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SEQUENTIALLY COHEN-MACAULAYNESS OF BIGRADED MODULES

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ABSTRACT. Let K be a field, $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ a standard bigraded polynomial ring, and M a finitely generated bigraded S-module. In this paper, we study the sequentially Cohen-Macaulayness of M with respect to $Q = (y_1, \ldots, y_n)$. We characterize the sequentially Cohen-Macaulayness of $L \otimes_K N$ with respect to Q as an S-module when L and N are non-zero finitely generated graded modules over $K[x_1, \ldots, x_m]$ and $K[y_1, \ldots, y_n]$, respectively. All hypersurface rings that are sequentially Cohen-Macaulay with respect to Q are classified.

1. Introduction. In **[13]**, Stanley introduced the notion of sequentially Cohen-Macaulayness for graded modules. This concept has since been studied by several authors; we refer the reader to **[3, 4, 6, 9, 8, 12, 14]**. In this paper, we define sequentially Cohen-Macaulayness for bigraded modules and introduce some new algebraic invariants which are relevant to this case. We let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a standard bigraded polynomial ring over a field K, M a finitely generated bigraded S-module. We set $Q = (y_1, \ldots, y_n)$. In **[10]**, M is said to be Cohen-Macaulay with respect to Q if $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M)$, where $\operatorname{cd}(Q, M)$ denotes the cohomological dimension of M with respect to Q.

We call a finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M by bigraded submodules M a Cohen-Macaulay filtration with respect to Q if:

(a) each quotient M_i/M_{i-1} is Cohen-Macaulay with respect to Q;
(b) $0 \leq \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \cdots < \operatorname{cd}(Q, M_r/M_{r-1})$.

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If M admits a Cohen-Macaulay filtration with respect to Q, then we say that M is sequentially Cohen-Macaulay with respect to Q. The usual notion of sequentially Cohen-Macaulayness results from our definition if we assume that $P = 0$.

A finite filtration $\mathcal{D} : 0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ of M by bigraded submodules is called the *dimension filtration* of M with respect to Q if D_{i-1} is the largest bigraded submodule of D_i for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$, for all $i = 1, \ldots, r$. In Section 2, we show that if M is sequentially Cohen-Macaulay with respect to Q, then the filtration $\mathcal{F}$ is uniquely determined and it is merely the dimension filtration of M with respect to Q, that is, $\mathcal{F} = \mathcal{D}$. We explicitly describe the structure of the submodules D_i in **[8]**. We also show that, if M is sequentially Cohen-Macaulay with respect to Q with $\operatorname{grade}(Q, M) > 0$ and $|K| = \infty$, then there exists a bihomogeneous M-regular element $y \in Q$ of degree $(0, 1)$ such that M/yM is sequentially Cohen-Macaulay with respect to Q, too. An example is given to show that the converse does not hold in general.

Let $K[x] = K[x_1, \ldots, x_m]$ and $K[y] = K[y_1, \ldots, y_n]$. In Section 3, we consider $L \otimes_K N$ as an S-module where L and N are two non-zero finitely generated graded modules over $K[x]$ and $K[y]$, respectively. We characterize the sequentially Cohen-Macaulayness of $L \otimes_K N$ with respect to Q as follows: $L \otimes_K N$ is a sequentially Cohen-Macaulay with respect to Q if and only if N is a sequentially Cohen-Macaulay $K[y]$-module.

In Section 4, we let $f \in S$ be a bihomogeneous element of degree (a, b) and consider the hypersurface ring $R = S/fS$. Note that, if $a, b > 0$, we have $\operatorname{grade}(Q, R) = n - 1$ and $\operatorname{cd}(Q, R) = n$; hence, R is not Cohen-Macaulay with respect to Q. Thus, it is natural to ask whether R is sequentially Cohen-Macaulay with respect to Q. We classify all hypersurface rings that are sequentially Cohen-Macaulay with respect to Q. In fact, we show that R is sequentially Cohen-Macaulay with respect to Q if and only if $f = h_1 h_2$ where $\deg(h_1) = (a, 0)$ with $a \geq 0$ and $\deg(h_2) = (0, b)$ with $b \geq 0$.

2. Preliminaries.

Let K be a field, and let

$$S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

be a standard bigraded polynomial ring over K, in other words, $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ for all i and j. We set $P = (x_1,\dots,x_m)$ and $Q = (y_1,\dots,y_n)$. Let M be a finitely generated bigraded S-module. We denote by $\mathrm{cd}(Q,M)$ the *cohomological dimension of M with respect to Q* which is the largest integer i for which $H^i_Q(M) \neq 0$. Note that $0 \le \mathrm{cd}(Q,M) \le n$.

Definition 2.1. We say M is *Cohen-Macaulay with respect to Q* if we have only one non vanishing local cohomology module with respect to Q. In [**10**], this was referred to as *relative Cohen-Macaulay with respect to Q*; here, we omit the word "relative" for simplicity.

We recall the following facts which will be used in the sequel.

Fact 2.2. Let M be a finitely generated bigraded S-module. Then

(a) $\mathrm{cd}(P,M) = \dim M/QM$ and $\mathrm{cd}(Q,M) = \dim M/PM$, see [**10,** formula 3].
(b) $\mathrm{grade}(Q,M) \le \dim M - \mathrm{cd}(P,M)$, and the equality holds if M is Cohen-Macaulay, see [**10,** formula 5];
(c) the exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated bigraded S-modules yields $\mathrm{cd}(Q,M) = \max\{\mathrm{cd}(Q,M'), \mathrm{cd}(Q,M'')\}$, see the general version of [**2,** Proposition 4.4];
(d) $\mathrm{cd}(Q,M) = \max\{\mathrm{cd}(Q,S/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Ass}(M)\}$, see the general version of [**2,** Corollary 4.6].

Definition 2.3. We call a finite filtration

$$\mathcal{F} : 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$$

of M by bigraded submodules a Cohen-Macaulay filtration with respect to Q if

(a) each quotient M_i/M_{i-1} is Cohen-Macaulay with respect to Q;
(b) $0 \le \mathrm{cd}(Q, M_1/M_0) < \mathrm{cd}(Q, M_2/M_1) < \cdots < \mathrm{cd}(Q, M_r/M_{r-1})$.

If M admits a Cohen-Macaulay filtration with respect to Q, then we say M is *sequentially Cohen-Macaulay with respect to Q*.

Observe that the ordinary definition of sequentially Cohen-Macaulay modules results from our definition if we assume that $P = 0$.

Remark 2.4. By applying Fact 2.2 (a) to the exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

one immediately has $\operatorname{cd}(Q, M_i) = \operatorname{cd}(Q, M_i/M_{i-1})$ for $i = 1, \dots, r$.

Example 2.5. Cohen-Macaulay modules with respect to Q are obvious examples of sequentially Cohen-Macaulay modules with respect to Q. Any module M such that $\operatorname{cd}(Q, M) \leq 1$ is sequentially Cohen-Macaulay with respect to Q. To show this, we may assume that M is not Cohen-Macaulay with respect to Q. Thus, $\operatorname{grade}(Q, M) = 0$ and $\operatorname{cd}(Q, M) = 1$. The filtration $0 = M_0 \subsetneq M_1 \subsetneq M_2 = M$, where $M_1 = H_Q^0(M)$ is a Cohen-Macaulay filtration with respect to Q.

Next, we show that the filtration $\mathcal{F}$ given in Definition 2.3 is unique. To do so, we need some preparation.

Lemma 2.6. *There is a unique largest bigraded submodule N of M for which* $\operatorname{cd}(Q, N) < \operatorname{cd}(Q, M)$.

Proof. Let $\sum$ be the set of all bigraded submodules L of M such that $\operatorname{cd}(Q, L) < \operatorname{cd}(Q, M)$. As M is a Noetherian S-module, $\sum$ has a maximal element with respect to inclusion, say N. Let T be an arbitrary element in $\sum$. Fact 2.2 (c) implies that $\operatorname{cd}(Q, T + N) < \operatorname{cd}(Q, M)$; hence, the maximality of N yields $T \subseteq N$. $\square$

Definition 2.7. A filtration $\mathcal{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ of M by bigraded submodules is called the *dimension filtration of M with respect to Q* if D_{i-1} is the largest bigraded submodule of D_i for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \dots, r$.

The dimension filtration introduced by Schenzel **[12]** is thus a dimension filtration with respect to the maximal ideal $\mathfrak{m} = P + Q$. A filtration $\mathcal{D}$ as in Definition 2.7 is unique by Lemma 2.6. In order to prove the uniqueness of an $\mathcal{F}$ as in Definition 2.3, we will show that

$\mathcal{F} = \mathcal{D}$. In [**7**], M is said to be *relatively unmixed* with respect to Q if $\operatorname{cd}(Q, M) = \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$.

Lemma 2.8. *Let N be a non-zero bigraded submodule of M. If M is Cohen-Macaulay with respect to Q, then* $\operatorname{cd}(Q, N) = \operatorname{cd}(Q, M)$.

Proof. Since M is Cohen-Macaulay with respect to Q, it follows from [**7,** Corollary 1.11] that M is relatively unmixed with respect to Q, i.e., $\operatorname{cd}(Q, M) = \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. As N is a non-zero submodule of M, we have $\operatorname{Ass}(N) \neq \emptyset$ and $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$. Thus, Fact 2.2 (d) implies that

$$\operatorname{cd}(Q, N) = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}(N)\} = \operatorname{cd}(Q, M),$$

as desired. □

Proposition 2.9. *Let $\mathcal{F}$ be a Cohen-Macaulay filtration of M with respect to Q and $\mathcal{D}$ be the dimension filtration of M with respect to Q. Then, $\mathcal{F} = \mathcal{D}$.*

Proof. Let

$$\mathcal{F} : 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$$

and

$$\mathcal{D} : 0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_s = M.$$

We will show that $r = s$ and $M_i = D_i$ for all i. By Remark 2.4, we have $\operatorname{cd}(Q, M_{i-1}) < \operatorname{cd}(Q, M_i)$ for all $i = 1, \ldots, r$. Hence, Definition 2.7 says that $M_{r-1} \subseteq D_{s-1}$. Assume that $M_{r-1} \subsetneq D_{s-1}$. Thus, D_{s-1}/M_{r-1} is a non-zero submodule of M/M_{r-1}. Since M/M_{r-1} is Cohen-Macaulay with respect to Q, it follows from Lemma 2.8 that $\operatorname{cd}(Q, D_{s-1}/M_{r-1}) = \operatorname{cd}(Q, M/M_{r-1}) = \operatorname{cd}(Q, M)$, where the second equality is yielded by Remark 2.4. Now, applying Fact 2.2 (c) to the exact sequence

$$0 \longrightarrow M_{r-1} \longrightarrow D_{s-1} \longrightarrow D_{s-1}/M_{r-1} \longrightarrow 0,$$

yields $\operatorname{cd}(Q, D_{s-1}) = \operatorname{cd}(Q, M)$, a contradiction. Thus, $M_{r-1} = D_{s-1}$. Continuing in this way, we get $r = s$ and $M_i = D_i$ for all i. Therefore, $\mathcal{F} = \mathcal{D}$. □

We conclude this section with Proposition 2.11, which gives us a class of sequentially Cohen-Macaulay with respect to Q. First, we have Lemma 2.10.

Lemma 2.10. *Let M be sequentially Cohen-Macaulay with respect to Q. If M is relatively unmixed with respect to Q, then M is Cohen-Macaulay with respect to Q.*

Proof. Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be the Cohen-Macaulay filtration with respect to Q. By Fact 3.3, we have $\operatorname{grade}(Q, M) = \operatorname{grade}(Q, M_1)$. Since M_1 is Cohen-Macaulay with respect to Q, it follows from [**7,** Corollary 1.11] that M_1 is relatively unmixed with respect to Q. Thus,

$$\operatorname{grade}(Q, M) = \operatorname{grade}(Q, M_1) = \operatorname{cd}(Q, M_1) = \operatorname{cd}(Q, S/\mathfrak{p}),$$

for all $\mathfrak{p} \in \operatorname{Ass}(M_1)$. As M is relatively unmixed with respect to Q and $\operatorname{Ass}(M_1) \subseteq \operatorname{Ass}(M)$, we have $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M)$, as desired. $\square$

Proposition 2.11. *Suppose that* $\operatorname{grade}(Q, M) > 0$ *and* $|K| = \infty$*. If M is sequentially Cohen-Macaulay with respect to Q, then there exists a bihomogeneous M-regular element $y \in Q$ of degree* $(0, 1)$ *such that M/yM is sequentially Cohen-Macaulay with respect to Q.*

Proof. We assume that M is sequentially Cohen-Macaulay and let $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be the Cohen-Macaulay filtration, with respect to Q. Since $\operatorname{grade}(Q, M) = \operatorname{grade}(Q, M_1) = \operatorname{cd}(Q, M_1) > 0$, it follows that $\operatorname{grade}(Q, M_i/M_{i-1}) = \operatorname{cd}(Q, M_i/M_{i-1}) > 0$ for all i. We set $N_i = M_i/M_{i-1}$. Thus, by [**10,** Corollary, 3.5], which is also valid for finitely many modules that are Cohen-Macaulay and have positive cohomological dimension with respect to Q. There exists a bihomogeneous element $y \in Q$ of degree $(0, 1)$ such that y is N_i-regular for all i and $\overline{N_i}$ is Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, \overline{N_i}) = \operatorname{cd}(Q, N_i) - 1$. Here, $\overline{L} = L/yL$ for any S-module L.

Consider the exact sequence:

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow N_i \longrightarrow 0 \quad \text{for all } i.$$

Since y is regular on N_i for all i, it follows that $\operatorname{Tor}_1^S(S/yS, N_i) = 0$ for all i. Hence, we obtain the exact sequence:

$$0 \longrightarrow \overline{M_{i-1}} \longrightarrow \overline{M_i} \longrightarrow \overline{N_i} \longrightarrow 0 \quad \text{for all } i.$$

Now, the filtration

$$\mathcal{G} : 0 = \overline{M_0} \subsetneq \overline{M_1} \subsetneq \cdots \subsetneq \overline{M_r} = M/yM$$

is the Cohen-Macaulay filtration for M/yM with respect to Q. In fact, $\overline{M_i}/\overline{M_{i-1}} \cong \overline{N_i}$ and

$$\operatorname{grade}(Q, \overline{N_i}) = \operatorname{grade}(Q, N_i) - 1 = \operatorname{cd}(Q, N_i) - 1 = \operatorname{cd}(Q, \overline{N_i}).$$

Hence, $\operatorname{grade}(Q, \overline{M_i}/\overline{M_{i-1}}) = \operatorname{cd}(Q, \overline{M_i}/\overline{M_{i-1}})$. As $\operatorname{cd}(Q, M_i/M_{i-1}) < \operatorname{cd}(Q, M_{i+1}/M_i)$ for all i, we have $\operatorname{cd}(Q, \overline{M_i}/\overline{M_{i-1}}) < \operatorname{cd}(Q, \overline{M_{i+1}}/\overline{M_i})$ for all i. □

The next example shows that the converse of Proposition 2.11 does not hold in general.

Example 2.12. Consider the hypersurface ring

$$R = K[x_1, x_2, y_1, y_2]/(f)$$

where $f = x_1y_1 + x_2y_2$. One has $\operatorname{grade}(Q, R) = 1$ and $\operatorname{cd}(Q, R) = 2$. By [**10,** Lemma 3.4], there exists a bihomogeneous R-regular element $y \in Q$ of degree $(0, 1)$ such that $\operatorname{cd}(Q, R/yR) = \operatorname{cd}(Q, R) - 1 = 1$ and, of course, $\operatorname{grade}(Q, R/yR) = \operatorname{grade}(Q, R) - 1 = 0$. Hence, R/yR is sequentially Cohen-Macaulay with respect to Q. On the other hand, R is not sequentially Cohen-Macaulay with respect to Q. Indeed, $\operatorname{Ass}(R) = \{(f)\}$ and $\operatorname{cd}(Q, R) = \operatorname{cd}(Q, S/(f))$ show that R is relatively unmixed with respect to Q. If R is sequentially Cohen-Macaulay with respect to Q, then by Lemma 2.10, R is Cohen-Macaulay with respect to Q, a contradiction.

3. Sequentially Cohen-Macaulayness of $L \otimes_K N$ with respect to Q. In this section, we characterize the sequentially Cohen-Macaulayness of $L \otimes_K N$ with respect to Q as an S-module where L and N are two non-zero finitely generated graded modules over $K[x]$ and $K[y]$, respectively. For the bigraded S-module M we define the bigraded Matlis-dual of M to be $M^\vee$, where the $(-i,-j)$th bigraded components of $M^\vee$ are given by $\operatorname{Hom}_K(M_{(i,j)},K)$. We set

$$M_k = M_{(k,*)} = \oplus_j M_{(k,j)}$$

and consider it to be a finitely generated graded $K[y]$-module.

Lemma 3.1. *Let M be a finitely generated bigraded S-module. If M is Cohen-Macaulay with respect to Q, $\operatorname{cd}(Q,M)=q$, then $(H_Q^q(M)^\vee)_{(k,*)}$ is a finitely generated Cohen-Macaulay $K[y]$-module of dimension q for all k.*

Proof. Note that

$$\begin{aligned}(H_Q^i(M)^\vee)_{(k,*)} &\cong (H_Q^i(M)_{(-k,*)})^\vee\\ &\cong (H^i_{(y_1,\dots,y_n)}(M_{(-k,*)}))^\vee\\ &\cong \operatorname{Ext}^{n-i}_{K[y]}(M_{(-k,*)},K[y](-n)).\end{aligned}$$

Since M is Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q,M)=q$, it follows from [**10,** Proposition 1.2] that $M_{(-k,*)}$ is a Cohen-Macaulay $K[y]$-module of dimension q, and the conclusion follows immediately. □

Lemma 3.2. *Let M be sequentially Cohen-Macaulay with respect to Q with Cohen-Macaulay filtration $\mathcal{F}$: $0=M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ with respect to Q. Then, we have*

$$H_Q^{q_i}(M)\cong H_Q^{q_i}(M_i)\cong H_Q^{q_i}(M_i/M_{i-1}),$$

where

$$q_i=\operatorname{cd}(Q,M_i)\quad \textit{for } i=1,\dots,r$$

and

$$H_Q^k(M)=0\quad \textit{for } k\notin\{q_1,\dots,q_r\}.$$

Proof. We proceed by induction on the length r of $\mathcal{F}$. The case $r=1$ is obvious.

Now, suppose that $r\ge 2$ and that the statement holds for sequentially Cohen-Macaulay modules with respect to Q with filtrations of length $< r$. We want to prove it for M which is sequentially Cohen-Macaulay with respect to Q and has the Cohen-Macaulay filtration $\mathcal{F}$ of length r. Note that M_{r-1}, which appears in the filtration $\mathcal{F}$ of M, is also sequentially Cohen-Macaulay with respect to Q. Thus, by the induction hypothesis, we have

$$H_Q^{q_i}(M_{r-1}) \cong H_Q^{q_i}(M_i) \cong H_Q^{q_i}(M_i/M_{i-1}) \quad \text{for } i = 1, \dots, r-1$$

and

$$H_Q^k(M_{r-1}) = 0 \quad \text{for } k \notin \{q_1, \dots, q_{r-1}\}.$$

Now, the exact sequence

$$0 \longrightarrow M_{r-1} \longrightarrow M \longrightarrow M/M_{r-1} \longrightarrow 0$$

yields $H_Q^{q_r}(M) \cong H_Q^{q_r}(M_r/M_{r-1})$ and $H_Q^t(M) \cong H_Q^t(M_{r-1})$ for $0 \leq t < q_r$. Therefore, the desired result follows. □

Fact 3.3. In the proof of Lemma 3.2, one observes that

$$\operatorname{grade}(Q, M_i) = q_1 \quad \text{for} \quad i = 1, \dots, r.$$

Theorem 3.4. *Let L and N be two non-zero finitely generated graded modules over $K[x]$ and $K[y]$, respectively. We set $M = L \otimes_K N$. Then, the following statements are equivalent*:

(a) *M is a sequentially Cohen-Macaulay S-module with respect to Q*;
(b) *N is a sequentially Cohen-Macaulay $K[y]$-module.*

Proof.

(a) $\Rightarrow$ (b). Let $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be the Cohen-Macaulay filtration with respect to Q. By Lemma 3.2, we have

$$H_Q^{q_i}(M) \cong H_Q^{q_i}(M_i) \cong H_Q^{q_i}(M_i/M_{i-1}),$$

where $q_i = \operatorname{cd}(Q, M_i) = \operatorname{cd}(Q, M_i/M_{i-1})$ for $i = 1, \dots, r$ and $H_Q^k(M) = 0$ for $k \notin \{q_1, \dots, q_r\}$. Note that

$$H_Q^{q_i}(M) \cong L \otimes_K H_Q^{q_i}(N) \quad \text{for } i = 1, \dots, r,$$

see also the proof of [**10,** Proposition 1.5]. Hence,

$$H_Q^{q_i}(M)^\vee \cong L^\vee \otimes_K H_Q^{q_i}(N)^\vee,$$

where $(-)^\vee$ is the Matlis-dual, see [**5,** Lemma 1.1]. We conclude that

$$\begin{aligned}(H_Q^{q_i}(M_i/M_{i-1})^\vee)_{(k,*)} &\cong (H_Q^{q_i}(M)^\vee)_{(k,*)}\\ &\cong (L^\vee)_k \otimes_K H_Q^{q_i}(N)^\vee\\ &\cong \operatorname{Ext}_{K[y]}^{n-q_i}(N, K[y])^t,\end{aligned}$$

where $t = \dim_K(L^\vee)_k$. Since each M_i/M_{i-1} is Cohen-Macaulay with respect to Q with $\operatorname{cd}(Q, M_i/M_{i-1}) = q_i$, it follows from the above isomorphisms and Lemma 3.1 that $\operatorname{Ext}_{K[y]}^{n-q_i}(N, K[y])$ is Cohen-Macaulay of dimension q_i for $i = 1, \dots, r$. If $k \notin \{q_1, \dots, q_r\}$, then $L \otimes_K H_Q^k(N) \cong H_Q^k(M) = 0$, and hence, $H_Q^k(N) = 0$. Thus, $\operatorname{Ext}_{K[y]}^{n-k}(N, K[y]) = 0$ for $k \notin \{q_1, \dots, q_r\}$. Therefore, the result follows from [**6,** Theorem 1.4].

(b) $\Rightarrow$ (a). Let N be a sequentially Cohen-Macaulay $K[y]$-module with the Cohen-Macaulay filtration $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$. Consider the filtration

$$0 = L \otimes_K N_0 \subseteq L \otimes_K N_1 \subseteq \cdots \subseteq L \otimes_K N_r = L \otimes_K N.$$

We claim this filtration is the Cohen-Macaulay filtration with respect to Q. First, we note that $L \otimes_K N_i \subsetneq L \otimes_K N_{i+1}$ for all i. Otherwise, we have $\dim N_i = \dim N_{i+1}$ by [**11,** Corollary 2.3], a contradiction. For all k and i we have the next isomorphisms

$$\begin{aligned}H_Q^k((L \otimes_K N_i)/(L \otimes_K N_{i-1})) &\cong H_Q^k(L \otimes_K (N_i/N_{i-1}))\\ &\cong L \otimes_K H_Q^k(N_i/N_{i-1}).\end{aligned}$$

The first isomorphism is standard, and for the second, see the proof of [**10,** Proposition 1.5]. We set $D_i = (L \otimes_K N_i)/(L \otimes_K N_{i-1})$ for all i. Thus, we have $\operatorname{cd}(Q, D_i) = \dim N_i/N_{i-1}$ for all i. This implies that $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all i. Also, each D_i is Cohen-Macaulay with respect to Q because N_i/N_{i-1} is Cohen-Macaulay for all i. □

4. Hypersurface rings that are sequentially Cohen-Macaulay with respect to Q. Let $f \in S$ be a bihomogeneous element of degree (a, b), and consider the hypersurface ring $R = S/fS$. We may write

$$f = \sum_{\substack{|\alpha|=a\\ |\beta|=b}} c_{\alpha\beta} x^\alpha y^\beta \quad \text{where } c_{\alpha\beta} \in K.$$

Note that R is a Cohen-Macaulay module of dimension $m+n-1$. Next, we summarize some observations.

Lemma 4.1. *Consider the hypersurface ring R defined above. Then, the statements hold*:

(a) *if $a = 0$ and $b > 0$, then R is Cohen-Macaulay with respect to P of $\operatorname{cd}(P, R) = m$ and Cohen-Macaulay with respect to Q of $\operatorname{cd}(Q, R) = n - 1$;*
(b) *if $a > 0$ and $b = 0$, then R is Cohen-Macaulay with respect to P of $\operatorname{cd}(P, R) = m - 1$ and Cohen-Macaulay with respect to Q of $\operatorname{cd}(Q, R) = n$;*
(c) *if $a > 0$ and $b > 0$, then $\operatorname{grade}(P, R) = m - 1$ and $\operatorname{cd}(P, R) = m$, and $\operatorname{grade}(Q, R) = n - 1$ and $\operatorname{cd}(Q, R) = n$.*

Proof. In order to prove (a), if $a = 0$, then we may write

$$f = \sum_{|\beta|=b} c_\beta y^\beta.$$

Fact 2.2 (a) implies that

$$\operatorname{cd}(P, R) = \dim S/(Q + (f)) = m$$

and

$$\operatorname{cd}(Q, R) = \dim S/(P + (f)) = n - 1.$$

On the other hand, by Fact 2.2 (b), we have

$$\operatorname{grade}(P, R) = \dim R - \operatorname{cd}(Q, R) = m + n - 1 - (n - 1) = m$$

and

$$\operatorname{grade}(Q, R) = \dim R - \operatorname{cd}(P, R) = m + n - 1 - m = n - 1.$$

Thus, the conclusions follow. Parts (b) and (c) are proved in the same way. □

Note that, if $a, b > 0$, then R is not Cohen-Macaulay with respect to Q. Thus, it is natural to ask whether R is sequentially Cohen-Macaulay with respect to Q. In the following, we classify all hypersurface rings that are sequentially Cohen-Macaulay with respect to Q. First, we have the next proposition.

Proposition 4.2. *Let $f \in S$ be a bihomogeneous element of degree (a, b) such that $f = h_1 h_2$, where*

$$h_1 = \sum_{|\alpha|=a} c_\alpha x^\alpha \quad \textit{with } c_\alpha \in K$$

and

$$h_2 = \sum_{|\beta|=b} c_\beta y^\beta \quad \textit{with } c_\beta \in K,$$

i.e., $\deg h_1 = (a, 0)$ *and* $\deg h_2 = (0, b)$. *Consider the hypersurface ring $R = S/fS$. Then, R is sequentially Cohen-Macaulay with respect to P and Q.*

Proof. We show that R is sequentially Cohen-Macaulay with respect to P. The argument for Q is similar. Consider the filtration $\mathcal{F}$: $0 = R_0 \subsetneq R_1 \subsetneq R_2 = R$ where $R_1 = h_2 S/fS$. We claim that this filtration is the Cohen-Macaulay filtration with respect to P. Observe that $R_2/R_1 \cong S/h_2 S$ is Cohen-Macaulay with respect to P with $\operatorname{cd}(P, R_2/R_1) = m$, by Lemma 4.1 (a). Now, consider the map

$$\varphi : S \longrightarrow h_2 S/fS$$

given by

$$g \longmapsto g h_2 + fS.$$

We obtain the isomorphism

$$S/h_1 S \cong h_2 S/fS \cong R_1/R_0.$$

Thus, R_1/R_0 is Cohen-Macaulay with respect to P with $\operatorname{cd}(P, R_1/R_0) = m-1$, by Lemma 4.1 (b). Therefore, $\mathcal{F}$ is the Cohen-Macaulay filtration of R with respect to P. □

For the proof of the main theorem, we recall the next results from [**8**].

Fact 4.3. Let $\mathcal{D} : 0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ be the dimension filtration of M with respect to Q. Then

(a) $D_i = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j$ for $i = 1, \dots, r-1$, where $0 = \bigcap_{j=1}^{s} N_j$ is a reduced primary decomposition of 0 in M with N_j $\mathfrak{p}_j$-primary for $j = 1, \dots, s$, and

$$B_{i,Q} = \{\mathfrak{p} \in \mathrm{Ass}(M) : \mathrm{cd}(Q, S/\mathfrak{p}) \leq \mathrm{cd}(Q, D_i)\};$$

(b) $\mathrm{Ass}(M/D_i) = \mathrm{Ass}(M) \setminus \mathrm{Ass}(D_i)$ for $i = 1, \dots, r$;
(c) $\mathrm{grade}(Q, M/D_{i-1}) = \mathrm{cd}(Q, D_i)$ for $i = 1, \dots, r$ if and only if M is sequentially Cohen-Macaulay with respect to Q.

Theorem 4.4. *Let $f \in S$ be a bihomogeneous element of degree (a, b), and let $R = S/fS$ be the hypersurface ring. Then, the next statements are equivalent*:

(a) *R is sequentially Cohen-Macaulay with respect to Q*;
(b) *$f = h_1 h_2$, where* $\deg h_1 = (a, 0)$ *with $a \geq 0$ and* $\deg h_2 = (0, b)$ *with $b \geq 0$.*

Proof.

(a) $\Rightarrow$ (b). Assume that R is not Cohen-Macaulay with respect to Q, see Lemma 4.1. Let

$$f = \prod_{i=1}^{r} f_i$$

be the unique factorization of f into bihomogeneous irreducible factors f_i with $\deg f_i = (a_i, b_i)$ for $i = 1, \dots, r$. Note that

$$\sum_{i=1}^{r} a_i = a \quad \text{and} \quad \sum_{i=1}^{r} b_i = b.$$

Our aim is to show that, for each f_i, we have $\deg f_i = (a_i, 0)$ with $a_i \geq 0$ or $\deg f_i = (0, b_i)$ with $b_i \geq 0$. Assume that this is not the case, and so there exists $1 \leq s \leq r$ such that $\deg f_s = (a_s, b_s)$ with $a_s, b_s > 0$. Thus, we may write that $\deg f_i = (a_i, 0)$ with $a_i \geq 0$ for $i = 1, \dots, s-1$, $\deg f_i = (a_i, b_i)$ with $a_i, b_i > 0$ for $i = s, s+1, \dots, t$ and $\deg f_i = (0, b_i)$ with $b_i \geq 0$ for $i = t+1, \dots, r$, and $t < r$. By Fact 4.3 (a), R has the dimension filtration

$$\mathcal{F} : 0 = (f)/(f) \subsetneq I/(f) \subsetneq R = S/(f)$$

with respect to Q, where

$$I = \bigcap_{i=1}^{t} (f_i).$$

Note that $\operatorname{cd}(Q, I/(f)) = n - 1$ by Fact 4.3 (b), Fact 2.2 (d) and $\operatorname{cd}(Q, R) = n$. As R is sequentially Cohen-Macaulay with respect to Q, we must have $\operatorname{grade}(Q, S/I) = \operatorname{cd}(Q, R)$ by Fact 4.3 (c). Since S/I is Cohen-Macaulay, it follows from Fact 2.2 (b) that

$$\operatorname{grade}(Q, S/I) = \dim S/I - \operatorname{cd}(P, S/I) = (m + n - 1) - m = n - 1,$$

a contradiction.

(b) $\Rightarrow$ (a). Follows from Proposition 4.2. □

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