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ATOMIC DECOMPOSITION OF MARTINGALE WEIGHTED LORENTZ SPACES WITH TWO-PARAMETER AND APPLICATIONS

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ABSTRACT. We introduce martingale weighted two-parameter Lorentz spaces and establish atomic decomposition theorems. As an application of atomic decomposition we obtain a sufficient condition for sublinear operators defined on martingale weighted Lorentz spaces to be bounded. Moreover, some interpolation properties with a function parameter of those spaces are obtained.

1. Introduction. It is well known that the method of atomic decompositions plays an important role in martingale theory and harmonic analysis. For instance, atomic decomposition is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities both in martingale theory and harmonic analysis. In **[7]**, Coifman used the Fefferman-Stein theory of H^P spaces **[9]** to decompose the functions of these spaces into basic building blocks (atoms). Coifman and Weiss have provided a comprehensive treatment of these ideas and many applications to harmonic analysis **[8]**. For one- and two-parameter martingale spaces, Weisz **[17]** gave some atomic decomposition theorems on martingale spaces and proved many important martingale inequalities and the duality theorems for martingale Hardy spaces with the aid of atomic decompositions. Hou and Ren **[11]** obtained some weak types of martingale inequalities through the use of atomic decompositions.

Atomic decompositions of Lorentz martingales were first studied by Jiao, et al., in **[12]**. In **[10]**, Ho investigated the atomic decomposition of Lorentz-Karamata martingale spaces using similar ideas as in **[12]**. Riyan and Shixin **[16]** obtained atomic decomposition for B-valued

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martingales in the two-parameter case and, in **[13]**, Li and Liu proved atomic decomposition theorems for two-parameter B-valued martingales in weak Hardy spaces. The technique of stopping times used in the one-parameter case is usually unsuitable for the case of two-parameter martingales, but the method of atomic decompositions deals with them in the same way. In this paper, by using the ideas of **[17]**, we prove the atomic decomposition theorem for martingale weighted Lorentz spaces. We obtain a sufficient condition for sublinear operators, defined on martingale weighted Lorentz spaces, to be bounded. Finally, we establish some interpolation theorems of these spaces with a function parameter.

2. Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The distribution function λ_f of a measurable function f on Ω is given by

$$\lambda_f(t) = P(\{w \in \Omega : |f(w)| > t\}), \quad t \geq 0,$$

and its decreasing rearrangement of f is the function $\widetilde{f}$ defined on $[0, \infty)$ by

$$\widetilde{f}(s) = \inf\{t > 0 : \lambda_f(t) \leq s\}, \quad s \geq 0.$$

Let $\varphi > 0$ be a non-negative and local integrable function on $[0, \infty)$. The classical Lorentz space $\Lambda_q(\varphi)$ is defined to be the collection of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_q(\varphi)} := \begin{cases} \left(\displaystyle\int_0^\infty (\widetilde{f}(t)\varphi(t))^q \, \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_s \widetilde{f}(s)\varphi(s), & q = \infty, \end{cases}$$

is finite. Moreover, integration by parts yields

$$\int_0^\infty (\widetilde{f}(t)\varphi(t))^q \, \frac{dt}{t} = q \int_0^\infty y^{q-1} \left\{ \int_0^{\lambda_f(y)} \varphi^q(t) \, \frac{dt}{t} \right\} dy, \quad 0 < q < \infty,$$

and hence,

$$\int_0^\infty (\widetilde{f}(t)\varphi(t))^q \, \frac{dt}{t} = q \int_0^\infty y^{q-1} w^q(\lambda_f(y)) \, dy,$$

where $w(t) = \{\int_0^t \varphi^q(s)\, ds/s\}^{1/q}$ is a positive, non-decreasing weight, see [**4**]. For $q = \infty$, we have

$$\|f\|_{\Lambda_\infty(\varphi)} = \sup_y y w(\lambda_f(y)) < \infty.$$

Recall that, for $0 < q \le \infty$, $\|\cdot\|_{\Lambda_q(\varphi)}$ is a quasi-norm if its *fundamental function* $w(t) = \{\int_0^t \varphi^q(s)\, ds/s\}^{1/q}$ satisfies the Δ_2-condition, $w(2t) \le Cw(t)$ for some $C > 0$, and, since w is a non-decreasing function, we have that $w(x+y) \le C(w(x) + w(y))$. Then

$$\begin{aligned}
\|f+g\|^q_{\Lambda_q(\varphi)} &= q \int_0^\infty y^{q-1} w^q(\lambda_{f+g}(y))\, dy \\
&\le q \int_0^\infty y^{q-1} w^q\left(\lambda_f\left(\frac{y}{2}\right) + \lambda_g\left(\frac{y}{2}\right)\right) dy \\
&\le C \int_0^\infty y^{q-1} w^q\left(\lambda_f\left(\frac{y}{2}\right) + w^q\left(\lambda_g\left(\frac{y}{2}\right)\right)\right) dy \\
&\le C(\|f\|^q_{\Lambda_q(\varphi)} + \|g\|^q_{\Lambda_q(\varphi)}).
\end{aligned}$$

These spaces play an important role in the theory of Banach spaces, and they have been the object of intensive investigation [**1, 2, 3, 5, 6**].

Let (A_0, A_1) denote a compatible quasi-Banach pair (i.e., A_0 and A_1 are quasi-Banach spaces, and both are continuously embedded in some Hausdorff topological vector space). For every $f \in A_0 + A_1$, and any $0 < t < \infty$, the so-called Peetre K-functional is defined by

$$K(t, f, A_0, A_1) = K(t, f) := \inf_{f_0 + f_1 = f} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\},$$

where $f_i \in A_i$, $i = 0, 1$. For $0 < q \le \infty$, and each measurable function ϱ, the real interpolation space $(A_0, A_1)_{\varrho,q}$ consists of all elements of $f \in A_0 + A_1$ such that the following quantity is finite:

$$\|f\|_{(A_0,A_1)_{\varrho,q}} := \begin{cases} \left(\displaystyle\int_0^\infty \left(\frac{K(t,f)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{t>0} \frac{K(t,f)}{\varrho(t)}, & q = \infty. \end{cases}$$

Let a and b be real numbers such that $a < b$. Following Persson's convention [**14**], we adopt the following notation. By $\varphi(t) \in Q[a, b]$, we mean that $\varphi(t)t^{-a}$ is non-decreasing and $\varphi(t)t^{-b}$ is non-increasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a,b)$, wherever $\varphi(t) \in Q[a+\epsilon, b-\epsilon]$ for some $\epsilon > 0$. By $\varphi(t) \in Q(a,-)$, or $\varphi(t) \in Q(-,b)$, we mean that $\varphi(t) \in Q(a,c)$, or $\varphi(t) \in Q(c,b)$, for some real number c.

In this paper, we shall consider the interpolation spaces $(A_0, A_1)_{\varrho,q}$ with a parameter function $\varrho = \varrho(t) \in Q(0,1)$, where A_0 and A_1 are martingale spaces. It is easy to see that $\varrho(t) = t^\theta$, $0 < \theta < 1$, belongs to $Q(0,1)$; thus, by replacing the measurable function $\varrho = \varrho(t)$ with t^θ, we obtain $(A_0, A_1)_{\theta,q}$. Let $0 < p_0$, $p_1 < \infty$, $p_0 \neq p_1$, $0 < q \leq \infty$, $\varrho \in Q(0,1)$. Then, by [**14,** Proposition 6.2], we know that

$$(L_{p_0}, L_{p_1})_{\varrho,q} = \Lambda_q(t^{1/p_0}/\varrho(t^{1/p_0-1/p_1})). \tag{2.1}$$

In order to prove our main results, we need the next lemma.

Lemma 2.1 ([**14**]). *Let $0 < q \leq \infty$, $0 < p < \infty$ and $\psi(t) \in Q(-,-)$. Let h be a positive and non-increasing function on $(0,\infty)$.*

(i) *If $\varphi(t) \in Q(-,0)$, then*

$$\bigg(\int_0^\infty (\varphi(t))^q \bigg(\int_0^t (h(u)\psi(u))^p \, \frac{du}{u}\bigg)^{q/p} \frac{dt}{t}\bigg)^{1/q} \leq C\bigg(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \, \frac{dt}{t}\bigg)^{1/q}.$$

(ii) *If $\varphi(t) \in Q(0,-)$, then*

$$\bigg(\int_0^\infty (\varphi(t))^q \bigg(\int_t^\infty (h(u)\psi(u))^p \, \frac{du}{u}\bigg)^{q/p} \frac{dt}{t}\bigg)^{1/q} \leq C\bigg(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \, \frac{dt}{t}\bigg)^{1/q}.$$

(*C depends only upon q and the constants involved in the definitions of φ and ψ.*)

Next, we state some basic facts and provide standard notation for two-parameter stochastic processes as may be found in [**17**]. Let us denote the set of non-negative integers and the set of integers, by $\mathbf{N}$ and $\mathbf{Z}$, respectively.

For $n, m \in \mathbf{N}^2$, $n = (n_1, n_2)$, $m = (m_1, m_2)$, $n \leq m$ means that $n_1 \leq m_1$ and $n_2 \leq m_2$; $n < m$ means that $n \leq m$ and $n \neq m$. Moreover, $n \ll m$ means that both of the inequalities $n_1 < m_1$ and $n_2 < m_2$ hold. For $n = (n_1, n_2) \in \mathbf{N}^2$, we set $n - 1 := (n_1 - 1, n_2 - 1)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n,\ n \in \mathbf{N}^2\}$ an increasing family of sub-σ-algebras of $\mathcal{F}$. We introduce the following σ-algebras:

$$\mathcal{F}_\infty = \sigma\Big(\bigcup_{n \in \mathbf{N}^2} \mathcal{F}_n\Big),\ \mathcal{F}_{n_1,\infty} = \sigma\Big(\bigcup_{k=0}^{\infty} \mathcal{F}_{n_1,k}\Big),\ \mathcal{F}_{\infty,n_2} = \sigma\Big(\bigcup_{k=0}^{\infty} \mathcal{F}_{k,n_2}\Big).$$

For the sake of simplicity, we assume that $\mathcal{F}_\infty = \mathcal{F}$ and define $\mathcal{F}_{-1} := \mathcal{F}_0$, $\mathcal{F}_{-1,-1} := \mathcal{F}_{0,0}$, $\mathcal{F}_{-1,i} := \mathcal{F}_{0,i}$ and $\mathcal{F}_{i,-1} := \mathcal{F}_{i,0}$ $(i \in \mathbf{N})$.

We denote by E, E_n, $E_{n_1,\infty}$ and E_{∞,n_2} the expectation operator and the conditional expectation operators with respect to $\mathcal{F}_n$ $(n \in \mathbf{N}^2 \cup \{\infty\})$, $\mathcal{F}_{n_1,\infty}$ and $\mathcal{F}_{\infty,n_2}$ $(n_1, n_2 \in \mathbf{N})$, respectively. For simplicity, we assume that $E_n f = 0$ when $n_1 = 0$ or $n_2 = 0$.

Suppose that $f = (f_n,\ n \in \mathbf{N}^2)$ is an integrable process. Then, f is a martingale if

- f is adapted to the filtration $(\mathcal{F}_n,\ n \in \mathbf{N}^2)$, i.e., each f_n is $\mathcal{F}_n$-measurable;
- $E[f_m|\mathcal{F}_n] = f_n$ for all $n \leq m$.

A martingale f is said to be L_p-*bounded* if

$$\sup_{n \in \mathbf{N}^2} \|f_n\|_p < \infty.$$

Recall that a stopping time τ relative to $(\mathcal{F}_n,\ n \in \mathbf{N}^2)$ is a random variable which maps Ω into the set of subspaces of $\mathbf{N}^2 \cup \{\infty\}$ such that the elements of $\tau(w)$ are incomparable for all $w \in \Omega$, i.e., if $(k,l), (n \cdot m) \in \tau(w)$, then neither $(k,l) \leq (n \cdot m)$ nor $(n \cdot m) \leq (k,l)$; of course, $(k,l) < \infty$ for all $k, l \in \mathbf{N}$, and $\{n \in \tau\} := \{w \in \Omega : n \in \tau(w)\} \in \mathcal{F}_n$ for every $n \in \mathbf{N}^2$. The maximal function of a martingale $f = (f_n,\ n \in \mathbf{N}^2)$ is denoted by

$$f_n^* := \sup_{m \leq n} |f_m|, \qquad f^* := \sup_{m \in \mathbf{N}^2} |f_m|.$$

For a martingale $f = (f_n,\ n \in \mathbf{N}^2)$ relative to $(\Omega, \mathcal{F}, P)$, denote the martingale differences by

$$d_m f := f_{m_1,m_2} - f_{m_1-1,m_2} - f_{m_1,m_2-1} + f_{m_1-1,m_2-1},$$

and $d_m f := 0$ if $m_1 = 0$ or $m_2 = 0$.

We define the square function and the conditional square function of f as follows:

$$S_m(f) := \Big(\sum_{n\le m} |d_n f|^2\Big)^{1/2}, \qquad S(f) := \Big(\sum_{n\in\mathbf{N}^2} |d_n f|^2\Big)^{1/2},$$

$$s_m(f) := \Big(\sum_{n\le m} E_{n-1}|d_n f|^2\Big)^{1/2}, \qquad s(f) := \Big(\sum_{n\in\mathbf{N}^2} E_{n-1}|d_n f|^2\Big)^{1/2}.$$

For $0 < q \le \infty$, martingale weighted Lorentz spaces as follows are defined by

$$\Lambda_q^*(\varphi) = \Big\{f = (f_n)_{n\in\mathbf{N}^2} : \|f\|_{\Lambda_q^*(\varphi)} := \|f^*\|_{\Lambda_q(\varphi)} < \infty\Big\},$$
$$\Lambda_q^s(\varphi) = \Big\{f = (f_n)_{n\in\mathbf{N}^2} : \|f\|_{\Lambda_q^s(\varphi)} := \|s(f)\|_{\Lambda_q(\varphi)} < \infty\Big\},$$
$$\Lambda_q^S(\varphi) = \Big\{f = (f_n)_{n\in\mathbf{N}^2} : \|f\|_{\Lambda_q^S(\varphi)} := \|S(f)\|_{\Lambda_q(\varphi)} < \infty\Big\}.$$

Note that, if $\varphi(t) = t^{1/p}$, then $\Lambda_q(\varphi) = L_{p,q}$, $\Lambda_q^*(\varphi) = H_{p,q}^*$, $\Lambda_q^s(\varphi) = H_{p,q}^s$ and $\Lambda_q^s(\varphi) = H_{p,q}^S$. In particular, if $\varphi(t) = t^{1/q}$, then $\Lambda_q(\varphi) = L_q$, $\Lambda_q^*(\varphi) = H_q^*$, $\Lambda_q^s(\varphi) = H_q^s$ and $\Lambda_q^S(\varphi) = H_q^S$. In what follows, C always denotes a constant, which may be different in different places. For two non-negative quantities A and B, by $A \lesssim B$, we mean that there exists a constant $C > 0$ such that $A \le CB$, and, by $A \approx B$, that $A \lesssim B$ and $B \lesssim A$. Throughout this article, we assume $w \in \Delta_2$, where w is the function defined by

$$w(t) = \Big(\int_0^t \varphi^q(s)\,\frac{ds}{s}\Big)^{1/q}, \quad q < \infty,$$

for a given weight φ in $\Lambda_q^s(\varphi)$.

3. Atomic decomposition. For two-parameter martingale spaces, Weisz obtained some atomic decomposition theorems which are used to establish important martingale inequalities and interpolation theorems for martingale Hardy spaces. In this section, using ideas from **[17]**, we establish the atomic decomposition theorem of martingale weighted Lorentz spaces.

Definition 3.1. A function $a \in L_r$ is called a (p, r) *atom* if there exists a stopping time ν such that

(1) $a_n := E_n a = 0$, if $\nu \not\lll n$;
(2) $\|a^*\|_r \le P\ (\nu \ne \infty)^{1/r-1/p}$, $0 < p \le r$, $1 < r \le \infty$.

Theorem 3.2. *If* $f = (f_n,\ n \in \mathbf{N}^2) \in \Lambda_q^s(\varphi)$, $0 < q \le \infty$, *then there exists a sequence* $\{(a^k, \nu_k)\}_{k\in\mathbf{Z}}$ *of* $(p, 2)$ *atoms,* $0 < p \le 2$, *such that*

$$\sum_{k=-\infty}^{\infty} \mu_k E_n a^k = f_n,$$

where $\mu_k = 2^{k+1}\sqrt{2}P(\nu_k \ne \infty)^{1/p}$ *and*

$$\|\{2^k w(P(\nu_k \ne \infty))\}_{k\in\mathbf{Z}}\|_{l_q} \lesssim \|f\|_{\Lambda_q^s(\varphi)}. \tag{3.1}$$

Moreover, if $0 < q \le 1$, *then*

$$\|f\|_{\Lambda_q^s(\varphi)} \approx \inf \|\{2^k w(P(\nu_k \ne \infty))\}_{k\in\mathbf{Z}}\|_{l_q},$$

where the infimum is taken over all the preceding decompositions of f.

Proof. Let $f = (f_n,\ n \in \mathbf{N}^2) \in \Lambda_q^s(\varphi)$. Set $F_k := \{s(f) > 2^k\}$ and, for any $k \in \mathbf{Z}$, define stopping times ν_k as $\nu_k := \inf\{n \in \mathbf{N}^2 : E_n\chi(F_k) > 1/2\}$. Now, for stopped martingale $f_n^\nu := \sum_{m\le n} \chi\ (\nu \not\lll m) d_m f$, we obtain

$$\begin{aligned}\sum_{k\in\mathbf{Z}}(f_n^{\nu_{k+1}} - f_n^{\nu_k}) &= \sum_{k\in\mathbf{Z}}\left(\sum_{m\le n}(\chi(\nu_{k+1} \not\lll m)\, d_m f - \chi(\nu_k \not\lll m)\, d_m f)\right)\\ &= \sum_{m\le n}\left(\sum_{k\in\mathbf{Z}}\chi(\nu_k \lll m \not\ggg \nu_{k+1})\, d_m f\right) = f_n.\end{aligned}$$

Put

$$a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

Obviously, $(a_n^k,\ n \in \mathbf{N}^2)$ is a martingale. It is easy to see that a^k is a $(p,2)$ atom corresponding to the stopping time ν_k, and

$$f_n = \sum_{k\in\mathbf{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) = \sum_{k\in\mathbf{Z}} \mu_k a_n^k = \sum_{k\in\mathbf{Z}} \mu_k E_n a^k.$$

Let $0 < q < \infty$. Applying Chebyshev's inequality, the equivalence between H_2^*, L_2 and $w \in \Delta_2$, we obtain

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} 2^{kq} w^q(P(\nu_k \neq \infty)) &= \sum_{k\in\mathbf{Z}} 2^{kp} w^q\Big(P\Big(\sup_n E_n\chi(F_k) > 1/2\Big)\Big) \\
&\leq \sum_{k\in\mathbf{Z}} 2^{kq} w^q\Big(4E\Big(\sup_n E_n\chi(F_k)\Big)^2\Big) \\
&\lesssim \sum_{k\in\mathbf{Z}} 2^{kq} w^q\Big(E\Big(\sup_n E_n\chi(F_k)\Big)^2\Big) \\
&\lesssim \sum_{k\in\mathbf{Z}} 2^{kq} w^q(P(F_k)) \\
&= \sum_{k\in\mathbf{Z}} 2^{kq} w^q(P(s(f) > 2^k)) \\
&\lesssim \sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} dy\, w^q(P(s(f) > 2^k)) \\
&\lesssim \sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} w^q(P(s(f) > y))\, dy \\
&\lesssim \int_0^\infty y^{q-1} w^q(P(s(f) > y))\, dy = \|f\|_{\Lambda_q^s(\varphi)}^q.
\end{aligned}$$

If $q = \infty$, then

$$2^k w(P(\nu_k \neq \infty)) \lesssim 2^k w(P(s(f) > 2^k)) \lesssim \|s(f)\|_{\Lambda_\infty(\varphi)} =: \|f\|_{\Lambda_\infty^s(\varphi)},$$

which implies $\sup_{k\in\mathbf{Z}} 2^k w(P(\nu_k \neq \infty)) \lesssim \|f\|_{\Lambda_\infty^s(\varphi)}$. The proof of the first part of Theorem 3.2 is complete. Further, we have

$$\begin{aligned}
&\sum_{n\in\mathbf{N}^2} E_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \\
&\qquad = \sum_{n\in\mathbf{N}^2} E_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \chi(s(f) \leq 2^{k+1})
\end{aligned}$$

$$+\sum_{n\in\mathbf{N}^2} E_{n-1}|d_nf|^2\chi(\nu_k \ll n \not\gg \nu_{k+1})\chi(s(f) > 2^{k+1})$$
$$= \mathbf{I} + \mathbf{II}.$$

We first estimate **I**:

$$\mathbf{I} \le \sum_{n\in\mathbf{N}^2} E_{n-1}|d_nf|^2\chi(\nu_k \ll n \not\gg \infty)\chi(s(f) \le 2^{k+1})$$
$$\le s(f)^2\ (\nu_k \ne \infty)\ \chi(s(f) \le 2^{k+1}) \le 4^{k+1}.$$

Taking the conditional expectation in **II** with respect to $\mathcal{F}_{n-1}$, we obtain

$$\mathbf{II} = \sum_{n\in\mathbf{N}^2} E_{n-1}|d_nf|^2\chi(\nu_k \ll n \not\gg \nu_{k+1})E_{n-1}\chi(s(f) > 2^{k+1}).$$

By the definition of ν_{k+1} we have $E_{n-1}\chi(s(f) > 2^{k+1}) \le 1/2$ if $\nu_{k+1} \not\ll n$. It follows that

$$\mathbf{II} \le 1/2 \sum_{n\in\mathbf{N}^2} E_{n-1}|d_nf|^2\chi(\nu_k \ll n \not\gg \nu_{k+1}).$$

Hence,

$$s(a_n^k)^2 = s\left(\frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}\right)^2 = \frac{1}{\mu_k^2}\sum_{n\in\mathbf{N}^2} E_{n-1}|d_nf|^2\chi(\nu_k \ll n \not\gg \nu_{k+1})$$
$$< P\ (\nu_k \ne \infty)^{-2/p}.$$

Consequently,

$$\|s(a^k)\|_\infty < P\ (\nu_k \ne \infty)^{-1/p}.$$

Since $a_n^k = E_na^k = 0$ on $\{\nu_k \not\ll n\}$, we have

$$\chi\,(\nu \not\ll n)E_{n-1}|d_na^k|^2 = E_{n-1}\,\chi\,(\nu \not\ll n)|d_na^k|^2 = 0.$$

Hence, $s(a^k) = 0$ on $\{\nu_k = \infty\}$. Thus,

$$P(s(a^k) > y) \le P(s(a^k) \ne 0) \le P\ (\nu_k \ne \infty).$$

Therefore, we obtain

$$\|a^k\|^q_{\Lambda_q^s(\varphi)} = q\int_0^\infty y^{q-1}w^q(P(s(a^k) > y))\,dy$$

$$
\begin{aligned}
&= q \int_0^{P(\nu_k \neq \infty)^{-1/p}} y^{q-1} w^q(P(s(a^k) > y))\, dy \\
&\leq q w^q(P(\nu_k \neq \infty)) \int_0^{P(\nu_k \neq \infty)^{-1/p}} y^{q-1} dy \\
&\leq w^q(P(\nu_k \neq \infty)) P(\nu_k \neq \infty)^{-q/p}.
\end{aligned}
$$

Finally, since, for $0 < q \leq 1$, the quasi-normed $\| \cdot \|_{\Lambda_p^s(\varphi)}$ is equivalent to a q-norm,

$$
\begin{aligned}
\|f\|^q_{\Lambda^s_q(\varphi)} &\leq \Big\| \sum_{k \in \mathbf{Z}} \mu_k s(a^k) \Big\|^q_{\Lambda_q(\varphi)} \leq \sum_{k \in \mathbf{Z}} \mu_k^q \|s(a^k)\|^q_{\Lambda_q(\varphi)} \\
&\leq \sum_{k \in \mathbf{Z}} \mu_k^q w^q(P(\nu_k \neq \infty)) P(\nu_k \neq \infty)^{-q/p} \\
&\lesssim \sum_{k \in \mathbf{Z}} 2^{kq} w^q(P(\nu_k \neq \infty)).
\end{aligned}
$$

The proof is complete. □

4. Sublinear operator on martingale spaces. As an application of atomic decompositions, we obtain some sufficient conditions which cause the sublinear operator to be bounded from the martingale weighted Lorentz spaces to weighted Lorentz spaces.

An operator $T : X \to Y$ is called a *sublinear operator* if it satisfies

$$
|T(f+g)| \leq |Tf| + |Tg|, \qquad |T(\alpha f)| \leq |\alpha||Tf|, \quad \alpha \in \mathbf{R},
$$

where X is a martingale space and Y is a measurable function space.

Definition 4.1. A function F is said to obey the $\triangle$-condition, often written as $F \in \triangle$, if there exists a positive constant b such that $F(xy) \leq bF(x)F(y)$ for arbitrary $x, y \geq 0$; and it obeys the ∇-condition, symbolically denoted as $F \in \bigtriangledown$, if there exists a positive constant B such that $F(x)F(y) \leq F(Bxy)$ for arbitrary $x, y \geq 0$, where $B \geq 1$, see **[15]**.

Theorem 4.2. *Let $w \in \triangle \cap \bigtriangledown$, and let $T : H_2^s \to L_2$ be a bounded sublinear operator. For every atom a of $(p, 2)$, $0 < p < 2$, if $Ta = 0$*

on $\{\nu_k = \infty\}$, *where* ν *is the stopping time associated with* a, *then*

$$\|Tf\|_{\Lambda_\infty(\varphi)} \le \|f\|_{\Lambda^s_\infty(\varphi)}, \quad f \in \Lambda^s_\infty(\varphi).$$

Proof. Let $f \in \Lambda^s_\infty(\varphi)$. Then, f has an atomic decomposition of $(p, 2)$ atoms as in Theorem 3.2. For any $y > 0$, choose $j \in \mathbf{Z}$ such that $2^{j-1} \le y < 2^j$, and let

$$f = \sum_{k\in\mathbf{Z}} \mu_k a^k = \sum_{k=-\infty}^{j-1} \mu_k a^k + \sum_{k=j}^{\infty} \mu_k a^k =: g + h.$$

We have

$$\{|Th| \neq 0\} \subset \bigcup_{k\ge j} \{T(a^k) \neq 0\} \subset \bigcup_{k\ge j} \{\nu_k \neq \infty\}$$

because $T(a^k) = 0$ on $\{\nu_k = \infty\}$. Since $|Th| \le \sum_{k=j}^{\infty} \mu_k |T(a^k)|$,

$$\begin{aligned} w(P(|Th| > y)) \le w(P(|Th| \neq 0)) &\lesssim \sum_{k=j}^{\infty} w(P(\nu_k \neq \infty)) \\ &\lesssim \sum_{k=j}^{\infty} w(P(s(f) > 2^k)) \\ &\le \sum_{k=j}^{\infty} 2^{-k} \|s(f)\|_{\Lambda_\infty(\varphi)}, \text{ by inequality (3.1)}, \\ &\lesssim y^{-1} \|s(f)\|_{\Lambda_\infty(\varphi)}. \end{aligned}$$

It follows, from the boundedness of T and $s(a^k) = 0$ on $\{\nu_k = \infty\}$, that

$$\begin{aligned} \|Tg\|_2 &\le C\|g\|_{H^s_2} = C\|s(g)\|_2 \\ &\le C \Big\| \sum_{k=-\infty}^{j-1} \mu_k s(a^k) \Big\|_2 \le C \sum_{k=-\infty}^{j-1} \mu_k \|s(a^k)\|_2 \\ &\le C \sum_{k=-\infty}^{j-1} \mu_k P(\nu_k \neq \infty)^{-1/p} P(\nu_k \neq \infty)^{1/2} \\ &\le C \sum_{k=-\infty}^{j-1} 2^k P(\nu_k \neq \infty)^{1/2}. \end{aligned}$$

Since $w \in \triangle \cap \bigtriangledown$, we have

$$\begin{aligned} w(P(|Tg| > y)) &\le w(y^{-2}\|Tg\|_2^2) \\ &\le w\Big(y^{-2}\Big(C\sum_{k=-\infty}^{j-1} 2^k P(\nu_k \neq \infty)^{1/2}\Big)^2\Big) \\ &\lesssim \Big(w\Big(y^{-1}\sum_{k=-\infty}^{j-1} 2^k P(\nu_k \neq \infty)^{1/2}\Big)\Big)^2, \quad \text{by } w \in \triangle, \\ &\lesssim \Big(\sum_{k=-\infty}^{j-1} y^{-1}2^k w(P(\nu_k \neq \infty)^{1/2})\Big)^2 \\ &\lesssim \Big(\sum_{k=-\infty}^{j-1} y^{-1}2^{k/2}2^{k/2} w(P(\nu_k \neq \infty)^{1/2})\Big)^2 \\ &\lesssim (y^{-1/2}\|s(f)\|_{\Lambda_\infty(\varphi)}^{1/2})^2, \quad \text{by } w \in \nabla, \\ &= y^{-1}\|s(f)\|_{\Lambda_\infty(\varphi)}. \end{aligned}$$

Then, it follows from the sublinearity of T that $|Tf| \le |Tg| + |Th|$ and

$$P(|Tf| > 2y) \le P(|Tg| + |Th| > 2y) \le P(|Tg| > y) + P(|Th| > y).$$

Thus, we obtain

$$w(P(|Tf|>2y)) \lesssim w(P(|Tg|>y)) + w(P(|Th|>y)) \lesssim y^{-1}\|s(f)\|_{\Lambda_\infty(\varphi)},$$

and, therefore, $T : \Lambda_\infty^s(\varphi) \to \Lambda_\infty(\varphi)$ is bounded. $\square$

Corollary 4.3. *Let $w \in \triangle \cap \bigtriangledown$. Then, the following imbeddings hold*:

$$\Lambda_\infty^s(\varphi) \hookrightarrow \Lambda_\infty^*(\varphi), \qquad \Lambda_\infty^s(\varphi) \hookrightarrow \Lambda_\infty^S(\varphi).$$

Proof. Let T be the maximal operator $Tf = f^*$. We know that $\|f^*\|_2 \le \|s(f)\|_2$, and T is sublinear. If a is a $(p, 2)$ atom with respect to the stopping time ν, then $Ta = a^* = 0$ on $\{\nu = \infty\}$. It follows from Theorem 4.2 that

$$\|f^*\|_{\Lambda_\infty(\varphi)} \le \|s(f)\|_{\Lambda_\infty^s(\varphi)}.$$

Using Theorem 4.2 and $\|S(f)\|_2 \le \|s(f)\|_2$, it similarly follows that

$$\|S(f)\|_{\Lambda_\infty(\varphi)} \le \|s(f)\|_{\Lambda_\infty^s(\varphi)}. \qquad \square$$

5. Interpolation. In this section, as another application, we apply atomic decompositions of two-parameter martingale weighted Lorentz spaces to the real interpolation between two-parameter martingale Hardy spaces. The next lemma follows from Theorem 3.2 by the atomic decomposition of $\Lambda_q^s(\varphi)$.

Lemma 5.1. *Let $f \in \Lambda_q^s(\varphi)$, $0 < q \le \infty$, $y > 0$, and fix $0 < p \le 1$. Then f can be decomposed into the sum of two martingales g and h, such that*

$$\|g\|_2 \le C_2\left[\left(\int_{\{s(f)\le y\}} s(f)^2 dP\right)^{1/2} + yP(s(f) > y)^{1/2}\right]$$

and

$$\|h\|_{H_p^s} \le C_p\left(\int_{\{s(f)>y\}} s(f)^p dP\right)^{1/p},$$

where the positive constant C_p depends only upon p.

Proof. The proof is similar to that of [**17,** Theorem 5.19]. □

Theorem 5.2. *Let $0 < p \le 1$, $0 < q \le \infty$ and $\varrho \in Q(0,1)$ be parameter functions. Then*

$$(H_p^s, L_2)_{\varrho,q} = \Lambda_q^s \frac{t^{1/p}}{\varrho(t^{1/p-1/2})}.$$

Proof. Let f be a function in $\Lambda_q^s(t^{1/p}/\varrho(t^{1/p-1/2}))$, and let $\widetilde{s}$ be the non-increasing rearrangement of $s = s(f)$. Set $1/\alpha = 1/p - 1/2$ and, for a fixed $t > 0$, consider $y := \widetilde{s}(t^\alpha)$. For this y, denote the two martingales in Lemma 5.1 by h_t and g_t. By the definition of the functional K,

$$K(t, f, H_p^s, L_2) \le \|h_t\|_{H_p^s} + t\|g_t\|_{L_2}.$$

By Lemma 5.1, we obtain

$$\|h_t\|_{H_p^s} \le C\left(\int_{\{s(f)>y\}} s(f)^p dP\right)^{1/p} = C\left(\int_0^{t^\alpha} \widetilde{s}(x)^p dx\right)^{1/p}. \tag{5.1}$$

Let $0 < q < \infty$. By **[14,** Lemma 1.1], $1/\varrho(t^{1/\alpha}) \in Q(-1/\alpha, 0)$, we have

$$
\begin{aligned}
(5.2)\quad \int_0^1 \left(\frac{\|h_t\|_{H_p^s}}{\varrho(t)}\right)^q \frac{dt}{t} &\le C\int_0^1 \left(\frac{1}{\varrho(t)}\right)^q \left(\int_0^{t^\alpha} (\widetilde{s}(x))^p dx\right)^{q/p} \frac{dt}{t} \\
&\le C\int_0^1 \left(\frac{1}{\varrho(t^{1/\alpha})}\right)^q \left(\int_0^{t} (\widetilde{s}(x))^p dx\right)^{q/p} \frac{dt}{t} \\
&\le C\int_0^1 \left(\frac{t^{1/p}\widetilde{s}(t)}{\varrho(t^{1/\alpha})}\right)^q \frac{dt}{t}, \quad \text{by Lemma 2.1,} \\
&= C\|s(f)\|^q_{\Lambda_q(t^{1/p}/\varrho(t^{1/\alpha}))}.
\end{aligned}
$$

It follows from Lemma 5.1 that

$$
(5.3)\quad \|g_t\|_2 \le C\left(\int_{\{s(f)\le \widetilde{s}(t^\alpha)\}} s(f)^2 dP\right)^{1/2} + C\widetilde{s}(t^\alpha)P(s > \widetilde{s}(t^\alpha))^{1/2}.
$$

Moreover,

$$
(5.4)\qquad P(s > \widetilde{s}(t^\alpha)) = P(\widetilde{s} > \widetilde{s}(t^\alpha)) \le t^\alpha.
$$

Since s and $\widetilde{s}$ have identical distributions, it follows from (5.3) and (5.4) that

$$
(5.5)\qquad \|g_t\|_2 \le C\left(\int_{t^\alpha}^1 \widetilde{s}(x)^2 dx\right)^{1/2} + C\widetilde{s}(t^\alpha)t^{\alpha/2}.
$$

Hence,

$$
\begin{aligned}
(5.6)\quad \int_0^1 \left(\frac{t\|g_t\|_2}{\varrho(t)}\right)^q \frac{dt}{t} &\le C\int_0^1 \left(\frac{t}{\varrho(t)}\right)^q \left(\int_{t^\alpha}^1 (\widetilde{s}(x))^2 dx\right)^{q/2} \frac{dt}{t} \\
&\quad + C\int_0^1 \left(\frac{t}{\varrho(t)}\right)^q \widetilde{s}(t^\alpha)^q t^{(\alpha q)/2} \frac{dt}{t} \\
&= \mathbf{I} + \mathbf{II}.
\end{aligned}
$$

We shall estimate **I** and **II** separately. First, $\varrho(t)t^{-\epsilon}$ is non-decreasing for some $\epsilon > 0$. Since $\varrho \in Q(0,1)$, it follows that $\varrho(t) \le C\varrho(4t)$ for $t > 0$. Moreover, $t^{1/\alpha}/\varrho(t^{1/\alpha}) \in Q(0, 1/\alpha)$ by **[14,** Lemma 1.1]; thus, we conclude that

$$\begin{aligned}
\mathbf{I} &\le C\int_0^1 \left(\frac{t^{1/\alpha}}{\varrho(t^{1/\alpha})}\right)^q \left(\int_t^1 (\widetilde{s}(x))^2 dx\right)^{q/2} \frac{dt}{t} \\
&\le C\int_0^1 \left(\frac{t^{1/\alpha}}{\varrho(t^{1/\alpha})}\right)^q \left(\int_{t/4}^1 (x^{1/2}\widetilde{s}(x))^r \frac{dx}{x}\right)^{q/r} \frac{dt}{t}, \quad \text{by } [\mathbf{17,}\ (5.14)], \\
&\le C\int_0^1 \left(\frac{t^{1/\alpha+1/2}}{\varrho(t^{1/\alpha})}\right)^q \widetilde{s}(t)^q \frac{dt}{t}, \quad \text{by Lemma 2.1}, \\
&= C\|s(f)\|^q_{\Lambda_q(t^{1/p}/\varrho(t^{1/\alpha}))},
\end{aligned}$$

where $r \le \min(2, q)$. It is clear that

$$\mathbf{II} \le C\int_0^1 \left(\frac{t^{1/\alpha}}{\varrho(t^{1/\alpha})}\right)^q \widetilde{s}(t)^q t^{q/2} \frac{dt}{t} = C\|s(f)\|^q_{\Lambda_q(t^{1/p}/\varrho(t^{1/\alpha}))}.$$

It follows from (5.2), (5.6) and the definition of the functional K,

$$\begin{aligned}
\|f\|_{(H_p^s, L_2)_{\varrho,q}} &= \left(\int_0^1 \left(\frac{K(t, X, H_p^s, L_2)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\le C\|f\|_{\Lambda_q^s(t^{1/p}/\varrho(t^{1/p-1/2}))}.
\end{aligned} \tag{5.7}$$

Thus, the first is included in the second.

Now, suppose that $f \in (H_p^s, L_2)_{\varrho,q}$. We consider the operator $T : f \mapsto s(f)$. The sublinear operators $T : L_2 \to L_2$ and $T : H_p^s \to L_p$ are bounded. By [**14,** Theorem 2.2], the operator

$$T : (H_p^s, L_2)_{\varrho,q} \longrightarrow (L_p, L_2)_{\varrho,q} = \Lambda_q\left(\frac{t^{1/p}}{\varrho(t^{1/p-1/2})}\right), \quad \text{by (2.1)},$$

is bounded. Hence,

$$\|f\|_{\Lambda_q^s(t^{1/p}/\varrho(t^{1/p-1/2}))} := \|s(f)\|_{\Lambda_q(t^{1/p}/\varrho(t^{1/p-1/2}))} \le C\|f\|_{(H_p^s, L_2)_{\varrho,q}}.$$

Thus, $f \in \Lambda_q^s(t^{1/p}/\varrho(t^{1/p-1/2}))$.

Suppose now that $q=\infty$. Since $\varrho\in Q(0,1)$, then $\varrho(t)t^{-\epsilon}$ is non-decreasing for some $\epsilon>0$. Therefore, we have

$$\begin{aligned}\sup_{t>0}\frac{\|h_t\|_{H_p^s}}{\varrho(t)} &\le C\sup_{t>0}\frac{(\int_0^{t^\alpha}\widetilde{s}(x)^p dx)^{1/p}}{\varrho(t)}, \quad \text{by (5.1)},\\ &\le C\sup_{t>0}\frac{(\int_0^{t}\widetilde{s}(x^\alpha)^p x^{\alpha-1} dx)^{1/p}}{\varrho(t)}\\ &\le C\sup_{x>0}\frac{x^{\alpha/p}\widetilde{s}(x^\alpha)}{\varrho(x)}\sup_{t>0}\frac{\varrho(t)t^{-\epsilon}(\int_0^t x^{p\epsilon-1}dx)^{1/p}}{\varrho(t)}\\ &\le C\|f\|_{\Lambda_\infty^s(t^{1/p}/\varrho(t^{1/\alpha}))}\end{aligned}$$

and

$$\begin{aligned}\sup_{t>0}\frac{t\|g_t\|_2}{\varrho(t)} &\le C\sup_{t>0}\frac{t}{\varrho(t)}\left(\int_{t^\alpha}^1(\widetilde{s}(x))^2dx\right)^{1/2}\\ &\quad + C\sup_{t>0}\frac{t}{\varrho(t)}\widetilde{s}(t^\alpha)t^{\alpha/2}, \quad \text{by (5.5)},\\ &= \mathbf{III}+\mathbf{IV}.\end{aligned}$$

As at the beginning of the proof, we will estimate **III** and **IV** separately. First, since $\varrho(t)t^{-1+\epsilon}$ is non-increasing for some $\epsilon>0$, it follows that

$$\begin{aligned}\mathbf{III} &\le C\sup_{t>0}\frac{t}{\varrho(t)}\left(\int_{t^\alpha/4}^1(x^{1/2}\widetilde{s}(x))^r\frac{dx}{x}\right)^{1/r}, \quad \text{by [\textbf{17,} (5.14)]},\\ &\le C\sup_{t>0}\frac{t}{\varrho(t)}\left(\int_{t}^1(x^{\alpha/2}\widetilde{s}(x^\alpha))^r\frac{dx}{x}\right)^{1/r}\\ &\le C\sup_{x>0}\frac{x^{\alpha/p}\widetilde{s}(x^\alpha)}{\varrho(x)}\sup_{t>0}\frac{t\varrho(t)t^{-1+\epsilon}(\int_t^1 x^{-\epsilon r}dx/x)^{1/r}}{\varrho(t)}\\ &\le C\|s(f)\|_{\Lambda_\infty(t^{1/p}/\varrho(t^{1/\alpha}))},\end{aligned}$$

where $0<r<2$. Moreover,

$$\mathbf{IV}\le C\sup_{t>0}\frac{t^{1/\alpha}}{\varrho(t^{1/\alpha})}\widetilde{s}(t)t^{1/2}=C\|s(f)\|^q_{\Lambda_\infty(t^{1/p}/\varrho(t^{1/\alpha}))}.$$

Therefore,

$$\|f\|_{(H_p^s,L_2)_{\varrho,\infty}}=\sup_{t>0}\frac{K(t,f,H_p^s,L_2)}{\varrho(t)}\le C\|f\|_{\Lambda_\infty^s(t^{1/p}/\varrho(t^{1/p-1/2}))}.$$

Hence, $\Lambda_\infty^s(t^{1/p}/\varrho(t^{1/p-1/2})) \subseteq (H_p^s, L_2)_{\varrho,\infty}$. In order to prove the converse, consider the operator $T: f \mapsto s(f)$. The sublinear operators $T: L_2 \to L_2$ and $T: H_p^s \to L_p$ are bounded. By [**14,** Theorem 2.2], the operator

$$T: (H_p^s, L_2)_{\varrho,\infty} \longrightarrow (L_p, L_2)_{\varrho,\infty} = \Lambda_\infty(t^{1/p}/\varrho(t^{1/p-1/2})), \quad \text{by (2.1)},$$

is bounded. Thus, we have

$$\|f\|_{\Lambda_\infty^s(t^{1/p}/\varrho(t^{1/p-1/2}))} := \|s(f)\|_{\Lambda_\infty(t^{1/p}/\varrho(t^{1/p-1/2}))} \leq C\|f\|_{(H_p^s, L_2)_{\varrho,\infty}}.$$

Hence, $(H_p^s, L_2)_{\varrho,\infty} \subseteq \Lambda_\infty^s(t^{1/p}/\varrho(t^{1/p-1/2}))$. The proof is complete. □

Corollary 5.3. *For $0 < \theta < 1$ and $0 < p_0 \leq 1$, if we take $\varrho(t) = t^\theta$ in Theorem 5.2, then*

$$(H_{p_0}^s, L_2)_{\theta,q} = H_{p,q}^s, \quad \tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{2}.$$

Theorem 5.4. *Let $\varphi_i(t) \in Q(1/2,-)$, $i = 0,\ 1$, $0 < p \leq 1$, $0 < q_0, q_1$, $q \leq \infty$ and $\varrho \in Q(0,1)$. Then*

(i) *$(\Lambda_{q_0}^s(\varphi_0), L_2)_{\varrho,q} = \Lambda_q^s(\varphi)$, where $\varphi(t) = \varphi_0(t)/\varrho(\varphi_0(t))$;*
(ii) *if $\varphi_1(t) \in Q(1/2, 1/p)$, then $(H_p^s, \Lambda_{q_1}^s(\varphi_1))_{\varrho,q} = \Lambda_q^s(\varphi)$, where $\varphi(t) = t^{1/p}/\varrho(t^{1/p}/\varphi_1(t))$;*
(iii) *if $\varphi_0(t)/\varphi_1(t) \in Q(0,-)$ or $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, then $(\Lambda_{q_0}^s(\varphi_0), \Lambda_{q_1}^s(\varphi_1))_{\varrho,q} = \Lambda_q^s(\varphi)$, where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$.*

Proof. First, we prove (iii). Put $\varrho_i(t) = t^{\alpha/p}/\varphi_i(t^\alpha)$ where $1/\alpha = 1/p - 1/2$, and choose α and p such that $\varrho_i(t) \in Q(0,1)$, $i = 0, 1$. According to [**14,** Corollary 4.4 (3)] and Theorem 5.2, we obtain

$$\begin{aligned}(\Lambda_{q_0}^s(\varphi_0), \Lambda_{q_1}^s(\varphi_1))_{\varrho,q} &= ((H_p^s, L_2)_{\varrho_0,q_0}, (H_p^s, L_2)_{\varrho_1,q_1})_{\varrho,q} \\ &= (H_p^s, L_2)_{\varrho_0\varrho(\varrho_1/\varrho_0),q} = \Lambda_q^s(\varphi),\end{aligned}$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$. In order to prove (ii), we first note that, by [**14,** Lemma 1.1], the condition $\varphi_1(t) \in Q(1/2, 1/p)$ implies that $\varrho_1(t) = t^{\alpha/p}/\varphi_i(t^\alpha) \in Q(0,1)$. Thus, the proof follows as above by using Theorem 5.2 and [**14,** Corollary 4.4 (2)]. In a similar way, we see that (i) is an easy consequence of Theorem 5.2 and [**14,** Corollary 4.4 (1)]. The proof is complete. □

The following result is a simple application of Theorem 5.4 (iii) by replacing parameter function $\varrho = \varrho(t)$ by t^θ.

Corollary 5.5. *Under the hypothesis of Theorem* 5.4 (iii), *we have*

$$(\Lambda^s_{q_0}(\varphi_0), \Lambda^s_{q_1}(\varphi_1))_{\theta,q} = \Lambda^s_q(\varphi_0^{1-\theta}\varphi_1^\theta).$$

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