^{[(1-\alpha)(2+\beta)+1]/[(2-\alpha)(2+\beta)]} \|\nabla^2 j\|_{L^2}\right)^{[2(2+\beta)]/(3+2\beta)} \\
&\le \frac{1}{16}\|\Lambda^\beta \nabla^2 j\|_{L^2}^2 + C\|j\|_{L^2}^{[2(1+\beta)]/(3+2\beta)} (1+\|\Lambda^\alpha \omega\|_{L^2}^2) \\
&\quad \cdot \left(\|\nabla^2 \omega\|_{L^2}^2 + \|\nabla^2 j\|_{L^2}^2\right)^{\nu_2},
\end{aligned}
$$

where ν_2 is given by

$$
\nu_2 = \frac{(3-2\alpha)(2+\beta)+1}{(2-\alpha)(3+2\beta)} < 1
$$

due to $\alpha+\beta>1$. Once again, due to $\alpha+\beta>1$, we can take p_2 satisfying

$$
\max\left\{0,\ \frac{1-\beta}{2}\right\} < \frac{1}{p_2} < \min\left\{\frac{1}{2},\ \frac{3-\beta}{2},\ \frac{\alpha}{2}\right\}, \tag{4.10}
$$

which, by making use of the Gagliardo-Nirenberg and Young inequalities, allows us to deduce:

(4.11)

$$
\begin{aligned}
J_5 &\lesssim \|\nabla b\|_{L^2} \|\nabla^2 j\|_{L^{p_2}} \|\nabla^2 \omega\|_{L^{(2p_2)/(p_2-2)}} \\
&\lesssim \|j\|_{L^2} \|\nabla^2 j\|_{L^{p_2}} \|\nabla^2 \omega\|_{L^{(2p_2)/(p_2-2)}} \\
&\lesssim \|j\|_{L^2} \|\Lambda^\beta j\|_{L^2}^{1-[(3-\beta)p_2-2]/(2p_2)} \|\Lambda^\beta \nabla^2 j\|_{L^2}^{[(3-\beta)p_2-2]/(2p_2)} \\
&\quad \cdot \|\nabla^2 \omega\|_{L^2}^{1-[2/(\alpha p_2)]} \|\Lambda^\alpha \nabla^2 \omega\|_{L^2}^{2/(\alpha p_2)} \\
&\le \frac{1}{16}\|\Lambda^\beta \nabla^2 j\|_{L^2}^2 + \frac{1}{16}\|\Lambda^\alpha \nabla^2 \omega\|_{L^2}^2 + C\|b\|_{L^2}^{[4(2\alpha-1)]/(2\beta-1)} \\
&\quad \cdot \|\nabla b\|_{L^2}^{[4(3-2\alpha)]/(2\beta-1)} \|\Lambda^\beta j\|_{L^2}^2 \\
&\le \frac{1}{16}\|\Lambda^\beta \nabla^2 j\|_{L^2}^2 + \frac{1}{16}\|\Lambda^\alpha \nabla^2 \omega\|_{L^2}^2 + C\|j\|_{L^2}^{(4\alpha p_2)/[\alpha(1+\beta)p_2+2\alpha-4]} \\
&\quad \cdot \|\Lambda^\beta j\|_{L^2}^{[2\alpha[2-(1-\beta)p_2]]/[\alpha(1+\beta)p_2+2\alpha-4]}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \|\nabla^2\omega\|_{L^2}^{[2\alpha[2-(1-\beta)p]]/[\alpha(1+\beta)p_2+2\alpha-4]} \\
&\le \frac{1}{16}\|\Lambda^\beta\nabla^2 j\|_{L^2}^2 + \frac{1}{16}\|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + C\|j\|_{L^2}^{(4\alpha p_2)/[\alpha(1+\beta)p_2+2\alpha-4]} \\
&\cdot (1+\|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2\omega\|_{L^2}^{\nu_3},
\end{aligned}
$$

where ν_3 is given by

$$
\nu_3 = \frac{2\alpha[2-(1-\beta)p]}{\alpha(1+\beta)p_2+2\alpha-4} < 1,
$$

due to $\alpha+\beta>1$. For J_6, we directly obtain

$$
J_6 \le C\|\nabla u\|_{L^\infty}\|\nabla^2 j\|_{L^2}^2. \tag{4.12}
$$

Plugging estimates (4.4)–(4.12) into (3.8) and absorbing the dissipative terms, we have:

(4.13)

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla^2\omega(t)\|_{L^2}^2 + \|\nabla^2 j(t)\|_{L^2}^2) + \|\Lambda^\alpha\nabla^2\omega\|_{L^2}^2 + \|\Lambda^\beta\nabla^2 j\|_{L^2}^2 \\
&\quad\le C\|\nabla u\|_{L^\infty}(\|\nabla^2\omega\|_{L^2}^2 + \|\nabla^2 j\|_{L^2}^2) \\
&\qquad + C(1+\|\Lambda^\beta j\|_{L^2}^2)(\|\nabla^2 j\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2)^{\nu_1} \\
&\qquad + C\|j\|_{L^2}^{[2(1+\beta)]/(3+2\beta)}(1+\|\Lambda^\alpha\omega\|_{L^2}^2)\big(\|\nabla^2\omega\|_{L^2}^2 + \|\nabla^2 j\|_{L^2}^2\big)^{\nu_2} \\
&\qquad + C\|j\|_{L^2}^{[4\alpha p_2]/[\alpha(1+\beta)p_2+2\alpha-4]}(1+\|\Lambda^\beta j\|_{L^2}^2)\|\nabla^2\omega\|_{L^2}^{\nu_3}.
\end{aligned}
$$

Making use of the logarithmic Sobolev inequality (3.1), we thus get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^2\omega(t)\|_{L^2}^2 + \|\nabla^2 j(t)\|_{L^2}^2) \\
&\quad\le C(T_0)\big(1+\|\omega\|_{\dot B^0_{\infty,\infty}}\log(1+M(t))\big)M(t) \\
&\qquad + C(1+\|\Lambda^\beta j\|_{L^2}^2)M(t)^{\nu_1} \\
&\qquad + C(1+M(t))^{(1+\beta)/(3+2\beta+\nu_2)}(1+\|\Lambda^\alpha\omega\|_{L^2}^2) \\
&\qquad + C(1+M(t))^{(2\alpha p_2\epsilon)/[\alpha(1+\beta)p_2+2\alpha-4+\nu_3]}(1+\|\Lambda^\beta j\|_{L^2}^2).
\end{aligned} \tag{4.14}
$$

Integrating over interval (T_0,t) and using the monotonicity of $M(t)$, we thus have

(4.15)

$$
\begin{aligned}
&1+M(t)-M(T_0)\\
&\quad\le C(T_0)\int_{T_0}^{t}\big(1+\|\omega(s)\|_{\dot{B}^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds\\
&\qquad+C(T_0)M(t)^{\nu_1}\int_{T_0}^{t}(1+\|\Lambda^{\beta}j\|_{L^2}^2)\,ds\\
&\qquad+C(T_0)(1+M(t))^{(1+\beta)/[3+2\beta+\nu_2]}\int_{T_0}^{t}(1+\|\Lambda^{\alpha}\omega\|_{L^2}^2)\,ds\\
&\qquad+C(T_0)(1+M(t))^{(2\alpha p_2\epsilon)/[\alpha(1+\beta)p_2+2\alpha-4+\nu_3]}\int_{T_0}^{t}(1+\|\Lambda^{\beta}j\|_{L^2}^2)\,ds\\
&\quad\le C(T_0)\int_{T_0}^{t}\big(1+\|\omega(s)\|_{\dot{B}^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds\\
&\qquad+C(T_0)(1+M(t))^{\nu_1+\epsilon}+C(T_0)(1+M(t))^{[(1+\beta)/(3+2\beta)]+\nu_2+\epsilon}\\
&\qquad+C(T_0)(1+M(t))^{(2\alpha p_2\epsilon)/[\alpha(1+\beta)p_2+2\alpha-4]+\nu_3+\epsilon}.
\end{aligned}
$$

Taking

$$
\epsilon=\frac{1}{2}\min\Big\{1-\nu_1,\frac{(1-\nu_2)(3+2\beta)}{1+\beta},\frac{(1-\nu_3)[\alpha(1+\beta)p_2+2\alpha-4]}{\alpha(3+\beta)p_2+2\alpha-4}\Big\}>0,
$$

we can thus obtain:

$$
\begin{aligned}
(4.16)\quad &1+M(t)-M(T_0)\\
&\le C(T_0)\int_{T_0}^{t}\big(1+\|\omega(s)\|_{\dot{B}^0_{\infty,\infty}}\log(1+M(s))\big)(1+M(s))\,ds\\
&\quad+C(T_0)(1+M(t))^{\gamma}+C(T_0),
\end{aligned}
$$

with some $\gamma\in(0,1)$. Therefore, taking advantage of the same arguments as used in the proof of Theorem 1.1, it is easy to show that the desired conclusion holds true. Thus, the proof of Theorem 1.2 is complete. □

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