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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A DOUBLY DEGENERATE PARABOLIC NON-DIVERGENCE FORM EQUATION

CHUNHUA JIN AND JINGXUE YIN

ABSTRACT. This paper is concerned with the asymptotic behavior of a doubly degenerate parabolic equation in non-divergence form. The proofs are divided into three cases according to exponent values of the source, and, by using different methods, we prove the stability of the steady states. We also expand the discussion of asymptotic stability for equations with the periodic source.

1. Introduction. In this paper, we consider the asymptotic behavior of solutions for a non-divergence form equation with Dirichlet boundary condition:

$$\frac{\partial u}{\partial t} = u^m \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q \quad (x,t) \in Q, \tag{1.1}$$

$$u(x,t)|_{\partial\Omega} = 0 \quad t \in \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x) > 0 \quad x \in \Omega, \tag{1.3}$$

where $Q = \Omega \times \mathbb{R}^+$, Ω is a bounded connected domain in $\mathbb{R}^N$, and $\partial\Omega \in C^{2+\alpha}$ with $0 < \alpha < 1$, $m \geq 1$, $p > 1$, $q, \lambda > 0$ are constants, $u_0(x) \in C^1(\overline{\Omega})$ with

$$\left.\frac{\partial u_0}{\partial \mathbf{n}}\right|_{\partial\Omega} > 0$$

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satisfying some compatibility conditions, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

Non-divergence form equations are often used to describe various physical phenomena, such as the diffusive process for biological species, the resistive diffusion phenomena in force-free magnetic fields, curve shortening flow, the spread of infectious disease, and so on, see [**3, 4, 6, 12**]. Compared to classical divergence form equations, this type of equation is, in some instances, a better descriptor of actual cases. For example, for a biological species, diffusion of the divergence form implies that the species is able to move to all locations within its environment with equal probability; however, if we consider this problem with objective conditions, the population density will affect the rate of diffusion, so a 'biased' diffusion equation is more realistic. For the non-divergence form diffusion, the diffusion rate is regulated by population density, that is, it increases for large populations and it decreases for small populations. Some properties of solutions for non-divergence form equations, such as existence, non-uniqueness, blow-up properties, etc., have been discussed by many authors. See, for example, [**6, 18, 20**] for equation (1.1) with $p = 2$, $m \geq 1$, and see [**23, 24**] for the case $m = 1$, $p > 1$. There have been numerous studies on asymptotic stability for the semilinear heat equation with nonlinear source, as shown in equation (1.1); see [**1, 2, 11, 13, 19**] for the corresponding problem with Dirichlet or Neumann boundary conditions. There are also papers regarding asymptotic stability for degenerate parabolic equations, see, for example [**16**]. As far as non-divergence form equations are concerned, only a few papers address such problems. Here, we refer to the work of Wiegner [**21**], who studied a typical case of equation (1.1), that is, the case $p = 2$, and obtained asymptotic stability for $q \leq m + 1$.

In the present paper, we focus on asymptotic stability of problem (1.1)–(1.3). Note that equation (1.1) is very different from some classical divergence form equations, such as the polytropic filtration equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u^\gamma|^{p-2}\nabla u^\gamma).$$

In fact, the polytropic filtration equation can be transformed into an equation similar to equation (1.1) but with exponent $m = 1 - 1/\gamma < 1$. If $m \geq 1$, equation (1.1) is a strictly non-divergence form equation. This type of equation may have some very different, even incredible, features. For example, an important characteristic of non-divergence form equations is the resistive diffusion property. It is well known that, for divergence form equations such as the porous medium or the p-Laplace equations, fast diffusions (including classical heat transformations) result in infinite propagation speeds of disturbances for which the solutions become everywhere positive for any nontrivial, nonnegative initial datum whenever $t > 0$. To the contrary, slow diffusive equations admit perturbation propagations with finite speeds; however, the support of solutions will continue to grow, and goes to infinity. In addition, the supports of solutions of nondivergence form equations will never expand, and even shrink. Such a property is an important factor for non-uniqueness of solutions. In fact, non-uniqueness has been discovered by Dal Passo and Luckhaus **[6]** for the special case $m = 1$, $p = 2$. Indeed, for any $T > 0$, they have constructed a weak solution with extinction time T.

In the present paper, we assume that the initial datum satisfies

$$u_0(x) > 0, \qquad \left.\frac{\partial u_0}{\partial \mathbf{n}}\right|_{\partial\Omega} > 0, \qquad u_0(x)|_{\partial\Omega} = 0,$$

which seems very restrictive. However, it is necessary, to some extent because, if

$$\operatorname{supp} u_0(x) = \Omega' \subset \Omega,$$

then one may consider the same problem in Ω' since the supports of solutions of nondivergence form equations never expand. However, a so-called maximal solution with constant support is uniquely determined by the initial datum. So, in some cases, especially for the asymptotic stability of positive steady states, we may restrict the discussion to maximal solutions.

The study begins with the existence of an elliptic problem. We show that, for the sub-critical case, the steady states in our situation are solutions of singular elliptic problems in most cases, which add to the difficulties. In addition, we will show that, for the sup-critical case, the solution may blow up, so we restrict our study to those initial values for which the solution of problem (1.1)–(1.3) remains uniformly bounded. Finally, by virtue of the results, we also expand upon the discussion of asymptotic stability for the equation with a periodic source.

Due to degeneracy and singularity, equation (1.1) may not have classical solutions in general, and hence, we consider nonnegative solutions of equation (1.1) in the weak sense. For different exponents m, p, q, regularity of the solutions may be different, for $q > m-1$, $u^{-m/2}u_t \in L^2$, see the proof in **[8]**. Here, E is merely the weakest space of these solutions.

Definition 1.1. A function $u \in E$ is said to be a weak super-solution of problem (1.1)–(1.3), provided that, for any $T > 0$, $0 \le \varphi \in C_0^1(Q_T)$,

$$\begin{cases} \iint_{Q_T} \frac{\partial u}{\partial t}\varphi\, dx\, dt + \iint_{Q_T} |\nabla u|^{p-2}\nabla u \nabla(u^m\varphi)\, dx\, dt \ge \lambda \iint_{Q_T} u^q \varphi\, dx\, dt, \\ u(x,t) \ge 0 & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) \ge u_0(x) & x \in \Omega, \end{cases}$$

where

$$E = \{u \in L^\infty(Q_T); u_t \in L^1(Q_T), \nabla u, |u|^{(m-1)/p} \cdot \nabla u \in L^p(Q_T)\}.$$

Replacing $\ge$ with $\le$ in the above inequalities, a weak sub-solution follows. Furthermore, if u is a weak super-solution as well as a weak sub-solution, then we call it a *weak solution* of problem (1.1)–(1.3).

2. Preliminaries. To study the asymptotic behavior of solutions we first consider the elliptic equation with Dirichlet boundary value condition:

$$\begin{cases} -w^\gamma \operatorname{div}(|\nabla w|^{p-2}\nabla w) = \lambda & x \in \Omega, \\ w(x)|_{\partial\Omega} = 0, \\ w(x) > 0 & x \in \Omega, \end{cases} \tag{2.1}$$

where $\gamma \ge 0$. In order to obtain the existence of positive solutions for the above problem, next we consider the regularized problem:

$$\begin{cases} -w^\gamma \operatorname{div}(|\nabla w|^{p-2}\nabla w) = \lambda & x \in \Omega, \\ w(x)|_{\partial\Omega} = \varepsilon, \\ w(x) > 0 & x \in \Omega. \end{cases} \tag{2.2}$$

First, a well-known comparison lemma is given which has been proved by many authors, see, for example **[15]**.

Lemma 2.1. *Assume that* $\varepsilon \le w_i \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $i = 1, 2$. *If* w_1 *and* w_2 *are the weak super- and sub-solutions of problem* (2.2), *respectively, then* $w_1 \ge w_2$.

Using Lemma 2.1, we obtain the next existence result.

Proposition 2.2. *If* $\gamma < N + 1$, *then problem* (2.1) *admits a unique solution* $w \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. *If* $\gamma \ge N + 1$, *then there exists a solution* $w \in L^\infty(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$, *which is obtained as a limit of an approximate process* (*called the* maximum solution).

Remark 2.3. From Proposition 2.2, we see that problem (2.1) admits a unique solution if $\gamma < N + 1$. If $\gamma > N + 1$, it is difficult to obtain uniqueness of solutions due to the lack of $W^{1,p}$ estimates for this case; thus, we define a maximum limit solution, which is the limit solution of regularized problem (2.2). In fact, we see that, for any $\varepsilon > 0$ corresponding to the solution w_ε, v a solution of condition (2.1), we have $v \le w_\varepsilon$. From the proof, we see that $w_\varepsilon \searrow w$; thus, we confirm $v \le w$.

Proof. Similar to the proof of Lemma 2.1, it is easy to see that the solution w_ε of problem (2.2) is increasing on ε. Furthermore, let ϕ, $\|\phi\|_\infty = 1$, be the first eigenfunction of the p-Laplacian equation corresponding to the homogeneous Dirichlet boundary value condition on Ω, and let λ_1 be the corresponding first eigenvalue. Moreover, take $\widetilde{\Omega} \supset \overline{\Omega}$, ψ with $\|\psi\|_\infty = 1$ the first eigenfunction of the p-Laplacian corresponding to the homogeneous Dirichlet boundary value condition on $\widetilde{\Omega}$, $\widetilde{\lambda}_1$ the corresponding first eigenvalue. Then,

$$-(k\phi)^\gamma \mathrm{div}(|\nabla(k\phi)|^{p-2}\nabla(k\phi)) = \lambda_1 (k\phi)^{\gamma+p-1} \le \lambda$$

is ensured by $k^{\gamma+p-1} \le \lambda/\lambda_1$ means that $k\phi$ is a sub-solution of problem (2.2) for appropriately small k. In addition, letting $\sigma = \inf_{x\in\Omega} \psi(x)$, we see that

$$-(K\psi)^\gamma \mathrm{div}(|\nabla(K\psi)|^{p-2}\nabla(K\psi)) = \widetilde{\lambda}_1 K^{\gamma+p-1}\psi^{\gamma+p-1} \ge \lambda$$

is ensured by

$$(K\sigma)^{\gamma+p-1} \ge \frac{\lambda}{\widetilde{\lambda}_1},$$

meaning that $K\psi$ is a super-solution of problem (2.2) if K is large enough. By Lemma 2.1, we see that $k\phi \le w_\varepsilon \le K\psi$. Furthermore, we see that, if $\gamma \le 1$, multiplying by $w_\varepsilon^{1-\gamma}$ on both sides and integrating over Ω yields

$$\int_\Omega |\nabla w_\varepsilon|^p \, dx \le \lambda \int_\Omega w_\varepsilon^{1-\gamma} \le \lambda K^{1-\gamma} |\Omega|,$$

if $1 < \gamma < N + 1$. Then,

$$\int_\Omega |\nabla w_\varepsilon|^p \, dx \le \lambda \int_\Omega (k\phi)^{1-\gamma}.$$

By **[7, 17]**, we see that there exists $d > 0$ such that $\phi(x) \ge d \operatorname{dist}(x, \partial\Omega)$, which implies that $\int_\Omega |\nabla w_\varepsilon|^p \, dx$ is uniformly bounded. If $\gamma \ge N + 1$, then, for any $\Omega' \subset\subset \Omega$ with $\eta = \min_{x\in\Omega'} k\phi > 0$, $\varepsilon \le \eta$, we have

$$\int_\Omega |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla (w_\varepsilon - \eta)_+ \, dx = \lambda \int_\Omega w_\varepsilon^{-\gamma} (w_\varepsilon - \eta)_+ \le \lambda k \eta^{-\gamma} |\Omega|,$$

where $u_+ = u$ if $u > 0$; otherwise, it is 0, which implies that

$$\int_{\Omega'} |\nabla w_\varepsilon|^p \, dx \le \lambda k \eta^{-\gamma} |\Omega|.$$

In addition, integrating the first equation of (2.2) over Ω, we also have

$$\int_\Omega |\nabla w_\varepsilon^{(p+\gamma-1)/p}|^p dx \le C\lambda |\Omega|.$$

Thus, there exists a function $w > 0$ in Ω since $w_\varepsilon \ge k\phi$ and $w \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ if $\gamma \le 1$, $w \in L^\infty(\Omega) \cap W_{\text{loc}}^{1,p}(\Omega)$ and $w^{(p+\gamma-1)/p} \in W_0^{1,p}(\Omega)$ if $\gamma > 1$, such that $w_\varepsilon \searrow w$, $\nabla w_\varepsilon \rightharpoonup \nabla w$ in the sense $L^p(\Omega)$ or $L_{\text{loc}}^p(\Omega)$ by the weak lower semi-continuity of the norm; namely, w is a weak solution of problem (2.1).

It remains to show uniqueness for $\gamma < N + 1$. Assume that w and v are two solutions of problem (2.1). Then we have

$$\int_\Omega \Big(|\nabla w|^{p-2} \nabla w - |\nabla v|^{p-2} \nabla v \Big) \nabla (w - v)_+ \, dx = \lambda \int_\Omega (w^{-\gamma} - v^{-\gamma})(w - v)_+ \, dx \le 0,$$

which means that $w \equiv v$. The proof is complete. $\square$

Remark 2.4. From the above proof, we see that $w_\varepsilon \searrow w$ and $w^{(p+\gamma-1)/p} \in W_0^{1,p}(\Omega)$, so that the trace of the solution is obtained as $\gamma w(x) \mid_{\partial\Omega} = 0$.

Consider the elliptic problem:

$$\begin{cases} -\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \lambda w^\gamma & x \in \Omega, \\ w(x) \mid_{\partial\Omega} = 0, & \\ w(x) > 0 & x \in \Omega. \end{cases} \tag{2.3}$$

For $0 < \gamma < p-1$, we have the following result.

Proposition 2.5. *Assume that* $0 < \gamma < p-1$. *Then, problem* (2.3) *admits a unique positive solution* $w \in C^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$.

Proof. We shall use the super- and sub-solutions method to prove existence, so we only need to construct a pair of super- and sub-solutions. Let ψ, ϕ be defined as in Proposition 2.2. We see that

$$-\mathrm{div}(|\nabla(k\phi)|^{p-2}\nabla(k\phi)) = \lambda_1(k\phi)^{p-1} \le \lambda(k\phi)^\gamma$$

is ensured by $k^{p-1-\gamma} \le \lambda/\lambda_1$, which means that $k\phi$ is a sub-solution of problem (2.3) for appropriately small k. In addition, let $\sigma = \inf_{x\in\Omega}\psi(x)$. We see that

$$-\mathrm{div}(|\nabla K\psi|^{p-2}\nabla K\psi) = \widetilde{\lambda}_1 K^{p-1}\psi^{p-1} \ge \lambda(K\psi)^\gamma$$

is ensured by $(K\sigma)^{p-1-\gamma} \ge \lambda/\widetilde{\lambda}_1$. Therefore, $K\psi$ is a super-solution of problem (2.3) if K is chosen large enough. By the super- and sub-solution iteration method, we see that problem (2.3) admits at least one positive solution w with $k\phi \le w \le K\psi$. By **[7, 17]**, we see that there exists $d > 0$ such that $\phi(x) \ge d\,\mathrm{dist}(x,\partial\Omega)$, which implies that $w(x) \ge d\,\mathrm{dist}(x,\partial\Omega)$. Moreover, by **[10]**, we also conclude that there exists $\beta(N,p\|w\|_\infty)$ with $0 < \beta < 1$ such that $w \in C^{1,\beta}(\overline{\Omega})$. Therefore, we also have $w(x) \le D\,\mathrm{dist}(x,\partial\Omega)$.

Next, we show uniqueness. Suppose, to the contrary, that there exists another positive solution $v \in C^{1,\beta}(\overline{\Omega})$. By the strong maximum principle **[7, 17]**, there exists a $d_1(v) > 0$ such that $v(x) \ge d_1\mathrm{dist}(x,\partial\Omega)$. On the other hand, we also have $v(x) \le D_1\mathrm{dist}(x,\partial\Omega)$

since $v \in C^{1,\beta}(\overline{\Omega})$. Thus, there exists a constant μ with $0 < \mu < 1$ such that $\mu v \leq w$. Let

$$\mu^* = \sup\{0 < \mu \leq 1; \mu v \leq w \text{ on } \Omega\}. \tag{2.4}$$

We have $\mu^* = 1$. Otherwise, from equation (2.4), we see that $\mu^* v \leq w$ on Ω. Direct calculation yields

$$\begin{aligned} -\mathrm{div}(|\nabla(\mu^* v)|^{p-2}\nabla(\mu^* v)) &= \lambda(\mu^*)^{p-1-q}(\mu^* v)^q \\ &\leq \lambda(\mu^*)^{p-1-q} w^q \\ &= -(\mu^*)^{p-1-q}\mathrm{div}(|\nabla w|^{p-2}\nabla w). \end{aligned}$$

Then, for $0 \leq \varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega (|\nabla(\mu^* v)|^{p-2}\nabla(\mu^* v) - (\mu^*)^{p-1-q}|\nabla w|^{p-2}\nabla w)\nabla\varphi \, dx \leq 0.$$

Taking

$$\varphi = (\mu^* v - \mu^{*(p-1-q)/(p-1)} w)_+,$$

we conclude that

$$\mu^* v \leq \mu^{*(p-1-q)/(p-1)} w < w.$$

Clearly, this contradicts the definition of μ^*. Hence, $\mu^* = 1$; namely, $w \leq v$. Interchanging the roles of w and v, we also have $v \leq w$. Uniqueness follows. $\square$

Remark 2.6. Similar to the above proof, one can also infer the same conclusions of Proposition 2.5 hold for the problem:

$$\begin{cases} -\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \lambda w^\gamma & x \in \Omega, \\ w(x) \mid_{\partial\Omega} = \varepsilon, & \\ w(x) > 0 & x \in \Omega, \end{cases} \tag{2.5}$$

where $0 < \gamma < p - 1$.

3. The asymptotic behavior of solutions. In this section, we consider the asymptotic behavior of solutions. Due to the special type of this equation, weak solutions of problem (1.1)–(1.3) may not be uniquely determined by the initial data. From **[6]**, we see that there exists a group of solutions which go to 0 in some finite time with the diffusion term replaced by $u\Delta u$. In fact, the family of solutions is primarily constructed by the non-expanding property of support sets of the solutions, while in **[9,** Lemma 2.3], we show this property for equation (1.1) for any $m, p > 1$. Thus, by using the same method, we can also construct a group of solutions which become extinct in finite times. Therefore, in this paper, we consider the asymptotic behavior of the maximal solution for problem (1.1)–(1.3), that is, the maximal of all of these solutions, which is obtained as a limited solution of a regularized problem.

Consider the following regularized problem:

$$(3.1)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = u^m \mathrm{div}((|\nabla u|^2 + \eta)^{(p-2)/2}\nabla u) + \lambda u^q && (x,t) \in Q_T,\\ &u(x,t) = \varepsilon && x \in \partial\Omega,\\ &u(x,0) = u_0(x) + \varepsilon = u_{0\varepsilon}(x) && x \in \Omega. \end{aligned}$$

From [**8**], we see that, for any fixed $\varepsilon > 0$, by approximation, problem (3.1) always admits a solution:

$$\varepsilon \le u_\varepsilon \in W_p^{1,0}(Q_T) \cap W_2^{0,1}(Q_T) \cap L^\infty(Q_T) \quad \text{for } \eta = 0.$$

Here,

$$\begin{aligned} W_p^{1,0}(Q_T) &= \{u \in L^p(Q_T); \nabla u \in L^p(Q_T)\},\\ W_2^{0,1}(Q_T) &= \{u \in L^2(Q_T); u_t \in L^2(Q_T)\}. \end{aligned}$$

Before proceeding, we verify the next comparison principle.

Lemma 3.1. *Let ω be a super-solution such that $\omega \ge \varepsilon$ in Q_T, and let $v \ge 0$ be a sub-solution of problem* (1.1)–(1.3). *If ω, $v \in W_p^{1,0}(Q_T) \cap W_2^{0,1}(Q_T)$ and $\omega(x,0) \ge v(x,0)$, then we have $\omega(x,t) \ge v(x,t)$.*

Proof. Since ω and v are the super- and sub-solutions of equation (1.1), respectively, then we have

$$\begin{aligned} \int_0^t \int_\Omega (\omega_t \varphi_\omega + |\nabla \omega|^{p-2}\nabla\omega\nabla(\omega^m\varphi_\omega))\,dx\,ds &\ge \lambda \int_0^t \int_\Omega \omega^q \varphi_\omega\,dx\,ds,\\ \int_0^t \int_\Omega (v_t \varphi_v + |\nabla v|^{p-2}\nabla v\nabla(v^m\varphi_v))\,dx\,ds &\le \lambda \int_0^t \int_\Omega v^q \varphi_v\,dx\,ds. \end{aligned}$$

Take

$$\varphi_\omega = \frac{\operatorname{sgn}_\delta((v-\omega)_+)}{\omega^m} \quad \text{and} \quad \varphi_v = \frac{\operatorname{sgn}_\delta((v-\omega)_+)}{v^m}.$$

Then, we have

$$\begin{aligned}\int_0^t \int_\Omega &(v^{-m} v_t - \omega^{-m}\omega_t) \operatorname{sgn}_\delta((v-\omega)_+)\, dx\, ds \\ &+ \int_0^t \int_\Omega (|\nabla v|^{p-2}\nabla v - |\nabla\omega|^{p-2}\nabla\omega)\nabla(\operatorname{sgn}_\delta((v-\omega)_+))\, dx\, ds \\ &\le \lambda \int_0^t \int_\Omega (v^{q-m} - \omega^{q-m}) \operatorname{sgn}_\delta((v-\omega)_+))\, dx\, ds,\end{aligned}$$

where $\operatorname{sgn}_\delta u = \operatorname{sgn}(u)\inf\{|u|/\delta, 1\}$. Note that the second term of the left side is equivalent to

$$\int_0^t \int_\Omega (|\nabla v|^{p-2}\nabla v - |\nabla\omega|^{p-2}\nabla\omega)\nabla(v-\omega)_+ \operatorname{sgn}'_\delta((v-\omega)_+)\, dx\, ds \ge 0.$$

Then, we have

$$\begin{aligned}\int_0^t \int_\Omega &(v^{-m} v_t - \omega^{-m}\omega_t) \operatorname{sgn}_\delta((v-\omega)_+)\, dx\, ds \\ &\le \lambda \int_0^t \int_\Omega (v^{q-m} - \omega^{q-m}) \operatorname{sgn}_\delta((v-\omega)_+))\, dx\, ds.\end{aligned} \tag{3.2}$$

Recalling the equality from [**6,** Lemma 2], we obtain

$$\begin{aligned}&\int_0^t \int_\Omega (u-v)_t \operatorname{sgn}_\delta(\Phi(u)-\Phi(v))_+\, dx\, ds \\ &= \int_\Omega \left(\int_0^u \operatorname{sgn}_\delta(\Phi(z)-\Phi(v))_+\, dz - \int_0^v \operatorname{sgn}_\delta(\Phi(0)-\Phi(z))_+\, dz \right)\Big|_0^t\, dx \\ &\qquad - \int_0^t \int_\Omega v_t \left(\int_0^u \operatorname{sgn}'_\delta(\Phi(z)-\Phi(v))(\Phi'(z)-\Phi'(v))_+\, dz \right) dx\, ds,\end{aligned} \tag{3.3}$$

where Φ is a monotone smooth, convex scalar-valued function. We first consider the case $m = 1$. Recall equation (3.2), and let $\widehat{\omega} = \ln\omega$, $\widehat{v} = \ln v$. Note that the effective integral domain is, in fact, $\Omega_{\{v>\omega\ge\varepsilon\}}$.

Thus, we have

$$
\begin{aligned}
&\int_0^t \int_\Omega (\widehat{v}_t - \widehat{\omega}_t)\,\mathrm{sgn}_\delta((e^{\hat v} - e^{\hat\omega})_+)\,dx\,ds \\
&\quad \geq \int_\Omega \left(\int_0^{\hat v(x,t)} \mathrm{sgn}_\delta(e^z - e^{\hat\omega(x,t)})_+ dz - \int_0^{\hat\omega(x,t)} \mathrm{sgn}_\delta(1 - e^z)_+ dz \right) dx \\
&\qquad - \int_0^t \int_\Omega \widehat{\omega}_t \left(\int_0^{\hat v} \mathrm{sgn}'_\delta(e^z - e^{\hat\omega})(e^z - e^{\hat\omega})_+\,dz \right) dx\,ds.
\end{aligned}
$$

Substituting into equation (3.2) yields

$$
\begin{aligned}
&\int_\Omega \left(\int_0^{\hat v(x,t)} \mathrm{sgn}_\delta(e^z - e^{\hat\omega(x,t)})_+\,dz - \int_0^{\hat\omega(x,t)} \mathrm{sgn}_\delta(1 - e^z)_+\,dz \right) dx \\
&\quad \leq \int_0^t \int_\Omega \widehat{\omega}_t \left(\int_0^{\hat v} \mathrm{sgn}'_\delta(e^z - e^{\hat\omega})(e^z - e^{\hat\omega})_+\,dz \right) dx\,ds \\
&\qquad + \lambda \int_0^t \int_\Omega (e^{(q-1)\hat v} - e^{(q-1)\hat\omega})\,\mathrm{sgn}_\delta((e^{\hat v} - e^{\hat\omega})_+)\,dx\,ds \\
&\quad \leq \int_0^t \int_\Omega \widehat{\omega}_t \left(\int_0^{\hat v} \mathrm{sgn}'_\delta(e^z - e^{\hat\omega})(e^z - e^{\hat\omega})_+\,dz \right) dx\,ds \\
&\qquad + \lambda(q-1)|v|_\infty^{q-1} \int_0^t \int_\Omega (\widehat{v} - \widehat{\omega})\,\mathrm{sgn}_\delta((e^{\hat v} - e^{\hat\omega})_+)\,dx\,ds.
\end{aligned}
$$

The term on the left converges to

$$
\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+\,dx,
$$

and the first term on the right converges to 0 as $\delta \to 0$. Letting $\delta \to 0$, we see that, if $q \leq 1$,

$$
\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+\,dx \leq 0,
$$

while, if $q > 1$,

$$
\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+\,dx \leq \lambda(q-1)|v|_\infty^{q-1} \int_0^t \int_\Omega (\widehat{v} - \widehat{\omega})_+\,dx\,ds.
$$

Gronwall's inequality gives

$$\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+ \, dx \le 0,$$

which implies that $\widehat{v}(x,t) \le \widehat{\omega}(x,t)$, that is, $v(x,t) \le \omega(x,t)$.

Next, we consider the case $m > 1$. Similar to equation (3.3), for any $0 < a < \max\{\|u\|_\infty, \|v\|_\infty\}$, we obtain

(3.4)

$$\begin{aligned}
&\int_0^t \int_\Omega (u-v)_t \operatorname{sgn}_\delta(\Phi(u) - \Phi(v))_+ \, dx \, ds \\
&\quad = \int_\Omega dx \bigg(\int_{-a}^u \operatorname{sgn}_\delta(\Phi(z) \\
&\qquad - \Phi(v))_+ \, dz - \int_{-a}^v \operatorname{sgn}_\delta(\Phi(-a) - \Phi(z))_+ \, dz \bigg) \bigg|_0^t \\
&\qquad - \int_0^t \int_\Omega v_t \bigg(\int_{-a}^u \operatorname{sgn}'_\delta(\Phi(z) - \Phi(v))(\Phi'(z) - \Phi'(v))_+ \, dz \bigg) \, dx \, ds.
\end{aligned}$$

Let

$$\widehat{\omega} = -\frac{1}{m-1}\, \omega^{1-m}, \qquad \widehat{v} = -\frac{1}{m-1}\, v^{1-m}.$$

Then, we have

$$\begin{aligned}
&\int_0^t \int_\Omega (\widehat{v}_t - \widehat{\omega}_t) \operatorname{sgn}_\delta(((1-m)\widehat{v})^{1/(1-m)} - ((1-m)\widehat{\omega})^{1/(1-m)})_+ \, dx \, ds \\
&\quad = \int_\Omega \bigg(\int_{-a}^{\hat{v}(x,t)} \operatorname{sgn}_\delta(((1-m)z)^{1/(1-m)} - ((1-m)\widehat{\omega})^{1/(1-m)})_+ \, dz \\
&\qquad - \int_{-a}^{\hat{\omega}(x,t)} \operatorname{sgn}_\delta(((m-1)a)^{1/(1-m)} - ((1-m)z)^{1/(1-m)})_+ dz \bigg) dx \\
&\qquad - \int_0^t \int_\Omega \widehat{\omega}_t \bigg(\int_{-a}^{\hat{v}} \operatorname{sgn}'_\delta((1-m)z)^{1/(1-m)} - ((1-m)\widehat{\omega})^{1/(1-m)} \bigg) \\
&\qquad \cdot (((1-m)z)^{m/(1-m)} - ((1-m)\widehat{\omega})^{m/(1-m)})_+ \, dz \bigg) \, dx \, ds.
\end{aligned}$$

Substituting into equation (3.2) yields

$$\int_\Omega \bigg(\int_{-a}^{\hat{v}(x,t)} \operatorname{sgn}_\delta \Big(((1-m)z)^{1/(1-m)} - ((1-m)\widehat{\omega})^{1/(1-m)} \Big)_+ \, dz$$

$$
\begin{aligned}
&- \int_{-a}^{\hat{\omega}(x,t)} \operatorname{sgn}_\delta(((m-1)a)^{1/(1-m)} - ((1-m)z)^{1/(1-m)})_+ \, dz\Big) dx \\
&\leq \int_0^t \int_\Omega \widehat{\omega}_t \Big(\int_{-a}^{\hat{v}} \operatorname{sgn}_\delta'(((1-m)z)^{1/(1-m)} - ((1-m)\widehat{\omega})^{1/(1-m)}) \\
&\qquad \cdot \Big(((1-m)z)^{m/(1-m)} - ((1-m)\widehat{\omega})^{m/(1-m)} \Big)_+ dz \Big) \, dx \, ds \\
&\quad + \lambda \int_0^t \int_\Omega \Big(((1-m)\widehat{v})^{(q-m)/(1-m)} - ((1-m)\widehat{\omega})^{(q-m)/(1-m)} \Big) \\
&\cdot \operatorname{sgn}_\delta \Big(((1-m)\widehat{v})^{(q-m)/(1-m)} - ((1-m)\widehat{\omega})^{(q-m)/(1-m)} \Big)_+ \, dx \, ds.
\end{aligned}
$$

The term on the left converges to

$$\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+ \, dx,$$

and the first term on the right converges to 0 as $\delta \to 0$. Letting $\delta \to 0$, we see that, if $q \leq m$, then

$$\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+ \, dx \leq 0,$$

while, if $q > m$,

$$\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+ \, dx \leq \lambda(q-m) \mid v\Big|_\infty^{q-1} \int_0^t \int_\Omega (\widehat{v} - \widehat{\omega})_+ \, dx \, ds.$$

Recalling Gronwall's inequality gives

$$\int_\Omega (\widehat{v}(x,t) - \widehat{\omega}(x,t))_+ \, dx \leq 0.$$

Summing up, we conclude that $v \leq \omega$. The proof is complete. $\square$

Using the comparison principle, we see that the solution of problem (3.1) is unique, and, for any one solution v of problem (1.1)–(1.3), $v \leq u_\varepsilon$. Letting ε go to 0, the limit solution u also satisfies $u \geq v$.

From this analysis, we see that solution u, obtained by an approximation process of the regularized problem, is the maximal solution of problem (1.1)–(1.3). In what follows, we may assume that the maximal solution is smooth enough since it is obtained by an approximation process of the regularized problem, and the solutions of the regularized problem are also smooth enough.

For simplicity, we define

$$J[u] = u_t - u^m \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \lambda u^q.$$

Clearly, we have $J(k\phi + \varepsilon) \leq 0$, $k\phi \leq u_{0\varepsilon}(x)$ and $J(K\psi) \geq 0$, $K\psi \geq u_{0\varepsilon}(x)$ for appropriately small $k > 0$ and appropriately large $K > 0$, $\varepsilon > 0$, where ϕ and ψ are defined as in the proof of Proposition 2.2. Consider the problems:

$$\begin{cases} J(u) = 0 & (x,t) \in Q, \\ u(x,t) = (K\psi - \varepsilon)e^{-t} + \varepsilon & x \in \partial\Omega, \\ u(x,0) = K\psi(x) & x \in \Omega. \end{cases} \tag{3.5}$$

$$\begin{cases} J(u) = 0 & (x,t) \in Q, \\ u(x,t) = \varepsilon & x \in \partial\Omega, \\ u(x,0) = k\phi + \varepsilon & x \in \Omega. \end{cases} \tag{3.6}$$

Similar to **[8]**, it is easy to conclude that the above problems admit unique solutions, respectively. Let ω_ε and v_ε be the solutions of problems (3.5) and (3.6), respectively. By comparison, we obtain

$$v_\varepsilon \leq u_\varepsilon \leq \omega_\varepsilon.$$

Furthermore, by equations (3.5) and (3.6), we see that

$$\left.\frac{\partial \omega_\varepsilon}{\partial t}\right|_{\partial\Omega} < 0, \qquad \frac{\partial \omega_\varepsilon}{\partial t}(x,0) \leq 0,$$
$$\left.\frac{\partial v_\varepsilon}{\partial t}\right|_{\partial\Omega} = 0, \qquad \frac{\partial v_\varepsilon}{\partial t}(x,0) \geq 0.$$

Consider the equation satisfied by $v_{\varepsilon t}$, $\omega_{\varepsilon t}$, that is,

$$h_t - (p-1)u^m \mathrm{div}(|\nabla u|^{p-2}\nabla h) - (mu^{m-1}\Delta_p u + \lambda q u^{q-1})h = 0.$$

Using the maximum principle, we obtain that $v_{\varepsilon t} \geq 0$ and $\omega_{\varepsilon t} \leq 0$, which implies that v_ε is monotone non-decreasing and ω_ε is monotone non-increasing. In addition, by comparison, we also see that

$$k\phi + \varepsilon \leq v_\varepsilon(x,t) \leq \omega_\varepsilon(x,t) \leq K\psi.$$

Thus, by Dini's theorem, there exist $v_\varepsilon(x)$ and $\omega_\varepsilon(x)$ with

$$k\phi + \varepsilon \le v_\varepsilon(x) \le \omega_\varepsilon(x) \le K\psi, \tag{3.7}$$

such that

$$v_\varepsilon(x,t) \nearrow v_\varepsilon(x), \qquad \omega_\varepsilon(x,t) \searrow \omega_\varepsilon(x), \tag{3.8}$$

uniformly. Furthermore, for any $\Omega' \subset\subset \Omega$, let $r = \min_{x\in\Omega'} k\phi(x)$. Thus, there exists a $T_0 > 0$ such that, when $2\varepsilon < r$, $(K\psi - \varepsilon)e^{-T_0} < r/2$. Multiplying the first equation of (3.5) by $(\omega_\varepsilon - r)_+$, and integrating over $(n, n+1)\times\Omega$ for any $n > T_0$, we obtain

$$\int_n^{n+1}\int_{\substack{x\in\Omega\\ \omega_\varepsilon\ge r}} (\omega_{\varepsilon t}(\omega_\varepsilon - r) + \omega_\varepsilon^m|\nabla\omega_\varepsilon|^p)\,dx\,dt \le \lambda\int_n^{n+1}\int_\Omega \omega_\varepsilon^{q+1}dx\,dt.$$

In addition, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big(\omega_\varepsilon(x,n+1) - r\Big)_+ dx + \int_n^{n+1}\int_{\substack{x\in\Omega\\ \omega_\varepsilon\ge r}} \omega_\varepsilon^m|\nabla\omega_\varepsilon|^p dx\,dt\\
&\quad\le \lambda\int_n^{n+1}\int_\Omega \omega_\varepsilon^{q+1}dx\,dt + \frac{1}{2}\int_\Omega (\omega_\varepsilon(x,n) - r)_+\,dx\\
&\quad\le M_1
\end{aligned}$$

and

$$\int_n^{n+1}\int_{\substack{x\in\Omega\\ \omega_\varepsilon\ge r}} |\nabla\omega_\varepsilon|^p dx\,dt \le M_2.$$

By use of the integral mean value theorem, there exists a $t_n \in [n, n+1]$ such that

$$\int_{\substack{x\in\Omega\\ \omega_\varepsilon\ge r}} |\nabla\omega_\varepsilon(x,t_n)|^p dx\,dt \le M_2,$$

where M_2 depends on r and is independent of ε. In addition, multiplying the first equation of (3.5) by $\omega_\varepsilon^{-m}(\omega_\varepsilon - r)_{+t}$, integrating over $(t_n, t)\times\Omega$ for any $t > t_n$, and noting that $\omega_{\varepsilon t} \le 0$, we obtain

$$\begin{aligned}
&\int_{t_n}^t\int_\Omega \omega_\varepsilon^{-m}|(\omega_\varepsilon - r)_{+t}|^2 dx\,dt + \frac{1}{p}\int_\Omega \Big|\nabla\Big(\omega_\varepsilon(x,t) - r\Big)_+\Big|^p dx\\
&\qquad\le \frac{1}{p}\int_{\substack{x\in\Omega\\ \omega_\varepsilon\ge r}} |\nabla\omega_\varepsilon(x,t_n)|^p dx,
\end{aligned}$$

which implies that there exists a constant M_3 independent of ε such that

$$\begin{aligned}\int_{t_n}^{\infty}\int_{\Omega'}\omega_\varepsilon^{-m}|\omega_{\varepsilon t}|^2dx\,dt+\sup_{t\geq T_0+1}\int_{\Omega'}|\nabla\omega_\varepsilon(x,t)|^pdx\\ \leq\int_{t_n}^{\infty}\int_{\substack{x\in\Omega\\ \omega_\varepsilon\geq r}}\omega_\varepsilon^{-m}|\omega_{\varepsilon t}|^2dx\,dt+\int_{\substack{x\in\Omega\\ \omega_\varepsilon\geq r}}|\nabla\omega_\varepsilon(x,t)|^pdx\\ \leq M_3.\end{aligned}\tag{3.9}$$

For equation (3.6), we note that $\partial v_\varepsilon/\partial\mathbf{n}\leq 0$. Then, when $q\geq m-1$, multiplying the equation by v_ε^{1-m} and integrating over Ω yields

$$\int_\Omega v_\varepsilon^{1-m}v_{\varepsilon t}\,dx+\int_\Omega|\nabla v_\varepsilon|^p\,dx\leq\lambda\int_\Omega v_\varepsilon^{q+1-m}dx.$$

Noting that $v_{\varepsilon t}\geq 0$, we have

$$\sup_{t\geq 0}\int_\Omega|\nabla v_\varepsilon|^p\,dx\leq\widehat{M}_1,\tag{3.10}$$

where $\widehat{M}_1$ is independent of ε. If $q<m-1$, we obtain

$$\sup_{t\geq 0}\int_\Omega v_\varepsilon^{m-1-q}|\nabla v_\varepsilon|^p\,dx\leq\widehat{M}_2,\tag{3.11}$$

where $\widehat{M}_2$ is independent of ε. Similar to equation (3.9), we also obtain, for any $\Omega'\subset\subset\Omega$,

$$\sup_{t\geq 0}\int_{\Omega'}|\nabla v_\varepsilon(x,t)|^pdx\,dt\leq\widehat{M}_3,\tag{3.12}$$

where $\widehat{M}_3$ is independent of ε. Furthermore, we also have

$$\int_0^t\int_\Omega v_\varepsilon^{-m}|v_{\varepsilon t}|^2dx\,dt\leq\widehat{M}_4,\quad\text{if } q>m-1,\tag{3.13}$$

$$\int_0^t\int_{\Omega'} v_\varepsilon^{-m}|v_{\varepsilon t}|^2dx\,dt\leq\widehat{M}_4,\quad\text{if } q\leq m-1,\tag{3.14}$$

where $\widehat{M}_4$ is independent of ε.

By these a priori estimates and combining with equation (3.8), we conclude (taking a subsequence if necessary) that

$$\nabla v_\varepsilon(x,t)\rightharpoonup\nabla\widetilde{v}_\varepsilon(x),\qquad\nabla\omega_\varepsilon(x,t)\rightharpoonup\nabla\widetilde{\omega}_\varepsilon(x),$$

in $L^p_{\text{loc}}(\Omega)$ as $t \to \infty$, which means that $\widetilde{v}_\varepsilon(x)$ and $\widetilde{\omega}_\varepsilon(x)$ are the solutions of problem

$$\begin{cases} -\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \lambda w^{q-m} & x \in \Omega, \\ w|_{\partial\Omega} = \varepsilon, & \\ w > 0 & x \in \Omega. \end{cases} \tag{3.15}$$

According to Propositions 2.2, 2.5 and Remark 2.6, we see that, when $q < m + p - 1$, equation (3.15) admits a unique solution, namely, $\widetilde{v}_\varepsilon(x) = \widetilde{\omega}_\varepsilon(x)$, denoted $\widetilde{w}_\varepsilon(x)$. Then, $u_\varepsilon(x,t) \to \widetilde{w}_\varepsilon(x)$ uniformly. Furthermore, we see that

$$k\phi(x) \le v_\varepsilon(x,t) \le u_\varepsilon(x,t) \le \omega_\varepsilon(x,t) \le K\psi(x).$$

By the above arguments, we see that, when $t \to \infty$, $u_\varepsilon(x,t)$ goes to a steady solution, that is, the solution of equation (3.15). Similar to **[8]**, it is not difficult to show that, when $\varepsilon \to 0$,

$$v_\varepsilon(x,t) \longrightarrow v, \qquad u_\varepsilon(x,t) \longrightarrow u, \qquad \omega_\varepsilon \longrightarrow \omega,$$

in the weak sense, and v, u and ω are solutions of problems (1.1)–(1.3), (3.5) and (3.6) with $\varepsilon = 0$, respectively. Repeating the above process with $\varepsilon = 0$, we see that all the estimates obtained above hold for $\varepsilon = 0$. We finally conclude that $u(x,t)$ goes to the solution of equation (3.15) with $\varepsilon = 0$, uniformly. Thus, we obtain the next result.

Theorem 3.2. *Assume that* $0 < q < m + p - 1$. *Then, the solution* $u(x,t)$ *of problem* (1.1)–(1.3) *obtained by an approximating process of the regularized problem* (3.1) *goes to the positive maximum steady solution* $w(x)$ *of problem* (2.3) *with* $\gamma = q - m$ *uniformly.*

In what follows, we consider the case $q = m + p - 1$. We obtain the next result.

Theorem 3.3. *Assume that* $q = m+p-1$. *Let* λ_1 *be the first eigenvalue of the p-laplacian with homogeneous boundary condition, and let* φ *be the corresponding eigenfunction with* $\|\varphi\|_\infty = 1$. *Then,*

(i) *when* $\lambda < \lambda_1$, *all of these solutions of problem* (1.1)–(1.3) *go to* 0;

(ii) *when* $\lambda = \lambda_1$*, if there exist* $\varepsilon > 0$ *such that* $u_0(x) > \varepsilon\varphi(x)$*, then the maximal solution of problem* (1.1)–(1.3) *goes to a steady state* $K\varphi(x)$ *for some* $K > 0$;

(iii) *when* $\lambda > \lambda_1$*, all positive solutions blow up in some finite time.*

Proof.

(i) Firstly, we consider the case $\lambda < \lambda_1$. We see that there exist $\widetilde{\lambda}$ with $\lambda < \widetilde{\lambda} < \lambda_1$ and a domain $\Omega \subset\subset \widetilde{\Omega}$ such that $\widetilde{\lambda}$ is the first eigenvalue of the p-laplacian with homogeneous boundary condition, and, correspondingly, $\widetilde{\psi}$ is the first eigenfunction with $\|\widetilde{\psi}\|_\infty = 1$. A simple calculation yields that $K\widetilde{\psi}$ is a super-solution of the regularized problem (3.1) for appropriately large $K > 0$. Then, we have $\varepsilon \le u_\varepsilon \le K\widetilde{\psi}$. Similar to the case $q < m+p-1$, we define ω_ε as in equation (3.5), and we conclude that $u_\varepsilon(x,t) \le \omega_\varepsilon(x,t)$. Furthermore, we also have $\omega_\varepsilon(x,t)$ is decreasing on t, denoted

$$\omega_\varepsilon(x) = \lim_{t\to\infty} \omega_\varepsilon(x,t).$$

Then,

$$\overline{\lim_{t\to\infty}}\, u_\varepsilon(x,t) \le \omega_\varepsilon(x).$$

Similarly, we obtain that $\omega_\varepsilon(x)$ is a solution of the steady problem (3.15). Letting $\varepsilon \to 0$, we conclude that

$$\omega(x) = \lim_{\varepsilon\to 0} \omega_\varepsilon(x)$$

is a solution of problem (3.15) with $\varepsilon = 0$. Clearly problem (3.15) has no nontrivial solution since $\lambda < \lambda_1$, that is, $\omega(x) = 0$. Thus, we have $u_\varepsilon(x,t) \to 0$ as $t \to \infty$, $\varepsilon \to 0$. By comparison, we see that, for any solution u of problem (1.1)–(1.3), $u(x,t) \le u_\varepsilon(x,t)$, which means that $u(x,t)$ goes to 0 uniformly as $t \to \infty$.

(ii) Now, we turn our attention to the case $\lambda = \lambda_1$. Let

$$F(\omega) = \frac{1}{p}\int_\Omega |\nabla\omega(x)|^p\,dx - \frac{\lambda}{p}\int_\Omega \omega^p(x)\,dx,$$

$$H(\omega) = \begin{cases} 1/(2-m)\int_\Omega \omega^{2-m}(x)\,dx & \text{if } m \ne 2,\\ \int_\Omega \ln\omega(x)\,dx & \text{if } m = 2.\end{cases}$$

Since $u_0 > \varepsilon\varphi(x)$, by comparison, $u \geq \varepsilon\varphi(x)$. Therefore, $F(u)$ and $H(u)$ are well defined. We see that

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{p}\int_\Omega u^p\, dx \geq \frac{\lambda_1 - \lambda}{p}\int_\Omega u^p\, dx = 0.$$

In addition, we also note that

$$\frac{dF(u)}{dt} = -\int_\Omega u^{-m}\, u_t^2\, dx \leq 0,$$

which implies that $F(u)(t)$ is decreasing. Let

$$\lim_{t\to\infty} F(u)(t) = a.$$

Then, we have $u_t \to 0$ as $t \to \infty$, which implies that u goes to 0 or the first eigenfunction $k\varphi(x)$. Hence, we have that $u(x)$ goes to the first eigenfunction $k\varphi(x)$ since $u \geq \varepsilon\varphi(x)$.

(iii) Finally, we give attention to the case $\lambda > \lambda_1$. A direct calculation yields that

$$\frac{dH}{dt} = -pF.$$

If $F(u_0) < 0$, from the decreasing property of F on t, we have $F(u(t)) \leq F(u_0) = -\delta$; then

$$H(u(t)) \geq H(u_0) + p\,\delta\, t.$$

If $m > 2$,

$$\int_\Omega u^{2-m} dx \leq \int_\Omega u_0^{2-m} dx - (m-2)\,\delta\, t.$$

Thus, there exists a $0 < T^* < +\infty$ such that $\int_\Omega u^{2-m} dx = 0$, which implies that u blows up in finite time. If $m < 2$, then

$$\frac{d^2H(u)}{dt^2} = -p\frac{dF(u)}{dt} = p\int_\Omega u^{-m} u_t^2\, dx.$$

Then, we have

$$\begin{aligned}\left(\frac{dH}{dt}\right)^2 &= \left(\int_\Omega u^{1-m} u_t\, dx\right)^2 \leq \int_\Omega u^{2-m}\, dx \int_\Omega u^{-m} u_t^2\, dx \\ &= \frac{2-m}{p}\,\frac{d^2H}{dt^2} H,\end{aligned}$$

which implies that

$$\frac{d^2}{dt^2} H^{1-p/(2-m)}(u) \leq 0.$$

Noting that $dH(u(0))/dt > 0$, $d^2H/dt^2 > 0$, we have

$$\frac{dH}{dt} \geq \frac{dH(u(0))}{dt} > 0.$$

Direct calculation gives

$$\begin{aligned} H^{(2-m-p)/(2-m)}(u)(t) \leq{}& H^{(2-m-p)/(2-m)}(u)(0) \\ &+ p\,\frac{m+p-2}{2-m} H^{-p/(2-m)}(u_0)F(u_0)t. \end{aligned}$$

Thus, there exists a $T^* < +\infty$ such that

$$H(u)(t) \longrightarrow \infty, \quad \text{as } t \to T^*,$$

while, if $m = 2$, we have

$$\frac{dH}{dt} = \int_\Omega u^{-1} u_t \, dx.$$

$$\frac{d^2H(u)}{dt^2} = -p\,\frac{dF(u)}{dt} = p \int_\Omega u^{-2} u_t^2 \, dx \geq p\left(\frac{dH}{dt}\right)^2.$$

Then, we have

$$\left(\frac{1}{H'}\right)' \leq -p.$$

Integrating from 0 to t yields

$$H'(u(t)) \geq \left(-\frac{1}{pF(u_0)} - pt\right)^{-1},$$

which means that

$$H(u(t)) \geq H(u_0) + \frac{1}{p} \ln \frac{c_0}{c_0 - pt},$$

where $c_0 = -1/(pF(u_0))$. Thus, there exists a $T^* < +\infty$ such that $H \to \infty$ as $t \to T^*$, that is, u blows up.

In what follows, we consider the general case $u_0(x) > 0$. It is known that there exists a $\Omega' \subset\subset \Omega$ such that the first eigenvalue $\lambda_1 < \lambda' < \lambda$

on Ω' and some function $\psi(x)$ with $\psi(x) < u_0(x)$ on Ω' such that

$$\widehat{F}(\psi) = \frac{1}{p}\int_{\Omega'} |\nabla\psi|^p\,dx - \frac{\lambda}{p}\int_{\Omega'} \psi^p\,dx < 0.$$

Denote the solution on Ω' by $\widetilde{u}$ with initial value $\psi(x)$. Similar to the above, we obtain the same results on Ω', while we clearly have $u \geq \widetilde{u}$. Thus, we conclude that there exists a $T^* < \infty$ such that $u(t) \to \infty$ as $t \to T^*$, that is, u blows up in finite time. $\square$

Lastly, we turn to the case $q > m + p - 1$. From **[8]**, we see that, when $F(u_0) < 0$, the solution will blow up in finite time. Thus, in what follows, we may assume that $F(u_0) \geq 0$.

From Sacks **[16]**, we define

$$\mathscr{A} = \{0 \leq \varphi \in C_0^1(\overline{\Omega}); -\mathrm{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = \lambda\varphi^{q-m}\}, \qquad \mathscr{A}^* = \mathscr{A}\backslash\{0\},$$

and let

$$\rho = \rho(\Omega) = \inf_{v\in\mathscr{A}^*} F(v),$$

with $\rho = +\infty$ if $\mathscr{A}^* = \emptyset$, and

$$\begin{aligned} L(v_0) = \{u(x) \in C(\overline{\Omega});\ &\text{there exists } t_n \to \infty \\ &\text{such that } u(\cdot, t_n, u_0) \Longrightarrow u(x) \text{ as } n \to \infty\}, \end{aligned}$$

and

$$B_\tau(u_0) = \{u(\cdot, t, u_0); t \geq \tau\}.$$

For any $v \in \mathscr{A}^*$, let

$$q^* = \frac{q - (m + p - 1)}{p(q + 1 - m)}.$$

Then,

$$F(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \frac{\lambda}{q+1-m}\int_\Omega v^{q+1-m}dx = q^*\int_\Omega |\nabla v|^p\,dx.$$

We also note that

$$\int_\Omega |\nabla v|^p\,dx = \lambda\int_\Omega v^{q+1-m}dx.$$

By the isotropic embedding theorem, we see that, for $q < Np/(N-p)+m-1$ if $p < N$, or $q < +\infty$ if $p \geq N$. We have

$$\int_\Omega |\nabla v|^p \, dx = \lambda \int_\Omega v^{q+1-m} dx \leq \lambda C\Big(\int_\Omega |\nabla v|^p \, dx\Big)^{(q+1-m)/p},$$

where C is a constant dependent only on m, p, q and Ω. Therefore, we further have

$$\int_\Omega |\nabla v|^p \, dx \geq (C\lambda)^{-p/(q-m-p+1)}.$$

Hence, we have

$$\rho \geq \frac{q-(m+p-1)}{p(q+1-m)}(C\lambda)^{-p/(q-m-p+1)},$$

while, if $q \geq Np/(N-p)+m-1$ for $p < N$, by **[14]**, we see that $\mathscr{A}^* = \emptyset$ if Ω is star-shaped. This means that $\rho\,(\Omega) = +\infty$. Summing up, we finally obtain $\rho > 0$.

Multiplying equation (1.1) by $u_t u^{-m}$ on both sides and integrating over $(0,t)\times\Omega$ yields

$$\int_0^t \int_\Omega u^{-m} u_t^2 \, dx \, d\tau + F(u)(t) = F(u_0), \tag{3.16}$$

which implies that $F(u)(t) \leq F(u_0)$ and is nonincreasing on t. Now, we have the next lemmas.

Lemma 3.4. *Assume that $F(u_0) \geq 0$ and $B_\tau(u_0) \leq C$. Then, $\mathscr{A} \cap L(u_0) \neq \emptyset$.*

Proof. We first show that $L(u_0) \neq \emptyset$. Since $B_\tau(u_0) \leq C$, the collection of functions $\{u(\cdot,t;u_0)\}_{t>\tau}$ is uniformly bounded, and hence, equicontinuous by [**16,** Theorem 2.2]. Therefore, by the Arzela-Ascoli theorem, there exists a $\{t_n\}$ with $t_n \to \infty$ and $v \in C(\overline{\Omega})$ such that $u(\cdot,t_n;u_0) \to v(x)$ uniformly, which means that $v \in L(u_0)$.

Let $F(v^*) = \min_{v\in L(u_0)} F(v)$. Clearly, $F(v^*)$ is well defined and $v^* \in L(u_0)$ since $F(v)$ is lower semi-continuous and bounded from below on $L(u_0)$. However, $v(\cdot,t;v^*) \in L(u_0)$ for t since $F(u(t))$ is decreasing on t, which means that

$$F(u(\cdot,t,v^*)) = F(v^*)$$

for any $t > 0$. From equation (3.16),

$$\|u^{-m/2}u_t\|_{L^2(\Omega)} \longrightarrow 0, \quad \text{as } t \to \infty;$$

thus, we necessarily have $v^* \in \mathscr{A}$. $\square$

Lemma 3.5. *If* $0 \in L(u_0)$, *then* $L(u_0) = \{0\}$.

Proof. Suppose to the contrary; namely, there is a $v \in L(u_0)$ with $v \not\equiv 0$. Take a domain $\widetilde{\Omega}$ such that $\Omega \subset\subset \widetilde{\Omega}$. Let $\widetilde{\psi}$ with $\|\widetilde{\psi}\|_\infty = 1$ be the first eigenfunction of the p-Laplacian corresponding to the Dirichlet boundary value condition on $\widetilde{\Omega}$, let $\widetilde{\lambda}$ be the corresponding first eigenvalue, and let $\sigma = \inf_{x\in\Omega} \widetilde{\psi}$. Let $\Psi = k\widetilde{\psi}$. After simple calculation, we conclude that Ψ is a super-solution of equation (1.1) on Ω if

$$k \le \left(\frac{\widetilde{\lambda}}{\lambda}\right)^{1/(q-(m+p-1))}.$$

Since $0 \in L(u_0)$, there exists $\{t_n\}$ with $t_n \to \infty$ such that $u(\cdot, t_n, u_0) \to 0$ uniformly in Ω. Let

$$\delta = \min\left\{\frac{\|v\|_\infty}{2}, \left(\frac{\widetilde{\lambda}}{\lambda}\right)^{1/(q-(m+p-1))}\right\}.$$

Take $k < \delta$. It is easy to see that there exists an $n_0 > 0$ such that, for any $n \ge n_0$,

$$\|u(\cdot, t_n, u_0)\|_{L^\infty(\Omega)} \le k\sigma \le \Psi.$$

By comparison, we conclude that

$$u(\cdot, t, u_0) \le \Psi \le \frac{\|v\|_\infty}{2}, \quad \text{for any } t > t_{n_0},$$

which contradicts $v \in L(u_0)$. $\square$

Theorem 3.6. *Assume that* $q > m + p - 1$, $0 \le u_0 \in L^\infty(\Omega)$. *Then,* $L(u_0) = \{0\}$ *if one of the following conditions is satisfied*:

(i) $B_\tau(u_0) \le C$ *and* $F(u_0) < \rho(\Omega)$;
(ii) *there exists* $\mu < 1$, $\omega \in \mathscr{A}^*$, *such that* $u_0 \le \mu\omega$;

(iii)
$$u_0 \le \left(\frac{2\lambda}{\widetilde{\lambda}}\right)^{-1/(q-(m+p-1))} \sigma,$$
for some domain $\widetilde{\Omega}$ *with* $\widetilde{\lambda}$, σ *defined as in Lemma* 3.5.

Proof. From Lemma 3.4, we see that it suffices to show that $B_\tau(u_0) \le C$. Then, $\mathscr{A}^* \cap L(u_0) = \emptyset$.

(i) From the definition of ρ and the monotonicity of $F(u)(t)$ on t, the conclusion is clear.

(ii) We note that
$$-\mathrm{div}(|\nabla(\mu\omega)|^{p-2}\nabla(\mu\omega)) = \mu^{m+p-1-q}\lambda(\mu\omega)^{q-m} \ge \lambda(\mu\omega)^{q-m},$$
which means that $\mu\omega$ is a super-solution. By comparison, we arrive at $u(\cdot, t, u_0) \le \mu\omega$. If $\mathscr{A}^* \cap L(u_0) \ne \emptyset$, then there exists a $v \in \mathscr{A}^* \cap L(u_0)$ with $v \le \mu\omega$. Let
$$\mu^* = \inf\{\mu < 1; v \le \mu\omega\}.$$
Then, $\mu^* < 1$, $v \le \mu^*\omega$. In addition, we note that
$$\begin{aligned} -\mathrm{div}(|\nabla(\mu^*\omega)|^{p-2}\nabla(\mu^*\omega)) &= \mu^{*m+p-1-q}\lambda(\mu^*\omega)^{q-m} \\ &\ge \lambda\mu^{*m+p-1-q}v^{q-m} \\ &= -\mu^{*m+p-1-q}\mathrm{div}(|\nabla v|^{p-2}\nabla v), \end{aligned}$$
which implies that
$$\mu^*\omega \ge \mu^{*(m+p-1-q)/(p-1)}v,$$
which contradicts the definition of μ^*.

(iii) Take $\widetilde{\Omega}$, $\widetilde{\psi}$, $\widetilde{\lambda}$, and σ as in Lemma 3.5, and let
$$\overline{u}(x,t) = f(t)\widetilde{\psi}(x),$$
where
$$f(t) = \left(a_0 + \frac{m+p-2}{2}\widetilde{\lambda}\sigma^{m+p-2}t\right)^{-1/(m+p-2)}.$$
Then,
$$\frac{\partial \overline{u}}{\partial t} \ge \overline{u}^m \mathrm{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) + \lambda\overline{u}^q \tag{3.17}$$

is equivalent to

$$f'(t) \geq f^{m+p-1}\widetilde{\psi}^{m+p-2}(\lambda(f\widetilde{\psi})^{q-(m+p-1)} - \widetilde{\lambda}).$$

Let

$$a_0 = \left(\frac{2\lambda}{\widetilde{\lambda}}\right)^{m+p-2/(q-(m+p-1))}.$$

A simple calculation yields that $\overline{u}$ satisfies equation (3.17). Then, for any $u_0(x) < \overline{u}(x,0)$, we have $u \leq \overline{u}$. Hence, $u(\cdot,t,u_0) \to 0$ as $t \to \infty$ since $\overline{u}(x,t) \to 0$ as $t \to \infty$ means that $L(u_0) = 0$. The proof is complete. □

Remark 3.7. From Theorem 3.6 (ii), we see that any nonnegative and nontrivial steady solutions are unstable, while, from Theorem 3.6 (iii), we see that 0 is an asymptotically stable steady solution.

4. Extension to the periodic source case. In previous work [**9**], we have proved the existence of time periodic solutions for the periodic source case. In this section, as an extension to the results obtained above, we consider asymptotic stability for the periodic source case.

Consider the equation:

$$\frac{\partial u}{\partial t} = u^m \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \alpha(x,t)u^q, \quad (x,t) \in Q, \tag{4.1}$$

with boundary and initial value conditions (1.2) and (1.3), where $\alpha(x,t)$ is smooth, strictly positive and periodic in time with periodic $\omega > 0$, and all other conditions are the same as equations (1.1)–(1.3).

First, consider the sub-critical case, that is, $q < m+p-1$. We have the following results.

Theorem 4.1. *Assume that $q < m+p-1$. Then, problem* (1.2), (4.1) *has minimal and maximal positive periodic solutions $u_*(x,t)$ and $u^*(x,t)$. Moreover, if $u(x,t)$ is the maximal solution of the initial boundary value problem* (1.2), (1.3), (4.1), *then, for any $\delta > 0$,*

$$u_*(x,t) - \delta \leq u(x,t) \leq u^*(x,t) + \delta$$

holds for $x \in \Omega$ and sufficiently large t.

Proof of Theorem 4.1. Consider the regularized problem

$$\frac{\partial u}{\partial t} = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \alpha(x,t)u^q \quad (x,t) \in Q_T, \tag{4.2}$$

$$u(x,t) = \varepsilon \quad x \in \partial\Omega, \tag{4.3}$$

$$u(x,0) = u_0(x) + \varepsilon = u_{0\varepsilon}(x) \quad x \in \Omega. \tag{4.4}$$

From [**8**], we see that, for any fixed small $\varepsilon > 0$, problem (4.2)–(4.4) always admits a solution $\varepsilon \le u_\varepsilon \in W_p^{1,0}(Q_T) \cap W_2^{0,1}(Q_T) \cap L^\infty(Q_T)$. In order to obtain the minimal and maximal periodic solutions of problem (4.2)–(4.4), we consider the problem:

$$\begin{cases} \partial u/\partial t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + Lu^q & (x,t) \in Q_T, \\ u(x,t) = \varepsilon & x \in \partial\Omega, \\ u(x,0) = u_0(x) + \varepsilon = u_{0\varepsilon}(x) & x \in \Omega. \end{cases} \tag{4.5}$$

$$\begin{cases} \partial u/\partial t = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + Su^q & (x,t) \in Q_T, \\ u(x,t) = \varepsilon & x \in \partial\Omega, \\ u(x,0) = u_0(x) + \varepsilon = u_{0\varepsilon}(x) & x \in \Omega. \end{cases} \tag{4.6}$$

From Section 3, we see that solutions $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$ of problems (4.5) and (4.6), respectively, go to the unique positive steady state $\Psi_\varepsilon(x)$, $\phi_\varepsilon(x)$ with $\Psi_\varepsilon(x) \ge \phi_\varepsilon(x)$. Let ψ with $\|\psi\|_\infty = 1$ be the first eigenfunction of the p-Laplacian equation corresponding to the Dirichlet boundary value condition on Ω, and let λ_1 be the corresponding first eigenvalue. Moreover, take $\widetilde{\Omega} \supset \overline{\Omega}$ and $\widetilde{\psi}$ with $\|\widetilde{\psi}\|_\infty = 1$, $\sigma = \inf_{x\in\Omega} \widetilde{\psi} > \varepsilon$, to be the first eigenfunction of the p-Laplacian equation corresponding to the Dirichlet boundary value condition on $\widetilde{\Omega}$. Let $\widetilde{\lambda}_1$ be the corresponding first eigenvalue. Then,

$$-\operatorname{div}(|\nabla(\rho\psi)|^{p-2}\nabla(\rho\psi)) = \lambda_1(\rho\psi)^{p-1} \le S(\rho\psi)^{q-m}$$

is ensured by

$$\rho^{m+p-1-q} \le \frac{S}{\lambda_1},$$

while

$$-\operatorname{div}(|\nabla(\widetilde{\rho}\widetilde{\psi})|^{p-2}\nabla(\widetilde{\rho}\widetilde{\psi})) = \widetilde{\lambda}_1(\widetilde{\rho}\widetilde{\psi})^{p-1} \ge L(\widetilde{\rho}\widetilde{\psi})^{q-m}$$

is ensured by

$$(\widetilde{\rho}\sigma)^{m+p-1-q} \geq \frac{L}{\widetilde{\lambda}_1}.$$

Take ρ sufficiently small and $\widetilde{\rho}$ sufficiently large; then, we have that $\widetilde{\rho}\widetilde{\psi}$ and $\rho\psi$ are both the super- and sub-solutions of the problems

$$\begin{cases} -\mathrm{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = L\varphi^{q-m} & x \in \Omega, \\ \varphi(x) = \varepsilon & x \in \partial\Omega, \end{cases} \tag{4.7}$$

and

$$\begin{cases} -\mathrm{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = S\varphi^{q-m} & x \in \Omega, \\ \varphi(x) = \varepsilon & x \in \partial\Omega. \end{cases} \tag{4.8}$$

By Remark 2.6, problem (4.7), (4.8) admits a positive solution $\overline{\phi}_\varepsilon$, $\underline{\phi}_\varepsilon$ with $\rho\psi \leq \underline{\phi}_\varepsilon \leq \overline{\phi}_\varepsilon \leq \widetilde{\rho}\widetilde{\psi}$. For the uniqueness of solutions, see Remark 2.6. We further have

$$\underline{\phi}_\varepsilon = \phi_\varepsilon, \qquad \overline{\phi}_\varepsilon = \Phi_\varepsilon.$$

By comparison, we see that the solution u_ε of problem (4.2)–(4.4) satisfies $\underline{u}_\varepsilon \leq u_\varepsilon \leq \overline{u}_\varepsilon$. Then, we have

$$\rho\psi \leq \underline{\phi}_\varepsilon \leq \varliminf_{t\to\infty} u_\varepsilon(x,t) \leq \varlimsup_{t\to\infty} u_\varepsilon(x,t) \leq \overline{\phi}_\varepsilon \leq \widetilde{\rho}\widetilde{\psi}, \tag{4.9}$$

for any positive initial value $u_0(x)$ satisfying the corresponding compatibility conditions. Thus, for any positive periodic solution $\widehat{u}_\varepsilon$ of problem (4.2)–(4.3), we have that

$$\rho\psi \leq \widehat{u}_\varepsilon(x,t) \leq \widetilde{\rho}\widetilde{\psi}.$$

Define the sequences $\{\underline{u}_n\}_n$, $\{\overline{u}_n\}_n$ by

$$\begin{cases} \underline{u}_{nt} - \underline{u}_n^m \mathrm{div}(|\nabla\underline{u}_n|^{p-2}\nabla\underline{u}_n) = \alpha(x,t)\underline{u}_{n-1}^q, \\ \underline{u}_n(x,t)|_\Omega = \varepsilon, \\ \underline{u}_n(x,0) = (\underline{u}_{n-1}(x,\omega) - \varepsilon)_+ + \varepsilon, \end{cases} \tag{4.10}$$

$$\begin{cases} \overline{u}_{nt} - \overline{u}_n^m \mathrm{div}(|\nabla\overline{u}_n|^{p-2}\nabla\overline{u}_n) = \alpha(x,t)\overline{u}_{n-1}^q, \\ \overline{u}_n(x,t)|_\Omega = (\widetilde{\rho}\widetilde{\psi} - \varepsilon)e^{-(n-1)\omega-t} + \varepsilon, \\ \overline{u}_n(x,0) = \overline{u}_{n-1}(x,\omega), \end{cases} \tag{4.11}$$

with $\underline{u}_0 = \rho\psi$, $\overline{u}_0 = \widetilde{\rho}\widetilde{\psi}$. Similar to Section 2, by recursivity, and combining with the above analysis, for any positive periodic solution $\widehat{u}_\varepsilon$ of problem (4.2)–(4.3), it is easy to obtain the solutions of problem (4.10), (4.11), satisfying that

$$\rho\psi \le \underline{u}_1 \le \cdots \le \underline{u}_n \le \cdots \le \widehat{u}_\varepsilon \le \overline{u}_n \le \cdots \le \overline{u}_1 \le \widetilde{\rho}\widetilde{\psi}. \tag{4.12}$$

Let

$$\underline{u}_{\varepsilon *} = \lim_{n\to\infty} \underline{u}_n(x,t), \qquad \overline{u}_\varepsilon^* = \lim_{n\to\infty} \overline{u}_n(x,t).$$

Similar to **[22]**, we obtain that $\underline{u}_{\varepsilon *}$ and $\overline{u}_\varepsilon^*$ are the periodic solutions of problem (4.2)–(4.3). Then, we have

$$\underline{u}_{\varepsilon *} \le \widehat{u}_\varepsilon \le \overline{u}_\varepsilon^*.$$

Letting ε go to 0, similar to **[22]**, we obtain that the limit functions

$$\underline{u}_* = \lim_{\varepsilon\to 0} \underline{u}_{\varepsilon *}, \qquad \overline{u}^* = \lim_{\varepsilon\to 0} \overline{u}_\varepsilon^*,$$

are the minimal and maximal positive periodic solutions.

On the other hand, by $\partial u_0/\partial \mathbf{n} > 0$ on $\partial\Omega$, we see that there must exist appropriately small $\rho > 0$ and large $\widetilde{\rho} > 0$ such that $\rho\psi \le u_\varepsilon(x,0) \le \widetilde{\rho}\widetilde{\psi}$ for sufficiently small $\varepsilon > 0$. Then, for the solution $u_\varepsilon(x,t)$ of problem (4.2)–(4.4), we also have

$$\underline{u}_{k+1}(x,t) \le u_\varepsilon(x,k\omega+t) \le \overline{u}_{k+1}(x,t). \tag{4.13}$$

Recalling equation (4.13), and letting $k\to\infty$, we obtain that

$$\underline{u}_{\varepsilon *}(x,t) \le \varliminf_{k\to\infty} u_\varepsilon(x,k\omega+t) \le \varlimsup_{k\to\infty} u_\varepsilon(x,k\omega+t) \le \overline{u}_\varepsilon^*(x,t).$$

Furthermore, noting the monotonicity of $\underline{u}_{\varepsilon *}$, $\overline{u}_\varepsilon^*$ and u_ε on ε, and letting $\varepsilon\to 0$, yields

$$\underline{u}_* \le \lim_{\varepsilon\to 0} \varliminf_{k\to\infty} u_\varepsilon(x,k\omega+t) \le \lim_{\varepsilon\to 0} \varlimsup_{k\to\infty} u_\varepsilon(x,k\omega+t) \le \overline{u}^*.$$

Denote $u^* = \lim_{\varepsilon\to 0} u_\varepsilon(x,t)$. We see that u^* is a solution of problem (1.2), (1.3), (4.1), see **[8]**. Then, we have

$$\underline{u}_*(x,t) \le \varliminf_{k\to\infty} u^*(x,k\omega+t) \le \varlimsup_{k\to\infty} u^*(x,k\omega+t) \le \overline{u}^*(x,t).$$

Thus, for any $\delta > 0$, there exists a $T_\delta > 0$, such that, for any $t > T_\delta$,

$$\underline{u}_*(x,t) - \delta \le u^*(x,t) \le \overline{u}^*(x,t) + \delta.$$

The proof is complete. □

In what follows, we consider the case of $q = m+p-1$. We have the next result.

Theorem 4.2. *Assume that $q = m+p-1$. Let λ_1 be the first eigenvalue of the p-Laplace equation with homogeneous Dirichlet boundary value condition. Then,*

(i) *when $\alpha(x,t) < \lambda_1$, all solutions of problem* (1.2), (1.3), (4.1) *go to* 0;
(ii) *when $\alpha(x,t) \equiv \lambda_1$, if there exist $\varepsilon > 0$ such that $u_0(x) > \varepsilon\varphi(x)$, then the maximal solution of problem* (1.2), (1.3), (4.1) *goes to a steady state $K\varphi(x)$ for some $K > 0$;*
(iii) *when $\alpha(x,t) > \lambda_1$, all positive solutions blow up.*

We show that Theorem 4.2 is a direct result of Theorem 3.3. In fact, consider problem (1.1)–(1.3). If $\alpha(x,t) < \lambda_1$, we choose λ with $\alpha(x,t) < \lambda < \lambda_1$. Denote the solution of problem (1.1)–(1.3) by u_λ. Clearly, u_λ is a super-solution of problem (1.2), (1.3), (4.1). By comparison, we see that $u \le u_\lambda$. Thus, $u \to 0$ since $u_\lambda \to 0$. Similarly, if $\alpha(x,t) > \lambda_1$, we choose λ such that $\alpha(x,t) > \lambda > \lambda_1$. It follows that $u \ge u_\lambda$. Therefore, u blows up.

We study the asymptotic stability of the zero solution for the sup-critical case. Consider the equation

$$\frac{\partial u}{\partial t} = u^m \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \|\alpha\|_\infty u^q, \quad (x,t) \in Q, \tag{4.14}$$

with initial and boundary value conditions (1.2), (1.3). Denote the maximal solution of the above problem by u^*, and the solution of problem (1.2), (1.3), (4.1) by u. Clearly, u^* is a super-solution of problem (1.2), (1.3), (4.1). By comparison, $u^* > u$. Denote

$$\widehat{F}(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \frac{\|\alpha\|_\infty}{q+1-m}\int_\Omega v^{q+1-m} dx.$$

Similarly to Section 3, we define $\widehat{\mathscr{A}}$, $\widehat{\mathscr{A}^*}$, $\widehat{\rho}$, $\widehat{L}(u_0)$ and $\widehat{B}_\tau(u_0)$ as:

$$\widehat{\mathscr{A}}=\{\varphi; 0\le\varphi\in W_0^{1,p}(\Omega)\cap L^{q-m+1}(\Omega), -\mathrm{div}(|\nabla\varphi|^{p-2}\nabla\varphi)=\|\alpha\|_\infty\varphi^{q-m}\},$$
$$\widehat{\mathscr{A}^*}=\widehat{\mathscr{A}}\setminus\{0\},\quad \widehat{\rho}=\widehat{\rho}(\Omega)=\inf_{v\in\widehat{\mathscr{A}^*}}\widehat{F}(v),\qquad \widehat{\rho}=+\infty\quad\text{if } \widehat{\mathscr{A}^*}=\emptyset,$$

$$\widehat{L}(u_0)=\{u(x);\text{there exists } t_n\to\infty \text{ such that, for some } r>0,\\ \text{in } L^r(\Omega),\text{when } n\to\infty, u^*(\cdot,t_n,u_0)\to u(x)\},$$

$$\widehat{B}_\tau(u_0)=\{u^*(\cdot,t,u_0);t\ge\tau\}.$$

By comparison, we have the next result.

Theorem 4.3. *Assume that* $q>m+p-1$, $u_0\in L^\infty(\Omega)$. *Then, when* $t\to\infty$, *the solution* $u(\cdot,t,u_0)$ *of problem* (1.2), (1.3), (4.1) *goes to* 0 *if one of the following conditions hold*:

(i) u^* *is bounded uniformly, and* $\widehat{F}(u_0)<\widehat{\rho}(\Omega)$;

(ii) *there exists* $\mu<1$, $\omega\in\widehat{\mathscr{A}^*}$, *such that* $u_0\le\mu\omega$;

(iii)
$$u_0\le\left(\frac{2\|\alpha\|_\infty}{\widetilde{\lambda}}\right)^{-1/(q-(m+p-1))}\sigma,$$

for some domain $\widetilde{\Omega}\supset\supset\Omega$, *where* $\widetilde{\lambda}$, σ *defined as in Section* 3, *Lemma* 3.5.

REFERENCES

1. N.D. Alikakos and R. Rostamian, *Large time behavior of solutions of Neumann boundary value problems for the porous medium equation*, Indiana Univ. Math. J. **30** (1981), 749–785.

2. ______, *Stabilization of solutions of the equation* $\partial u/\partial t=\Delta\phi(u)-\beta(u)$, Nonlin. Anal. **6** (1982), 637–647.

3. L.J.S. Allen, *Persistence and extinction in single-species reaction-diffusion models*, Bull. Math. Biol. **45** (1983), 209–227.

4. S. Angenent, *On the formation of singularities in the curve shortening flow*, J. Diff. Geom. **33** (1991), 601–633.

5. M. Bertsch and P. Bisegna, *Blow-up of solutions of a nonlinear parabolic equation in damage mechanics*, Europ. J. Appl. Math. **8** (1997), 89–123.

6. R. Dal Passo and S. Luckhaus, *A degenerate diffusion problem not in divergence form*, J. Diff. Eq. **69** (1987), 1–14.

7. J. García-Melián and J. Sabina de Lis, *Maximum and comparison principles for operators involving the p-Laplacian*, J. Math. Anal. Appl. **218** (1998), 49–65.

8. C.H. Jin and J.X. Yin, *Critical exponent of a doubly degenerate parabolic equation in non-divergence form with reaction sources*, Chinese Ann. Math. **30** (2009), 525–538.

9. ———, *Periodic solutions of a non-divergent diffusion equation with nonlinear sources*, Disc. Cont. Dynam. **17** (2012), 101–126.

10. Gary M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlin. Anal. Th. Meth. Appl. **12** (1988), 1203–1219.

11. P.L. Lions, *Asymptotic behavior of some nonlinear heat equations*, Phys. Nonlin. Phenom. **5** (1982), 293–306.

12. B.C. Low, *Resistive diffusion of force-free magnetic fields in a passive medium*, Astrophys. J. **81** (1973), 209–226.

13. H. Matano, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Kyoto University, Publ. RIMS **15** (1979), 401–454.

14. M. Ôtani, *Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations*, J. Funct. Anal. **76** (1988), 140–159.

15. T. Peter, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Eq. **8** (1983), 773–817.

16. P.E. Sacks, *Global behavior for a class of nonlinear evolution equations*, SIAM J. Math. Anal. **16** (1985), 233–250.

17. J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

18. C.P. Wang and J.X. Yin, *Shrinking self-similar solutions of a nonlinear diffusion equation with nondivergence form*, J. Math. Anal. Appl. **289** (2004), 387–404.

19. F. Wesslir, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.

20. M. Wiegner, *Blow-up for solutions of some degenerate parabolic equations*, Diff. Integ. Eq. **7** (1994), 1641–1647.

21. ———, *A degenerate diffusion equation with a nonlinear source term*, Nonlin. Anal. Th. Meth. Appl. **28** (1997), 1977–1995.

22. J.X. Yin and C.H. Jin, *Periodic solutions of the evolutionary p-Laplacian with nonlinear sources*, J. Math. Anal. Appl. **368** (2010), 604–622.

23. W. Zhou and Z. Wu, *Some results on a class of degenerate parabolic equations not in divergence form*, Nonlin. Anal. **60** (2005), 863–886.

24. W. Zhou and Z. Yao, *Behaviors of solutions for a singular diffusion equation*, J. Math. Anal. Appl. **327** (2007), 611–619.

South China Normal University, School of Mathematical Sciences, Guangzhou 510631, China
Email address: **jinchhua@126.com**

South China Normal University, School of Mathematical Sciences, Guangzhou 510631, China
Email address: **yjx@scnu.edu.cn**