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PROLONGATION OF SYMMETRIC KILLING TENSORS AND COMMUTING SYMMETRIES OF THE LAPLACE OPERATOR

JEAN-PHILIPPE MICHEL, PETR SOMBERG AND JOSEF ŠILHAN

ABSTRACT. We determine the space of commuting symmetries of the Laplace operator on pseudo-Riemannian manifolds of constant curvature and derive its algebra structure. Our construction is based on Riemannian tractor calculus, allowing us to construct a prolongation of the differential system for symmetric Killing tensors. We also discuss some aspects of its relation to projective differential geometry.

1. Introduction. The Laplace operator is one of the cornerstones of geometrical analysis on pseudo-Riemannian manifolds. There exists a close relationship between spectral properties of the Laplace operator and local as well as global invariants of the underlying pseudo-Riemannian manifold.

The question of conformal symmetries of the Yamabe-Laplace operator Δ_Y on conformally flat spaces has been solved [**8**]. A differential operator D is a conformal symmetry of Δ_Y, provided $[\Delta_Y, D] \in (\Delta_Y)$, where (Δ_Y) is the left ideal generated by Δ_Y in the algebra of differential operators. These D operators are called *conformal symmetries* because they preserve the kernel of Δ_Y. On a given flat conformal manifold M, there is a bijection between the vector space of symmetric conformal Killing tensors and the quotient of the space of conformal symmetries by (Δ_Y). Note that the space of symmetric conformal Killing tensors is the solution space of a conformally invariant system

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of overdetermined partial differential equations, which is locally finite-dimensional.

In the present paper, we classify the commuting symmetries of the Laplace operator Δ on pseudo-Riemannian manifolds of constant curvature, i.e., manifolds locally isometric to a space form. In fact, the Laplace operator differs from the Yamabe-Laplace operator by a multiple of the identity operator; therefore, both operators share the same commuting symmetries. The eigenspaces of the Laplace operator are preserved by commuting symmetries, i.e., by linear differential operators D commuting with the Laplace operator:

$$[\Delta, D] = 0.$$

The vector space of commuting symmetries is generated by Killing vector fields and their far reaching generalization called symmetric Killing tensors, or Killing tensors for short. Their composition as differential operators provides an algebraic structure that we shall determine.

Killing 2-tensors on pseudo-Riemannian manifolds are the most studied among Killing tensors, and they play a key role in the separation of variables of the Laplace equation. The construction of commuting symmetries out of Killing 2-tensors is well known in a number of geometrical situations **[7]**, particularly on constant curvature manifolds. Higher Killing tensors give integrals of motion for the geodesic equation and contribute to its integrability. They can be regarded as hidden symmetries of the underlying pseudo-Riemannian manifold. Killing tensors themselves are solutions of an invariant system of PDEs, and trace-free Killing tensors are special examples of conformal Killing tensors.

As a technical tool, we introduce, and to a certain extent develop, the Riemannian tractor calculus, focusing mainly on manifolds of constant curvature. This allows a uniform description of the prolongation of the invariant system of PDEs for Killing tensors and plays a key role in our analysis of the correspondence between commuting symmetries of the Laplace operator and Killing tensors. In particular, we obtain an explicit version of the identification in **[15]**, see also **[17]**, of the space of fixed valence Killing tensors with a representation of the general linear group.

The Riemannian tractor calculus can be interpreted as the tractor calculus for projective parabolic geometry in a scale corresponding to a metric connection in the projective class of affine connections. Restricting to locally flat special affine connections, Einstein metric connections in the projective class correspond to manifolds of constant curvature [**11**]. Since the Killing equations on symmetric tensor fields are projectively invariant [**9**], we may use invariant tractor calculus in projective parabolic geometry to construct commuting symmetries. Projective invariance explains that the space of Killing tensors carries a representation of the general linear group.

As for the style of presentation and exposition, we have attempted to make the paper accessible to a broad audience with basic knowledge in Riemannian geometry. Following this perspective, the structure of our paper follows. After setting the conventions in Section 2, we introduce the rudiments of Riemannian tractor calculus in Section 3. The core of the article is in Section 4, where we construct the prolongation of the differential system for Killing tensors and derive the space of differential operators preserving the spectrum of the Laplace operator. Afterwards, we determine the underlying structure of associative algebra on this space, induced by the composition of differential operators. We compute explicit formulas for commuting symmetries of order at most 3. In special cases, we compare commuting symmetries with conformal symmetries constructed in [**8**]. In Section 5, we interpret our results in terms of the holonomy reduction of a Cartan connection in projective parabolic geometry and its restriction on a curved orbit equipped with an Einstein metric.

2. Notation and conventions. Let (M,g) be a smooth pseudo-Riemannian manifold. Throughout the paper, we employ Penrose's abstract index notation and use $\mathcal{E}^a$ to denote the space of smooth sections of the tangent bundle TM on M, and $\mathcal{E}_a$ for the space of smooth sections of the cotangent bundle T^*M. We also use $\mathcal{E}$ for the space of smooth functions. All tensors considered are assumed to be smooth. With abuse of notation, we will often use the same symbols for the bundles and their spaces of sections. The metric g_{ab} will be used to identify TM with T^*M. We shall assume that the manifold M has dimension $n \geq 2$.

An index which appears twice, once raised and once lowered, indicates the contraction. Square brackets $[\cdots]$ will denote skew-symmetrization of enclosed indices, while round brackets $(\cdots)$ will indicate symmetrization.

We set ∇ for the Levi-Civita connection corresponding to g_{ab}. Then, the Laplacian Δ is given by

$$\Delta = g^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b.$$

Since the Levi-Civita connection is torsion-free, the Riemannian curvature $R_{ab}{}^c{}_d$ is given by

$$[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^{\,d},$$

where $[\cdot,\cdot]$ indicates the commutator bracket. The Riemannian curvature can be decomposed in terms of the totally trace-free Weyl curvature C_{abcd}, and the symmetric Schouten tensor P_{ab},

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathrm{P}_{b]d} + 2g_{d[b}\mathrm{P}_{a]c}. \tag{2.1}$$

We will refer to P_{ab} as a Riemannian Schouten tensor to distinguish from the projective Schouten tensor which will be introduced later in this paper. We define $\mathrm{J} := \mathrm{P}^a{}_a$, such that

$$\mathrm{J} = \frac{\mathrm{Sc}}{2(n-1)},$$

with Sc the scalar curvature.

Throughout the paper, we work (if not stated otherwise) on manifolds of constant curvature, i.e., locally symmetric spaces with parallel curvature

$$R_{abcd} = \frac{4}{n}\mathrm{J}g_{c[a}g_{b]\,d},$$

cf., **[18]**. Thus, the function J is constant. In signature (p,q), M is then locally isomorphic to G/H, where $G = SO(p+1,q)$ and $H = SO(p,q)$ if $J > 0$, $G = SO(p,q+1)$, $H = SO(p-1,q+1)$ if $J < 0$, $G = E(p,q)$ and $H = SO(p,q)$ if $J = 0$. Here, we denote the group of pseudo-Euclidean motions on $\mathbb{R}^{p,q}$ by $E(p,q)$.

3. Tractor calculus in Riemannian geometry. The notion of associated tractor bundles is well known in the category of parabolic geometries. We refer to **[5]** for a review with many applications. In this section, we introduce and develop rudiments of a class of tractor bundles in the category of pseudo-Riemannian manifolds, in close analogy with tractor calculi in parabolic geometries.

We assume M has constant curvature, i.e.,

$$R_{abcd} = \frac{4}{n}\mathrm{J} g_{c[a}g_{b]d},$$

with J constant. We define the *Riemannian standard tractor bundle* or *standard tractor bundle*, for short,

$$\mathcal{T} := \mathcal{L} \oplus TM$$

where $\mathcal{L}$ denotes the trivial bundle over M.

The Levi-Civita connection ∇_a induces a connection on $\mathcal{T}$, which is trivial on $\mathcal{L}$. The *tractor connection* is another connection on $\mathcal{T}$, also denoted (with an abuse of notation) by ∇_a, and defined by

$$\nabla_a \begin{pmatrix} f \\ \mu^b \end{pmatrix} = \begin{pmatrix} \nabla_a f - \mu_a \\ \nabla_a \mu^b + \frac{2}{n}\mathrm{J} f \delta_a^{\ b} \end{pmatrix}, \tag{3.1}$$

where $f \in \mathcal{E}$, $\mu^b \in \mathcal{E}^b$. In the first line, we use the isomorphism $TM \cong T^*M$. The dual connection on the dual bundle:

$$\mathcal{T}^* := T^*M \oplus \mathcal{L},$$

also denoted by ∇_a, is given by

$$\nabla_a \begin{pmatrix} \nu_b \\ f \end{pmatrix} = \begin{pmatrix} \nabla_a \nu_b + f g_{ab} \\ \nabla_a f - \frac{2}{n}\mathrm{J}\nu_a \end{pmatrix}, \tag{3.2}$$

where $\nu_b \in \mathcal{E}_b$ and $f \in \mathcal{E}$. Direct computation shows that the curvature of the tractor connection ∇ is trivial, i.e., the tractor connection ∇ is flat. Note that this connection differs from that induced by the Cartan connection [**5,** subsection 1.5].

The bundle $\mathcal{T}$ is equipped with the symmetric bilinear form $\langle\ ,\ \rangle$,

$$\left\langle \begin{pmatrix} f \\ \mu^b \end{pmatrix}, \begin{pmatrix} \overline{f} \\ \overline{\mu}^b \end{pmatrix} \right\rangle = \frac{2}{n}\mathrm{J} f\overline{f} + \mu^b \overline{\mu}_b \tag{3.3}$$

which is invariant with respect to the tractor connection ∇. For $\mathrm{J} \neq 0$, this form is non-degenerate and called the *tractor metric*. Then, it yields an isomorphism $\mathcal{T} \cong \mathcal{T}^*$.

We define

$$\mathcal{A} := \bigwedge^2 \mathcal{T} = TM \oplus \bigwedge^2 TM$$

as the *adjoint tractor bundle*, and we extend the tractor connection from $\mathcal{T}$ to $\mathcal{A}$ by the Leibniz rule. Similarly, we obtain an induced tractor connection on

$$\mathcal{A}^* = \bigwedge^2 T^*M \oplus T^*M.$$

Explicitly, these connections are given by the formulas

$$\nabla_a \begin{pmatrix} \varphi^b \\ \psi^{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a \varphi^b - 2\psi_a{}^b \\ \nabla_a \psi^{bc} + \frac{2}{n} \mathrm{J} \delta_a^{[b} \varphi^{c]} \end{pmatrix} \tag{3.4}$$

and

$$\nabla_a \begin{pmatrix} \mu_{bc} \\ \omega_b \end{pmatrix} = \begin{pmatrix} \nabla_a \mu_{bc} + 2g_{a[b}\omega_{c]} \\ \nabla_a \omega_b - \frac{4}{n} \mathrm{J} \mu_{ab} \end{pmatrix},$$

where $(\varphi^b, \psi^{bc}) \in \Gamma(\mathcal{A})$, i.e., $\varphi^b \in \mathcal{E}^b$ and $\psi^{bc} \in \mathcal{E}^{[bc]}$, and $(\mu_{bc}, \omega_b) \in \Gamma(\mathcal{A}^*)$, i.e., $\mu_{bc} \in \mathcal{E}_{[bc]}$ and $\omega_b \in \mathcal{E}_b$.

We extend the tractor connection to the tensor product bundle

$$\Big(\bigotimes \mathcal{T}\Big) \otimes \Big(\bigotimes \mathcal{T}^*\Big)$$

by the Leibniz rule. The resulting connection is again flat, denoted by ∇ and called the tractor connection. Further, the tractor and Levi-Civita connections induce the connection

$$\nabla_a : \mathcal{E}_{b\cdots d} \otimes \Gamma(W) \longrightarrow \mathcal{E}_{ab\cdots d} \otimes \Gamma(W) \tag{3.5}$$

for any tractor subbundle

$$W \subseteq \Big(\bigotimes \mathcal{T}\Big) \otimes \Big(\bigotimes \mathcal{T}^*\Big),$$

i.e., any subbundle preserved by the tractor connection. This coupled Levi-Civita tractor connection allows for extending all natural operators, e.g., the Laplace operator Δ, to tensor-tractor bundles.

The invariant pairing on $\mathcal{A}$ induced by equation (3.3) is given by the formula

$$\left\langle \begin{pmatrix} \varphi^b \\ \psi^{bc} \end{pmatrix}, \begin{pmatrix} \overline{\varphi}^b \\ \overline{\psi}^{bc} \end{pmatrix} \right\rangle = \frac{1}{n} \mathrm{J} \varphi^a \overline{\varphi}_a + \psi^{ab} \overline{\psi}_{ab}. \tag{3.6}$$

For $\mathrm{J} \neq 0$, this defines a metric on $\mathcal{A}$ and $\mathcal{A} \cong \mathcal{A}^*$. Moreover, there is a Lie algebra structure

$$[\cdot, \cdot] : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A},$$

given by

$$\left[\begin{pmatrix} \varphi^b \\ \psi^{bc} \end{pmatrix}, \begin{pmatrix} \overline{\varphi}^b \\ \overline{\psi}^{bc} \end{pmatrix} \right] = \begin{pmatrix} \varphi_r \overline{\psi}^{ra} - \overline{\varphi}_r \psi^{ra} \\ -2\psi^{r[b} \overline{\psi}_r{}^{c]} - \frac{1}{n} \mathrm{J} \varphi^{[b} \overline{\varphi}^{c]} \end{pmatrix}, \tag{3.7}$$

which is also invariant under the tractor connection.

Remark 3.1. Tractor connections can be defined on any Riemannian manifold. For example, we can define ∇ on $\mathcal{A}$ by

$$\nabla_a \begin{pmatrix} \varphi^b \\ \psi^{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a \varphi^b - 2\psi_a{}^b \\ \nabla_a \psi^{bc} + \frac{1}{2} R^{bc}{}_{as} \varphi^s \end{pmatrix}$$

for $\varphi^b \in \mathcal{E}^b$ and $\psi^{bc} \in \mathcal{E}^{[bc]}$. This definition originates in the work of Kostant [**14**]. We observe that, for a Killing vector field $k^a \in \mathcal{E}^a$, its prolongation

$$K = \begin{pmatrix} k^a \\ \frac{1}{2} \nabla^{[a} k^{b]} \end{pmatrix} \in \Gamma(\mathcal{A}) \tag{3.8}$$

is parallel for the tractor connection. Hence, any isometry is locally determined by its first jet.

We shall use abstract index notation for the adjoint tractor bundle as follows: $\Gamma(\mathcal{T})$ will be denoted by $\mathcal{E}^A$ and $\Gamma(\mathcal{A})$ will be denoted by $\mathcal{E}^{\mathbf{A}}$ where $\mathbf{A} = [A^1 A^2]$. Similarly, $\mathcal{E}_A = \Gamma(\mathcal{A}^*)$ and $\mathcal{E}_{\mathbf{A}} = \Gamma(\mathcal{A}^*)$, that is, we use boldface capital indices for an adjoint tractor bundle and its dual.

There is a convenient way to treat the bundles $\mathcal{T}$ and $\mathcal{T}^*$, based on the so-called *injectors*, or tensor-tractor frame, denoted by Y^A, Z_a^A

for $\mathcal{T}$ and denoted by Y_A, Z^a_A for $\mathcal{T}^*$. These are defined by

$$\begin{pmatrix} f \\ \mu^b \end{pmatrix} = Y^A f + Z^A_b \mu^b , \quad \begin{pmatrix} \nu_b \\ f \end{pmatrix} = Z^b_A \nu_b + Y_A f, \tag{3.9}$$

and their contractions are

$$Y^A Y_A = 1, \quad Z^A_a Z^b_A = \delta^{\,b}_{\,a} \quad \text{and} \quad Y^A Z^b_A = Z^A_a Y_A = 0.$$

The covariant derivatives in equations (3.1) and (3.2) are then encoded in covariant derivatives of these injectors:

$$\begin{aligned} \nabla_c Y^A &= \frac{2}{n} \mathrm{J} Z^A_a \delta^{\,a}_{\,c}, & \nabla_c Z^A_a &= -Y^A g_{ac}, \\ \nabla_c Z^a_A &= -\frac{2}{n} \mathrm{J} Y_A \delta^{\,a}_{\,c}, & \nabla_c Y_A &= Z^a_A g_{ca}. \end{aligned} \tag{3.10}$$

We denote tractor pairing equation (3.3) by $h_{AB} \in \mathcal{E}_{(AB)}$, which has the explicit form:

$$h_{AB} = \frac{2}{n} \mathrm{J}\, Y_A Y_B + Z^a_A Z^b_B g_{ab}. \tag{3.11}$$

Injectors for the adjoint tractor bundle $\mathcal{E}^{\mathbf{A}}$ are

$$\mathbb{Y}^{\mathbf{A}}_a = Y^{[A^1} Z^{A^2]}_a \quad \text{and} \quad \mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} = Z^{[A^1}_{a^1} Z^{A^2]}_{a^2};$$

injectors for the dual bundle $\mathcal{E}_{\mathbf{A}}$ are

$$\mathbb{Y}^a_{\mathbf{A}} = Y_{[A^1} Z^a_{A^2]} \quad \text{and} \quad \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} = Z^{a^1}_{[A^1} Z^{a^2}_{A^2]},$$

that is,

$$\begin{pmatrix} \varphi^b \\ \psi^{\mathbf{a}} \end{pmatrix} = \varphi^b \mathbb{Y}^{\mathbf{A}}_b + \psi^{\mathbf{a}} \mathbb{Z}^{\mathbf{A}}_{\mathbf{a}}, \qquad \begin{pmatrix} \mu_{\mathbf{a}} \\ \omega_b \end{pmatrix} = \mu_{\mathbf{a}} \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} + \omega_b \mathbb{Y}^{\,b}_{\mathbf{A}}, \tag{3.12}$$

where $\mathbf{a} = [a^1 a^2]$. The only nonzero contractions are

$$\mathbb{Y}^{\mathbf{A}}_a \mathbb{Y}^b_{\mathbf{A}} = \frac{1}{2} \delta^{\,b}_{\,a} \quad \text{and} \quad \mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} \mathbb{Z}^{\mathbf{c}}_{\mathbf{A}} = \delta^{\,c^1}_{\,[a^1} \delta^{\,c^2}_{\,a^2]}.$$

The covariant derivatives (3.4) are then equivalent to

$$\begin{aligned} \nabla_c \mathbb{Y}^{\mathbf{A}}_b &= \frac{2\mathrm{J}}{n} \mathbb{Z}^{\mathbf{A}}_{ab} \delta^{\,a}_{\,c}, & \nabla_c \mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} &= -2 \mathbb{Y}^{\mathbf{A}}_{[a^2} g_{a^1]c}, \\ \nabla_c \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} &= -\frac{4\mathrm{J}}{n} \mathbb{Y}^{[a^2}_{\mathbf{A}} \delta^{\,a^1]}_{\,c}, & \nabla_c \mathbb{Y}^b_{\mathbf{A}} &= \mathbb{Z}^{ab}_{\mathbf{A}} g_{ca}, \end{aligned} \tag{3.13}$$

and the pairing (3.6) on $\mathcal{E}^{\mathbf{A}}$ can be written:

$$h_{\mathbf{AB}} = \frac{4}{n} \mathrm{J}\, \mathbb{Y}^a_{\mathbf{A}} \mathbb{Y}^b_{\mathbf{B}} g_{ab} + \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} \mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} g_{a^1 b^1} g_{a^2 b^2} \,. \tag{3.14}$$

A crucial component of our construction is the differential operator

$$\mathbb{D}_{\mathbf{A}} : \mathcal{E}_{b_1 \cdots b_s} \otimes \Gamma(W) \longrightarrow \mathcal{E}_{b_1 \cdots b_s} \otimes \Gamma\Big(\mathcal{A}^* \otimes W\Big) \tag{3.15}$$

for a tractor subbundle

$$W \subseteq \Big(\bigotimes \mathcal{T}\Big) \otimes \Big(\bigotimes \mathcal{T}^*\Big).$$

This operator is closely related to the so-called fundamental derivative [**5**]. It is defined as follows: for $f \in \Gamma(W)$, we set

$$\mathbb{D}_{\mathbf{A}} f = \begin{pmatrix} 0 \\ 2\nabla_a f \end{pmatrix} \in \Gamma(\mathcal{A}^* \otimes W), \tag{3.16}$$

and, for $\varphi_b \in \mathcal{E}_b$, we set

$$\mathbb{D}_{\mathbf{A}} \varphi_b = \begin{pmatrix} 2 g_{b[a^1} \varphi_{a^2]} \\ 2\nabla_a \varphi_b \end{pmatrix} \in \mathcal{E}_b \otimes \Gamma(\mathcal{A}^*). \tag{3.17}$$

Then, we extend $\mathbb{D}_{\mathbf{A}}$ to all tensor-tractor bundles by the Leibniz rule. Using injectors (3.12), formulas (3.16) and (3.17) are given by

$$\begin{aligned} \mathbb{D}_{\mathbf{A}} f &= 2\mathbb{Y}^a_{\mathbf{A}} \nabla_a f, \\ \mathbb{D}_{\mathbf{A}} \varphi_b &= 2\mathbb{Y}^a_{\mathbf{A}} \nabla_a \varphi_b + \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} 2 g_{b[a^0} \varphi_{a^1]} \end{aligned} \tag{3.18}$$

where $\mathbf{a} = [a^1 a^2]$.

Theorem 3.2. *Let M be a manifold of constant curvature. The operator $\mathbb{D}_{\mathbf{A}}$ commutes with the coupled Levi-Civita tractor connection ∇_c,*

$$\nabla_c \mathbb{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}} \nabla_c : \mathcal{E}_{b_1 \cdots b_s} \otimes \Gamma(W) \longrightarrow \mathcal{E}_{c b_1 \cdots b_s \mathbf{A}} \otimes \Gamma(W).$$

Proof. Since the tractor connection is flat, it is sufficient to prove the statement for W equal to the trivial line bundle. We present two versions of the proof.

First, one can easily show by direct computation using equations (3.13) and (3.18) that the explicit formulas for the compositions $\nabla_c \mathbb{D}_{\mathbf{A}}$ and $\mathbb{D}_{\mathbf{A}} \nabla_c$ are the same when acting on $f \in \mathcal{E}$ and $\varphi_b \in \mathcal{E}_b$. Hence, the formulas agree on any tensor bundle.

Alternatively, recall that, for a Killing vector field $k^a \in \mathcal{E}^a$, its prolongation $K^{\mathbf{A}} \in \Gamma(\mathcal{A})$ is parallel, see equation (3.8). We further observe that $L_k = K^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$ is the Lie derivative along k^a when acting on tensor bundles. Since L_k commutes with the covariant derivative, and the space of Killing vector fields on manifolds with constant curvature has dimension equal to

$$\dim(\mathcal{A}) = n + \frac{1}{2}n(n-1),$$

the statement follows. □

As a consequence of Theorem 3.2, $\mathbb{D}_{\mathbf{A}}$ also commutes with Laplace operator Δ on functions, forms, etc. We will now make this result more general and precise. Assume that

$$F : \Gamma(U_1) \longrightarrow \Gamma(U_2)$$

is a Riemannian invariant linear differential operator, acting between tensor bundles U_1 and U_2. It can be written in terms of the metric, the Levi-Civita connection ∇ and the curvature J. Replacing ∇ in the formula for F by the coupled Levi-Civita tractor connection, we obtain the operator

$$F^{\nabla} : \Gamma(U_1 \otimes W) \longrightarrow \Gamma(U_2 \otimes W)$$

for any tractor subbundle

$$W \subseteq \left(\bigotimes \mathcal{T}\right) \otimes \left(\bigotimes \mathcal{T}^*\right).$$

Note that, since the tractor connection is flat, the curvature of the coupled Levi-Civita tractor connection agrees with the curvature of the Levi-Civita connection. Using Theorem 3.2 and $\nabla_a \mathrm{J} = \nabla_a g = 0$, we obtain the following.

Corollary 3.3. *Let*

$$F : \Gamma(U_1) \longrightarrow \Gamma(U_2)$$

be a Riemannian invariant linear differential operator on the manifold M. Then, $\mathbb{D}_{\mathbf{A}}$ commutes with F^{∇}, i.e.,

$$\mathbb{D}_{\mathbf{A}} \circ F^{\nabla} = F^{\nabla} \circ \mathbb{D}_{\mathbf{A}} : \Gamma(U_1 \otimes W) \longrightarrow \Gamma(U_2 \otimes \mathcal{A}^* \otimes W).$$

4. Commuting symmetries of the Laplace operator.

Definition 4.1. Let U be a tensor bundle, and let

$$F : \Gamma(U) \longrightarrow \Gamma(U)$$

be a linear differential operator on M. A *commuting symmetry* of operator F is a linear differential operator $\mathcal{D}$ fulfilling $\mathcal{D}F = F\mathcal{D}$.

We are interested in commuting symmetries of the Laplace operator $F = \Delta$ on functions. They form a subalgebra of the associative algebra of linear differential operators acting on $\mathcal{E}$. The exposition in the rest of this section closely follows that given in [**8**] for conformal symmetries. For a comparison of both types of symmetries, see Remark 4.16.

Let ℓ be a non-negative integer. A linear ℓth order differential operator acting on functions can be written:

$$\mathcal{D} = V^{a_1\cdots a_\ell}\nabla_{a_1}\cdots\nabla_{a_\ell} + \text{LOTS}, \tag{4.1}$$

where LOTS stands for *lower order terms* in $\mathcal{D}$, and its principal symbol $V^{a_1\cdots a_\ell}$ is symmetric in its indices $V^{a_1\cdots a_\ell} = V^{(a_1\cdots a_\ell)}$.

Definition 4.2. A Killing tensor on M is a symmetric tensor field $V^{a_1\cdots a_\ell}$, fulfilling the first order differential equation

$$\nabla^{(a_0}V^{a_1\cdots a_\ell)} = 0. \tag{4.2}$$

The vector space of all Killing tensors of valence ℓ will be denoted by $\mathcal{K}_\ell$.

Since differential equation (4.2) is overdetermined, the space $\mathcal{K}_\ell$ is finite-dimensional. Note that the symmetric product of two Killing tensors is again a Killing tensor, such that $\bigoplus_{\ell\leq 0}\mathcal{K}_\ell$ is a commutative graded algebra.

Theorem 4.3. *Let $\mathcal{D}$ be an ℓ*th *order commuting symmetry of the Laplace operator. Then, the principal symbol $V^{a_1\cdots a_\ell}$ of $\mathcal{D}$ is a Killing tensor of valence ℓ.*

Proof. When $\mathcal{D}$ is of the form (4.1), we compute

$$\Delta\mathcal{D} - \mathcal{D}\Delta = 2(\nabla^b V^{a_1\cdots a_\ell})\nabla_b\nabla_{a_1}\cdots\nabla_{a_\ell} + \text{LOTS}, \tag{4.3}$$

and the claim follows. □

The converse of this statement is covered in the next theorem.

Theorem 4.4. *There exists a linear map*:

$$V^{a_1\cdots a_\ell} \longmapsto \mathcal{D}^V,$$

from symmetric tensor fields to differential operators, such that the principal symbol of $\mathcal{D}^V$ is $V^{a_1\cdots a_\ell}$ and $\mathcal{D}^V\Delta = \Delta\mathcal{D}^V$ if V is a Killing tensor.

The proof of Theorem 4.4 is postponed to the next section, where the Riemannian prolongation connection for symmetric powers of the adjoint tractor bundle is constructed. This allows explicit computation of the symmetry operators $\mathcal{D}^V$.

Combining both theorems, we deduce a linear bijection between the space of Killing tensors and the space of commuting symmetries of Δ. More explicitly, the space of 0th order commuting symmetries is the space of constants, the space of first order commuting symmetries contains in addition the Killing vector fields, and by induction, the space of ℓth order commuting symmetries contains the space of $(\ell-1)$th order commuting symmetries together with a copy of the space $\mathcal{K}_\ell$ of Killing tensors of valence ℓ. In particular, the dimension of the vector space of ℓth order symmetry operators is finite and equal to

$$\dim\mathcal{K}_0 + \dim\mathcal{K}_1 + \cdots + \dim\mathcal{K}_\ell.$$

4.1. Prolongation for Killing tensors. Let $k^a \in \mathcal{E}^a$ be a Killing vector field. In Remark 3.1, we observed that its prolongation

$$K = \left(k^a, \frac{1}{2}\nabla^{[a}k^{b]}\right) \in \Gamma(\mathcal{A})$$

is parallel for the tractor connection. Our aim is to construct analogous prolongation for Killing tensors.

Lemma 4.5. *If $k^{a_1\cdots a_\ell}\in\mathcal{E}^{(a_1\cdots a_\ell)}$ is a Killing tensor, then*

$$\nabla^{(a_1}\nabla^{|[c}k^{d]|a_2\cdots a_\ell)}=-\frac{2(\ell+1)}{n}\mathrm{J}\,g^{(a_1|[c}k^{d]|a_2\cdots a_\ell)},\tag{4.4}$$

where the notation $|\cdots|$ means that the enclosed indices c, d are excluded from the symmetrization.

Proof. Straightforward computation. □

The prolongation of $k^{a_1\cdots a_\ell}$ is a section $K\in\Gamma(\otimes^\ell\mathcal{A})$. First, we observe the following:

Lemma 4.6. *The differential operator*

$$\begin{aligned}&\Pi:\mathcal{E}^{(a_1\cdots a_\ell)}\longrightarrow\mathcal{E}^{(a_1\cdots a_{\ell-1})}\otimes\Gamma(\mathcal{A}),\\ &(\Pi\sigma)^{a_1\cdots a_{\ell-1}\mathbf{B}}=\begin{pmatrix}\sigma^{ca_1\cdots a_{\ell-1}}\\ \frac{1}{\ell+1}\nabla^{[c}\sigma^{d]a_1\cdots a_{\ell-1}}\end{pmatrix}\in\mathcal{E}^{(a_1\cdots a_{\ell-1})}\otimes\Gamma(\mathcal{A})\end{aligned}\tag{4.5}$$

satisfies, for all $\sigma^{a_1\cdots a_\ell}\in\mathcal{E}^{(a_1\cdots a_\ell)}$,

$$\nabla^{(a_0}\sigma^{a_1\cdots a_\ell)}=0\Longleftrightarrow\nabla^{(a_0}(\Pi\sigma)^{a_1\cdots a_{\ell-1})\mathbf{B}}=0.\tag{4.6}$$

Proof. Using equation (3.4), we compute

$$\nabla^b(\Pi\sigma)^{a_1\cdots a_{\ell-1}\mathbf{B}}=\begin{pmatrix}\nabla^b\sigma^{ca_1\cdots a_{\ell-1}}-\frac{2}{\ell+1}\nabla^{[b}\sigma^{c]a_1\cdots a_{\ell-1}}\\ \frac{1}{\ell+1}\nabla^b\nabla^{[c}\sigma^{d]a_1\cdots a_{\ell-1}}+\frac{2}{n}\mathrm{J}\,g^{b[c}\sigma^{d]a_1\cdots a_{\ell-1}}\end{pmatrix}.$$

Observe the "top slot" on the right side is equal to

$$\frac{\ell}{\ell+1}\nabla^b\sigma^{ca_1\cdots a_{\ell-1}}+\frac{1}{\ell+1}\nabla^c\sigma^{ba_1\cdots a_{\ell-1}},$$

which, after symmetrization $(ba_1\cdots a_{\ell-1})$, yields exactly $\nabla^{(c}\sigma^{ba_1\cdots a_{\ell-1})}$. This proves implication $\Leftarrow$ of equation (4.6), and also that if

$$\nabla^{(a_0}\sigma^{a_1\cdots a_\ell)}=0,$$

then the "top slot" of $\nabla^{(a_0}(\Pi\sigma)^{a_1\cdots a_{\ell-1})}$ vanishes. Since the bottom slot vanishes by Lemma 4.5, implication $\Rightarrow$ in equation (4.6) follows as well. □

Considering ∇ in formula (4.5) as the coupled Levi-Civita tractor connection, we obtain the operator

$$\Pi : \mathcal{E}^{(a_1\cdots a_\ell)} \otimes \Gamma(W) \longrightarrow \mathcal{E}^{(a_1\cdots a_{\ell-1})} \otimes \Gamma(\mathcal{A} \otimes W),$$

where

$$W \subseteq \left(\bigotimes \mathcal{T}\right) \otimes \left(\bigotimes \mathcal{T}^*\right)$$

is a tractor subbundle. Its iteration

$$\Pi^{(\ell)} : \mathcal{E}^{(a_1\cdots a_\ell)} \longrightarrow \Gamma(\otimes^\ell \mathcal{A}), \tag{4.7}$$

yields the prolongation for Killing tensors.

Proposition 4.7. *Let $\sigma^{a_1\cdots a_\ell} \in \mathcal{E}^{(a_1\cdots a_\ell)}$. The operator $\Pi^{(\ell)}$ satisfies*

$$\nabla^{(a_0}\sigma^{a_1\cdots a_\ell)} = 0 \Longleftrightarrow \nabla(\Pi^{(\ell)}\sigma) = 0. \tag{4.8}$$

Proof. Since the tractor connection is flat, we have the analogue of equation (4.6):

$$\nabla^{(a_0}\sigma^{a_1\cdots a_\ell)\bullet} = 0 \Longleftrightarrow \nabla^{(a_0}(\Pi\sigma)^{a_1\cdots a_{\ell-1})\bullet} = 0,$$

for all

$$\sigma^{a_1\cdots a_\ell\bullet} \in \mathcal{E}^{(a_1\cdots a_\ell)} \otimes \Gamma(W),$$

where $\bullet$ denotes an unspecified tractor index. By iteration, we obtain equation (4.8). $\square$

The symmetries of tractor $\Pi^{(\ell)}\sigma$ are best understood in the language of Young diagrams. Setting $\mathcal{T} = \square$, we have $\mathcal{A} = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}$ and we set

$$\boxtimes^\ell \mathcal{A} := \overbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \cdots & \; \\ \hline \; & \; & \cdots & \; \\ \hline \end{array}}^{\ell} \subseteq S^\ell \mathcal{A}, \tag{4.9}$$

where $S^\ell \mathcal{A} \subset \bigotimes^\ell \mathcal{A}$ is the subspace of symmetric tensors. For instance, we have

$$S^2\mathcal{A} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array} \bigoplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array},$$

or in other words,

$$\frac{1}{2}\Big(V \otimes W + W \otimes V\Big) = V \boxtimes W + V \wedge W, \tag{4.10}$$

for any $V, W \in \mathcal{A}$, with $\wedge$ the wedge product in $\bigwedge \mathcal{T}$.

Proposition 4.8. *The map* $\Pi^{(\ell)}$*, defined in equation* (4.7)*, is valued in* $\Gamma(\boxtimes^\ell \mathcal{A})$*.*

Proof. Let $\sigma^{a_1\cdots a_\ell} \in \mathcal{E}^{(a_1\cdots a_\ell)}$. Using abstract indices $\mathbf{B}_i = [B_i^1 B_i^2]$, we have

$$(\Pi^{(\ell)}\sigma)^{B_1^1 B_1^2 \cdots B_\ell^1 B_\ell^2} \in \mathcal{E}^{\mathbf{B}_1\cdots\mathbf{B}_\ell} = \Gamma(\otimes^\ell \mathcal{A}).$$

First, we prove that $(\Pi^{(\ell)}\sigma)^{\mathbf{B}_1\cdots\mathbf{B}_\ell}$ is symmetric in indices $\mathbf{B}_1, \ldots, \mathbf{B}_\ell$, i.e, $\Pi^{(\ell)}\sigma \in \Gamma(S^\ell\mathcal{A})$. In fact, it is sufficient to show the symmetry in two neighboring indices $\mathbf{B}_i$ and $\mathbf{B}_{i+1}$. To do this, we show that, for all $\sigma^{a_1\cdots a_\ell \bullet} \in \mathcal{E}^{(a_1\cdots a_\ell)\bullet}$ with $\ell \geq 2$, $(\Pi^{(2)}\sigma)^{a_1\cdots a_{\ell-2}\mathbf{B}_1\mathbf{B}_2\bullet}$ is symmetric in $\mathbf{B}_1$ and $\mathbf{B}_2$. This follows from the explicit formula

$$\begin{aligned}(\Pi^{(2)}\sigma)^{a_1\cdots a_{\ell-2}\mathbf{B}\mathbf{C}\bullet} &= \mathbb{Y}_b^{\mathbf{B}}\mathbb{Y}_c^{\mathbf{C}}\sigma^{a_1\cdots a_{\ell-2}bc\bullet} \\ &+ \frac{1}{\ell+1}\Big[\mathbb{Y}_b^{\mathbf{B}}\mathbb{Z}_{\mathbf{c}}^{\mathbf{C}}\nabla^{c^1}\sigma^{a_1\cdots a_{\ell-2}bc^2\bullet} + \mathbb{Z}_{\mathbf{b}}^{\mathbf{B}}\mathbb{Y}_c^{\mathbf{C}}\nabla^{b^1}\sigma^{a_1\cdots a_{\ell-2}b^2c\bullet}\Big] \\ &+ \frac{1}{\ell}\mathbb{Z}_{\mathbf{b}}^{\mathbf{B}}\mathbb{Z}_{\mathbf{c}}^{\mathbf{C}}\Big[\frac{1}{\ell+1}\nabla^{b^1}\nabla^{c^1}\sigma^{a_1\cdots a_{\ell-2}b^2c^2\bullet} + \frac{2}{n}\mathrm{J}g^{b^1c^1}\sigma^{a_1\cdots a_{\ell-2}b^2c^2\bullet}\Big],\end{aligned} \tag{4.11}$$

obtained from equation (4.5) after short computation. Here,

$$\mathbf{B} = [B^1B^2], \qquad \mathbf{C} = [C^1C^2],$$
$$\mathbf{b} = [b^1b^2] \quad \text{and} \quad \mathbf{c} = [c^1c^2].$$

It suffices to prove that

$$(\Pi^{(\ell)}\sigma)^{B_1^1 B_1^2 \cdots B_\ell^1 B_\ell^2} \in \mathcal{E}^{\mathbf{B}_1\cdots\mathbf{B}_\ell}$$

vanishes after skew-symmetrization over any triple of indices B_i^j. Since $(\Pi^{(\ell)}\sigma)^{\mathbf{B}_1\cdots\mathbf{B}_\ell}$ is symmetric in tractor form indices $\mathbf{B}_i$, it is sufficient to consider only two triples of indices: either B_1^1, B_1^2, B_2^1 or B_1^1, B_2^1, B_3^1. Elementary representation theory shows that the third symmetric power of $\mathcal{A}$ has the decomposition

$$S^3\left(\ydiagram{1,1}\right) = \ydiagram{1,1,1,1,1,1} \oplus 2\,\ydiagram{2,2,1,1} \oplus \ydiagram{3,3}. \tag{4.12}$$

Hence, if we skew over three factors of the standard tractor bundle in $S^3\mathcal{E}^{\mathbf{A}}$, the result will in fact be skew symmetric in at least four factors of the standard tractor bundle. As a result, it is sufficient to consider only skew symmetrization over indices B_1^1, B_1^2, B_2^1 of

$$(\Pi^{(\ell)}\sigma)^{B_1^1B_1^2\cdots B_\ell^1B_\ell^2}.$$

Using equation (4.11) with $\ell = 2$, straightforward computation shows that

$$(\Pi^{(\ell)}\sigma)^{[B_1^1B_1^2B_2^1]B_2^2\cdots B_\ell^1B_\ell^2} = 0. \qquad \square$$

The proposition can also be proved using invariant techniques in parabolic geometries, known as "BGG machinery," applied to the case of projective parabolic geometry. We refer to the next section for further discussion on this relation.

Next, we obtain the main result of this section.

Theorem 4.9. *The map $\Pi^{(\ell)}$ induces a bijective correspondence between the space $\mathcal{K}_\ell$ of Killing ℓ-tensors and the space of parallel sections of the tractor bundle $\boxtimes^\ell\mathcal{A}$.*

Proof. If σ_ℓ is a Killing ℓ-tensor, then $\Pi^{(\ell)}\sigma_\ell$ is a parallel section of $\boxtimes^\ell\mathcal{A}$, by Propositions 4.7 and 4.8. It remains to prove that, if F is a non-vanishing parallel section of $\boxtimes^\ell\mathcal{A}$, then $F = \Pi^{(\ell)}\sigma_\ell$ for some Killing ℓ-tensor σ_ℓ.

As a section of $\Gamma(S^\ell\mathcal{A})$, F has the form

$$F^{\mathbf{A}_1\cdots\mathbf{A}_\ell} = \sum_{i=0}^{\ell} \mathbb{Y}^{(\mathbf{A}_1}_{a_1}\cdots\mathbb{Y}^{\mathbf{A}_i}_{a_i}\mathbb{Z}^{\mathbf{A}_{i+1}}_{\mathbf{c}_{i+1}}\cdots\mathbb{Z}^{\mathbf{A}_\ell)}_{\mathbf{c}_\ell}(\sigma_i)^{a_1\cdots a_i\mathbf{c}_{i+1}\cdots\mathbf{c}_\ell}, \tag{4.13}$$

where $(\mathbf{A}_1\cdots\mathbf{A}_\ell)$ denotes the symmetrization over the form tractor indices, and *not* over the standard tractor indices. Here,

$$(\sigma_i)^{a_1\cdots a_i\mathbf{c}_{i+1}\cdots\mathbf{c}_\ell} \in \mathcal{E}^{a_1\cdots a_i\mathbf{c}_{i+1}\cdots\mathbf{c}_\ell},$$

where a_i are indices of the tangent bundle whereas $\mathbf{c}_i = [c_i^1 c_i^2]$ are form indices. Since $F \in \Gamma(\boxtimes^\ell \mathcal{A})$, the skew symmetrization over any triple of indices of

$$(\sigma_i)^{a_1 \cdots a_i [c_{i+1}^1 c_{i+1}^2] \cdots [c_\ell^1 c_\ell^2]}$$

vanishes.

First, we show that $\sigma_\ell = 0$ implies $F = 0$. To do this, we assume that

$$\sigma_{i_0+1} = \cdots = \sigma_\ell = 0,$$

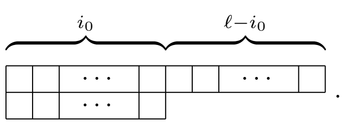
and prove that $\sigma_{i_0} = 0$, with $0 \leq i_0 < \ell$. The tractor form $\nabla^b F^{\mathbf{A}_1 \cdots \mathbf{A}_\ell}$ can be written as in equation (4.13), and it follows from equation (3.13) that

$$\begin{aligned}\nabla^b F^{\mathbf{A}_1 \cdots \mathbf{A}_\ell} = 2(\ell - i_0) \mathbb{Y}_{a_1}^{(\mathbf{A}_1} \cdots \mathbb{Y}_{a_{i_0+1}}^{\mathbf{A}_{i_0+1}} \mathbb{Z}_{\mathbf{c}_{i_0+2}}^{\mathbf{A}_{i_0+2}} \cdots \mathbb{Z}_{\mathbf{c}_\ell}^{\mathbf{A}_\ell)} \\ \cdot (\sigma_{i_0})^{(a_1 \cdots a_{i_0} a_{i_0+1}) b \mathbf{c}_{i_0+2} \cdots \mathbf{c}_\ell} \\ + \text{ terms with at most } i_0 \text{ of } \mathbb{Y}\text{'s.}\end{aligned}$$

Thus,

$$(\sigma_{i_0})^{(a_1 \cdots a_{i_0} a_{i_0+1}) b \mathbf{c}_{i_0+2} \cdots \mathbf{c}_\ell} = 0.$$

On the other hand, symmetries of F imply that symmetries of $(\sigma_{i_0})^{a_1 \cdots a_{i_0} \mathbf{c}_{i_0+1} \cdots \mathbf{c}_\ell}$ correspond to the Young diagram:



Hence,

$$(\sigma_{i_0})^{(a_1 \cdots a_{i_0} a_{i_0+1}) b \mathbf{c}_{i_0+2} \cdots \mathbf{c}_\ell} = 0$$

means

$$(\sigma_{i_0})^{a_1 \cdots a_{i_0} \mathbf{c}_{i_0+1} \cdots \mathbf{c}_\ell} = 0,$$

as intended.

Next, we show that the tensor field $(\sigma_\ell)^{a_1 \cdots a_\ell}$ is Killing. Similarly as above, computing the $\mathbb{Y}_{a_1}^{(\mathbf{A}_1} \cdots \mathbb{Y}_{a_\ell}^{\mathbf{A}_\ell)}$-summand of $\nabla^b F^{\mathbf{A}_1 \cdots \mathbf{A}_\ell}$ (which is 0), one easily concludes that $\nabla^{(b} (\sigma_\ell)^{a_1 \cdots a_\ell)} = 0$. Details are left to the reader. Finally, since the difference $F - \Pi^{(\ell)} \sigma_\ell \in \Gamma(\boxtimes^\ell \mathcal{A})$ is parallel and the $\mathbb{Y}^{(\mathbf{A}_1}_{a_1} \cdots \mathbb{Y}^{\mathbf{A}_\ell)}_{a_\ell}$-summand of $F - \Pi^{(\ell)}\sigma_\ell$ vanishes, it follows from the first part of the proof that $F - \Pi^{(\ell)}\sigma_\ell = 0$. □

4.2. Construction of commuting symmetries. Let

$$V^{a_1\cdots a_\ell} \in \mathcal{E}^{(a_1\cdots a_\ell)}$$

be a symmetric tensor and let $\Pi^{(\ell)}$ be the map defined in equation (4.7). We define the differential operator $\mathcal{D}^V$ of order ℓ by

$$\mathcal{D}^V := \langle \Pi^{(\ell)} V, \mathbb{D}^{(\ell)} \rangle : \Gamma(U) \longrightarrow \Gamma(U), \tag{4.14}$$

where

$$\mathbb{D}^{(\ell)} : \Gamma(U) \longrightarrow \bigotimes^{\ell} \Gamma(\mathcal{A}^*) \otimes \Gamma(U)$$

is the ℓth iteration of operator (3.15).

Lemma 4.10. *Differential operator $\mathcal{D}^V$ has principal symbol $V^{a_1\cdots a_\ell}$.*

Proof. Extending vertical notation for elements in

$$\Gamma(\mathcal{A}) = \begin{matrix} \mathcal{E}^a \\ \bigoplus \\ \mathcal{E}^{[ab]} \end{matrix} \quad \text{and} \quad \Gamma(\mathcal{A}^*) = \begin{matrix} \mathcal{E}_{[ab]} \\ \bigoplus \\ \mathcal{E}_a \end{matrix}$$

to sections in the tensor products $S^\ell \mathcal{A}$ and $S^\ell \mathcal{A}^*$, we obtain

$$\Pi^{(\ell)} V = \begin{matrix} V^{a_1\dots a_\ell} \\ \bigoplus \\ \vdots \end{matrix} \in \begin{matrix} \mathcal{E}^{a_1\dots a_\ell} \\ \bigoplus \\ \vdots \end{matrix}$$

and

$$\mathbb{D}^{(\ell)} u = \begin{matrix} \vdots \\ \bigoplus \\ \nabla_{a_1} \dots \nabla_{a_\ell} u \end{matrix} \in \begin{matrix} \vdots \\ \bigoplus \\ \mathcal{E}_{a_1\dots a_\ell} \bigotimes \Gamma(U). \end{matrix}$$

Thus, the contraction $\langle \Pi^{(\ell)} V, \mathbb{D}^{(\ell)} u \rangle$ has the leading term

$$V^{a_1\cdots a_\ell} \nabla_{a_1} \cdots \nabla_{a_\ell} u.$$

□

We consider a Riemannian invariant linear differential operator

$$F : \Gamma(U) \longrightarrow \Gamma(U),$$

acting on a tensor bundle U.

Theorem 4.11. *Let $k^{a_1\cdots a_\ell} \in \mathcal{E}^{(a_1\cdots a_\ell)}$ be a Killing tensor. Then, the differential operator $\mathcal{D}^k$ is a commuting symmetry of F with principal symbol $k^{a_1\cdots a_\ell}$.*

Proof. By Lemma 4.10, $\mathcal{D}^k$ has principal symbol $k^{a_1\cdots a_\ell}$.

Let $u \in \Gamma(U)$, and let

$$K := \Pi^{(\ell)} k \in \Gamma\Big(\bigotimes^{\ell} \mathcal{A}\Big)$$

be the prolongation of the Killing ℓ-tensor k. Then, we obtain

$$F\,\mathcal{D}^k u = F\langle K, \mathbb{D}^{(\ell)} u\rangle = \langle K, F^{\nabla}\mathbb{D}^{(\ell)} u\rangle = \langle K, \mathbb{D}^{(\ell)} F u\rangle = \mathcal{D}^k F u,$$

where we have used Proposition 4.7 (which implies $\nabla K = 0$) in the second equality and Corollary 3.3 in the third equality. Recall that the operator F^{∇} is given by the same formula as F, but ∇ is interpreted as the coupled Levi-Civita tractor connection in F^{∇}. $\square$

Corollary 4.12. *Assume that $k^{a_1\cdots a_\ell} \in \mathcal{E}^{(a_1\cdots a_\ell)}$ is a Killing tensor. Then, $\mathcal{D}^k$ is a commuting symmetry of the Laplacian $\Delta : \mathcal{E} \to \mathcal{E}$ with principal symbol $k^{a_1\cdots a_\ell}$.*

4.3. Algebraic structure on the space of commuting symmetries of the Laplace operator $\Delta : \mathcal{E} \to \mathcal{E}$. Let $\mathcal{B}$ be the algebra of commuting symmetries of Δ. Theorems 4.3 and 4.4 allow us to identify the vector space of commuting symmetries of Δ,

$$\mathcal{B} \simeq \bigoplus_{\ell=0}^{\infty} \mathcal{K}_\ell. \tag{4.15}$$

In order to study the algebra structure on $\mathcal{B}$, some basic notation is needed. Depending on the curvature, the Lie group of isometries is $G = SO(p+1,q)$, $G = SO(p,q+1)$ or $G = E(p,q)$. For all possibilities, the Lie algebra of isometries is denoted by $\mathfrak{g} = \mathrm{Lie}(G)$, and the identifications

$$\mathfrak{so}(p+1,q) \simeq \wedge^2 \mathbb{R}^{p+1,q},$$
$$\mathfrak{so}(p,q+1) \simeq \wedge^2 \mathbb{R}^{p,q+1},$$

and

$$\mathrm{Lie}(E(p,q)) \simeq \wedge^2 (\mathbb{R}^{p,q} \oplus \mathbb{R})$$

are used. In the last case, the identification is deduced from the representation of $E(p,q)$ on $\mathbb{R}^{p,q} \oplus \mathbb{R}$, induced by the standard group morphism

$$\mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n \longrightarrow \mathrm{GL}(n+1,\mathbb{R}).$$

The space of parallel sections of $\mathcal{A}$ is isomorphic to $\mathfrak{g}$, and it is easy to verify that the Lie bracket on $\mathfrak{g}$ is isomorphic to bracket (3.7). Via the induced identification of the symmetric product $S^\ell \mathfrak{g}$ with parallel sections of $S^\ell \mathcal{A}$, we define the subspace $\boxtimes^\ell \mathfrak{g} \subseteq S^\ell \mathfrak{g}$ as follows:

$$\boxtimes^\ell \mathfrak{g} \cong \{\text{parallel sections of } \ \boxtimes^\ell \mathcal{A}\},$$

where $\boxtimes^\ell \mathcal{A}$ is defined in equation (4.9). According to equation (4.10), we have

$$V \boxtimes W = \frac{1}{2}\Big(V \otimes W + W \otimes V\Big) - V \wedge W,$$

for any $V, W \in \mathfrak{g}$. From Theorem 4.9, we deduce that $\boxtimes^\ell \mathfrak{g}$ is isomorphic, as a $\mathfrak{g}$-module, to the space $\mathcal{K}_\ell$ of Killing ℓ-tensors. Hence, we have the $\mathfrak{g}$-module isomorphism

$$\mathcal{B} \simeq \bigoplus_{\ell=0}^{\infty} \boxtimes^\ell \mathfrak{g}.$$

Theorem 4.13. *The symmetry algebra $\mathcal{B}$ is isomorphic to the tensor algebra*

$$\bigoplus_{i=0}^{\infty} \otimes^i \mathfrak{g} \tag{4.16}$$

modulo the two-sided ideal $\mathcal{I}$, generated by

$$V \otimes W - V \boxtimes W - \frac{1}{2}[V,W], \quad V, W \in \mathfrak{g}. \tag{4.17}$$

Proof. First we compute the compositions $\mathcal{D}^k\mathcal{D}^{\check{k}}$, where $k^a, \check{k}^a \in \Gamma(TM)$ are Killing vector fields. Set $K = \Pi k^a$ and $\check{K} = \Pi\check{k}^a$ where $K, \check{K} \in \mathfrak{g}$. Since $\nabla\check{K} = 0$, definition (3.18) of $\mathbb{D}_{\mathbf{B}}$ acting on functions yields

$$\begin{aligned}\mathcal{D}^k\mathcal{D}^{\check{k}} &= K^{\mathbf{B}}\mathbb{D}_{\mathbf{B}}\check{K}^{\mathbf{C}}\mathbb{D}_{\mathbf{C}} \\ &= \left[\frac{1}{2}(K^{\mathbf{B}}\check{K}^{\mathbf{C}} + K^{\mathbf{C}}\check{K}^{\mathbf{B}}) + \frac{1}{2}(K^{\mathbf{B}}\check{K}^{\mathbf{C}} - K^{\mathbf{C}}\check{K}^{\mathbf{B}})\right]\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{C}} \\ &= (K \boxtimes \check{K})^{\mathbf{BC}}\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{C}} + \frac{1}{2}[K, \check{K}]^{\mathbf{B}}\mathbb{D}_{\mathbf{B}}.\end{aligned}$$

In the last equality, to deal with the symmetrized term, we use decomposition (4.10) and the identity $\mathbb{D}_{[B^1B^2}\mathbb{D}_{C^1C^2]} = 0$, which can be easily verified. To deal with the skew-symmetrized term, we use equation (3.7).

The computation of $\mathcal{D}^k\mathcal{D}^{\check{k}}$ shows that all elements of the form (4.17) are in the ideal. Since there is a vector space isomorphism

$$\left(\bigoplus_{\ell=0}^{\infty} \otimes^\ell \mathfrak{g}\right)/\mathcal{I} \cong \bigoplus_{\ell=0}^{\infty} \boxtimes^\ell \mathfrak{g},$$

it remains to show that elements in $\boxtimes^\ell\mathfrak{g} \cong \mathcal{K}_\ell$ indeed give rise to nonzero ℓth order symmetries. This follows from Corollary 4.12. The proof is complete. $\square$

Passage from tensor algebra to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ means to substitute

$$V \otimes W = \frac{1}{2}\Big(V \otimes W + W \otimes V\Big) - \frac{1}{2}\Big(V \otimes W - W \otimes V\Big)$$

and quotient through the two-sided ideal generated by

$$V \otimes W - W \otimes V = [V, W], \quad V, W \in \mathfrak{g}.$$

Accordingly, we obtain the following.

Corollary 4.14. *The symmetry algebra $\mathcal{B}$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ modulo the two-sided ideal generated by $V \wedge W$ for $V, W \in \mathfrak{g}$, or equivalently, by*

$$V \otimes W + W \otimes V - 2V \boxtimes W, \quad V, W \in \mathfrak{g}. \tag{4.18}$$

4.4. Examples of commuting symmetries. The recursion tractor formula (4.14) for commuting symmetries $\mathcal{D}^k$ can be transformed into an explicit formula for $\mathcal{D}^k$, expressed in terms of the Levi-Civita connection ∇ and the curvature J, by equations (3.18) and (4.6). In what follows, we compute the explicit commuting symmetries up to order 3 acting on $\mathcal{E}$.

We use tractor form indices

$$\mathbf{A} = [A^1A^2], \qquad \mathbf{B} = [B^1B^2], \qquad \mathbf{C} = [C^1C^2],$$

and form indices

$$\mathbf{a} = [a^1a^2], \qquad \mathbf{b} = [b^1b^2], \qquad \mathbf{c} = [c^1c^2].$$

For a Killing vector field $k^a \in \mathcal{E}^a$, we have

$$K^{\mathbf{A}} = (\Pi k)^{\mathbf{A}} = \mathbb{Y}^{\mathbf{A}}_a k^a + \frac{1}{2}\mathbb{Z}^{\mathbf{A}}_{\mathbf{a}}\nabla^{[a^1}k^{a^2]}, \qquad \mathbb{D}_{\mathbf{A}}f = 2\mathbb{Y}^a_{\mathbf{A}}\nabla_a f. \tag{4.19}$$

Hence, the symmetry $\mathcal{D}^k f = K^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}f = k^a\nabla_a f$ coincides with the Lie derivative along the Killing vector field k^a.

For a Killing 2-tensor $k^{bc} \in \mathcal{E}^{(bc)}$, we obtain from equation (4.11),

$$\begin{aligned} K^{\mathbf{AB}} = (\Pi^{(2)}k)^{\mathbf{BC}} &= \mathbb{Y}^{\mathbf{B}}_b\mathbb{Y}^{\mathbf{C}}_c k^{bc} + \frac{1}{3}\big(\mathbb{Y}^{\mathbf{B}}_b\mathbb{Z}^{\mathbf{C}}_{\mathbf{c}} + \mathbb{Y}^{\mathbf{C}}_b\mathbb{Z}^{\mathbf{B}}_{\mathbf{c}}\big)\nabla^{c^1}k^{c^2b} \\ &+ \frac{1}{2}\mathbb{Z}^{\mathbf{B}}_{\mathbf{b}}\mathbb{Z}^{\mathbf{C}}_{\mathbf{c}}\left[\frac{1}{3}\nabla^{b^1}\nabla^{c^1}k^{b^2c^2} + \frac{2}{n}\mathrm{J}g^{b^1c^1}k^{b^2c^2}\right]. \end{aligned} \tag{4.20}$$

Since

$$\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{C}}f = 4\mathbb{Y}^b_{\mathbf{B}}\mathbb{Z}^{\mathbf{c}}_{\mathbf{C}}g_{bc^0}\nabla_{c^1}f + 4\mathbb{Y}^b_{\mathbf{B}}\mathbb{Y}^c_{\mathbf{C}}\nabla_b\nabla_c f,$$

by equation (3.18), we obtain

$$\mathcal{D}^k = K^{\mathbf{BC}}\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{C}}f = k^{bc}\nabla_b\nabla_c f + (\nabla_r k^{rc})\nabla_c f.$$

Note that $K^{\mathbf{BC}}h_{\mathbf{BC}}$ is a constant, and, using $\nabla^a k^r{}_r = -2\nabla_r k^{ar}$, which follows from

$$3g_{bc}\nabla^{(a}k^{bc)} = \nabla^a k^r{}_r + 2\nabla_r k^{ar} = 0,$$

a short computation reveals its value

$$K^{\mathbf{BC}}h_{\mathbf{BC}} = \frac{1}{4}\Big[-\nabla_r\nabla_s\sigma^{rs} + \frac{2(n+1)}{n}\mathrm{J}k^r{}_r\Big].$$

Thus, the modification of $\mathcal{D}^k$ by any multiple of

$$\nabla_r \nabla_s k^{rs} - \frac{2(n+1)}{n} \mathrm{J} k^r{}_r,$$

is again a symmetry of Δ. This means that there is no unique formula, written in terms of k^{ab}, for a symmetry. This is in contrast with the case of conformal symmetries [**8**].

Now, we consider Killing 3-tensors $k^{abc} \in \mathcal{E}^{(abc)}$. Then,

$$g_{cd} \nabla^{(a} k^{bcd)} = 3\nabla_r k^{rab} + 3\nabla^a k^{br}{}_r = 0,$$

and, applying ∇_a, we obtain $\nabla_r \nabla_s k^{rsa} + \Delta k^{ar}{}_r = 0$. Summarizing, we obtain

$$\nabla_r k^{rab} = -\nabla^a k^{br}{}_r, \quad \nabla_r \nabla_s k^{rsa} = -\Delta k^{ar}{}_r \quad \text{and} \quad \nabla_r k^{rs}{}_s = 0,$$

where the last equality is the trace of $\nabla_r k^{rab} + \nabla^a k^{br}{}_r = 0$. Now, computing $\Pi^{(3)} k$, which requires the application of Π in equation (4.5) to equation (4.11), results in

$$\begin{aligned} K^{\mathbf{ABC}} = \left(\Pi^{(3)} k\right)^{\mathbf{ABC}} &= \mathbb{Y}_a^{\mathbf{A}} \mathbb{Y}_b^{\mathbf{B}} \mathbb{Y}_c^{\mathbf{C}} k^{abc} \\ &+ \frac{1}{4}\left(\mathbb{Y}_a^{\mathbf{A}} \mathbb{Y}_b^{\mathbf{B}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{C}} + \mathbb{Y}_a^{\mathbf{C}} \mathbb{Y}_b^{\mathbf{A}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{B}} + \mathbb{Y}_a^{\mathbf{B}} \mathbb{Y}_b^{\mathbf{C}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{A}}\right) \nabla^{c^1} k^{c^2 ab} \\ &+ \frac{1}{3}\left(\mathbb{Y}_a^{\mathbf{A}} \mathbb{Z}_{\mathbf{b}}^{\mathbf{B}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{C}} + \mathbb{Y}_a^{\mathbf{C}} \mathbb{Z}_{\mathbf{b}}^{\mathbf{A}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{B}} + \mathbb{Y}_a^{\mathbf{B}} \mathbb{Z}_{\mathbf{b}}^{\mathbf{C}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{A}}\right) \\ &\times \left[\frac{1}{4} \nabla^{b^1} \nabla^{c^1} k^{b^2 c^2 a} + \frac{2}{n} \mathrm{J} g^{b^1 c^1} k^{b^2 c^2 a}\right] + \mathbb{Z}_{\mathbf{a}}^{\mathbf{A}} \mathbb{Z}_{\mathbf{b}}^{\mathbf{B}} \mathbb{Z}_{\mathbf{c}}^{\mathbf{C}} \psi_{\mathbf{abc}} \end{aligned}$$

for some $\psi_{\mathbf{abc}}$, which we do not need to compute. Furthermore,

$$\begin{aligned} \mathbb{D}_{\mathbf{A}} \mathbb{D}_{\mathbf{B}} \mathbb{D}_{\mathbf{C}} f &= 8 \mathbb{Y}_{\mathbf{A}}^a \mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} \mathbb{Z}_{\mathbf{C}}^{\mathbf{c}} g_{ab^1} g_{b^2 c^1} \nabla_{c^2} f \\ &+ 16 \mathbb{Y}_{\mathbf{A}}^a \mathbb{Y}_{\mathbf{B}}^b \mathbb{Z}_{\mathbf{C}}^{\mathbf{c}} g_{c^1 (a} \nabla_{b)} \nabla_{c^2} f \\ &+ 8 \mathbb{Y}_{\mathbf{A}}^a \mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} \mathbb{Y}_{\mathbf{C}}^c g_{ab^1} \nabla_c \nabla_{b^2} f \\ &+ 8 \mathbb{Y}_{\mathbf{A}}^a \mathbb{Y}_{\mathbf{B}}^b \mathbb{Y}_{\mathbf{C}}^c \left[\nabla_a \nabla_b \nabla_c f - \frac{4}{n} \mathrm{J} g_{b[a} \nabla_{c]} f\right], \end{aligned} \tag{4.21}$$

by equation (3.18). Combining the previous two displays yields

$$\begin{aligned}\mathcal{D}^k = K^{\mathbf{ABC}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{C}}f &= k^{abc}\nabla_a\nabla_b\nabla_c f \\ &+ \frac{3}{2}(\nabla_r k^{rbc})\nabla_b\nabla_c f \\ &+ \frac{1}{4}(\nabla_r\nabla_s k^{rsc})\nabla_c f - \frac{n-1}{2n}\mathrm{J}k^{cr}{}_r\nabla_c f.\end{aligned} \tag{4.22}$$

By construction, the vector field $(\Pi^{(2)}k)^{a\mathbf{BC}}h_{\mathbf{BC}}$ is Killing. Using equations (3.14), (4.11) and (4.21), one easily computes

$$(\Pi^{(2)}k)^{a\mathbf{BC}}h_{\mathbf{BC}} = -\frac{1}{12}\left[\nabla_r\nabla_s k^{rsa} - \frac{4(n+2)}{n}\mathrm{J}k^{ar}{}_r\right].$$

Thus, symmetry $\mathcal{D}^k$ can be modified by a multiple of the operator

$$\left[\nabla_r\nabla_s k^{rsa} - \frac{4(n+2)}{n}\mathrm{J}k^{ar}{}_r\right]\nabla_a f.$$

Remark 4.15. Let $\ell \in \mathbb{N}$ and $k \in \mathcal{K}_\ell$. If the curvature of metric g vanishes, i.e., M is locally isomorphic to the pseudo-Euclidean space $\mathbb{R}^{p,q}$, straightforward computation shows that

$$\mathcal{D}^k = \sum_{i=0}^{\ell}\frac{1}{2^i}\binom{\ell}{i}(\nabla_{a_1}\cdots\nabla_{a_i}k^{a_1\cdots a_\ell})\nabla_{a_{i+1}}\cdots\nabla_{a_\ell}$$

is a commuting symmetry of Δ. This can also be deduced from properties of the Weyl quantization of $T^*\mathbb{R}^{p,q}$, namely, $\mathcal{D}^k$ coincides with the Weyl quantization of k, and the symplectic equivariance of the Weyl quantization, see, e.g., **[10]**, yields the equalities $[\Delta, \mathcal{D}^k] = [g,k]_S = 0$. Here, $[\cdot,\cdot]_S$ denotes the Schouten bracket of symmetric tensors and $\mathcal{D}^{[g,k]_S} = 0$ is equivalent to the Killing equation.

Remark 4.16. Let $\ell \in \mathbb{N}$ and $k \in \mathcal{K}_\ell$. If k is trace-free, then straightforward computation shows that

$$\mathcal{D}^k = k^{a_1\cdots a_\ell}\nabla_{a_1}\cdots\nabla_{a_\ell}$$

is a commuting symmetry of Δ. The trace-free Killing tensors are those which are also conformal, and this allows for comparison of our results with the work of Eastwood **[8]**. Out of conformal Killing tensors V, he explicitly built conformal symmetries of the Laplacian, i.e., differential operators $\mathcal{D}_1^V$ and $\mathcal{D}_2^V$ with principal symbol V such that $\mathcal{D}_2^V \Delta = \Delta \mathcal{D}_1^V$. The lower order terms involve divergences and contractions of V with the trace-free Ricci tensor. On a space of constant curvature, with $V = k$ a trace-free Killing tensor, both divergences and contractions vanish, see, e.g., equation (4.21), and we obtain $\mathcal{D}_1^k = \mathcal{D}_2^k = \mathcal{D}^k$. In **[8]**, the symmetries built out of trace-free Killing tensors are commuting symmetries of the Laplacian and coincide with the symmetries constructed in our article. Note that trace components correspond to trivial conformal symmetries in the sense of **[8]**. This prevents comparison of our results with those of **[8]** for general Killing tensors.

5. Riemannian geometry via projective geometry. Overdetermined equations for Killing tensors are projectively invariant **[9]**, so it is natural to consider their prolongation within the framework of projective geometry. As this is an example of parabolic geometry, we can employ the general invariant theory for this class of structures, **[5]**. We shall observe that several results obtained in the previous section then follow immediately.

Recall that we are interested in manifolds of constant curvature. These are conformally flat, and thus, projectively flat as well, see equation (5.1), that is, we will consider locally flat projective structures.

5.1. Tractor calculus in projective geometry. We shall briefly recall invariant calculus on projective manifolds, see **[1]** for more details. A projective structure on a manifold M is given by a class $[\nabla]$ of *special* affine connections with the same geodesics as unparametrized curves, where *special* indicates that there is a parallel volume form for every connection in $[\nabla]$. These connections are parametrized by nowhere vanishing sections of projective density bundles $\mathcal{E}(1)$. We shall also assume orientability, characterized by a compatible volume form

$$\epsilon_{a^1 \cdots a^n} \in \mathcal{E}_{[a^1 \cdots a^n]}(n+1) \cong \mathcal{E},$$

parallel for every affine connection in $[\nabla]$. The decomposition of the curvature of ∇ is

$$R_{ab}{}^{c}{}_{d} = \overline{C}_{ab}{}^{c}{}_{d} + 2\delta^{\,c}_{[a}\overline{\mathrm{P}}_{b]d}, \tag{5.1}$$

where $\overline{\mathrm{P}}_{ab}$ is the projective Schouten tensor and $\overline{C}_{ab}{}^{c}{}_{d} = C_{ab}{}^{c}{}_{d}$, that is, conformal and projective Weyl tensors coincide. Note that, for the

Levi-Civita connection ∇ of an Einstein metric g, the curvature is also of the form (2.1), and the relation between projective and Riemannian Schouten tensors is $\overline{\mathrm{P}}_{ab} = 2\mathrm{P}_{ab}$, see **[11]**.

We define the standard tractor bundle and its dual by their spaces of sections $\overline{\mathcal{E}}^A$ and $\overline{\mathcal{E}}_A$, respectively, as

$$\overline{\mathcal{E}}^A = \begin{array}{c} \mathcal{E}^a(-1) \\ \oplus \\ \mathcal{E}(-1) \end{array} \qquad \text{and} \qquad \overline{\mathcal{E}}_A = \begin{array}{c} \mathcal{E}(1) \\ \oplus \\ \mathcal{E}_a(1) \end{array};$$

see **[1]** for the meaning of the semi-direct product $\oplus$. The choice of a connection in the class $[\nabla]$ turns the previous display into the direct sum decomposition. These bundles are equipped with the *projectively invariant tractor connection* which we denote by $\overline{\nabla}$. Choosing ∇ in the projective class, $\overline{\nabla}$ is explicitly given by the formulas

(5.2)
$$\overline{\nabla}_a \begin{pmatrix} \nu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \delta_a{}^b \rho \\ \nabla_a \rho - \overline{\mathrm{P}}_{ab} \nu^b \end{pmatrix} \quad \text{and} \quad \overline{\nabla}_a \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \overline{\mathrm{P}}_{ab} \sigma \end{pmatrix};$$

see **[1]** for details. Here, $\nu^a \in \mathcal{E}^a(-1)$, $\rho \in \mathcal{E}(-1)$, $\sigma \in \mathcal{E}(1)$ and $\mu_a \in \mathcal{E}_a(1)$. We extend the connection $\overline{\nabla}$ to the tensor products of $\overline{\mathcal{E}}^A$ by the Leibniz rule. Also note that the structure of the tractor bundle $\overline{\mathcal{E}}^{[AB]}$ and of its dual is given by

$$\overline{\mathcal{E}}^{[AB]} = \begin{array}{c} \mathcal{E}^{[ab]}(-2) \\ \oplus \\ \mathcal{E}^a(-2) \end{array} \qquad \text{and} \qquad \overline{\mathcal{E}}_{[AB]} = \begin{array}{c} \mathcal{E}_a(2) \\ \oplus \\ \mathcal{E}_{[ab]}(2). \end{array} \tag{5.3}$$

In what follows, we shall use the tractor bundle

$$\overline{\mathcal{E}}_A{}^B = \overline{\mathcal{E}}_A \otimes \overline{\mathcal{E}}^B = \begin{array}{c} \mathcal{E}^a \\ \oplus \\ \mathcal{E}_a{}^b \oplus \mathcal{E}, \\ \oplus \\ \mathcal{E}_a \end{array} \tag{5.4}$$

where the trace-free part of $\overline{\mathcal{E}}_A{}^B$ is isomorphic to the projective adjoint tractor bundle. Analogously to equation (3.15), we define the projectively invariant differential operator

(5.5)
$$\overline{\mathbb{D}}_A{}^B : \mathcal{E}_{b_1\cdots b_s}(w) \otimes \overline{\mathcal{E}}_{C\cdots D}{}^{E\cdots F} \longrightarrow \mathcal{E}_{b_1\cdots b_s}(w) \otimes \overline{\mathcal{E}}_A{}^B \otimes \overline{\mathcal{E}}_{C\cdots D}{}^{E\cdots F},$$

as follows. Acting on $f \in \mathcal{E}(w)$ and $\varphi_a \in \mathcal{E}_a$, $\overline{\mathbb{D}}_A{}^B$ is given by

$$\overline{\mathbb{D}}_A{}^B f = \begin{pmatrix} 0 \\ 0 \,|\, wf \\ \nabla_a f \end{pmatrix}, \qquad \overline{\mathbb{D}}_A{}^B \varphi_c = \begin{pmatrix} 0 \\ \delta_c{}^b \varphi_a \,|\, -\varphi_c \\ \nabla_a \varphi_c \end{pmatrix}, \tag{5.6}$$

for an affine connection ∇ in the projective class. The formula for $\overline{\mathbb{D}}_A{}^B f$, $f \in \overline{\mathcal{E}}_{C\cdots D}{}^{E\cdots F}$, is formally the same as for $f \in \mathcal{E}(0)$, where we interpret ∇ as the coupled affine-tractor connection. Then, we extend $\overline{\mathbb{D}}_A{}^B$ to the general case by the Leibniz rule.

Henceforth, we assume the manifold M is projectively flat, i.e., the projective Weyl tensor vanishes. In particular, this means that the tractor connection $\overline{\nabla}$ is flat.

Let $F : \Gamma(U_1) \to \Gamma(U_2)$ be a projectively invariant linear differential operator, acting between tensor bundles U_1 and U_2. Then, F can be written in terms of an affine connection ∇. Regarding ∇ in the formula for F as the coupled affine tractor connection, we obtain the operator

$$F^{\overline{\nabla}} : \Gamma(\mathcal{A}^* \otimes U_1) \longrightarrow \Gamma(\mathcal{A}^* \otimes U_2).$$

Adapting the proof of Theorem 3.2 to the projective setting, we obtain the analogue of Corollary 3.3.

Theorem 5.1. *Let $F : \Gamma(U_1) \to \Gamma(U_2)$ be a projectively invariant linear differential operator over a projectively flat manifold. Then, $\overline{\mathbb{D}}_A{}^B$ commutes with F, i.e.,*

$$\overline{\mathbb{D}} \circ F = F^{\overline{\nabla}} \circ \overline{\mathbb{D}} : \Gamma(U_1) \longrightarrow \overline{\mathcal{E}}_A{}^B \otimes \Gamma(U_2).$$

As an example, consider the projectively invariant differential operator

$$\nabla_{(a}\nabla_{b)} + \overline{\mathrm{P}}_{ab} : \mathcal{E}(1) \longrightarrow \mathcal{E}_{(ab)}(1), \tag{5.7}$$

see e.g., [**4**]. Projective invariance and Theorem 5.1 imply

$$\begin{aligned} \overline{\mathbb{D}}_{A_1}^{\ B_1} \cdots \overline{\mathbb{D}}_{A_\ell}^{\ B_\ell} \big(\nabla_{(a}\nabla_{b)} + \overline{\mathrm{P}}_{ab}\big) \\ = \big(\nabla_{(a}\nabla_{b)} + \overline{\mathrm{P}}_{ab}\big) \overline{\mathbb{D}}_{A_1}^{\ B_1} \cdots \overline{\mathbb{D}}_{A_\ell}^{\ B_\ell}, \end{aligned} \tag{5.8}$$

where ∇ on the right side denotes the coupled affine tractor connection.

5.2. Killing tensors in projective geometry. Let $\ell \in \mathbb{N}$. We shall focus on the PDE

$$\nabla_{(a_0} k_{a_1 \cdots a_\ell)} = 0, \qquad k_{a_1 \cdots a_\ell} \in \mathcal{E}_{(a_1 \cdots a_\ell)}(2\ell), \tag{5.9}$$

which is projectively invariant [**9**].

Setting $\overline{\mathcal{E}}_A = \begin{array}{|c|}\hline \ \\ \hline\end{array}$, we have $\overline{\mathcal{E}}_{[AB]} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, and we set

$$\boxtimes^\ell \overline{\mathcal{E}}_{[AB]} := \overbrace{\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\ \hline \ & \ & \cdots & \ \\ \hline\end{array}}^{\ell} S^\ell \overline{\mathcal{E}}_{[AB]}.$$

There exists a linear map

$$\overline{\Pi}^{(\ell)} : \mathcal{E}_{a_1 \cdots a_\ell}(2\ell) \longrightarrow \boxtimes^\ell \overline{\mathcal{E}}_{[AB]},$$

characterized by curved Casimir operators, see [**6**], which takes the form

$$\overline{\Pi}^{(\ell)} : k_{a_1 \cdots a_\ell} \longmapsto \overline{K}_{[A_1 B_1] \cdots [A_\ell B_\ell]} = \begin{array}{c} k_{a_1 \cdots a_\ell} \\ \oplus \\ \vdots \end{array} \in \begin{array}{c} \mathcal{E}_{a_1 \cdots a_\ell}(2\ell) \\ \oplus \\ \vdots \end{array} \tag{5.10}$$

and such that $k_{a_1 \cdots a_\ell}$ is a solution of equation (5.9) if and only if $\overline{K}_{[A_1 B_1] \cdots [A_\ell B_\ell]}$ is $\overline{\nabla}$-parallel. Note that the unspecified terms (indicated by vertical dots) of $\overline{K}_{[A_1 B_1] \cdots [A_\ell B_\ell]}$ are differential in $k_{a_1 \cdots a_\ell}$, i.e., the map $\overline{\Pi}^{(\ell)}$ is given by a differential operator. In fact, this is an example of a splitting operator, see e.g., [**6**] for details. It yields an analog of Theorem 4.9.

Proposition 5.2. [2]. *Let $(M,[\nabla])$ be a projectively flat manifold. The map $\overline{\Pi}^{(\ell)}$ induces a bijective correspondence*

$$\text{(5.11)}\quad \{\textit{solutions } k_{a_1\cdots a_\ell} \textit{ of equation } (5.9)\} \xleftrightarrow{1-1} \{\overline{\nabla} - \textit{parallel sections of } \boxtimes^\ell \overline{\mathcal{E}}_{[AB]}\}.$$

If the Levi-Civita connection of a metric g pertains to the projective class $[\nabla]$, the latter proposition gives a description of Killing tensors for the metric g via the map

$$\text{(5.12)}\qquad V^{(a_1\cdots a_\ell)} \longmapsto g_{a_1b_1}\cdots g_{a_1b_1}V^{(a_1\cdots a_\ell)} \in \mathcal{E}_{(b_1\cdots b_\ell)}(2\ell).$$

Indeed, this map gives a bijection between Killing ℓ-tensors and solutions of equation (5.9).

5.3. Construction of symmetries. Now, assume that there is a Levi-Civita connection ∇ in the projective class $[\nabla]$, such that the associated metric g_{ab} has constant curvature, i.e.,

$$R_{abcd} = \frac{4}{n}\mathrm{J}g_{c[a}g_{b]d}$$

with J parallel. Then, a short computation based on equations (3.4) and (5.2) shows that $\overline{\nabla}$ on $\overline{\mathcal{E}}^{[AB]}$, respectively, $\overline{\mathcal{E}}_{[AB]}$, agrees with the Riemannian tractor connection ∇ on $\mathcal{E}^{\mathbf{A}}$, respectively, $\mathcal{E}_{\mathbf{A}}$. Moreover, the tractor section

$$\text{(5.13)}\qquad h^{AB} = \begin{pmatrix} g^{ab} \\ 0 \\ \frac{2}{n}\mathrm{J}\end{pmatrix} \in \overline{\mathcal{E}}^{(AB)} = \begin{matrix} \mathcal{E}^{(ab)}(-2) \\ \oplus \\ \mathcal{E}^{a}(-2) \\ \oplus \\ \mathcal{E}(-2)\end{matrix} \cong \begin{matrix} \mathcal{E}^{(ab)} \\ \oplus \\ \mathcal{E}^{a} \\ \oplus \\ \mathcal{E}\end{matrix}$$

is parallel, cf., **[11]**. The isomorphism $\cong$ corresponds to the choice of connection $\nabla \in [\nabla]$, and in particular, trivializes density bundles. Here, g^{ab} is the inverse of g_{ab} and

$$\mathrm{J} = g^{ab}\mathrm{P}_{ab} = \frac{1}{2}g^{ab}\overline{\mathrm{P}}_{ab}.$$

Summarizing, we shall consider the Riemannian manifold (M,g) as the corresponding locally flat projective manifold $(M,[\nabla])$ with the distinguished parallel section h^{AB}. This is an example of holonomy reduction of Cartan connections **[3]** for the projective Cartan connection associated to $(M,[\nabla])$.

For $\mathrm{J}\neq 0$, note that h^{AB} is non-degenerate, hence, a tractor metric. A direct computation gives the next display and Lemma 5.3:

$$h^{P[A}\overline{\mathbb{D}}_P{}^{B]}g^{ab} = h^{P[A}\overline{\mathbb{D}}_P{}^{B]}g_{ab} = 0. \tag{5.14}$$

Lemma 5.3. *The explicit formula for the differential operator*

$$h^{P[A}\overline{\mathbb{D}}_P{}^{B]} : \mathcal{E}(w) \longrightarrow \mathcal{E}^{[AB]}(w),$$

written in terms of the Levi-Civita connection ∇, does not depend on $w\in\mathbb{R}$.

We are now ready to construct the commuting symmetries of the Laplace operator. The metric g allows for identification of a tensor $V^{a_1\cdots a_\ell}\in\mathcal{E}^{(a_1\cdots a_\ell)}$ with an element in $\mathcal{E}_{(a_1\cdots a_\ell)}(2\ell)$, see equation (5.12), and we denote by $\overline{V}\in\boxtimes^\ell\,\overline{\mathcal{E}}_{[AB]}$ the corresponding tractor, obtained via the map $\overline{\Pi}^{\,\ell}$, see equation (5.10). We consider the operators

$$\overline{\mathcal{D}}^V := h^{A_1C_1}\cdots h^{A_\ell C_\ell}\overline{V}_{A_1B_1\cdots A_\ell B_\ell}\overline{\mathbb{D}}_{C_1}{}^{B_1}\cdots\overline{\mathbb{D}}_{C_\ell}{}^{B_\ell}, \tag{5.15}$$

acting on any tensor-tractor bundle U.

Lemma 5.4. *The principal symbol of the differential operator $\overline{\mathcal{D}}^V$ is the symmetric ℓ-tensor V.*

Proof. The proof is analogous to the proof of Theorem 4.11. Writing tractor sections in vertical notation, see equation (5.3), we can refer to their "top" or "bottom" parts. The "top" part of $\overline{V}_{A_1B_1\cdots A_\ell B_\ell}$ is $g_{a_1b_1}\cdots g_{a_1b_1}V^{(a_1\cdots a_\ell)}$, cf., equations (5.10) and (5.12). On the other hand, elementary computation using equations (5.6) and (5.13) shows that the bottom part (and the leading term) of $h^{C[A}\overline{\mathbb{D}}_C{}^{B]}f$ is equal to $g^{ab}\nabla_b f$ for any section f of a tensor bundle. Therefore, the bottom part of the composition

$$h^{C_1[A_1}\overline{\mathbb{D}}_{C_1}{}^{B_1]}\cdots h^{C_\ell[A_\ell}\overline{\mathbb{D}}_{C_\ell}{}^{B_\ell]}f$$

is equal to

$$g^{a_1b_1}\nabla_{b_1}\cdots g^{a_\ell b_\ell}\nabla_{b_\ell}f.$$

This completes the proof of Lemma 5.4. $\square$

Theorem 5.5. *Let (M, g) be a pseudo-Riemannian manifold of constant curvature, with Levi-Civita connection ∇, and let $(M, [\nabla])$ be the corresponding locally flat projective manifold. Then, if k is a Killing ℓ-tensor, the operator*

$$\overline{\mathcal{D}}^k : \mathcal{E} \longrightarrow \mathcal{E},$$

defined by equation (5.15)*, is a commuting symmetry of the Laplace operator $\Delta = g^{ab}\nabla_a\nabla_b$.*

Proof. Let $\overline{K} \in \boxtimes^\ell \overline{\mathcal{E}}_{[AB]}$ be the parallel tractor associated to k via the composition of maps (5.11) and (5.12). Since the tractor metric h is also parallel with respect to the projective tractor connection $\overline{\nabla}$, it follows from equations (5.8) and (5.14) that

$$\overline{\mathcal{D}}^k\big(g^{ab}(\nabla_{(a}\nabla_{b)} + \overline{\mathrm{P}}_{ab})\big) = \big(g^{ab}(\nabla_{(a}\nabla_{b)} + \overline{\mathrm{P}}_{ab})\big)\overline{\mathcal{D}}^k : \mathcal{E}(+1) \longrightarrow \mathcal{E}(-1),$$

where we consider $g^{ab} \in \mathcal{E}^{(ab)}(-2)$. The operator

$$\overline{\mathcal{D}}^k : \mathcal{E}(w) \longrightarrow \mathcal{E}(w),$$

expressed in terms of ∇, does not depend on $w \in \mathbb{R}$ by Lemma 5.3. Observing that $g^{ab}\overline{\mathrm{P}}_{ab}$ is parallel for ∇, the theorem follows. $\square$

Remark 5.6. Projectively invariant overdetermined operators, as the operator defined in equation (5.7), are discussed in [**9**]. They allow for analogous construction of symmetries for other Riemannian linear differential operators

$$F : \Gamma(U) \longrightarrow \Gamma(U).$$

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University of Liège, Grande Traverse, 12, Sart-Tilman, B-4000 Liège, Belgium
Email address: **jpmichel82@gmail.com**

Mathematical Institute, Charles University, Sokolovská 83, 186 75, Prague 8, Czech Republic
Email address: **somberg@karlin.mff.cuni.cz**

Institute of Mathematics and Statistics, Masaryk University, Building 08, Kotlářská 2, Brno, 611 37 Czech Republic
Email address: **silhan@math.muni.cz**