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NEVANLINNA UNIQUENESS OF LINEAR DIFFERENCE POLYNOMIALS

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ABSTRACT. In this paper, we investigate shared value problems related to an entire function $f(z)$ of hyper-order less than one and its linear difference polynomial $L(f) = \sum_{i=1}^{k} a_i f(z+c_i)$, where $a_i, c_i \in \mathbb{C}$. We give sufficient conditions in terms of weighted value sharing and truncated deficiencies, which imply that $L(f) \equiv f$.

1. Introduction. Difference Nevanlinna theory has emerged as a result of recent interest in value distribution and growth of meromorphic solutions of difference equations, see, e.g., **[2, 5]**. Resulting development of new tools in value distribution theory suited to study solutions of difference equations have enabled the study of the general value distribution properties of meromorphic functions from a new perspective. A new active direction of study has been uniqueness problems of meromorphic functions and their shifts, see, e.g., **[1, 8, 9, 11, 12]**.

Here, and throughout the rest of this paper, a meromorphic function is assumed to be meromorphic in the whole complex plane. The basic notions of Nevanlinna theory of meromorphic functions are assumed to be known to the reader, see e.g., **[7, 10, 15]**. An exception to the standard notation is that $S(r,f)$ is defined to be any quantity of the growth $o(T(r,f))$ as $r \to \infty$, outside of an exceptional set of finite logarithmic measure. This differs from the usual convention, where the exceptional set is assumed to be of finite linear measure. The family of all small meromorphic functions with respect to f, i.e., of the growth $S(r,f)$, is denoted by $S(f)$. Moreover, $\widehat{S}(f) = S(f) \cup \{\infty\}$.

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Heittokangas, et al., proved that, if a finite-order meromorphic function $f(z)$ and $f(z+\eta)$ share three distinct periodic functions $a_j \in \widehat{S}(f)$, $j = 1, 2, 3$, with period η CM, then f is a periodic function with period η, see [**8,** Theorem 2.1 (a)]. They also showed that the 3 CM assumption can be replaced by 2 CM + 1 IM, and the same conclusion holds, see [**9,** Theorem 2]. Chen and Yi [**1**] considered the case where $f(z)$ and $\Delta f(z)$ share three distinct values a, b, ∞ CM as follows.

Theorem A ([**1**])**.** *Let $f(z)$ be a transcendental meromorphic function such that its order of growth $\sigma(f)$ is not an integer or infinite, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share three distinct finite values a, b, ∞ CM, then $f(z+\eta) \equiv 2f(z)$.*

In the case of only one CM value, but with the function f being entire and additionally having a finite Borel exceptional value, Chen and Yi obtained the following theorem.

Theorem B ([**1**])**.** *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share the value a CM, then $a = 0$ and*

$$\frac{f(z+\eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant.

An immediate question which arises upon comparing the aforementioned results of Heittokangas, et al., to Theorems A and B is, "can the CM condition in these theorems be weakened to IM?" Another question is, "can we extend these results in a natural way to general linear operators, rather than just the difference $\Delta f(z)$ or the shift operator?"

The purpose of this paper is to study these problems from the point of view of weighted value sharing. In order to explain exactly what we intend to do, first we need to introduce some additional notation.

Let l be a non-negative integer or infinite. Denote by $E_l(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that f and g share (a,l). It is easy to see that, if f and g share (a,l), then f and g share (a,p) for $0 \leq p \leq l$. Also, we note that f and g share the value a IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively.

Let p be a positive integer and $a \in \mathbf{C} \cup \{\infty\}$. We use $N_{p)}(r, 1/(f-a))$ to denote the counting function of the zeros of $f-a$, whose multiplicities are not greater than p, $N_{(p+1}(r,1/(f-a))$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not less than $p+1$, and we use $\overline{N}_{p)}(r,1/(f-a))$ and $\overline{N}_{(p+1}(r,1/(f-a))$ to denote their corresponding reduced counting functions (ignoring multiplicities), respectively. We use $\overline{E}_{p)}(a,f)$ $(\overline{E}_{(p+1}(a,f))$ to denote the set of zeros of $f-a$ with multiplicities $\leq p$ $(\geq p+1)$ (ignoring multiplicity), respectively. We also use $N_p(r,1/(f-a))$ to denote the counting function of the zeros of $f-a$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then, by defining the truncated deficiency as

$$\delta_p(a,f) = 1 - \limsup_{r\to+\infty} \frac{N_p(r,1/(f-a))}{T(r,f)},$$

it follows that $\delta_p(a,f) \geq \delta(a,f)$, where $\delta(a,f)$ is the usual Nevanlinna deficiency of f.

Our results give sufficient conditions in terms of weighted value sharing and truncated deficiencies for a transcendental entire function of relatively slow growth to be mapped to itself by a linear difference operator.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function with hyper-order less than* 1*, and let a_j, $c_j \in \mathbb{C}$ be constants such that $L(f) := \sum_{j=1}^{k} a_j f(z+c_j) \not\equiv 0$. Assume that $f(z)-1$ and $L(f)-1$ share value $(0,l)$. Then,*

$$L(f) \equiv f, \tag{1.1}$$

if one of the following assumptions holds:

(i) $l \geq 2$, *and*

$$\delta_2(0,f) + \delta(0,f) + \delta(1,f) > 1; \tag{1.2}$$

(ii) $l = 1$, *and*

$$\frac{1}{2}\,\delta_2(0,f) + \frac{3}{4}\,\delta(0,f) + \frac{1}{2}\,\delta(1,f) > \frac{3}{4}; \tag{1.3}$$

(iii) $l = 0$, *i.e.*, $f - 1$ *and* $L(f) - 1$ *share the value* 0 IM, *and*

$$\delta_2(0,f) + 3\delta(0,f) + \Theta(0,f) + \delta(1,f) > 4. \tag{1.4}$$

In Theorem B, it was assumed that $a = 0$ is a Borel exceptional value of an entire function f, which immediately implies that f is of regular growth. Thus, for any $0 < 2\varepsilon < \rho - \lambda$, it follows by the definition of Borel exceptional value that

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N_2\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) < r^{\lambda+\varepsilon} < r^{\rho-\varepsilon} = o(T(r,f)),$$

where λ is the exponent of convergence of the zeros of $f(z)$, and ρ is the order of $f(z)$. However, this implies that

$$\delta_2(0,f) = \delta(0,f) = \Theta(0,f) = 1,$$

which means that, in fact, all of the conditions (1.2), (1.3) and (1.4) are satisfied, provided that $a = 0$ is a Borel exceptional value of f.

Equation (1.1) also implies that f is a solution to a linear difference equation with constant coefficients. Therefore, the exact form of f can be, at least in principle, determined by using the characteristic equation for linear difference equations.

The remaining two theorems give a different set of sufficient conditions for the same assertion.

Theorem 1.2. *Let f and $L(f)\,(\not\equiv 0)$ be defined as in Theorem* 1.1. *Assume that $f - 1$ and $L(f) - 1$ share value $(0, l)$ and $\overline{E}_{(i}(0,f) \subseteq \overline{E}_{(i}(0, L(f))$, $i \geq 3$. Then,*

$$L(f) \equiv f,$$

if one of the following assumptions holds:

(i) $l \geq 2$, *and*

$$2\delta_2(0, f) + \delta(1, f) > 1; \tag{1.5}$$

(ii) $l = 1$, *and*

$$\frac{5}{4}\,\delta_2(0, f) + \frac{1}{2}\,\delta(1, f) > \frac{3}{4}; \tag{1.6}$$

(iii) $l = 0$, *i.e.*, $f - 1$ *and* $L(f) - 1$ *share the value* 0 IM, *and*

$$2\delta_2(0, f) + \frac{1}{2}\Theta(0, f) + \frac{1}{2}\,\delta(1, f) > 2. \tag{1.7}$$

If, instead of assuming that $i \geq 3$ as in Theorem 1.2, we consider the more general case $i \geq 2$, we must impose slightly stronger conditions to obtain the same assertion.

Theorem 1.3. *Let* f *and* $L(f)(\not\equiv 0)$ *be defined as in Theorem* 1.1. *Assume that* $f - 1$ *and* $L(f) - 1$ *share the value* $(0, l)$ *and* $\overline{E}_{(i}(0, f) \subseteq \overline{E}_{(i}(0, L(f)))$, $i \geq 2$. *Then,*

$$L(f) \equiv f,$$

if one of the following assumptions holds:

(i) $l \geq 2$, *and*

$$2\delta_2(0, f) + \delta(1, f) > 1; \tag{1.8}$$

(ii) $l = 1$, *and*

$$\delta_2(0, f) + \frac{1}{4}\Theta(0, f) + \frac{1}{2}\,\delta(1, f) > \frac{3}{4}; \tag{1.9}$$

(iii) $l = 0$, *i.e.*, $f - 1$ *and* $L(f) - 1$ *share the value* 0 IM, *and*

$$\delta_2(0, f) + \frac{3}{2}\Theta(0, f) + \frac{1}{2}\,\delta(1, f) > 2. \tag{1.10}$$

2. Lemmas. A difference analogue of the lemma on the logarithmic derivative for finite-order meromorphic functions was proved independently by Halburd and Korhonen [**3,** Theorem 2.1], [**4,** Theorem 2.1] and Chiang and Feng [**2,** Theorem 2.4, Corollary 2.6].

The next lemma due to Halburd, Korhonen and Tohge [**6**] is an extension of these results to the case of hyper-order less than one.

Lemma 2.1 ([**6**]). *Let f be a non-constant meromorphic function, $\varepsilon > 0$ and $c \in \mathbb{C}$. If $\varsigma(f) = \varsigma < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right),$$

for all r outside of a set of finite logarithmic measure.

Suppose that f and g are two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with order p and simultaneously a 1-point of g with order q. We denote by $N_L(r, 1/(f-1))$ the counting function of those 1-points of f where $p > q$, by $N_E^{1)}(r, 1/(f-1))$ the counting function of those 1-points of f where $p = q = 1$, and by $N_E^{(2}(r, 1/(f-1))$ the counting function of those 1-points of f where $p = q \geq 2$. Each point in these functions is counted only once. Similarly, we can define $N_L(r, 1/(g-1))$, $N_E^{1)}(r, 1/(g-1))$ and $N_E^{(2}(r, 1/(g-1))$.

With this notation in hand, we can state the following auxiliary result.

Lemma 2.2 ([**14**]). *Let f and g be two nonconstant meromorphic functions, and let*

$$\Delta = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.1}$$

If f and g share 1 IM *and $\Delta \not\equiv 0$, then*

$$N_E^{1)}\left(r, \frac{1}{f-1}\right) \leq N(r, \Delta) + S(r,f) + S(r,g). \tag{2.2}$$

The following basic inequalities, by [**6,** Lemma 8.3], are frequently used in value distribution theory for differences.

Lemma 2.3. *Let $f(z)$ be a non-constant meromorphic function with hyper-order less than* 1*,* $c \in \mathbb{C}$*. Then,*

$$\begin{aligned} N\left(r, \frac{1}{f(z+c)}\right) &\leq N\left(r, \frac{1}{f(z)}\right) + S(r,f), \\ N(r, f(z+c)) &\leq N(r, f(z)) + S(r,f), \\ \overline{N}\left(r, \frac{1}{f(z+c)}\right) &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f), \\ \overline{N}(r, f(z+c)) &\leq \overline{N}(r, f(z)) + S(r,f). \end{aligned}$$

The next lemmas and remark are needed in the proofs of Theorems 1.1–1.3.

Lemma 2.4. *Let f be a non-constant meromorphic function with hyper-order less than* 1*, and let $L(f)$ ($\not\equiv 0$) be defined as in Theorem* 1.1*. Then,*

$$N\left(r, \frac{1}{L(f)}\right) \leq T(r, L(f)) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f), \tag{2.3}$$

and

$$N\left(r, \frac{1}{L(f)}\right) \leq N\left(r, \frac{1}{f}\right) + (k-1)N(r, f(z)) + S(r,f). \tag{2.4}$$

Proof. From Lemma 2.1, we have

$$m\left(r, \frac{1}{f}\right) = m\left(r, \frac{L(f)}{f} \cdot \frac{1}{L(f)}\right) \leq m\left(r, \frac{1}{L(f)}\right) + S(r,f).$$

By the first fundamental theorem, we have

$$T(r,f) - N\left(r, \frac{1}{f}\right) \leq T(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) + S(r,f).$$

Thus, we obtain

$$N\left(r, \frac{1}{L(f)}\right) \leq T(r, L(f)) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f).$$

This proves (2.3).

By using Lemmas 2.1 and 2.3, we have

$$\begin{aligned}T(r,L(f)) &= m(r,L(f))+N(r,L(f))\\&\le m\left(r,\frac{L(f)}{f}\right)+m(r,f)+\sum_{j=1}^{k}N(r,a_jf(z+c_j))\\&\le m(r,f)+kN(r,f)+S(r,f)\\&\le T(r,f)+(k-1)N(r,f)+S(r,f).\end{aligned}$$

From this and (2.3), we obtain (2.4). Thus, Lemma 2.4 is proved. □

Lemma 2.5. *Let f and $L(f)\,(\not\equiv 0)$ be defined as in Theorem* 1.1. *Suppose that*

$$\overline{E}_{(i}(0,f)\subseteq \overline{E}_{(i}(0,L(f))),\quad i\ge 3. \tag{2.5}$$

Then,

$$N_2\left(r,\frac{1}{L(f)}\right)\le T(r,L(f))-T(r,f)+N_2\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.6}$$

and

$$N_2\left(r,\frac{1}{L(f)}\right)\le N_2\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.7}$$

Proof. From (2.3), we have

$$\begin{aligned}&N_2\left(r,\frac{1}{L(f)}\right)+\sum_{j=3}^{\infty}\overline{N}_{(3}\left(r,\frac{1}{L(f)}\right)\\&\le T(r,L(f))-T(r,f)+N_2\left(r,\frac{1}{f}\right)+\sum_{j=3}^{\infty}\overline{N}_{(3}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Since $\overline{E}_{(i}(0,f) \subseteq \overline{E}_{(i}(0,L(f))$, $i \geq 3$, we have

$$\begin{aligned}N_2\left(r,\frac{1}{L(f)}\right) &\leq T(r,L(f)) - T(r,f) + N_2\left(r,\frac{1}{f}\right)\\ &\quad + \sum_{j=3}^{\infty}\overline{N}_{(3}\left(r,\frac{1}{f}\right) - \sum_{j=3}^{\infty}\overline{N}_{(3}\left(r,\frac{1}{L(f)}\right) + S(r,f)\\ &\leq T(r,L(f)) - T(r,f) + N_2\left(r,\frac{1}{f}\right) + S(r,f).\end{aligned}$$

Thus, (2.6) holds. By the same arguments as above, we obtain (2.7) from (2.4). □

Remark 2.6. Suppose (2.5) also holds for $i = 2$, i.e., (2.5) holds for $i \geq 2$. Then, we have the following inequalities:

$$\overline{N}\left(r,\frac{1}{L(f)}\right) \leq T(r,L(f)) - T(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + S(r,f)$$

and

$$\overline{N}\left(r,\frac{1}{L(f)}\right) \leq \overline{N}\left(r,\frac{1}{f}\right) + S(r,f).$$

3. Proof of Theorem 1.1. From the conditions of Theorem 1.1, we know that f and $L(f)$ share $(1,l)$. From the proof of Lemma 2.4, we have

$$T(r,L(f)) = O(T(r,f)) + S(r,f). \tag{3.1}$$

Let Δ be defined by (2.1). We discuss the following two cases.

Case 1. $\Delta \equiv 0$. By integration, we obtain from (2.1) that

$$\frac{1}{f-1} = \frac{A}{L(f)-1} + B, \tag{3.2}$$

where A $(\neq 0)$ and B are constants. From (3.2), we get that f and $L(f)$ share 1 CM.

From (3.2), we also have

$$\tag{3.3} \begin{aligned} f &= \frac{(B+1)L(f)+(A-B-1)}{BL(f)+(A-B)}, \\ L(f) &= \frac{(B-A)f+(A-B-1)}{Bf-(B+1)}. \end{aligned}$$

We discuss the following three subcases.

Subcase 1.1. Suppose that $B \neq 0, -1$. From (3.3) and $L(f)$ entire, we have that $(B+1)/B$ is a Picard value of f. From this and the second fundamental theorem, we have

$$\tag{3.4} \begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-((B+1)/B)}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f), \end{aligned}$$

and thus,

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \leq N_2\left(r, \frac{1}{f}\right) + S(r,f) \\ &\leq N\left(r, \frac{1}{f}\right) + S(r,f) \leq T(r,f) + S(r,f). \end{aligned}$$

By the definition of deficiency, we have that $\delta(0,f) = \delta_2(0,f) = \Theta(0,f) = 0$, which contradicts assumptions (1.2), (1.3), (1.4).

Subcase 1.2. Suppose that $B = 0$. From (3.3), we have

$$\tag{3.5} f = \frac{L(f)+A-1}{A}, \qquad L(f) = Af-(A-1).$$

If $A \neq 1$, from (3.5), we obtain

$$\overline{N}\left(r, \frac{1}{f-(A-1)/A}\right) = \overline{N}\left(r, \frac{1}{L(f)}\right).$$

From this, Lemma 2.4 and the second fundamental theorem, we have

$$\begin{aligned} 2T(r,f) &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-\frac{A-1}{A}}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{L(f)}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f)\\ &\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f), \end{aligned}$$

which gives that

$$\Theta(0,f)+\delta(0,f)+\delta(1,f)\le 1. \tag{3.6}$$

The definition of deficiency then implies

$$\Theta(0,f)\ge \delta_2(0,f)\ge \delta(0,f). \tag{3.7}$$

Combining this with (3.6) and (3.7), we have

$$\delta_2(0,f)+\delta(0,f)+\delta(1,f)\le \Theta(0,f)+\delta(0,f)+\delta(1,f)\le 1,$$

when $l\ge 2$; and

$$\begin{aligned} \frac{1}{2}\delta_2(0,f)+\frac{3}{4}\delta(0,f)+\frac{1}{2}\delta(1,f) &\le \frac{1}{2}\Theta(0,f)+\frac{3}{4}\delta(0,f)+\frac{1}{2}\delta(1,f)\\ &< \frac{3}{4}\Theta(0,f)+\frac{3}{4}\delta(0,f)+\frac{3}{4}\delta(1,f)\\ &\le \frac{3}{4}, \end{aligned}$$

when $l=1$; as well as

$$\begin{aligned} &\delta_2(0,f)+3\delta(0,f)+\Theta(0,f)+\delta(1,f)\\ &\quad\le \Theta(0,f)+3\delta(0,f)+\Theta(0,f)+\delta(1,f)\\ &\quad< 3\Theta(0,f)+3\delta(0,f)+3\delta(1,f)\\ &\quad\le 3, \end{aligned}$$

when $l=0$. This contradicts assumptions (1.2), (1.3) and (1.4). Thus, $A=1$. From (3.5), we have $f\equiv L(f)$.

Subcase 1.3. Supposing that $B=-1$, it follows from (3.3) that

$$f=\frac{A}{-L(f)+(A+1)},\qquad L(f)=\frac{(A+1)f-A}{f}. \tag{3.8}$$

If $A \neq -1$, we obtain from (3.8) that

$$\overline{N}\left(r, \frac{1}{f-(A/(A+1))}\right) = \overline{N}\left(r, \frac{1}{L(f)}\right).$$

By the same reasoning as discussed in Subcase 1.2, we obtain a contradiction. Hence, $A = -1$. From (3.8), we obtain

$$f \cdot L(f) \equiv 1. \tag{3.9}$$

Since f is entire, from (3.9), we get that 0 is a Picard exceptional value of f. Thus, from Lemma 2.1, we have

$$2T(r,f) = T\left(r, \frac{1}{f^2}\right) = m\left(r, \frac{L(f)}{f}\right) + N\left(r, \frac{L(f)}{f}\right) = S(r,f),$$

which is a contradiction.

Case 2. $\Delta \not\equiv 0$. By Lemma 2.2, we know that (2.2) holds. By the second fundamental theorem, we obtain

$$\begin{aligned} T(r,f) &+ T(r,L(f)) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{L(f)}\right) + \overline{N}\left(r, \frac{1}{L(f)-1}\right) \\ &\quad - N_0\left(r, \frac{1}{f'}\right) - N_0\left(r, \frac{1}{(L(f))'}\right) + S(r,f) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{L(f)-1}\right) \\ &= 2N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) + 2N_L\left(r, \frac{1}{f-1}\right) \\ &\quad + 2N_L\left(r, \frac{1}{L(f)-1}\right) + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + S(r,f) \\ &\leq N(r,\Delta) + N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) + 2N_L\left(r, \frac{1}{f-1}\right) \\ &\quad + 2N_L\left(r, \frac{1}{L(f)-1}\right) + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + S(r,f) \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&\leq \overline{N}_{(2}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{L(f)}\right) + 3N_L\left(r, \frac{1}{f-1}\right) \\
&\quad + 3N_L\left(r, \frac{1}{L(f)-1}\right) + N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) \\
&\quad + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{(L(f))'}\right) + S(r,f),
\end{aligned}
$$

where $N_0(r, (1/f'))$ denotes the counting function corresponding to the zeros of f', which are not the zeros of f and $f-1$, and $N_0(r, 1/(L(f))')$ denotes the counting function corresponding to the zeros of $(L(f))'$, which are not the zeros of $L(f)$ and $L(f)-1$.

Subcase 2.1. $l \geq 2$. It is easy to see that

$$
\begin{aligned}
&3N_L\left(r, \frac{1}{f-1}\right) + 3N_L\left(r, \frac{1}{L(f)-1}\right) \\
&\quad + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) \\
&\leq N\left(r, \frac{1}{f-1}\right) + S(r,f).
\end{aligned} \tag{3.12}
$$

From (3.10), (3.11) and (3.12), we have

$$
\begin{aligned}
T(r,f) + T(r, L(f)) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{L(f)}\right) \\
+ N\left(r, \frac{1}{f-1}\right) + S(r,f).
\end{aligned}
$$

Thus, from (2.3), we obtain that

$$
\begin{aligned}
T(r,f) + T(r, L(f)) \leq N_2\left(r, \frac{1}{f}\right) + T(r, L(f)) - T(r,f) \\
+ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r,f).
\end{aligned}
$$

Therefore, we have

$$
2T(r,f) \leq N_2\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r,f),
$$

which contradicts (1.2).

Subcase 2.2. $l = 1$. Now

$$\begin{aligned}(3.13)\qquad & 2N_L\left(r, \frac{1}{L(f)-1}\right) + 3N_L\left(r, \frac{1}{f-1}\right) \\ & + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) \\ & \leq N\left(r, \frac{1}{f-1}\right) + S(r,f),\end{aligned}$$

and, by Lemmas 2.1, 2.4 and (3.1),

$$\begin{aligned} N_L\left(r, \frac{1}{L(f)-1}\right) & \leq \frac{1}{2}N\left(r, \frac{L(f)}{(L(f))'}\right) \leq \frac{1}{2}T\left(r, \frac{L(f)}{(L(f))'}\right) \\ & = \frac{1}{2}T\left(r, \frac{(L(f))'}{L(f)}\right) + O(1) \\ & \leq \frac{1}{2}N\left(r, \frac{(L(f))'}{L(f)}\right) + S(r, L(f)) + O(1) \\ & \leq \frac{1}{2}\overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,f) \leq \frac{1}{2}N\left(r, \frac{1}{f}\right) + S(r,f).\end{aligned}$$

By combining the above inequalities, and using the same method as in Subcase 2.1, we get

$$2T(r,f) \leq N_2\left(r, \frac{1}{f}\right) + \frac{3}{2}N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r,f),$$

which contradicts (1.3).

Subcase 2.3. $l = 0$. Then,

$$\begin{aligned}(3.14)\qquad & N_L\left(r, \frac{1}{L(f)-1}\right) + 2N_L\left(r, \frac{1}{f-1}\right) \\ & + 2N_E^{(2}\left(r, \frac{1}{f-1}\right) + N_E^{1)}\left(r, \frac{1}{L(f)-1}\right) \\ & \leq N\left(r, \frac{1}{f-1}\right) + S(r,f).\end{aligned}$$

From Lemmas 2.1, 2.4 and (3.1), we have

$$N_L\left(r, \frac{1}{L(f)-1}\right) \leq N\left(r, \frac{L(f)}{(L(f))'}\right) \leq N\left(r, \frac{(L(f))'}{L(f)}\right) + S(r,f)$$

$$\leq \overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,f) \leq N\left(r, \frac{1}{f}\right) + S(r,f),$$

and thus,

$$(3.15)\quad 2N_L\left(r, \frac{1}{L(f)-1}\right) + N_L\left(r, \frac{1}{f-1}\right) \leq 2N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

Combining (3.10), (3.11) and (3.14) with (3.15), we have

$$(3.16)\quad T(r,f) + T(r,L(f)) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{f-1}\right) + 2N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

From Lemma 2.4, it follows that

$$N_2\left(r, \frac{1}{L(f)}\right) \leq T(r,L(f)) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f).$$

Substituting this into (3.16), we have

$$2T(r,f) \leq N_2\left(r, \frac{1}{f}\right) + 3N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r,f),$$

which contradicts (1.4).

The proof has now been completed. □

4. Proof of Theorem 1.2. From the conditions of Theorem 1.1, we have that f and $L(f)$ share $(1,l)$ and (3.1). Let Δ be defined by (2.1). We discuss the following two cases.

Case 1. $\Delta \not\equiv 0$. By a similar method as used in the proof of Theorem 1.1, we know that (2.2), (3.10) and (3.11) hold.

Subcase 1.1. $l \geq 2$. We have

$$T(r,f)+T(r,L(f)) \leq N_2\left(r, \frac{1}{f}\right)+N_2\left(r, \frac{1}{L(f)}\right)+N\left(r, \frac{1}{f-1}\right)+S(r,f).$$

Thus, from Lemma 2.5, we obtain that

$$\begin{aligned} T(r,f)+T(r,L(f)) \leq N_2\left(r,\frac{1}{f}\right) &+ T(r,L(f)) - T(r,f) \\ &+ N_2\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + S(r,f). \end{aligned}$$

Therefore, we have

$$2T(r,f) \leq 2N_2\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + S(r,f),$$

which contradicts (1.5).

Subcase 1.2. $l = 1$. Noting that (3.13) holds, by Lemmas 2.1, 2.5 and (3.1), we have

$$\begin{aligned} N_L\left(r,\frac{1}{L(f)-1}\right) &\leq \frac{1}{2}N\left(r,\frac{L(f)}{(L(f))'}\right) \leq \frac{1}{2}N\left(r,\frac{(L(f))'}{L(f)}\right) + S(r,f) \\ &\leq \frac{1}{2}\overline{N}\left(r,\frac{1}{L(f)}\right) + S(r,f) = \frac{1}{2}N_1\left(r,\frac{1}{L(f)}\right) + S(r,f) \\ &\leq \frac{1}{2}N_2\left(r,\frac{1}{L(f)}\right) + S(r,f) \leq \frac{1}{2}N_2\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned}$$

Using the same method as in Subcase 2.1, we obtain

$$2T(r,f) \leq \frac{5}{2}N_2\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + S(r,f),$$

which contradicts (1.6).

Subcase 1.3. $l = 0$. Noting that (3.14) holds, from Lemmas 2.1, 2.5 and (3.1), we have

$$\begin{aligned} N_L\left(r,\frac{1}{L(f)-1}\right) &\leq N\left(r,\frac{L(f)}{(L(f))'}\right) \leq N\left(r,\frac{(L(f))'}{L(f)}\right) + S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{L(f)}\right) + S(r,f) = N_1\left(r,\frac{1}{L(f)}\right) + S(r,f) \\ &\leq N_2\left(r,\frac{1}{L(f)}\right) + S(r,f) \leq N_2\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned}$$

Thus, we have

$$2N_L\left(r, \frac{1}{L(f)-1}\right) + N_L\left(r, \frac{1}{f-1}\right) \leq 2N_2\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f). \tag{4.1}$$

Combining (3.10), (3.11) and (3.14) with (4.1), we have

$$\begin{aligned} T(r, f) + T(r, L(f)) \leq{}& N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{L(f)}\right) \\ &+ N\left(r, \frac{1}{f-1}\right) + 2N_2\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \tag{4.2}$$

From Lemma 2.5, we have

$$N_2\left(r, \frac{1}{L(f)}\right) \leq T(r, L(f)) - T(r, f) + N_2\left(r, \frac{1}{f}\right) + S(r, f).$$

Substituting this into (4.2), we have

$$2T(r, f) \leq 4N_2\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r, f),$$

which contradicts (1.7).

Case 2. $\Delta \equiv 0$. By following a similar method as in the proof of Theorem 1.1, we have that (3.2) and (3.3) hold, and that f and g share 1 CM. We discuss the following three subcases.

Subcase 2.1. Suppose that $B \neq 0, -1$. Following the same method as Theorem 1.1, Subcase 1.1, we obtain a contradiction.

Subcase 2.2. Suppose that $B = 0$. Then, we have that (3.5) holds. If $A \neq 1$, from (3.5), we obtain

$$\overline{N}\left(r, \frac{1}{f - ((A-1)/A)}\right) = \overline{N}\left(r, \frac{1}{L(f)}\right).$$

From this, Lemma 2.5 and the second fundamental theorem, we have

$$\begin{aligned} 2T(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - ((A-1)/A)}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{L(f)}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) \end{aligned}$$

$$\leq \overline{N}\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f),$$

which contradicts assumptions (i)–(iii) of Theorem 1.2. Thus, $A = 1$. From (3.5), we have $f \equiv L(f)$.

Subcase 2.3. Suppose that $B = -1$. By the same reasoning as in Subcase 2.2, we obtain a contradiction.

This completes the proof of Theorem 1.2. $\square$

5. Proof of Theorem 1.3. The method of the proof is similar to that used for proving Theorem 1.2. Therefore, we only give a draft proof of Theorem 1.3.

From the conditions of Theorem 1.3, we have that f and $L(f)$ share $(1, l)$ and (3.1). Let Δ be defined as in (2.1). We discuss the following two cases.

Case 1. $\Delta \not\equiv 0$. By a similar method as that used in the proof of Theorem 1.1, we know that (2.2), (3.10) and (3.11) hold.

Subcase 1.1. $l \geq 2$. By a similar method as that used in the proof of Theorem 1.2, Subcase 1.1, we obtain a contradiction.

Subcase 1.2. $l = 1$. Noting that (3.13) holds by Lemma 2.1, Remark 2.6 and (3.1), we have

$$\begin{aligned} N_L\left(r, \frac{1}{L(f)-1}\right) &\leq \frac{1}{2}N\left(r, \frac{L(f)}{(L(f))'}\right) \leq \frac{1}{2}N\left(r, \frac{(L(f))'}{L(f)}\right) + S(r,f) \\ &\leq \frac{1}{2}\overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,f) \leq \frac{1}{2}\overline{N}\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

Using the same method as in Theorem 1.1, Subcase 2.1, we obtain

$$2T(r,f) \leq 2N_2\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + \frac{1}{2}\overline{N}\left(r, \frac{1}{f}\right) + S(r,f),$$

which contradicts (1.9).

Subcase 1.3. $l = 0$. Noting that (3.14) holds, from Lemma 2.1, Remark 2.6 and (3.1), we have

$$N_L\left(r, \frac{1}{L(f)-1}\right) \leq N\left(r, \frac{L(f)}{(L(f))'}\right) \leq N\left(r, \frac{(L(f))'}{L(f)}\right) + S(r,f)$$

$$\leq \overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,f) \leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f)$$

and

$$N_L\left(r, \frac{1}{f-1}\right) \leq N\left(r, \frac{f}{f'}\right) \leq N\left(r, \frac{f'}{f}\right)+S(r,f) \leq \overline{N}\left(r, \frac{1}{f}\right)+S(r,f).$$

Thus, we have

$$2N_L\left(r, \frac{1}{L(f)-1}\right) + N_L\left(r, \frac{1}{f-1}\right) \leq 3\overline{N}\left(r, \frac{1}{f}\right) + S(r,f). \tag{5.1}$$

Combining (3.10), (3.11) and (3.14) with (5.1), it follows that

$$\begin{aligned} T(r,f) + T(r,L(f)) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{L(f)}\right) \\ + N\left(r, \frac{1}{f-1}\right) + 3\overline{N}\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned} \tag{5.2}$$

Lemma 2.5 then yields

$$N_2\left(r, \frac{1}{L(f)}\right) \leq T(r,L(f)) - T(r,f) + N_2\left(r, \frac{1}{f}\right) + S(r,f).$$

Substituting this into (5.2), we have

$$2T(r,f) \leq 2N_2\left(r, \frac{1}{f}\right) + 3\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r,f),$$

which contradicts (1.10).

Case 2. $\Delta \equiv 0$. By following a method similar to the proof of Theorem 1.1, we have that (3.2) and (3.3) hold, and that f and g share 1 CM. We discuss the following three subcases.

Subcase 2.1. Suppose that $B \neq 0, -1$. Following the same method as in the Proof of Theorem 1.1, Subcase 1.1, we obtain a contradiction.

Subcase 2.2. Suppose that $B = 0$. Then we have that (3.5) holds. If $A \neq 1$, from (3.5), we obtain

$$\overline{N}\left(r, \frac{1}{f - ((A-1)/A)}\right) = \overline{N}\left(r, \frac{1}{L(f)}\right).$$

From this, Remark 2.6 and the second fundamental theorem, we have

$$\begin{aligned}2T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-((A-1)/A)}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{L(f)}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f)\\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f).\end{aligned}$$

This gives that

$$2\Theta(0,f)+\Theta(1,f)\leq 1.$$

Combining this with

$$1\geq\Theta(a,f)\geq\delta_2(a,f)\geq\delta(a,f)\geq 0,\quad a\in\mathbb{C}, \tag{5.3}$$

we have

$$2\delta_2(0,f)+\delta(1,f)\leq 2\Theta(0,f)+\Theta(1,f)\leq 1,$$

which contradicts (1.8) when $l\geq 2$;

$$\begin{aligned}&\delta_2(0,f)+\frac{1}{4}\Theta(0,f)+\frac{1}{2}\,\delta(1,f)\\ &\leq \frac{1}{4}\Theta(0,f)+\Theta(0,f)+\frac{1}{2}\Theta(1,f)\\ &\leq \frac{1}{4}+\frac{1}{2}\leq\frac{3}{4},\end{aligned}$$

which contradicts (1.9) when $l=1$;

$$\begin{aligned}&\delta_2(0,f)+\frac{3}{2}\Theta(0,f)+\frac{1}{2}\,\delta(1,f)\\ &\leq \delta_2(0,f)+\frac{1}{2}\Theta(0,f)+\Theta(0,f)+\frac{1}{2}\Theta(1,f)\\ &\leq \frac{3}{2}+\frac{1}{2}=2,\end{aligned}$$

which contradicts (1.10) when $l=0$. Thus, $A=1$. From (3.5), we have $f\equiv L(f)$.

Subcase 2.3. Suppose that $B=-1$. By the same reasoning as discussed in Subcase 2.2, we obtain a contradiction.

This completes the proof of Theorem 1.3. □

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