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SOME EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

H.R. MARASI, H. AFSHARI AND C.B. ZHAI

ABSTRACT. In this paper, we study the existence and uniqueness of positive solutions for some nonlinear fractional partial differential equations via given boundary value problems by using recent fixed point results for a class of mixed monotone operators with convexity.

1. Introduction. Fractional differential equations have recently been of great interest, due primarily to the intensive development of the theory of fractional calculus itself, as well as the application of constructions in various sciences such as physics, mechanics, chemistry, engineering, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, etc. For details, see [**3, 6, 7, 18, 22, 24, 25, 30, 31**] and the references therein. There are several definitions of a fractional derivative of order $\alpha > 0$. The most commonly used definitions are Riemann-Liouville and Caputo.

Definition 1.1 ([**18, 25**]). For a continuous function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,$$

where $n-1 < \alpha < n$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

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Definition 1.2 ([**18, 25**])**.** The Riemann-Liouville fractional derivative of order α for a continuous function f is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}}\, ds, \quad n = [\alpha]+1,$$

where the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.3 ([**18, 25**])**.** Let $[a,b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a,b],\mathbb{R})$ is defined by

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds,$$

whenever the integral exists.

Note that most papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations use the techniques of nonlinear analysis, such as fixed point results, the Leray-Schauder theorem, stability, etc, see, for example, [**1, 2, 5, 8, 12, 13, 14, 16, 20, 27, 29, 33, 34, 35, 36, 37, 38**] and the references therein. The existence of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by several authors, see [**8, 21, 35**] and the references therein. However, there are few results on uniqueness.

Recently, Zhai [**32**] proved some results on a class of mixed monotone operators with convexity. Following [**32**], we give existence and uniqueness results for equations of the form

$$\frac{D^{\alpha}}{Dt}\, u(s,t) + f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\, u(s,t)\Big) = 0, \quad 0<\alpha<1, \tag{1.1}$$

and

$$\frac{{}^{c}D^{\alpha}}{Dt}\, u(s,t) + f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\, u(s,t)\Big) = 0, \quad 1<\alpha<2, \tag{1.2}$$

via given boundary conditions where no conditions are prescribed on u as a function of s.

Fractional partial differential equations are important in a variety of application areas, including biology, chemistry, economics, mechanics, seismology, etc. They can be used to describe important phenomena in many fields of science and engineering, such as damping law, diffusion processes, etc. In [**15**], a more general fractional partial differential equation for seepage flow without the assumption of continuity is presented. A fractional diffusion equation has been introduced in [**23**] to describe diffusion in special types of porous media. The propagation of mechanical diffusive waves in viscoelastic media is governed by fractional wave equations [**19**]. A reaction diffusion equation of fractional order appears in the modeling of the evolution of the bacterium *Bacillus subtilis*, which grows on the surface of thin agar plates [**9**]. Host-parasitoid systems may be found in many experimental and theoretical investigations in ecology [**4**]. Indeed, many physical problems are governed by fractional differential equations of the forms (1.1)–(1.2). For example, Giona and Roman [**11**] studied a fractional differential equation of the form

$$\begin{gathered}\frac{D^\alpha}{Dt}\, u(x,t) = -Ax^{-\beta}\, \frac{\partial u(x,t)}{\partial x}, \\ 0 < \alpha \le \frac{1}{2}; \quad A > 0, \quad \beta \ge 0,\end{gathered} \tag{1.3}$$

and also considered its application in transport phenomena for describing diffusion on random fractal structures. In [**28**], the same authors discussed a more general form

$$\begin{gathered}\frac{D^\alpha}{Dt} u(x,t) = -A\left[\frac{\partial u(x,t)}{\partial x} + \frac{k}{x} u(x,t)\right], \\ 0 < \alpha \le \frac{1}{2}; \quad A > 0, \quad k \in R,\end{gathered} \tag{1.4}$$

for diffusion in isotropic and homogeneous fractal structures. In connection with phenomena between the heat and wave equations, Fujita [**10**] considered the cauchy problem for the equation

$$\begin{gathered}\frac{D^\alpha}{Dt}\, u(x,t) = \frac{D^\beta}{Dx}\, u(x,t), \\ 1 \le \alpha, \quad \beta \le 2; \quad 0 < t < T, \quad x \in R,\end{gathered} \tag{1.5}$$

and showed the existence and uniqueness of the solution. One may find a variety of applications in [**17, 25, 26**].

In the sequel, we present some basic concepts in ordered Banach spaces and a fixed-point theorem which will be used later. For details, we refer the reader to **[12, 13]**.

Suppose that $(E, \|\cdot\|)$ is a Banach space with the zero element denoted by θ. A non-empty closed convex set $P \subseteq E$ is a cone if it satisfies:

(i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$,
(ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

We suppose that E is partially ordered by P, that is, $x \leq y$ if and only if $y - x \in P$. A cone P is called *normal* if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $||x|| \leq N||y||$. Also, we define an ordered interval by

$$[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\} \quad \text{for all } x_1, x_2 \in E.$$

We say that an operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$. A is called a *positive operator* if $A(x) \geq \theta$ for any $x \geq \theta$.

Definition 1.4 (**[12, 13]**). $A : P \times P \to P$ is said to be a *mixed monotone operator* if $A(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i, $i = 1, 2 \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. The element $x \in P$ is called a *fixed point* of A if $A(x, x) = x$.

Theorem 1.5 (**[32]**). *Let E be a real Banach space, and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator satisfying*:

(i) *for $c \in (0, 1)$, $x, y \in P$, there exists $\alpha(c, x, y) \in (1, \infty)$ such that $A(cx, y) \leq c^{\alpha(c,x,y)} A(x, y)$;*
(ii) *there exist $u_0, v_0 \in P, r \in (0, 1)$, such that $u_0 \leq rv_0$, $A(u_0, v_0) \geq u_0$, $A(v_0, u_0) \leq v_0$.*

Then, A has a unique fixed point u^* in $[u_0, rv_0]$. Moreover, successively constructing the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \qquad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [u_0, rv_0]$, we have

$$\|x_n - u^*\| \longrightarrow 0, \qquad \|y_n - u^*\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

2. Main results. We study the existence and uniqueness of solutions for fractional differential equations on a partially ordered Banach space with two types of boundary conditions and two types of fractional derivatives, Riemann-Liouville and Caputo.

2.1. Existence results for the fractional differential equation with the Riemann-Liouville fractional derivative. First, we study the existence and uniqueness of positive solutions for the fractional differential equation

$$\frac{D^\alpha}{Dt}\,u(s,t)+f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\,u(s,t)\Big)=0, \tag{2.1}$$
$$0<\epsilon<T,\quad T\ge 1,\quad t\in[\epsilon,T],\quad 0<\alpha<1,\quad s\in[a,b],$$

subject to condition

$$u(s,\eta)=u(s,T),\qquad (s,\eta)\in[a,b]\times(\epsilon,T), \tag{2.2}$$

where D^α is the Riemann-Liouville fractional derivative of order α, $a,b\in(0,\infty)$, $a<b$.

Let $E=C([a,b]\times[\epsilon,T])$ be the Banach space of continuous functions on $[a,b]\times[\epsilon,T]$ with the sup norm, and set

$$P=\{y\in C([a,b]\times[\epsilon,T]):\min_{(s,t)\in[a,b]\times[\epsilon,T]} y(s,t)\ge 0\}.$$

Then, P is a normal cone.

Lemma 2.1. *Let $(s,t)\in[a,b]\times[\epsilon,T]$, $(s,\eta)\in[a,b]\times(\epsilon,t)$ and $0<\alpha<1$. Then, the equation*

$$\frac{D^\alpha}{Dt}\,u(s,t)+f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\,u(s,t)\Big)=0,$$

with boundary condition $u(s,\eta)=u(s,T)$, has a solution u_0 if and only if u_0 is a solution of the fractional integral equation

$$u(s,t)=\int_\epsilon^T G(t,\xi)f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi,$$

where

$$G(t,\xi)=\begin{cases}[t^{\alpha-1}(\eta-\xi)^{\alpha-1}-t^{\alpha-1}(T-\xi)^{\alpha-1}]/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)\\ \quad -(t-\xi)^{\alpha-1}/\Gamma(\alpha), \qquad \epsilon\le\xi\le\eta\le t\le T,\\ [-t^{\alpha-1}-(T-\xi)^{\alpha-1}]/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)\\ \quad -(t-\xi)^{\alpha-1}/\Gamma(\alpha), \qquad \epsilon\le\eta\le\xi\le t\le T,\\ [-t^{\alpha-1}(T-\xi)^{\alpha-1}]/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)\\ \qquad\qquad \epsilon\le\eta\le t\le\xi\le T.\end{cases}$$

Proof. From

$$\frac{D^\alpha}{Dt}\,u(s,t)+f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\,u(s,t)\Big)=0$$

and the boundary condition, we can see that

$$u(s,t)-c_1t^{\alpha-1}=-I_\epsilon^\alpha f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\,u(s,t)\Big).$$

By the definition of a fractional integral, we obtain

$$u(s,t)=c_1t^{\alpha-1}-\int_\epsilon^\eta\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi,$$
$$u(s,\eta)=c_1T^{\alpha-1}-\int_\epsilon^\eta\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi$$

and

$$u(s,T)=c_1T^{\alpha-1}-\int_\epsilon^T\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi.$$

Since $u(s,\eta)=u(s,T)$, we obtain

$$c_1=\frac{1}{\eta^{\alpha-1}-T^{\alpha-1}}\int_\epsilon^\eta\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi$$
$$-\frac{1}{\eta^{\alpha-1}-T^{\alpha-1}}\int_\epsilon^T\frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi.$$

Hence,

$$\begin{aligned} u(s,t) = &\frac{t^{\alpha-1}}{\eta^{\alpha-1}-T^{\alpha-1}}\int_{\epsilon}^{\eta}\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi \\ &-\frac{t^{\alpha-1}}{\eta^{\alpha-1}-T^{\alpha-1}}\int_{\epsilon}^{T}\frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi \\ &-\int_{\epsilon}^{t}\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi \\ =&\int_{\epsilon}^{T}G(t,\xi)f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi. \end{aligned}$$

This completes the proof. □

Now, we are ready to state and prove the first main result.

Theorem 2.2. *Let $0<\epsilon<T$ be given, and*

- (H1) $f(s,t,u(s,t),v(s,t))\in C([a,b]\times[\epsilon,T],[0,\infty),[0,\infty))$ *is increasing in u and decreasing in v.*
- (H2) $\partial/\partial s$ *is positive and, for $c\in(0,1)$, u, $v\in P$, there exists $\alpha(c,u,v)\in(1,\infty)$ such that*

$$f(s,t,cu(s,t),v(s,t))\le c^{\alpha(c,u,v)}f(s,t,u(s,t),v(s,t))$$

and

$$f(s,t,u(s,t),v(s,t))=0$$

whenever $G(s,t)<0$.

- (H3) *There exist $u_0,v_0\in P$ and $r\in(0,1)$ such that*

$$\begin{aligned} &u_0(s,t)\le rv_0(s,t),\\ &\int_{\epsilon}^{T}G(t,\xi)f\Big(s,\xi,u_0(s,\xi),\frac{\partial}{\partial s}v_0(s,\xi)\Big)\,d\xi\ge u_0(s,t),\\ &\int_{\epsilon}^{T}G(t,\xi)f\Big(s,\xi,u_0(s,\xi),\frac{\partial}{\partial s}v_0(s,\xi)\Big)\,d\xi\le v_0(s,t), \end{aligned}$$

for $(s,t)\in([a,b]\times[\epsilon,T])$.

Then, equation (2.1)*, with boundary condition* (2.2)*, has a unique solution* $u^* \in [u_0, rv_0]$*. Moreover, for the sequences*

$$u_{n+1} = \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,u_n(s,\xi),\frac{\partial}{\partial s}\,u_n(s,\xi)\Big)\,d\xi,$$
$$v_{n+1} = \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,v_n(s,\xi),\frac{\partial}{\partial s}v_n(s,\xi)\Big)\,d\xi,$$

$n = 0, 1, \ldots,$ *we have* $\|u_n - u^*\| \to 0$ *and* $\|v_n - u^*\| \to 0$.

Proof. By using Lemma 2.1, the problem is equivalent to the integral equation

$$u(s,t) = \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,u(s,\xi)\Big)\,d\xi,$$

where

$$G(t,\xi) = \begin{cases} t^{\alpha-1}(\eta-\xi)^{\alpha-1} - t^{\alpha-1}(T-\xi)^{\alpha-1}/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha) \\ \quad -(t-\xi)^{\alpha-1}/\Gamma(\alpha), \qquad \epsilon \le \xi \le \eta \le t \le T, \\ -t^{\alpha-1}-(T-\xi)^{\alpha-1}/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)-(t-\xi)^{\alpha-1}/\Gamma(\alpha) \\ \qquad\qquad \epsilon \le \eta \le \xi \le t \le T, \\ -t^{\alpha-1}(T-\xi)^{\alpha-1}/(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha) \\ \qquad\qquad \epsilon \le \eta \le t \le \xi \le T. \end{cases}$$

Define the operator $A : P \times P \to P$ by:

$$A(u(s,t), v(s,t)) = \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}\,v(s,\xi)\Big)\,d\xi.$$

Then, u is a solution for the problem if and only if $u = A(u,u)$. It is easy to see that the operator A is increasing in u and decreasing in v on P. On the other hand, for $c \in (0,1)$, $s,t \in P$, there exists $\alpha(c,s,t) \in (1,\infty)$ such that

$$\begin{aligned} A(cu(s,t), v(s,t)) &= \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,cu(s,\xi),\frac{\partial}{\partial s}v(s,\xi)\Big)\,d\xi \\ &\le c^{\alpha} \int_{\epsilon}^{T} G(t,\xi) f\Big(s,\xi,u(s,\xi),\frac{\partial}{\partial s}v(s,\xi)\Big)\,d\xi \\ &= c^{\alpha} A(u(s,t), v(s,t)). \end{aligned}$$

Therefore, A satisfies all conditions of Theorem 1.5, and so, the operator A has a unique positive solution (u^*, u^*) such that $A(u^*, u^*) = u^*$. This completes the proof. $\square$

Example 2.3. Let $0 < \epsilon < 1$ be given. Consider the periodic boundary value problem

$$D^{1/3}u(s,t) + g(s,t)[u(s,t)]^2 - \left[\frac{\partial}{\partial s}u(s,t)\right]^2 = 0,$$
$$(s,t) \in [a,b] \times [\epsilon, 1], \quad u(s,\eta) = u(s,1), \quad (s,\eta) \in [a,b] \times (\epsilon, t),$$

where $g(s,t)$ is continuous on $[a,b] \times [\epsilon, 1]$, with

$$\gamma_1 = \min_{(s,t)\in[a,b]\times[\epsilon,1]} g(s,t) > 0, \qquad \gamma_2 = \max_{(s,t)\in[a,b]\times[\epsilon,1]} g(s,t) > 0.$$

In addition, let

$$M_1 = \min_{\substack{t\in[\epsilon,1]\\ \eta\in[\epsilon,1]}} \frac{-t^{-2/3}(1-\epsilon)^{1/3} - (t-\epsilon)^{1/3}(\eta^{-2/3}-1)}{\Gamma(4/3)(\eta^{-2/3}-1)}$$

and

$$M_2 = \max_{\eta\in[\epsilon,1]} \frac{\epsilon^{-2/3}\eta^{1/3}}{\Gamma(4/3)(\eta^{-2/3}-1)},$$

such that $\gamma_1 \le 1/M_1$. Set

(2.3)
$$G(t,\xi) = \begin{cases} t^{-2/3}(\eta-\xi)^{-2/3} - t^{-2/3}(1-\xi)^{-2/3}/(\eta^{-2/3}-1^{-2/3})\Gamma(1/3) \\ \quad -(t-\xi)^{-2/3}/\Gamma(1/3), \qquad \epsilon \le \xi \le \eta \le t \le 1, \\ -t^{-2/3} - (1-\xi)^{-2/3}/(\eta^{-2/3}-1^{-2/3})\Gamma(1/3) \\ \quad -(t-\xi)^{-2/3}/\Gamma(1/3), \qquad \epsilon \le \eta \le \xi \le t \le 1, \\ -t^{-2/3}(1-\xi)^{-2/3}/(\eta^{-2/3}-1^{-2/3})\Gamma(1/3) \\ \qquad\qquad \epsilon \le \eta \le t \le \xi \le 1. \end{cases}$$

Then,

$$\int_\epsilon^1 G(t,\xi)\,d\xi = \frac{t^{-2/3}(\eta-\epsilon)^{1/3} - t^{-2/3}(1-\epsilon)^{1/3} - (t-\epsilon)^{1/3}(\eta^{-2/3}-1)}{\Gamma(4/3)(\eta^{-2/3}-1)}.$$

Also,

$$f(s,t,u(s,t),v(s,t)) = g(s,t)[u(s,t)]^2 - \left[\frac{\partial}{\partial s}\, v(s,t)\right]^2$$

is increasing in u and decreasing in v. For $c \in (0,1)$, consider $1 < \alpha(c,s,t) < 2$. Then we have

$$\begin{aligned} f\Big(s,t,cu(s,t),\frac{\partial}{\partial s}u(s,t)\Big) &= g(s,t)[cu(s,t)]^2 - \left[\frac{\partial}{\partial s}u(s,t)\right]^2 \\ &\le c^{\alpha(c,s,t)}\left(g(s,t)[u(s,t)]^2 - \left[\frac{\partial}{\partial s}u(s,t)\right]^2\right) \\ &= c^{\alpha(c,s,t)}\Big(f\Big(s,t,u(s,t),\frac{\partial}{\partial s}u(s,t)\Big)\Big). \end{aligned}$$

Now, set $u_0 = 1$, $r = 1/2$ and $2 \le v_0 \le 1/\gamma_2 M_2$. Note that u_0 and v_0 are real numbers. Then we have

$$\begin{aligned} u_0 &\le \frac{1}{2}v_0, \\ A(u_0,v_0) &= \int_\epsilon^1 G(t,\xi) f\Big(s,\xi,cu_0(s,\xi),\frac{\partial}{\partial s}v_0(s,\xi)\Big)\, d\xi \\ &= \int_\epsilon^1 G(t,\xi)g(s,t)[u_0(s,t)]^2 d\xi = \int_\epsilon^1 G(t,\xi)g(s,t)\, d\xi \\ &\ge \gamma_1 M_1 \ge 1 = u_0, \\ A(v_0,u_0) &= \int_\epsilon^1 G(t,\xi) f\Big(s,\xi,v_0(s,\xi),\frac{\partial}{\partial s}\, u_0(s,\xi)\Big)\, d\xi \\ &= \int_\epsilon^1 G(t,\xi)g(s,t)[v_0(s,t)]^2\, d\xi \le \gamma_2 M_2 {v_0}^2 \le v_0. \end{aligned}$$

Thus, by Theorem 2.2, the problem has a unique solution in $[1,(1/2)v_0]$.

2.2. Existence results for the fractional differential equation with the Caputo fractional derivative. We study the existence and uniqueness of a positive solution for the fractional differential equation

$$\frac{{}^cD^\alpha}{Dt}\, u(s,t) + f\Big(s,t,u(s,t),\frac{\partial}{\partial s}\, u(s,t)\Big) = 0, \qquad (2.4)$$
$$((s,t) \in [a,b]\times[0,T]), \quad T \ge 1, \quad 1 < \alpha < 2,$$

with boundary conditions

$$u(s,0) = \beta_1 u(s,\eta), \qquad u(s,T) = \beta_2 u(s,\eta), \\ ((s,\eta) \in [a,b] \times (0,t),\ 0 < \beta_1 < \beta_2 < 1), \tag{2.5}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α. Let $E = C([a,b] \times [0,T])$ be the Banach space of continuous functions on $[a,b] \times [0,T]$ with the sup norm and

$$P = \{y \in C([a,b] \times [0,T]) : \min_{((s,t) \in [a,b] \times [0,T])} y(s,t) \geq 0\}.$$

Then, P is a normal cone. Similar to the proof of Lemma 2.1, we can prove the next result.

Lemma 2.4. *Let $1 < \alpha < 2$, $T \geq 1$, $(s,t) \in ([a,b] \times [0,T])$, $((s,\eta) \in [a,b] \times (0,t))$ and $0 < \beta_1 < \beta_2 < 1$. Then the equation*

$$\frac{{}^cD^\alpha}{Dt}\, u(s,t) + f\Big(s,t,u(s,t), \frac{\partial}{\partial s}\, u(s,t)\Big) = 0, \tag{2.6}$$

with boundary conditions $u(s,0) = \beta_1 u(s,\eta)$, and $u(s,T) = \beta_2 u(s,\eta)$ has a solution u_0 if and only if u_0 is a solution of the fractional integral equation

$$u(s,t) = \int_0^T G(t,\xi) f\Big(s,\xi,u(s,\xi), \frac{\partial}{\partial s}\, u(s,\xi)\Big)\, d\xi, \tag{2.7}$$

where

$$G(t,\xi) = \begin{cases} L/T\Gamma(\alpha) & 0 \leq \xi \leq \eta \leq t \leq T, \\ t(T-\xi)^{\alpha-1} - T(t-\xi)^{\alpha-1}/T\Gamma(\alpha) & 0 \leq \eta \leq \xi \leq t \leq T, \\ t(T-\xi)^{\alpha-1}/T\Gamma(\alpha) & 0 \leq \eta \leq t \leq \xi \leq T, \end{cases}$$

and

$$L = [\beta_1 T + t(\beta_2 - \beta_1)](\eta - \xi)^{\alpha-1} + t(T-\xi)^{\alpha-1} - T(t-\xi)^{\alpha-1}.$$

Theorem 2.5. *Let $T \geq 1$, $0 < \epsilon < T$, and the following hold:*

(H4) $f(s,t,u(s,t),v(s,t)) \in C([a,b] \times [\epsilon,T],[0,\infty],[0,\infty])$ *is increasing in u and decreasing in v.*

(H5) *$\partial/\partial s$ is positive and for $c \in (0,1)$, u, $v \in P$, there exists $\alpha(c,u,v) \in (1,\infty)$ such that*

$$f(s,t,u(s,t),v(s,t)) \leq c^{\alpha(c,u,v)} f(s,t,u(s,t),v(s,t))$$

and

$$f(s,t,u(s,t),v(s,t)) = 0,$$

whenever $G(s,t) < 0$.

(H6) *There exist u_0, $v_0 \in P$ and $r \in (0,1)$ such that*

$$u_0(s,t) \leq r v_0(s,t),$$
$$\int_\epsilon^T G(t,\xi) f\Big(s,\xi,u_0(s,\xi), \frac{\partial}{\partial s}\, v_0(s,\xi)\Big)\, d\xi \geq u_0(s,t),$$
$$\int_\epsilon^T G(t,\xi) f\Big(s,\xi,u_0(s,\xi), \frac{\partial}{\partial s}\, v_0(s,\xi)\Big)\, d\xi \leq v_0(s,t),$$

for $(s,t) \in ([a,b] \times [\epsilon,T])$.

Then, equation (2.4) *with boundary conditions* (2.5) *has a unique solution $u^* \in [u_0, rv_0]$. Moreover, for the sequences*

$$u_{n+1} = \int_\epsilon^T G(t,\xi) f\Big(s,\xi,u_n(s,\xi), \frac{\partial}{\partial s}\, u_n(s,\xi)\Big)\, d\xi,$$
$$v_{n+1} = \int_\epsilon^T G(t,\xi) f\Big(s,\xi,v_n(s,\xi), \frac{\partial}{\partial s} v_n(s,\xi)\Big)\, d\xi,$$

$n = 0,1,\ldots$, we have $\|u_n - u^*\| \to 0$ and $\|v_n - u^*\| \to 0$.

Proof. Similar to the proof of Theorem 2.2, we can show that, for the operator A defined by

$$A(u(s,t),v(s,t)) = \int_\epsilon^T G(t,\xi) f\Big(s,\xi,u(s,\xi), \frac{\partial}{\partial s}\, v(s,\xi)\Big)\, d\xi,$$

$A(u(s,t),u(s,t)) \geq 0$ for all $u \in P$ and $(s,t) \in [a,b] \times [0,1]$. Also, A satisfies the conditions of Theorem 1.5; therefore, it has a unique positive fixed point u^*. The use of Lemma 2.4 completes the proof. $\square$

Remark 2.6. By using the fixed point theorem for mixed monotone operators with convexity, we established the uniqueness of positive solutions to fractional partial differential equation BVPs. The method, as well as the existence and uniqueness result to fractional partial differential equations, is relatively new to the literature.

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University of Tabriz, Department of Applied Mathematics, Tabriz, Iran
Email address: marasi@tabrizu.ac.ir

University of Bonab, Department of Mathematics, Bonab, Iran
Email address: afshari@bonabu.ac.ir

Shanxi University, School of Mathematical Sciences, Taiyuan, Shanxi 030006, P.R. China
Email address: cbzhai@sxu.edu.cn