ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 2, 2017

INVARIANTLY COMPLEMENTED AND AMENABILITY IN BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

ALI GHAFFARI AND SOMAYEH AMIRJAN

ABSTRACT. In this paper, among other things, we show that there is a close connection between the existence of a bounded projection on some Banach algebras associated to a locally compact group G and the existence of a left invariant mean on $L^\infty(G)$. A necessary and sufficient condition is found for a locally compact group to possess a left invariant mean.

1. Introduction. For a locally compact group G, $L^1(G)$ is its group algebra and $L^\infty(G)$ is the dual of $L^1(G)$. The theory of projections on group algebras has been extensively studied in such papers as [**7, 9, 12, 21, 23**]. Several authors have also studied the weak* closed left translation invariant complemented subspace of $L^\infty(G)$, see [**4, 5, 7**]. Recall that a subspace X of $L^\infty(G)$ is said to be *complemented* if there exists a bounded projection P from $L^\infty(G)$ onto X. A subspace X of $L^\infty(G)$ is called *invariantly complemented* if there exists a projection P from $L^\infty(G)$ onto X which commutes with the left translation, i.e., $P : L^\infty(G) \to X$ such that $P(L_x f) = L_x P(f)$ for all $x \in G$ and $f \in L^\infty(G)$ [**7**]. Rosenthal proved [**17**] that, if G is an abelian locally compact group and X is a weak* closed translation invariant complemented subspace of $L^\infty(G)$, then X is invariantly complemented in $L^\infty(G)$.

We say that X is *topologically invariantly complemented* in $L^\infty(G)$ if X is the range of a bounded projection P on $L^\infty(G)$ such that $P(\varphi * f) = \varphi * P(f)$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Note that this is a generalization of the notion of a topologically invariant mean on

2010 AMS *Mathematics subject classification.* Primary 43A07, Secondary 43A22.

Keywords and phrases. Amenability, Banach algebra, group algebra, homomorphism, operator, projection, weak* topology.

Received by the editors on February 6, 2014, and in revised form on June 15, 2015.

DOI:10.1216/RMJ-2017-47-2-445

$L^\infty(G)$, since $\{c1_G;\ c \in \mathbb{C}\}$ is topologically invariantly complemented in $L^\infty(G)$ if and only if there exists a topologically invariant mean on $L^\infty(G)$ [**15**]. Bekka proved [**1**] that, if X is a weak* closed left translation invariant subspace of $L^\infty(G)$, then X is topologically invariantly complemented in $L^\infty(G)$ if and only if X is invariantly complemented in $L^\infty(G)$.

The closed ideals I of $L^1(G)$ for which $I^\perp$ is complemented in $L^\infty(G)$ have been classified by Rosenthal [**17**] and Liu, van Rooij and Wong [**14**]. It turns out that these ideals are exactly those of $L^1(G)$, which possess bounded approximate identity [**14**]. Rudin [**19**] used an averaging argument to show that an ideal I of $L^1(G)$ is complemented if and only if there exists a projection $P : L^1(G) \to I$ which commutes with convolution, i.e., $P(\varphi * \psi) = \varphi * P(\psi)$ for all $\varphi, \psi \in L^1(G)$. Wood proved this fact for compact non-abelian groups (see [**22,** Theorem 4.6]).

The aim of this paper is to go further and generalize the above result to the collection of bounded linear maps on some Banach algebras associated to a locally compact group. We relate the amenability of a locally compact group G with the existence of projections in $\mathcal{B}(LUC(G))$. We also completely determined the weak* closed left translation invariant subspace X of $LUC(G)$ which is the range of a weak*–weak* continuous projection P on $LUC(G)$ commuting with left translations. Finally, we study the concept of approximately complemented subspaces of Banach algebras associated to a locally compact group.

2. Notation and preliminary results. Throughout this paper, G denotes a locally compact group with a fixed left Haar measure dx. For any subset A of G, 1_A denotes the characteristic function of A. Let $L^\infty(G)$ be the algebra of essentially bounded measurable complex-valued functions on G. The second dual $L^1(G)^{**}$ of $L^1(G)$ is a Banach algebra with the first Arens product [**3**]. G is *amenable* if there exists $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1_G) = 1$ and $m(L_x f) = m(f)$ for every $x \in G$, $f \in L^\infty(G)$, where $L_x f(y) = f(xy)$, $y \in G$. All abelian groups and all compact groups are amenable. The free group on two generators is not amenable [**15**].

A bounded linear operator T from $L^\infty(G)$ into $L^\infty(G)$ is said to commute with convolution if $T(\varphi * f) = \varphi * T(f)$ for all $\varphi \in L^1(G)$ and

$f \in L^\infty(G)$. In this case, T also commutes with left translations, i.e., $T(L_x f) = L_x T(f)$ for all $x \in G$ and $f \in L^\infty(G)$, see [**8,** Lemma 2]. Let $C_b(G)$ denote the Banach algebra of bounded continuous complex-valued functions on G, and let $C_0(G)$ be the closed subspace of $C_b(G)$ consisting of all functions in $C_b(G)$ which vanishes at infinity. Then its dual $C_0(G)^*$ identifies with all the complex regular Borel measures on G, denoted by $M(G)$. For $\mu \in M(G)$ and $f \in C_0(G)$, the formula

$$\langle \widetilde{\mu}, f\rangle = \int f(x^{-1})\, d\mu(x)$$

defines an element of $M(G)$ with $\|\widetilde{\mu}\| = \|\mu\|$. Let $LUC(G)$ be the space of all $f \in C_b(G)$ such that the mapping $x \mapsto L_x f$ from G into $C_b(G)$ is continuous. Then $LUC(G)$ is a C^*-subalgebra of $C_b(G)$ invariant under translations. It is known that $L^\infty(G)L^1(G) = LUC(G)$ and that $f\varphi = \widetilde{\varphi} * f$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$ [**6**]. Given $G \in LUC(G)^*$, $f \in LUC(G)$, let $Gf \in LUC(G)$ be given by $Gf(x) = \langle G, L_x f\rangle$. Given $F \in LUC(G)^*$, let FG (the Arens product of F, G) be defined by $\langle FG, f\rangle$. Then $LUC(G)^*$ with respect to this product becomes a Banach algebra.

Information about the Arens product and about $LUC(G)$ may be found in [**6**] (although the reader should be warned that $LUC(G)$ is defined as the space of right uniformly continuous functions). If $x \in G$, δ_x will denote either the point-measure at x in $M(G)$, or the point-evaluation linear functional in X^* when X is a subspace of $C_b(G)$.

Among the elements of $LUC(G)^*$ are the measures δ_x for $x \in G$. These do not appear in $L^1(G)^{**}$. Moreover, δ_e is an identity in $LUC(G)^*$, and $L^1(G)^{**}$ has a right identity [**3**].

A subspace $X \subseteq L^\infty(G)$ is called *left translation invariant* if $L_x f \in X$ for all $f \in X$ and $x \in G$. Let X be a left translation invariant subspace of $LUC(G)$. The collection of all bounded linear maps $T : LUC(G) \to X$ which commutes with left translations will be denoted by $\mathrm{Hom}(LUC(G), X)$. If A is a Banach algebra, $\mathcal{B}(A)$ will denote the Banach algebra of all bounded linear operators from A to A.

3. Main results. Lau proved [**7**] that G is amenable if and only if every left translation invariant W^*-subalgebra of $L^\infty(G)$ is invariantly complemented. It was also shown by Lau and Losert [**9**] that G is amenable if and only if, whenever X is a non-degenerate left Banach

G-module and L is a weak* closed G-invariant subspace of X which is complemented in X, then there exists a projection P of X^* onto L such that $P(f\cdot x) = P(f)\cdot x$ for all $x \in G$ and $f \in X^*$ (see **[10]** for this terminology). In the next theorem, a necessary and sufficient condition is given for a locally compact group G to have a left invariant mean.

Theorem 3.1. *Let G be a locally compact group. The following conditions are equivalent*:

(i) *G is amenable*;
(ii) $\mathrm{Hom}(LUC(G))$ *is the range of a bounded projection P on $\mathcal{B}(LUC(G))$ such that $P(I) = I$, $\|P\| = 1$ and $P(TL_y) = P(L_yT)$ for all $T \in \mathcal{B}(LUC(G))$ and $y \in G$.*

Proof.

(i) $\Rightarrow$ (ii). Let m be an invariant mean on $L^\infty(G)$ **[16]**. Let $\langle\ \rangle$ denote the pairing between $L^\infty(G)$ and $L^1(G)$. For $T \in \mathcal{B}(LUC(G))$, $f \in LUC(G)$ and $\psi \in L^1(G)$ the mapping $x \mapsto \langle L_{x^{-1}}TL_x(f), \psi\rangle$ is a bounded continuous function on G. Define an operator $P : \mathcal{B}(LUC(G)) \to \mathcal{B}(LUC(G))$ by

$$\langle P(T)(f), \psi\rangle = m(x \longmapsto \langle L_{x^{-1}}TL_x(f), \psi\rangle)$$

for $f \in LUC(G)$ and $\psi \in L^1(G)$. We claim that P is a bounded projection of $\mathcal{B}(LUC(G))$ onto $\mathrm{Hom}(LUC(G))$ and that $P(L_yT) = P(TL_y)$ for all $y \in G$. It is easy to see that $\|P\| \le 1$, $P(I) = I$, and so $\|P\| = 1$. To see that $P(\mathcal{B}(LUC(G)) \subseteq \mathrm{Hom}(LUC(G))$, let $T \in \mathcal{B}(LUC(G))$ and $y \in G$. For every $f \in LUC(G)$ and $\psi \in L^1(G)$, we have

$$\begin{aligned}\langle P(T)L_y(f), \psi\rangle &= \langle P(T)(L_yf), \psi\rangle\\ &= m(x \longmapsto \langle L_{x^{-1}}TL_x(L_yf), \psi\rangle)\\ &= m(x \longmapsto \langle L_yL_{(yx)^{-1}}TL_{yx}(f), \psi\rangle)\\ &= m(x \longmapsto \langle L_{(yx)^{-1}}TL_{yx}(f), L_{y^{-1}}\psi\rangle)\\ &= m(x \longmapsto \langle L_{x^{-1}}TL_x(f), L_{y^{-1}}\psi\rangle)\\ &= \langle P(T)(f), L_{y^{-1}}\psi\rangle\\ &= \langle L_yP(T)(f), \psi\rangle.\end{aligned}$$

Since this holds for all $f \in LUC(G)$ and $\psi \in L^1(G)$, we conclude that $P(T)L_y = L_yP(T)$. This shows that $P(\mathcal{B}(LUC(G)) \subseteq \mathrm{Hom}(LUC(G))$. To see that P is a bounded projection of $\mathcal{B}(LUC(G))$ onto $\mathrm{Hom}(LUC(G))$, it suffices to show that $T \in \mathrm{Hom}(LUC(G))$ implies $P(T) = T$. To see this, let $T \in \mathrm{Hom}(LUC(G))$, $f \in LUC(G)$ and $\psi \in L^1(G)$. Then

$$\begin{aligned}\langle P(T)(f), \psi\rangle &= m(x \longmapsto \langle L_{x^{-1}}TL_x(f), \psi\rangle)\\ &= m(x \longmapsto \langle T(f), \psi\rangle)\\ &= \langle T(f), \psi\rangle.\end{aligned}$$

Consequently, $P(T) = T$ for all $T \in \mathrm{Hom}(LUC(G))$. Finally, to see $P(TL_y) = P(L_yT)$ for all $y \in G$ and $T \in \mathcal{B}(LUC(G))$, let $y \in G$, $T \in \mathcal{B}(LUC(G))$, $f \in LUC(G)$ and $\psi \in L^1(G)$. We have

$$\begin{aligned}\langle P(L_yT)(f), \psi\rangle &= m(x \longmapsto \langle L_{x^{-1}}L_yTL_x(f), \psi\rangle)\\ &= m(x \longmapsto \langle L_{(xy^{-1})^{-1}}TL_yL_{xy^{-1}}(f), \psi\rangle)\\ &= m(x \longmapsto \langle L_{x^{-1}}TL_yL_x(f), \psi\rangle)\\ &= \langle P(TL_y)(f), \psi\rangle.\end{aligned}$$

This shows that $P(L_yT) = P(TL_y)$ for all $T \in \mathcal{B}(LUC(G))$ and $y \in G$.

(ii) $\Rightarrow$ (i). For $f \in LUC(G)$, we consider the mapping $\lambda_f : LUC(G) \to LUC(G)$ defined by $\lambda_f(g) = f \cdot g$, $g \in LUC(G)$. If $f \in LUC(G)$ and $x \in G$,

$$\begin{aligned}\lambda_{L_xf}(g) &= L_xf \cdot g = L_x(f \cdot L_{x^{-1}}g)\\ &= L_x(\lambda_f(L_{x^{-1}}g)) = L_x\lambda_fL_{x^{-1}}(g)\end{aligned}$$

for all $g \in LUC(G)$. We conclude that $\lambda_{L_xf} = L_x\lambda_fL_{x^{-1}}$.

Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ in $\{\psi \in L^1(G);\ \|\psi\|_1 = 1,\ \psi \geq 0\}$ **[6]**. For $f \in LUC(G)$, define $m(f) = \lim_\alpha\langle P(\lambda_f)(1_G), e_\alpha\rangle$. Since $P(I) = I$, we have

$$m(1_G) = \lim_\alpha\langle P(\lambda_{1_G})(1_G), e_\alpha\rangle = \lim_\alpha\langle I(1_G), e_\alpha\rangle = 1.$$

On the other hand, $\|P\| = 1$ and $\|\lambda_f\| \leq \|f\|$ for all $f \in LUC(G)$. It follows that $\|m\| = 1$. This shows that m is a mean on $LUC(G)$ **[16]**. To show that m is a left invariant mean on $LUC(G)$, let $f \in LUC(G)$ and $x \in G$. Since $P(TL_x) = L_xP(T)$ for all $T \in \mathcal{B}(LUC(G))$ and

$x \in G$, we have

$$\begin{aligned} m(L_x f) &= \lim_\alpha \langle P(\lambda_{L_x f})(1_G), e_\alpha \rangle \\ &= \lim_\alpha \langle P(L_x \lambda_f L_{x^{-1}})(1_G), e_\alpha \rangle \\ &= \lim_\alpha \langle P(L_{x^{-1}} L_x \lambda_f)(1_G), e_\alpha \rangle \\ &= \lim_\alpha \langle P(\lambda_f)(1_G), e_\alpha \rangle = m(f). \end{aligned}$$

Therefore, m is a left invariant mean on $LUC(G)$, and so G is amenable [**16**]. This completes the proof. □

Recall that the Banach space $LUC(G)^*$ is a Banach algebra. Among the elements of $LUC(G)^*$ are the unit point masses δ_x for $x \in G$. Let X be a subspace of $LUC(G)^*$ such that $\delta_x F \in X$ for all $F \in X$ and $x \in G$. The collection of all bounded linear maps $T : LUC(G)^* \to X$ such that $T(\delta_x F) = \delta_x T(F)$ for all $F \in LUC(G)^*$ and $x \in G$, will be denoted by $\mathrm{Hom}(LUC(G)^*, X)$.

Theorem 3.2. *Let G be a locally compact group. Assume that G is amenable as discrete. Let X be a weak* closed subspace of $LUC(G)^*$ such that $\delta_x F \in X$ for all $F \in X$ and $x \in G$. Let P be a bounded projection of $LUC(G)^*$ onto X. Then there exists a bounded projection $\mathcal{P}$ of $\mathcal{B}(LUC(G)^*)$ onto* $\mathrm{Hom}(LUC(G)^*, X)$.

Proof. We first show that there exists a bounded projection P' of $LUC(G)^*$ onto X such that $P'(\delta_y F) = \delta_y P'(F)$ for all $F \in LUC(G)^*$ and $y \in G$. We can prove this part by using an argument similar to that of the proof of Theorem 1.1 in [**17**]. Let m be an invariant mean on $l^\infty(G)$. Now, consider a bounded linear operator P' of $LUC(G)^*$ into $LUC(G)^*$ defined by

$$\langle P'(F), f \rangle = m(x \longmapsto \langle P(\delta_x F), \delta_x * f \rangle), \\ F \in LUC(G)^*, \quad f \in LUC(G).$$

Then, for any $y \in G$, $f \in LUC(G)$ and $F \in LUC(G)^*$, we have

$$\begin{aligned} \langle P'(\delta_y F), f \rangle &= m(x \longmapsto \langle P(\delta_{xy} F), \delta_x * f \rangle) \\ &= m(x \longmapsto \langle P(\delta_{xy} F), \delta_{xy} * \delta_{y^{-1}} * f \rangle) \\ &= m(x \longmapsto \langle P(\delta_x F), \delta_x * \delta_{y^{-1}} * f \rangle) \end{aligned}$$

$$= \langle P'(F), \delta_{y^{-1}} * f\rangle$$
$$= \langle \delta_y P'(F), f\rangle.$$

We conclude that $P'(\delta_y F) = \delta_y P'(F)$ for all $y \in G$ and $F \in LUC(G)^*$.

We next show that P' is a bounded projection of $LUC(G)^*$ onto X. Clearly, $\|P'\| \leq \|P\|$. Now fix $F \in LUC(G)^*$. If every weak* continuous linear functional $\widehat{f}$ on $LUC(G)^*$ for which $\widehat{f}(X) = 0$, also satisfies

$$\langle P(\delta_x F), \delta_x * f\rangle = \langle \widehat{f}, \delta_{x^{-1}} P(\delta_x F)\rangle = 0, \quad x \in G.$$

Then, by the Hahn-Banach theorem, $P'(F)$ also belongs to X **[18]**. This shows that $P'(LUC(G)^*) \subseteq X$. It is easy to see that $P'(F) = F$ for all $F \in X$. We conclude that P' is a bounded projection of $LUC(G)^*$ onto X. Thus, without loss of generality, we may assume that P is a bounded projection of $LUC(G)^*$ onto X and $P(\delta_y F) = \delta_y P(F)$ for all $y \in G$ and $F \in LUC(G)^*$.

Define $\mathcal{P} : \mathcal{B}(LUC(G)^*) \to \mathcal{B}(LUC(G)^*)$ by

$$\langle \mathcal{P}(T)(F), f\rangle = m(x \longmapsto \langle P(T(\delta_x F)), \delta_x * f\rangle),$$

where $T \in \mathcal{B}(LUC(G)^*)$, $F \in LUC(G)^*$ and $f \in LUC(G)$. Obviously $\mathcal{P}$ is a bounded linear operator of $\mathcal{B}(LUC(G)^*)$ into $\mathcal{B}(LUC(G)^*)$.

To see that $\mathcal{P}$ is a projection of $\mathcal{B}(LUC(G)^*)$ onto $\mathrm{Hom}(LUC(G)^*, X)$, it suffices to show that $\mathcal{P}(\mathcal{B}(LUC(G)^*)) \subseteq \mathrm{Hom}(LUC(G)^*, X)$ and that $T \in \mathrm{Hom}(LUC(G)^*, X)$ implies that $\mathcal{P}(T) = T$. For the first assertion, let $T \in \mathcal{B}(LUC(G)^*)$ and $y \in G$. We have

$$\begin{aligned}\langle \mathcal{P}(T)(\delta_y F), f\rangle &= m(x \longmapsto \langle P(T(\delta_{xy} F)), \delta_x * f\rangle)\\ &= m(x \longmapsto \langle P(T(\delta_{xy} F)), \delta_{xy} * \delta_{y^{-1}} * f\rangle)\\ &= m(x \longmapsto \langle P(T(\delta_x F)), \delta_x * \delta_{y^{-1}} * f\rangle)\\ &= \langle \delta_y \mathcal{P}(T)(F), f\rangle,\end{aligned}$$

where $F \in LUC(G)^*$ and $f \in LUC(G)$. Since this holds for all $F \in LUC(G)^*$ and $f \in LUC(G)$, we conclude that $\mathcal{P}(\mathcal{B}(LUC(G)^*)) \subseteq \mathrm{Hom}(LUC(G)^*)$. Note that, if $T \in \mathcal{B}(LUC(G)^*)$, $F \in LUC(G)^*$ and $\mathcal{P}(T)(F) \notin X$, there is an $f \in LUC(G)$ for which $\langle \mathcal{P}(T)(F), f\rangle \neq 0$ and

$$\langle P(T(\delta_x F)), \delta_x * f\rangle = \langle \delta_{x^{-1}} P(T(\delta_x F)), f\rangle = 0$$

for all $x \in G$ [**18**]. It follows that $\langle \mathcal{P}(T)(F), f\rangle = 0$, which is a contradiction. This shows that $\mathcal{P}(\mathcal{B}(LUC(G)^*)) \subseteq \mathrm{Hom}(LUC(G)^*, X)$. For the second assertion, suppose $T \in \mathrm{Hom}(LUC(G)^*, X)$ and $F \in LUC(G)^*$. Then $T(F) \in X$, and so $P(T(F)) = T(F)$. We have

$$\begin{aligned}\langle \mathcal{P}(T)(F), f\rangle &= m(x \longmapsto \langle P(T(\delta_x F)), \delta_x * f\rangle)\\ &= m(x \longmapsto \langle P(\delta_x T(F)), \delta_x * f\rangle)\\ &= m(x \longmapsto \langle \delta_x P(T(F)), \delta_x * f\rangle)\\ &= m(x \longmapsto \langle P(T(F)), f\rangle)\\ &= \langle P(T(F)), f\rangle\\ &= \langle T(F), f\rangle,\end{aligned}$$

for all $f \in LUC(G)$. Therefore, $\mathcal{P}(T) = T$. Consequently, $\mathcal{P}$ is a bounded projection of $\mathcal{B}(LUC(G)^*)$ onto $\mathrm{Hom}(LUC(G)^*, X)$. □

Theorem 3.3. *Let G be a locally compact group. Assume that G is amenable as discrete. Let X be a closed subspace of $\mathcal{B}(LUC(G)^*)$ (in the weak* operator topology) such that $\lambda_x T \lambda_{x^{-1}} \in X$ for all $x \in G$ and $T \in X$; here, λ_x is the left translation operator in $\mathcal{B}(LUC(G)^*)$ defined by $\lambda_x(F) = \delta_x F$. Let P be a bounded projection of $\mathcal{B}(LUC(G)^*)$ onto X. Then there exists a bounded projection from* $\mathrm{Hom}(LUC(G)^*)$ *onto* $\mathrm{Hom}(LUC(G)^*) \cap X$.

Proof. Let m be an invariant mean on $l^\infty(G)$ [**15**], and let P be a bounded projection of $\mathcal{B}(LUC(G)^*)$ onto X. Define $P' : \mathrm{Hom}(LUC(G)^*) \to \mathrm{Hom}(LUC(G)^*) \cap X$ by

$$\langle P'(T)(F), f\rangle = m(x \longmapsto \langle P(T)(\delta_x F), \delta_x * f\rangle),$$

where $T \in \mathrm{Hom}(LUC(G)^*)$, $F \in LUC(G)^*$ and $f \in LUC(G)$. It is not hard to see that P' is a bounded projection of $\mathrm{Hom}(LUC(G)^*)$ onto $\mathrm{Hom}(LUC(G)^*) \cap X$. □

In [**7**], Lau studied conditions where a weak* closed left translation invariant subspace in $L^\infty(G)$ of a compact group G is the range of a weak*–weak* continuous projection on $L^\infty(G)$ commutes with left translation. In the next theorem, we characterize the weak* closed left translation invariant subspace X of $LUC(G)$ which is the range of a weak*–weak* continuous projection P on $LUC(G)$ commuting with left translations.

Theorem 3.4. *Let G be a locally compact group. A $\sigma(LUC(G), L^1(G))$ closed left translation invariant subspace X of $LUC(G)$ is the range of a $\sigma(LUC(G), L^1(G)) - \sigma(LUC(G), L^1(G))$ continuous projection P on $LUC(G)$ commuting with left translations if and only if $X = \rho_\mu^*(LUC(G))$ for an idempotent $\mu \in M(G)$; here, ρ_μ is the right translation operator in $\mathcal{B}(L^1(G))$ defined by $\rho_\mu(\varphi) = \varphi * \mu$.*

Proof. Let X be a $\sigma(LUC(G), L^1(G))$ closed left translation invariant subspace of $LUC(G)$. Let $P : LUC(G) \to X$ be a $\sigma(LUC(G), L^1(G))$-$\sigma(LUC(G), L^1(G))$ continuous projection onto X such that $P(L_x f) = L_x P(f)$ for all $x \in G$ and $f \in LUC(G)$. Let $\mathcal{P} : L^\infty(G) \to L^\infty(G)$ be defined as

$$\langle \mathcal{P}(f), \varphi \rangle = \langle \delta_e, P(\widetilde{\varphi} * f) \rangle,$$

where $\widetilde{\varphi}(x) = \Delta(x^{-1})\varphi(x^{-1})$; here, Δ is the modular function on G. Since P commutes with left translation, we have $P(\varphi * f) = \varphi * P(f)$ for all $\varphi \in L^1(G)$ and $f \in LUC(G)$ [**16**]. If $\varphi, \psi \in L^1(G)$ and $f \in L^\infty(G)$, then

$$\begin{aligned} \langle \mathcal{P}(\varphi * f), \psi \rangle &= \langle \delta_e, P(\widetilde{\psi} * (\varphi * f)) \rangle \\ &= \langle \mathcal{P}(f), \widetilde{\varphi} * \psi \rangle \\ &= \langle \varphi * \mathcal{P}(f), \psi \rangle. \end{aligned}$$

Since this relation holds for all $\psi \in L^1(G)$, we conclude that $\mathcal{P}(\varphi * f) = \varphi * \mathcal{P}(f)$ for each $\varphi \in L^1(G)$ and each $f \in L^\infty(G)$.

Let $\{f_\alpha\}_{\alpha \in I}$ be a net in $L^\infty(G)$ converging to $f \in L^\infty(G)$ in the weak* topology of $L^\infty(G)$. For $\psi \in L^1(G)$, $\widetilde{\psi} * f_\alpha \to \widetilde{\psi} * f$ in the $\sigma(LUC(G), L^1(G))$ topology of $LUC(G)$. By assumption, $P(\widetilde{\psi} * f_\alpha) \to P(\widetilde{\psi} * f)$ in the $\sigma(LUC(G), L^1(G))$ topology. Since $L^1(G)$ has a bounded approximate identity, Cohen' s factorization theorem implies that each $\psi \in L^1(G)$ has the form $\psi_1 * \psi_2$ for $\psi_1, \psi_2 \in L^1(G)$. Hence, $\mathcal{P}(f_\alpha) \to \mathcal{P}(f)$ in the weak* topology of $L^\infty(G)$.

Let $\mathcal{P}^* : L^\infty(G)^* \to L^\infty(G)^*$ be the adjoint operator of $\mathcal{P}$, i.e., $\mathcal{P}^*$ is the bounded linear operator of $L^\infty(G)^*$ into $L^\infty(G)^*$ which satisfies $\langle \mathcal{P}^*(F), f \rangle = \langle F, \mathcal{P}(f) \rangle$ for all $F \in L^\infty(G)^*$ and $f \in L^\infty(G)$.

We conclude that $\mathcal{P}^*(\varphi) \in L^\infty(G)^*$ is weak* continuous, and so $\mathcal{P}^*(\varphi) \in L^1(G)$ for all $\varphi \in L^1(G)$ [**18**].

It is easy to see that $\mathcal{P}^*(\varphi * \psi) = \varphi * \mathcal{P}^*(\psi)$ for all $\varphi, \psi \in L^1(G)$. By [**3,** Theorem 3.3.40], there exists a $\mu \in M(G)$ such that $\mathcal{P}^*(\varphi) = \varphi * \mu$ for all $\varphi \in L^1(G)$. If $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, we have

$$\langle \mathcal{P}(f), \varphi \rangle = \langle f, \mathcal{P}^*(\varphi) \rangle = \langle f, \varphi * \mu \rangle = \langle \rho_\mu^*(f), \varphi \rangle.$$

This shows that $\mathcal{P}(f) = \rho_\mu^*(f)$ for all $f \in L^\infty(G)$. It is easily verified that $P(f) = \mu f$ for all $f \in LUC(G)$, that μ is idempotent and $X = \rho_\mu^*(LUC(G))$.

To prove the converse, let $X = \rho_\mu^*(LUC(G))$ for an idempotent $\mu \in M(G)$. Let $\{f_\alpha\}_{\alpha \in I}$ be a net in $LUC(G)$, and let $\{\mu f_\alpha\}_{\alpha \in I}$ converge to $f \in LUC(G)$ in the $\sigma(LUC(G), L^1(G))$ topology. It is not hard to see that $\mu * \mu f_\alpha = \mu f_\alpha \to \mu f$ in the $\sigma(LUC(G), L^1(G))$ topology. We conclude that X is $\sigma(LUC(G), L^1(G))$ closed. Let P be the bounded projection from $LUC(G)$ onto X defined by $P(f) = \mu f$. We easily see that P is $\sigma(LUC(G), L^1(G))$-$\sigma(LUC(G), L^1(G))$ continuous and that $P(L_x f) = L_x P(f)$ for all $x \in G$ and $f \in LUC(G)$. $\square$

Corollary 3.5. *Let G be a locally compact group. Let X be a weak* closed, left translation invariant, complemented subspace of $L^\infty(G)$. Then $X = \rho_\mu^*(L^\infty(G))$ for an idempotent $\mu \in M(G)$ if any one of the following conditions hold*:

(i) *there exists a weak*–weak* continuous projection P from $L^\infty(G)$ onto X which commutes with convolution*;
(ii) *G is compact.*

Note that $\varphi * f \in X$ for all $\varphi \in L^1(G)$ and $f \in X$, see [**10,** Lemma 2].

Proof.

(i) See Theorem 3.4 and its proof.

(ii) Let P be a bounded projection from $L^\infty(G)$ onto X commuting with left translation (see [**19,** Theorem 1]). By [**1,** Theorem 1], X is topologically invariantly complemented in $L^\infty(G)$. Since any bounded linear operator from $L^\infty(G)$ into $L^\infty(G)$ which commutes with convolution is weak*–weak* continuous (see [**10,** Lemma 4]), by (i), $X = \rho_\mu^*(L^\infty(G))$ for an idempotent $\mu \in M(G)$. $\square$

Remark 3.6. Let G be a locally compact group. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ vanishing at infinity. For an extensive study of $L_0^\infty(G)$, see Lau and Pym [**11**]. As shown in [**11**], for any $F \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$, $Ff \in L_0^\infty(G)$. We shall regard $L_0^\infty(G)^*$ as a Banach algebra with the first Arens multiplication. It is known that $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$, see [**11,** Theorem 2.11]. Let X be a subspace of $L_0^\infty(G)$ such that $Ff \in X$ for all $F \in L_0^\infty(G)^*$ and $f \in X$. Let P be a bounded projection from $L_0^\infty(G)$ onto X commuting with convolutions. Let $F \in L_0^\infty(G)^*$ and $\{e_\alpha\}_{\alpha\in I}$ be a bounded approximate identity for $L^1(G)$ [**6**]. Then,

$$\begin{aligned}\langle P(Ff), \varphi\rangle &= \lim_\alpha \langle P(Ff), e_\alpha * \varphi\rangle = \lim_\alpha \langle \widetilde{e}_\alpha * P(Ff), \varphi\rangle \\ &= \lim_\alpha \langle P(\widetilde{e}_\alpha * Ff), \varphi\rangle = \lim_\alpha \langle \widetilde{e}_\alpha FP(f), \varphi\rangle \\ &= \lim_\alpha \langle FP(f), e_\alpha * \varphi\rangle = \langle FP(f), \varphi\rangle,\end{aligned}$$

for all $\varphi \in L^1(G)$. This shows that $P(Ff) = FP(f)$ for all $F \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$. Now, let G be a compact group, and let X be a weak* closed left translation invariantly complemented subspace of $L^\infty(G)$. Then there exists a bounded projection P from $L^\infty(G)$ onto X such that $P(Ff) = FP(f)$ for all $F \in L^\infty(G)^*$ and $f \in L^\infty(G)$.

Theorem 3.7. *Let G be a locally compact group. Assume that G is amenable as discrete. Then the following conditions are equivalent*:

(i) *G is discrete*;
(ii) *any bounded projection P from $L^\infty(G)$ onto a weak* closed left translation invariant subspace X of $L^\infty(G)$ which commutes with left translation also commutes with convolution.*

Proof. Clearly (i) implies (ii).

(ii) $\Rightarrow$ (i). We assume to the contrary that G is non-discrete. Let m be a left invariant mean on $L^\infty(G)$ which is not a topologically left invariant mean, see [**16,** Proposition 22.3]. We consider the weak* closed subspace X of $L^\infty(G)$ consisting of constant functions. Define $P : L^\infty(G) \to X$ by $P(f) = \langle m, f\rangle 1_G$, $f \in L^\infty(G)$. Then, as readily checked, $\|P\| \le 1$, and P is a projection of $L^\infty(G)$ onto X commuting with left translations. Finally, let $f \in L^\infty(G)$ and $\varphi \in P^1(G)$ be such that $\langle m, \varphi * f\rangle \ne \langle m, f\rangle$; here, $P^1(G)$ is the set of all probability measures in $L^1(G)$. Then

$$P(\varphi * f) = \langle m, \varphi * f\rangle 1_G \neq \langle m, f\rangle 1_G = P(f).$$

We conclude that P does not commute with convolution. This is a contradiction. □

In [**10**], Lau and Losert proved that a locally compact group G is amenable if and only if, whenever X is a weak* closed left translation invariant complemented subspace of $L^\infty(G)$, X is invariantly complemented. Also, as shown by Lau [**7**], if G is an amenable locally compact group, then any weak* closed self-adjoint left translation invariant subalgebra of $L^\infty(G)$ is the range of a bounded projection commuting with left translations.

In the following, we define approximately complemented subspaces, and we obtain the other version of above facts.

Definition 3.8. Let E be a normed space. Then a subspace F of E is called *approximately complemented* in E if there is a net $\{P_\alpha\}_{\alpha\in I}$ of bounded operators from E into F such that $\lim_\alpha P_\alpha(f) = f$ uniformly on bounded subsets of F.

Theorem 3.9. *Let G be an amenable locally compact group, and let X be a closed subspace of $LUC(G)$ such that $L_x f \in X$ for all $f \in X$ and $x \in G$. If X is approximately complemented in $LUC(G)$, then there is a net of bounded operators $P'_\beta : LUC(G) \to X$ such that $\lim_\beta P'_\beta(f) = f$ uniformly on bounded subsets of X and, for every compact s*

$$\lim_\beta \|L_a P'_\beta(f) - P'_\beta(L_a f)\| = 0$$

uniformly for $a \in K$ and $f \in F$.

Proof. Let $\{P_\alpha\}_{\alpha\in I}$ be a net of bounded operators from $LUC(G)$ into X such that $\lim_\alpha P_\alpha(f) = f$ uniformly on bounded subsets of X. For $\varphi \in P^1(G)$ and $\alpha \in I$, we define an operator P_α^φ on $LUC(G)$ by

$$P_\alpha^\varphi(f)(y) = \int \varphi(x) P_\alpha(L_x f)(x^{-1}y)\, dx.$$

Since $x \mapsto L_{x^{-1}} P_\alpha(L_x f)$ is a continuous map from G into $LUC(G)$, $P_\alpha^\varphi(f)$ is well defined by **[2]** and that this integral defines a bounded linear operator from $LUC(G)$ into X.

Let K be a compact subset of G, $\alpha \in I$, and let $\epsilon > 0$. By **[16,** Lemma 6.13], there exists a $\varphi \in P^1(G)$ such that

$$\|\delta_a * \varphi - \varphi\|_1 < \frac{\epsilon}{\|P_\alpha\| + 1}$$

whenever $a \in K$. For every $a \in K$, we have

$$\begin{aligned}
&|P_\alpha^\varphi(L_a f)(y) - L_a P_\alpha^\varphi(f)(y)| \\
&\quad = \left| \int \varphi(x)(P_\alpha(L_{ax} f)(x^{-1}y) - P_\alpha(L_x f)(x^{-1}ay))\, dx \right| \\
&\quad = \left| \int (\varphi(a^{-1}x) - \varphi(x)) P_\alpha(L_x f)(x^{-1}ay)\, dx \right| \\
&\quad \le \|P_\alpha\| \|f\| \|\delta_a * \varphi - \varphi\|_1 \\
&\quad < \|f\| \epsilon,
\end{aligned}$$

whenever $f \in LUC(G)$. We consider the directed set $J = \mathcal{K} \times I \times (0,1)$ where, for $\beta = (K, \alpha, \epsilon) \in J$,

$$\beta' = (K', \alpha', \epsilon') \in J, \quad \beta' \succeq \beta$$

in the cases $K \subseteq K'$ and $\alpha' \succeq \alpha$ and $\epsilon' \le \epsilon$ (here $\mathcal{K}$ is the family of compact subsets of G). For each $\beta = (K, \alpha, \epsilon)$, there exists $\varphi_\beta \in P^1(G)$ such that

$$\|\delta_a * \varphi_\beta - \varphi_\beta\|_1 < \frac{\epsilon}{\|P_\alpha\| + 1} \quad \text{for all } a \in K.$$

We define $P'_\beta : LUC(G) \to X$ by $P'_\beta(f) = P_\alpha^{\varphi_\beta}(f)$.

Let K_0 be a compact subset of G, $\epsilon_0 > 0$, and let $\alpha_0 \in I$. For every $\beta = (K, \alpha, \epsilon) \succeq (K_0, \alpha_0, \epsilon_0) = \beta_0$, we have

$$\|P'_\beta(L_a f) - L_a P'_\beta(f)\| < \|f\| \epsilon \le \|f\| \epsilon_0$$

for every $a \in K$ and $f \in LUC(G)$. This shows that $\lim_\beta \|P'_\beta(L_a f) - L_a P'_\beta(f)\| = 0$ uniformly on every compact subset K of G and every bounded subset F of X.

Now, let F be a bounded subset of X and $\epsilon > 0$. Obviously, $\{L_x f;\ f \in F,\ x \in G\}$ is a bounded subset of X. By assumption, there exists $\alpha_0 \in I$ such that $\|P_\alpha(L_x f) - L_x f\| < \epsilon$ for all $\alpha \succeq \alpha_0$, $x \in G$ and $f \in F$. Put $\beta_0 = (\{e\}, \alpha_0, \epsilon)$, and let $\beta \succeq \beta_0$. For $\psi \in L^1(G)$, we have

$$\begin{aligned} |\langle P'_\beta(f) - f, \psi\rangle| &= \left| \int\int \varphi_\beta(x)(P_\alpha(L_x f)(x^{-1}y) - f(y))\psi(y)\,dx\,dy \right| \\ &= \left| \int\int \varphi_\beta(x)(P_\alpha(L_x f)(y) - f(xy))\psi(xy)\,dy\,dx \right| \\ &\leq \int \varphi_\beta(x)\|P_\alpha(L_x f) - L_x f\|\|\psi\|_1 dx \\ &\leq \epsilon\|\psi\|_1. \end{aligned}$$

Let G be an amenable locally compact group, and let X be a $\sigma(LUC(G), L^1(G))$ closed approximately complemented subspace of $LUC(G)$ such that $L_x f \in X$ for all $f \in X$ and $x \in G$. Then there is a net of bounded operators $\mathcal{P}_\beta : \mathcal{B}(LUC(G)) \to \mathcal{B}(LUC(G), X)$ such that $\mathcal{P}_\beta(T) = T$ uniformly on bounded subsets of $\mathcal{B}(LUC(G))$ and, for every compact set K of G and every bounded set $\mathcal{F}$ of $\mathcal{B}(LUC(G), X)$,

$$\lim_\beta \|L_a \mathcal{P}_\beta(T) - \mathcal{P}_\beta(L_a T)\| = 0$$

uniformly for $a \in K$ and $T \in \mathcal{F}$. We conclude that $\lim_\beta P'_\beta(f) = f$ uniformly on bounded subsets of X. □

Remark 3.10. Recall that a closed subspace F of a Banach space X is called *weakly complemented* in X if

$$F^\perp = \{f \in X^*;\ \langle f, x\rangle = 0 \quad \text{for all } x \in F\}$$

is complemented in X^*. It is easy to see that every complemented subspace is weakly complemented. It is known that c_0 is weakly complemented in l^∞, but not complemented, see **[20,** Exercise 2.3.3]. Denote by $L^1([0,1])$ the Banach space of all integrable functions defined on $[0,1]$. This has a subspace isomorphic to l^2 **[13]**. This subspace is approximately complemented in $L^1([0,1])$, but it is not weakly complemented in $L^1([0,1])$ **[24]**. Therefore, this subspace is not complemented in $L^1([0,1])$.

Theorem 3.11. *Let G be an amenable locally compact group, and let X be a weak* closed approximately complemented subspace of $LUC(G)$ such that $L_x f \in X$ for all $f \in X$ and $x \in G$. Then there is a net of* *bounded operators*

$$\mathcal{P}_\beta : \mathcal{B}(LUC(G)) \longrightarrow \mathcal{B}(LUC(G), X), \quad \beta \in J,$$

such that $\lim_\beta \mathcal{P}_\beta(T) = T$ *uniformly on bounded subsets of* $\mathcal{B}(LUC(G))$ *and, for every compact set* K *of* G *and every bounded set* $\mathcal{F}$ *of* $\mathcal{B}(LUC(G), X)$*,*

$$\lim_\beta \|L_a \mathcal{P}_\beta(T) - \mathcal{P}_\beta(L_a T)\| = 0$$

uniformly for $a \in K$ *and* $T \in \mathcal{F}$*.*

Proof. First, observe that $L_x T \in \mathcal{B}(LUC(G), X)$ for $x \in G$ and $T \in \mathcal{B}(LUC(G), X)$, since $L_x f \in X$ for all $x \in G$ and $f \in X$. If X is approximately complemented, there is a net of bounded operators $P_\beta : LUC(G) \to X$, $\beta \in J$, such that $\lim_\beta \|P_\beta(f) - f\| = 0$ uniformly on bounded subsets of X and, for every compact set K of G and every bounded set F of X,

$$\lim_\beta \|L_a P_\beta(f) - P_\beta(L_a f)\| = 0$$

uniformly for $a \in K$ and $f \in F$, see Theorem 3.9. For $\beta \in J$ and $T \in \mathcal{B}(LUC(G))$, we now set $\langle \mathcal{P}_\beta(T)(f), \varphi \rangle = \langle P_\beta(T(f)), \varphi \rangle$ whenever $f \in LUC(G)$ and $\varphi \in L^1(G)$. It is easy to see that $\mathcal{P}_\beta(T) \in \mathcal{B}(LUC(G), X)$ for all $T \in \mathcal{B}(LUC(G))$. Therefore, given a bounded set $\mathcal{F} \subseteq \mathcal{B}(LUC(G), X)$ and an $\epsilon > 0$, there is a $\beta_0 \in J$ such that

$$\|P_\beta(T(f)) - T(f)\| < \epsilon$$
$$\text{for all } \beta \succeq \beta_0,\ T \in \mathcal{F} \text{ and } f \in b(LUC(G));$$

here, $b(LUC(G))$ denotes the closed unit ball in $LUC(G)$. For every $\beta \succeq \beta_0$, $T \in \mathcal{F}$ and $f \in b(LUC(G))$ we have

$$\begin{aligned}
&|P_\alpha^\varphi(L_a f)(y) - L_a P_\alpha^\varphi(f)(y)| \\
&\quad = \left| \int \varphi(x)(P_\alpha(L_{ax} f)(x^{-1}y) - P_\alpha(L_x f)(x^{-1}ay))\, dx \right| \\
&\quad = \left| \int (\varphi(a^{-1}x) - \varphi(x)) P_\alpha(L_x f)(x^{-1}ay)\, dx \right| \\
&\quad \le \|P_\alpha\| \|f\| \|\delta_a * \varphi - \varphi\|_1 \\
&\quad < \|f\| \epsilon,
\end{aligned}$$

whenever $\varphi \in L^1(G)$. This shows that $\|\mathcal{P}_\beta(T) - T\| < \epsilon$ for all $\beta \succeq \beta_0$ and $T \in \mathcal{F}$.

Now, let $\mathcal{F}$ be a bounded subset of $\mathcal{B}(LUC(G), X)$. Given a compact set $K \subseteq G$ and $\epsilon > 0$, from Theorem 3.9, there is a $\beta_0 \in J$ such that $\|P_\beta(L_aT(f)) - L_aP_\beta(T(f))\| < \epsilon$ for all $\beta \succeq \beta_0$, $T \in \mathcal{F}$, $a \in K$ and $f \in b(LUC(G))$. For $\beta \succeq \beta_0$, $T \in \mathcal{F}$ and $f \in b(LUC(G))$, we have

$$\begin{aligned}
&|\langle L_a\mathcal{P}_\beta(T)(f) - \mathcal{P}_\beta(L_aT)(f), \varphi\rangle| \\
&\quad = |\langle L_aP_\beta(T(f)), \varphi\rangle - \langle P_\beta(L_aT(f)), \varphi\rangle| \\
&\quad \leq \|L_aP_\beta(T(f)) - P_\beta(L_aT(f))\|\|\varphi\|_1 \\
&\quad < \|\varphi\|_1\epsilon,
\end{aligned}$$

whenever $\varphi \in L^1(G)$. We conclude that $\|L_a\mathcal{P}_\beta(T) - \mathcal{P}_\beta(L_aT)\| < \epsilon$ for all $\beta \succeq \beta_0$, $a \in K$ and $T \in \mathcal{F}$. This completes the proof. □

REFERENCES

1. M.E.B. Bekka, *Complemented subspace of $L^\infty(G)$, ideals of $L^1(G)$ and amenability*, Monatsh. Math. **109** (1990), 195–203.

2. N. Bourbaki, *Elements de mathématique*, **25**, *Première partie*, Livre VI: *Intégration, Chapitre* 6: *Intégration vectorielle*, Act. Sci. Ind. **1281**, Hermann, Paris, 1959.

3. H.G. Dales, *Banach algebra and automatic continuity*, Lond. Math. Soc. Mono. **24**, Clarendon Press, Oxford, 2000.

4. B. Forrest, *Amenability and bounded approximate identities in ideals of $A(G)$*, Illinois J. Math. **34** (1990), 1–25.

5. A. Ghaffari, *Projections onto invariant subspaces of some Banach algebras,* Acta Math. Sinica **24** (2008), 1089–1096.

6. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, Volume I, Springer Verlag, Berlin, 1963; Volume II, Springer Verlag, Berlin, 1970.

7. A.T. Lau, *Invariantly complemented subspaces of $L^\infty(G)$ and amenable locally group*, Illinois J. Math. **26** (1982), 226–235.

8. ______, *Operators which commute with convolution on subspaces of $L^\infty(G)$*, Colloq. Math. **39** (1978), 351–359.

9. A.T. Lau and V. Losert, *Complementation of certain subspaces of $L^\infty(G)$ of a locally compact group*, Pacific J. Math. **141** (1990), 295–310.

10. ______, *Weak*-closed complemented invariant subspaces of $L_\infty(G)$ and amenable locally compact groups*, Pacific J. Math. **123** (1986), 149–159.

11. A.T. Lau and J. Pym, *Concerning the second dual of the group algebra of a locally compact group*, J. Lond. Math. Soc. **41** (1990), 445–460.

12. A.T. Lau and A. Ulger, *Characterization of closed ideals with bounded approximate identities in commutative Banach algebras, complemented subspaces of the group von Neumann algebras and applications*, Trans. Amer. Math. Soc. **366** (2014), 4151–4171.

13. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, I, Springer-Verlag, Berlin, 1977.

14. T.S. Lui, A. van Rooij and J.K. Wang, *Projections and approximate identities for ideals in group algebras*, Trans. Amer. Math. Soc. **175** (1973), 469–482.

15. A.L.T. Paterson, *Amenability*, Amer. Math. Soc. Math. Surv. Mono. **29**, Providence, Rhode Island, 1988.

16. J.P. Pier, *Amenable locally compact groups*, John Wiley And Sons, New York, 1984.

17. H.P. Rosenthal, *Projections onto translation invariant subspaces of* $L^p(G)$, Mem. Amer. Math. Soc. **63** (1966), 84 pages.

18. W. Rudin, *Functional analysis*, McGraw Hill, New York, 1991.

19. ______, *Projections on invariant subspaces*, Proc. Amer. Math. Soc. **13** (1962), 429–432.

20. V. Runde, *Lectures on amenability*, Lect. Notes Math. **1774**, Springer-Verlag, Berlin, 2002.

21. M. Takahashi, *Remarks on certain complemented subspaces on groups*, Hokkaido Math. J. **13** (1984), 260–270.

22. P.J. Wood, *Invariant complementation and projectivity in the Fourier algebra*, Proc. Amer. Math. Soc. **131** (2002), 1881–1890.

23. ______, *Complemented ideals in the Fourier algebra of a locally compact group*, Proc. Amer. Math. Soc. **128** (1999), 445–451.

24. Y. Zhang, *Approximate complementation and its applications in studying ideals of Banach algebras*, Math. Scand. **92** (2003), 301–308.

Semnan University, Department of Mathematics, P.O. Box 35195-363, Semnan, Iran
Email address: **aghaffari@semnan.ac.ir**

Semnan University, Department of Mathematics, P.O. Box 35195-363, Semnan, Iran
Email address: **somayehamirjan@yahoo.com**