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THE CLASSIFICATION OF INFINITE ABELIAN GROUPS WITH PARTIAL DECOMPOSITION BASES IN $L_{\infty\omega}$

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ABSTRACT. We consider the class of abelian groups with partial decomposition bases, which includes groups classified by Ulm, Warfield, Stanton and others. We define an invariant and classify these groups in the language $L_{\infty\omega}$, or equivalently, up to partial isomorphism. This generalizes a result of Barwise and Eklof and builds on Jacoby's classification of local groups with partial decomposition bases in $L_{\infty\omega}$.

1. Introduction. This is the second of two papers based on a 1980 doctoral dissertation [**4**] that has not previously been published in a readily available form but has been the starting point for recent work, specifically [**5, 6, 7, 8, 9, 10**]. Independently, Göbel, et al. [**3**] explored the same topic and proved similar results. Here, we extend to the global case the local case covered in [**11**]. This paper corrects and clarifies the original and streamlines some of the proofs.

Ulm's theorem [**15**] defines invariants that classify countable torsion abelian groups up to isomorphism. Warfield [**17**] developed new invariants that, along with the Ulm invariants, serve to classify a class of local mixed abelian groups. This was extended to the global case by Stanton [**14**]. Barwise and Eklof [**1**] looked at the classification problem in the language $L_{\infty\omega}$ and classified all torsion abelian groups up to $L_{\infty\omega}$-equivalence using modified Ulm invariants.

The first author unified these two generalizations of Ulm's theorem by defining a class of groups that includes the Warfield groups and that may be classified in $L_{\infty\omega}$, first addressing the local case [**11**].

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The defining property of these groups is the existence of what we call a partial decomposition basis, a generalization of the concept of decomposition basis that is preserved under $L_{\infty\omega}$-equivalence. This paper builds on the previous work to study this class of mixed abelian groups and to define invariants that classify these groups up to $L_{\infty\omega}$-equivalence.

Section 2 presents the background material, including the definitions of $L_{\infty\omega}$, the Ulm invariants and decomposition basis. The third section reviews the concept of partial decomposition basis and defines the modified Warfield invariant $\widehat{w}$ for the global case. It is proved that this invariant is independent of the choice of partial decomposition basis. Section 4 proves the classification theorem for groups with partial decomposition bases in $L_{\infty\omega}$.

The word "group" used in this paper will mean abelian group and "rank" will mean torsion-free rank.

2. Background.

2.1. Algebraic preliminaries. In the following definitions, we fix a group G and a prime p.

If α is an ordinal, we define $p^\alpha G$ by induction on α as follows: Let

$$pG = \{px : x \in G\},$$

$$p^\alpha G = p(p^\beta G) \quad \text{if } \alpha = \beta + 1$$

for some ordinal β and

$$p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$$

if α is a limit ordinal. We define

$$p^\infty G = \bigcap_{\alpha\in \mathrm{Ord}} p^\alpha G.$$

We define the *p-height* of x, $|x|_p$, for $x \in G$, to be the unique ordinal α such that $x \in p^\alpha G$ and $x \notin p^{\alpha+1}G$ if it exists, and the symbol ∞ otherwise. We let $G[p]$ denote $\{x \in G : px = 0\}$, and write $p^\alpha G[p]$ for $(p^\alpha G)[p]$.

For each ordinal α, we define the *Ulm invariant*

$$u_p(\alpha, G) = \dim p^\alpha G[p]/p^{\alpha+1}G[p]$$

as a $\mathbb{Z}/(p)$-vector space, and $u_p(\infty, G) = \dim p^\infty G[p]$. We define $\widehat{u}_p(\alpha, G) = \min\{u_p(\alpha, G), \omega\}$ for α an ordinal or ∞. Barwise and Eklof **[1]** proved that the invariants $\widehat{u}(\alpha, G)$ classify all torsion abelian groups in $L_{\infty\omega}$.

We say a sequence $(\alpha_i)_{i\in\omega}$ is an *Ulm sequence* if each α_i is either an ordinal or the symbol ∞ and, for all i, if $\alpha_i = \infty$, then $\alpha_{i+1} = \infty$, and if $\alpha_i \neq \infty$, then $\alpha_{i+1} > \alpha_i$. If x is an element of a group G, $U_p(x)$, the *p-Ulm sequence of* x is $(|p^i x|_p)_{i\in\omega}$. We call the Ulm sequences (α_i) and (β_i) *equivalent*, written $(\alpha_i) \sim (\beta_i)$ if there are positive integers m and n such that $\alpha_{i+n} = \beta_{i+m}$ for all $i \geq 0$.

We say $X \subseteq G$ is a *decomposition set* if X is an independent set of elements of infinite order and, for all $x_1, \ldots, x_n \in X$, $a_1, \ldots, a_n \in \mathbb{Z}$, and p a prime,

$$|a_1x_1 + \cdots + a_nx_n|_p = \min_{1\leq i\leq n}\{|a_ix_i|_p\}.$$

We let $\langle X\rangle$ denote the subgroup generated by X. If S is a subgroup of G, we let S^0 denote $\{x \in G : ax \in S \text{ for some } a \in \mathbb{Z}\setminus\{0\}\}$. If X is a decomposition set and $G = \langle X\rangle^0$, we say that X is a *decomposition basis* for G. Summands of simply presented groups are called *Warfield groups*. Warfield groups have decomposition bases. For G a local group with a decomposition basis X and e an equivalence class of Ulm sequences, we define the Warfield invariant, $w(e, G)$ = the cardinality of $\{x \in X : U(x) \in e\}$. Warfield **[17]** proved that this is independent of the choice of X and that these invariants, along with the Ulm invariants, serve to classify local Warfield groups up to isomorphism.

2.2. The language $L_{\infty\omega}$. The results of this paper will be considered in light of the language of infinitary logic known as $L_{\infty\omega}$. This is an extension of the familiar language of first order logic to allow infinite conjunctions and disjunctions, see **[1, 11]**. Since we are referencing groups, we include 0, + and − in the language L.

We say groups G and H are *$L_{\infty\omega}$-equivalent*, written $G \equiv_\infty H$ if they satisfy the same sentences of $L_{\infty\omega}$. Karp's theorem characterizes

$L_{\infty\omega}$-equivalence in terms of partial isomorphisms having the following back-and-forth property.

Theorem 2.1 ([**13**]). *Let G and H be groups. Then the following are equivalent*:

(i) $G \equiv_\infty H$;
(ii) *There is a non-empty set I of isomorphisms on finitely generated subgroups of G into H such that, if $f \in I$ and $x \in G$, $y \in H$, respectively, then f extends to a map $f' \in I$ such that $x \in$ domain(f') ($y \in$ range(f'), respectively).*

If (ii) holds, we say that G and H are *partially isomorphic*, which we represent by $I : G \cong_p H$ or simply $G \cong_p H$. This theorem, which Karp proved for general models, allows us to view the groups from either an algebraic or a logical perspective.

3. The partial decomposition basis and the invariant. In [**11**], we defined our class for modules over a principal ideal domain, so it applies both to the local case ($\mathbb{Z}_p$-modules, where $\mathbb{Z}_p$ is the integers localized at p) and the global case ($\mathbb{Z}$-modules). We say $\mathcal{C}$ is a *partial decomposition basis* for the module G if

(i) $\mathcal{C}$ is a nonempty collection of finite subsets of G;
(ii) if $X \in \mathcal{C}$, then X is a decomposition set;
(iii) if $X \in \mathcal{C}$ and $x \in G$, then there is a $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y \rangle^0$.

If $G \cong_p H$ and G has a partial decomposition basis, then so does H, see [**11,** Theorem 3.2].

Lemma 3.1. *Let G be a module over $\mathbb{Z}$ or $\mathbb{Z}_p$ with partial decomposition basis $\mathcal{C}$. Then G has a partial decomposition basis $\mathcal{C}'$ such that*

(i) *if $Y \subseteq G$ is a decomposition set and $\langle Y \rangle = \langle X \rangle$ for some $X \in \mathcal{C}'$, then $Y \in \mathcal{C}'$*;
(ii) *if $x_1, \ldots, x_n \in X$ for some $X \in \mathcal{C}'$ and $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$, then $\{a_1x_1, \ldots, a_nx_n\} \in \mathcal{C}'$.*

Proof. Define $\mathcal{C}_n$ by induction on n. Let $\mathcal{C}_0 = \mathcal{C}$. If n is odd, let

$$\mathcal{C}_n = \{Y \subseteq G : Y \text{ is a decomposition set and } \langle Y \rangle = \langle X \rangle \text{ for some } X \in \mathcal{C}_{n-1}\}.$$

If $n > 0$ is even, let

$$\mathcal{C}_n = \{\{a_1x_1, \ldots, a_mx_m\} : a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\} \text{ and } x_1, \ldots, x_m \in X \text{ for some } X \in \mathcal{C}_{n-1}\}.$$

Let $\mathcal{C}' = \bigcup \mathcal{C}_n$. Then it may be verified by induction on n that each $\mathcal{C}_n$ is a partial decomposition basis, $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$, $\mathcal{C}_n$ satisfies (i) if n is odd and (ii) if n is even, and hence, $\mathcal{C}'$ is a partial decomposition basis satisfying (i) and (ii). □

Given an equivalence class of Ulm sequences e and a local group G with partial decomposition basis $\mathcal{C}$, let

$$\widehat{w}_{\mathcal{C}}(e, G) = \text{the maximum } n \text{ s.t. there is an } X \in \mathcal{C} \text{ and } x_1, \ldots, x_n \in X$$

such that $U(x_i) \in e$ for $1 \leq i \leq n$, if such a maximum exists, and ω otherwise. This is independent of the choice of $\mathcal{C}$ and invariant under partial isomorphism [**11**].

We will need the following results from [**11,** Theorems 3.3, 4.7].

Theorem 3.2. *Let G be a module over a principal ideal domain R which has partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$. Let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then there are decomposition sets X' and Y' such that $X \subseteq X'$, $Y \subseteq Y'$, X' and Y' are unions of ascending chains of elements of $\mathcal{C}$ and $\mathcal{D}$, respectively, and $\langle X' \rangle^0 = \langle Y' \rangle^0$.*

Theorem 3.3. *Let G and H be $\mathbb{Z}_p$-modules with partial decomposition bases. Then $G \cong_p H$ if and only if, for every α, an ordinal or ∞, and equivalence class e of Ulm sequences, $\widehat{u}(\alpha, G) = \widehat{u}(\alpha, H)$ and $\widehat{w}(e, G) = \widehat{w}(e, H)$. In that case, if $\mathcal{C}$ and $\mathcal{D}$ are partial decomposition bases of G and H, respectively, satisfying Lemma* 3.1 (ii)*, then $I : G \cong_p H$ may be taken as the set of all maps $f : S \to T$ for which there exist $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ satisfying the properties*:

(i) *S and T are finitely generated submodules of G and H, respectively*;

(ii) f *is a height-preserving isomorphism*;
(iii) $X \subseteq S \subseteq \langle X \rangle^0$ *and* $Y \subseteq T \subseteq \langle Y \rangle^0$;
(iv) $f(X) = Y$.

We define an analog of the Warfield invariant for the global case. Let A be an $\omega \times \omega$ matrix $[\alpha_{p,i}]$ indexed over the primes and nonnegative integers. Then we say A is an *Ulm matrix* if, for each prime p, the row $(\alpha_{p,i})_{i \in \omega}$ is an Ulm sequence. If x is an element of a group G, it has an associated Ulm matrix

$$U(x) = [\, |p^i x|_p \,],$$

whose rows $U_p(x)$ are p-Ulm sequences of x.

For $A = [\alpha_{p,i}]$ an Ulm matrix and q a prime, we define $qA = [\beta_{p,i}]$, where, for all i, $\beta_{q,i} = \alpha_{q,i+1}$ and $\beta_{p,i} = \alpha_{p,i}$ for all $p \neq q$. Now, we may define by induction nA for any positive integer n in the obvious way. We say two Ulm matrices are *equivalent* if $mA = nB$ of some positive integers m and n. Note that A and B are equivalent if and only if the pth rows are equivalent for finitely many primes p and identical for all other primes p. Note that, as in the local case, $U(x)$ and $U(y)$ are equivalent if and only if $U(ax) = U(by)$ for some $a, b \in \mathbb{Z} \setminus \{0\}$.

We might expect, by an analog of local classification, that equivalence classes would be the basis of invariants that classify groups with partial decomposition bases. This is not the case. Warfield **[17]** cites an example of four countable rank 1 groups A_1, A_2, A_3, and A_4 such that $U(x_i)$ are all in different equivalence classes, where $x_i \in A_i$ is an element of infinite order for $1 \leq i \leq 4$, but $A_1 \oplus A_2 \cong A_3 \oplus A_4$. Thus, the decomposition bases $\{x_1, x_2\}$ and $\{x_3, x_4\}$ give different values for the obvious analog of the local invariant, showing it is not invariant under $\cong$, let alone $\cong_p$.

This problem was faced in generalizing the classification of Warfield modules up to isomorphism to the global case. This was solved by Stanton **[14]** with the introduction of new invariants based on the concept of compatibility.

We say that two Ulm matrices A and B are *compatible*, written $A \sim B$, if there are positive integers m and n such that $mA \geq B$ and $nB \geq A$, where we say $[\alpha_{p,i}] \geq [\beta_{p,i}]$ if $\alpha_{p,i} \geq \beta_{p,i}$ for every prime p and $i < \omega$. It is easy to verify that this is an equivalence relation. We call each equivalence class a *compatibility class*. Note that, if A and B are compatible, the pth rows are equal for all but finitely primes p. Now, we will define the invariant. Let G be a group with partial decomposition basis $\mathcal{C}$, c a compatibility class, p a prime and e an equivalence class of Ulm sequences. Then, we let

$$\widehat{w}_{\mathcal{C}}(c,p,e,G) = \text{the largest } n, \text{ if it exists, such that there are } X \in \mathcal{C},\ x_1,\ldots,x_n \in X, \text{ with } U(x_i) \in c \text{ and } U_p(x_i) \in e \text{ for } 1 \le i \le n.$$

If no such n exists, we let $\widehat{w}_{\mathcal{C}}(c,p,e,G) = \omega$. This is an analog of Stanton's invariant, also called the Warfield invariant, for a group G with decomposition basis X:

$$w(c,p,e,G) = |\{x \in X : U(x) \in c \text{ and } U_p(x) \in e\}|.$$

The next theorem allows us to drop the subscript $\mathcal{C}$.

Theorem 3.4. *If G has partial decomposition bases $\mathcal{C}$ and $\mathcal{C}'$, then for any compatibility class c, prime p and equivalence class e of Ulm sequences, $\widehat{w}_{\mathcal{C}}(c,p,e,G) = \widehat{w}_{\mathcal{C}'}(c,p,e,G)$. In fact, if $\widehat{w}_{\mathcal{C}}(c,p,e,G) \ge n$ and $Y \in \mathcal{C}'$, there is a $\widetilde{Y} \in \mathcal{C}'$ such that $Y \subseteq \widetilde{Y}$ and $\widetilde{Y}$ contains distinct elements $y_1,\ldots,y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for $1 \le i \le n$.*

Proof. Let c, p and e be given, and suppose $\widehat{w}_{\mathcal{C}}(c,p,e,G) \ge n$. Then, by definition, there is an $X \in \mathcal{C}$ containing elements $x_1,\ldots,x_n$ such that $U(x_i) \in c$ and $U_p(x_i) \in e$ for $1 \le i \le n$. Let Y be as given and choose X' and Y' as in Theorem 3.2. Then X' and Y' are both decomposition bases for $\langle X'\rangle^0 = \langle Y'\rangle^0$. Stanton **[14]** proved that $w(c,p,e,G)$ is independent of the choice of decomposition basis, so $w(c,p,e,\langle Y'\rangle^0) \ge n$. Thus, Y' contains elements $y_1,\ldots,y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for $1 \le i \le n$. Choose $\widetilde{Y} \in \mathcal{C}'$, $\widetilde{Y} \subseteq Y'$, containing Y and $y_1,\ldots,y_n$. □

Corollary 3.5. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{C}'$, respectively. Let c be a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Suppose that $\widehat{w}(c,p,e,G) = \widehat{w}(c,p,e,H)$, $X \in \mathcal{C}$, $Y \in \mathcal{C}'$, and $\widehat{w}(c,p,e,\langle X\rangle^0) >$*

$\widehat{w}(c,p,e,\langle Y\rangle^0)$. Then, there is a $Y' \in \mathcal{C}'$ such that $Y \subseteq Y'$ and a $y \in Y' \setminus Y$ such that $U(y) \in c$ and $U_p(y) \in e$.

Proof. Let $n = \widehat{w}(c,p,e,\langle Y\rangle^0)$. Then

$$\widehat{w}(c,p,e,H) = \widehat{w}(c,p,e,G) \geq \widehat{w}(c,p,e,\langle X\rangle^0) > \widehat{w}(c,p,e,\langle Y\rangle^0) = n,$$

so, by Theorem 3.4, there is a $Y' \in \mathcal{C}'$ such that $Y \subseteq Y'$ and Y' contains $n+1$ elements y such that $U(y) \in c$ and $U_p(y) \in e$. Since Y contains only n such elements, one is in $Y' \setminus Y$. $\square$

Note that, if G is a group with a decomposition basis X, then the set of all finite subsets of X is a partial decomposition basis, so $\widehat{w}(c,p,e,G) = \min\{w(c,p,e,G),\omega\}$. Also, if $G \cong_p H$ and G has a partial decomposition basis, then we may verify that $\widehat{w}(c,p,e,G) = \widehat{w}(c,p,e,H)$ for all c, p and e, cf., **[11]**. Thus, we have defined values that are invariant under $\cong_p$.

4. The global classification theorem. As in the local case, we will prove the classification theorem by an extension argument. In particular, suppose G and H have partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, as in Lemma 3.1 and $\widehat{w}(c,p,e,G) = \widehat{w}(c,p,e,H)$ for all c, p and e. Suppose also that $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, where $f(X) = Y$ for some injective and height-preserving f. If $X \cup \{x\} \in \mathcal{C}$, we would like to extend f by finding a $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$, $U(x) \sim U(y)$ and $U_p(x) \sim U_p(y)$ for all primes p. Corollary 3.5 allows us to choose such a y for each prime p.

Now, we will prove that it is possible to choose a single y that works for all primes. First, we will need a result of Stanton **[14,** Lemma 7].

Lemma 4.1. *Let X be a decomposition basis for a group G, let x_1 and x_2 be elements of X with compatible Ulm matrices, and let p be a prime. Then there are elements y_1 and y_2 in $\langle X\rangle$ such that $U_p(y_1) = U_p(x_2)$, $U_p(y_2) = U_p(x_1)$ and $U_q(y_1) = U_q(x_1)$, $U_q(y_2) = U_q(x_2)$ for all primes $q \neq p$. Moreover,*

$$Y = (X \setminus \{x_1,x_2\}) \cup \{y_1,y_2\}$$

is a decomposition basis and $\langle X\rangle = \langle Y\rangle$.

Lemma 4.2. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, as in Lemma* 3.1. *Suppose $\widehat{w}(c,p,e,H) = \widehat{w}(c,p,e,G)$ for every compatibility class c, prime p and equivalence class e of Ulm sequences. Then, if $X \cup \{x\} \in \mathcal{C}$, $Y \in \mathcal{D}$ and $\widehat{w}(c,p,e,\langle X\rangle^0) = \widehat{w}(c,p,e,\langle Y\rangle^0)$ for all c, p and e, then there is a $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and $\widehat{w}(c,p,e,\langle X \cup \{x\}\rangle^0) = \widehat{w}(c,p,e,\langle Y \cup \{y\}\rangle^0)$ for all c, p and e. In fact, $U(x)$ and $U(y)$ are equivalent.*

Proof. Let p_0 be an arbitrary prime. Let c be the compatibility class of $U(x)$ and e_0 the equivalence class of $U_{p_o}(x)$. Then, by Corollary 3.5, there is a $z \in H$ such that $z \notin Y$, $U(z) \in c$, $U_{p_0}(z) \in e_0$ and $Y \cup \{z\} \in \mathcal{D}$. But, then $U(z)$ and $U(x)$ are compatible, so $U_p(z)$ and $U_p(x)$ are equivalent except for finitely many primes, say $p_1, \ldots, p_n$. We will prove by induction on n that z can be replaced by an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$, $U(y) \in c$, and $U_p(y) \sim U_p(x)$ for all primes p. If $n = 0$, then $U_p(x)$ is equivalent to $U_p(z)$ for all primes p as required.

Suppose this is true for $n-1$. Let e_n be the equivalence class of $U_{p_n}(x)$. Then, by assumption, $U_{p_n}(z) \notin e_n$, so

$$\widehat{w}(c,p_n,e_n,\langle Y \cup \{z\}\rangle^0) = \widehat{w}(c,p_n,e_n,\langle Y\rangle^0) < \widehat{w}(c,p_n,e_n,\langle X \cup \{x\}\rangle^0).$$

Thus, by Corollary 3.5, there is a $z' \in H$ such that $Y \cup \{z, z'\} \in \mathcal{D}$, $U(z') \in c$ and $U_{p_n}(z') \in e_n$. By Lemma 4.1, there are y and $y' \in H$ such that $Y \cup \{y, y'\}$ is a decomposition set and $\langle Y, y, y'\rangle = \langle Y, z, z'\rangle$, $U_{p_n}(y) = U_{p_n}(z')$, $U_{p_n}(y') = U_{p_n}(z)$ and, for all $q \neq p_n$, $U_q(y) = U_q(z)$ and $U_q(y') = U_q(z')$. Then, in particular,

$$U_{p_n}(y) = U_{p_n}(z') \sim U_{p_n}(x),$$
$$U_{p_0}(y) = U_{p_0}(z) \sim U_{p_0}(x)$$

and

$$U_q(y) \sim U_q(x) \quad \text{for all } q \notin \{p_0, \ldots, p_n\}.$$

Thus, $U_p(y) \sim U_p(x)$ for all but at most $n-1$ primes p. Since $Y \cup \{y, y'\} \in \mathcal{D}$, $Y \cup \{y\} \in \mathcal{D}$. The result follows by induction. □

Lemma 4.3. *Let G be a group, X a decomposition basis for G and S a finitely generated subgroup of G such that $S \cap \langle X\rangle = \langle S \cap X\rangle$. Then, if $y \in X$ and $y \notin S$, there is a positive integer n such that, for all $m \in \mathbb{Z}$,*

$s \in S$ and p prime,

$$|mny + s|_p = \min\{|mny|_p, |s|_p\}.$$

Proof. This lemma was proved in the local case [**11,** Lemma 4.6]. We will localize to use that result. Let p be a prime and

$$\varphi_p : G \longrightarrow G_p = G \otimes \mathbb{Z}_p$$

the natural map defined by $x \mapsto x \otimes 1$. Then, by transfinite induction on heights, it is easy to see that, for all $x \in G$, $|x|_p = |\varphi_p(x)|_p$. Now suppose $\{x_1, \ldots, x_n\}$ is a decomposition set and $a_i/b_i \in \mathbb{Z}_p$ with $(b_i, p) = 1$ for $1 \leq i \leq n$. Then, if we multiply through by $c = \prod_{i=1}^n b_i$, we see that

$$\left| \frac{a_1}{b_1}(x_1 \otimes 1) + \cdots + \frac{a_n}{b_n}(x_n \otimes 1) \right|_p = \min_{1 \leq j \leq n} \left\{ \left| \frac{a_j}{b_j}(x_j \otimes 1) \right|_p \right\},$$

and independence can be proved similarly, so $\{x_1 \otimes 1, \ldots, x_n \otimes 1\}$ is a decomposition set. Thus, the image of X under φ is a decomposition basis.

Since X is a decomposition basis for G and S is finitely generated, there is an integer $k \neq 0$ such that $ks \in \langle X \rangle$ for all $s \in S$. Let p be a prime dividing k. We localize at p. Suppose $y \otimes 1 \in S \otimes \mathbb{Z}_p$. Then, $m(y \otimes 1) = s' \otimes 1$ for some $s' \in S$ and $m \in \mathbb{Z} \setminus \{0\}$, and so, $my \in S$. But, then $my \in S \cap \langle X \rangle = \langle S \cap X \rangle$, and so, $y \in S \cap X$, contradicting $y \notin S$. Thus, $y \otimes 1 \notin S \otimes \mathbb{Z}_p$, so we may apply the local version of this lemma to G_p to get an $n_p \in \mathbb{Z}$ such that

$$|rp^{n_p}(y \otimes 1) + s \otimes 1|_p = \min\{|rp^{n_p}(y \otimes 1)|_p, |s \otimes 1|_p\}$$

for all $r \in \mathbb{Z}_p$, $s \in S$.

Now let $n = \prod_{p|k} p^{n_p}$. Then, for any p dividing k,

$$\begin{aligned} |mny + s|_p &= |mn(y \otimes 1) + s \otimes 1|_p \\ &= \min\{|mn(y \otimes 1)|_p, |s \otimes 1|_p\} \\ &= \min\{|mny|_p, |s|_p\} \end{aligned}$$

for all $m \in \mathbb{Z}$ and $s \in S$.

Now suppose that p does not divide k and $s \in S$. Then $ks \in \langle X \rangle$, say $ks = a_1x_1 + \cdots + a_nx_n$. But $S \cap \langle X \rangle = \langle S \cap X \rangle$, so $ks \in \langle S \cap X \rangle$

and $x_1, \ldots, x_n \in S$. In particular, none of $x_1, \ldots, x_n$ is y. Then, since p does not divide k,

$$s \otimes 1 = \frac{a_1}{k}(x_1 \otimes 1) + \cdots + \frac{a_n}{k}(x_n \otimes 1)$$

is in G_p. But $\{x_1 \otimes 1, \ldots, x_n \otimes 1, y \otimes 1\}$ is a decomposition set, so

$$\begin{aligned} |mny + s|_p &= |mn(y \otimes 1) + s \otimes 1|_p \\ &= \min\{|mn(y \otimes 1)|_p, |s \otimes 1|_p\} \\ &= \min\{|mny|_p, |s|_p\}, \end{aligned}$$

as required. □

Warfield [**16,** 1.16] proved the next local-global theorem, which will allow us to extend local mappings to global.

Theorem 4.4. *Let G and H be groups, S and T subgroups of G and H, respectively, such that G/S and H/T are torsion and $f : S \to T$ a homomorphism. Suppose, for every prime p, the induced map $f_p : S \otimes \mathbb{Z}_p \to T \otimes \mathbb{Z}_p$ extends to a homomorphism $g(p) : G \otimes \mathbb{Z}_p \to H \otimes \mathbb{Z}_p$. Then, f extends to a homomorphism $g : G \to H$ such that $g_p = g(p)$ for all primes p. In addition, if $g(p)$ is an isomorphism for each p, then so is g.*

Now we may prove the main result of this paper, the classification in $L_{\infty\omega}$.

Theorem 4.5. *Let G and H be groups with partial decomposition bases. Then $G \cong_p H$ if and only if, for every ordinal α, compatibility class c, prime p and equivalence class e of Ulm sequences, $\widehat{u}_p(\alpha, G) = \widehat{u}_p(\alpha, H)$, $\widehat{w}(c, p, e, G) = \widehat{w}(c, p, e, H)$ and $\widehat{u}_p(\infty, G) = \widehat{u}_p(\infty, H)$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ be the partial decomposition bases for G and H, respectively. Suppose $I : G \cong_p H$. Then, $G \equiv_\infty H$ by Theorem 2.1. Given α, there is a sentence of $L_{\infty\omega}$ that says "$u(\alpha, G) \geq n$" [**1,** Lemma 2.2], so $\widehat{u}(\alpha, G) = \widehat{u}(\alpha, H)$ for all α an ordinal or ∞. By [**11,** Theorem 3.2], $\{f(X) : X \in \mathcal{C}, f \in I, X \subseteq \text{domain}(f)\}$ is a partial decomposition basis for H. Suppose, for some compatibility class c, prime p and equivalence class e, $\widehat{w}(c, p, e, G) \geq n$, say $x_1, \ldots, x_n \in X$,

$X \in \mathcal{C}$, and for all $1 \leq i \leq n$, $U(x_i) \in c$ and $U_p(x_i) \in e$. Since I is non-empty, we may choose some element in I and extend it to $f \in I$ with $\{x_1, \ldots, x_n\} \subseteq \text{domain}(f)$. Since f is height-preserving [**11,** Lemma 3.1], for all $1 \leq i \leq n$, $U(f(x_i)) = U(x_i)$, so $U(f(x_i)) \in e$ and $U_p(f(x_i)) \in e$. It follows that $\widehat{w}(c, p, e, H) \geq n$. The other direction follows from symmetry.

Now suppose all the invariants are equal. By Lemma 3.1, we may assume $\mathcal{C}$ has the following properties:

(1) if $Y \subseteq G$ is a finite decomposition set and $\langle Y \rangle = \langle X \rangle$ for some $X \in \mathcal{C}$, then $Y \in \mathcal{C}$;
(2) if $x_1, \ldots, x_n \in X$ for some $X \in \mathcal{C}$, $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$, then $\{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$,

and similarly for $\mathcal{C}'$.

Now let I be the set of $f : S \to T$ with associated $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$ such that:

(i) S and T are finitely generated subgroups of G and H respectively;
(ii) f is a height-preserving isomorphism;
(iii) $f(X) = Y$;
(iv) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$.

I is not empty since it contains the zero function with associated X and Y empty. Let $f : S \to T$ be an element of I and $x \in G \setminus S$. To prove that f extends to x, we need consider only two cases as in the local case: x has a nonzero multiple in S and $X \cup \{x\} \in \mathcal{C}$.

Case 1. Suppose x has a nonzero multiple in S. Let $G' = \langle S, x \rangle$ and $H' = T^0$. Then G'/S and H'/T are torsion. Let p be given. Then $\mathcal{C}$ and $\mathcal{C}'$ induce partial decomposition bases for $G \otimes \mathbb{Z}_p$ and $H \otimes \mathbb{Z}_p$, respectively, under the map $\varphi_p(x) = x \otimes 1$. Furthermore, for any equivalence class e of Ulm sequences and compatibility class c of Ulm matrices, we have $\widehat{w}(e, G \otimes \mathbb{Z}_p) \geq \widehat{w}(c, p, e, G)$. If $c' \neq c$, then, by Theorem 3.4, we have $\widehat{w}(e, G \otimes \mathbb{Z}_p) \geq \widehat{w}(c, p, e, G) + \widehat{w}(c', p, e, G)$.

Thus, repeated application of Theorem 3.4 yields

$$\begin{aligned}\widehat{w}(e, G \otimes \mathbb{Z}_p) &= \min\Big\{\sum_c \widehat{w}(c,p,e,G), \omega\Big\}\\ &= \min\Big\{\sum_c \widehat{w}(c,p,e,H), \omega\Big\} = \widehat{w}(e, H \otimes \mathbb{Z}_p).\end{aligned}$$

Also, $\widehat{u}_p(\alpha, G \otimes \mathbb{Z}_p) = \widehat{u}_p(\alpha, H \otimes \mathbb{Z}_p)$ for any α an ordinal or ∞, since φ_p preserves p-heights, cf., [**2,** Part 2, Lemmas 13 and 16]). Thus, by the local classification theorem, Theorem 3.3, $G \otimes \mathbb{Z}_p \cong_p H \otimes \mathbb{Z}_p$ and the induced map $f_p : S \otimes \mathbb{Z}_p \to T \otimes \mathbb{Z}_p$ is in I_p, the system of partial isomorphisms constructed in Theorem 3.3. By the definition of partial isomorphisms, f_p extends to $g(p) \in I_p$ with domain $\langle S, x\rangle \otimes \mathbb{Z}_p$. Thus, by Theorem 4.4, f extends to $g : \langle S, x\rangle \to \langle T, y\rangle$ for some $y \in H$ such that $g_p = g(p)$ for all primes p. Furthermore, $g(p) : \langle S, x\rangle \otimes \mathbb{Z}_p \to \langle T, y\rangle \otimes \mathbb{Z}_p$ is an isomorphism and so g is as well. Also, g is height-preserving since $g_p = g(p)$ is p-height-preserving for any p. Thus, g satisfies conditions (i) and (ii). Additionally, g satisfies (iii) and (iv) with the original X and Y.

Case 2. Suppose $X \cup \{x\} \in \mathcal{C}$. By Lemma 4.2, there is a $y \in H$ such that $U(x)$ is equivalent to $U(y)$ and $Y \cup \{y\} \in \mathcal{C}'$. Thus, $U(x') = U(y')$ for some x' and y', multiples of x and y, respectively. Apply Lemma 4.3 to the group $\langle S, x'\rangle^0$, the subgroup S and the decomposition basis $X' = X \cup \{x'\}$. This is possible since $S \cap \langle X'\rangle = \langle X\rangle = \langle S \cap X'\rangle$. Apply it similarly to $\langle T, y'\rangle^0$, T and $Y' = Y \cup \{y'\}$. Thus, there is an n such that

$$|mnx' + s|_p = \min\{|mnx'|_p, |s|_p\}$$

and

$$|mny' + t|_p = \min\{|mny'|_p, |t|_p\}$$

for all $m \in \mathbb{Z}$, p prime, $s \in S$ and $t \in T$. Let

$$\begin{aligned}S'' &= \langle S, nx'\rangle,\\ T'' &= \langle T, ny'\rangle,\\ X'' &= X \cup \{nx'\},\\ Y'' &= Y \cup \{ny'\},\end{aligned}$$

and define $g : S'' \to T''$ extending f by $g(nx') = ny'$. It is easy to verify that g satisfies conditions (i)–(iv) on I. Thus, as we showed in Case 1, there is a $g' \in I$ extending g with x in its domain. This completes Case 2.

Now, for any $x \in G$, we first find $X' \supseteq X$, $X' \in \mathcal{C}$, $x \in \langle X' \rangle^0$, and successively extend f to each element of X' by Case 2 then to x by Case 1. By symmetry, we may extend f to any y in H. This proves I is a system of partial isomorphisms. $\square$

This theorem is in fact a generalization of the theorem of Barwise and Eklof.

Corollary 4.6 ([**1**]). *Let G and H be torsion groups. Then, $G \equiv_\infty H$ if and only if $\widehat{u}(\alpha, G) = \widehat{u}(\alpha, H)$ for all α an ordinal or ∞.*

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