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# TATE COHOMOLOGY OF GORENSTEIN FLAT MODULES WITH RESPECT TO SEMIDUALIZING MODULES

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ABSTRACT. We study Tate cohomology of modules over a commutative Noetherian ring with respect to semidualizing modules. First, we show that the class of modules admitting a Tate $\mathcal{F}_C$-resolution is exactly the class of modules in $\mathcal{B}_C$ with finite $\mathcal{GF}_C$-projective dimension. Then, the interaction between the corresponding relative and Tate cohomologies of modules is given. Finally, we give some new characterizations of modules with finite $\mathcal{F}_C$-projective dimension.

**1. Introduction.** Tate cohomology originated from the study of representations of finite groups. It was created in the 1950s, based on Tate's observation that the $\mathbb{Z}$G-module $\mathbb{Z}$ with trivial action admits a complete projective resolution, and has recently been revitalized by a number of authors (see, for example, [**1, 2, 7, 23**]).

Over a commutative Noetherian ring $R$, a finitely generated $R$-module $C$ is *semidualizing* [**12**] if the natural homothety morphism $R \to \operatorname{Hom}(C,C)$ is an isomorphism and $\operatorname{Ext}^{i\geqslant 1}(C,C) = 0$. Furthermore, a semidualizing module $C$ is *dualizing* if it has finite injective dimension. A semidualizing $R$-module $C$ gives rise to several distinguished classes of modules; for instance, one class $\mathcal{P}_C$ ($\mathcal{F}_C$) of $C$-projective ($C$-flat) modules and another class $\mathcal{GP}_C$ ($\mathcal{GF}_C$) of $C$-Gorenstein projective ($C$-Gorenstein flat) modules. Detailed definitions can be found in Section 2.

In [**23**], Sather-Wagstaff, Sharif and White generalized the work of Avramov and Martsinkovsky [**2**] to arbitrary abelian categories. As

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an application, they proved that an $R$-module $M$ has a Tate $\mathcal{P}_C$-resolution if and only if $M \in \mathcal{B}_C$ with $\mathcal{GP}_C$-pd$(M) < \infty$ (see **[23,** Theorem A, Lemma 2.9]). Based on Tate $\mathcal{P}_C$-resolutions, one may define the Tate cohomology of modules in $\mathcal{B}_C$ with finite $C$-Gorenstein projective dimension. It is natural to consider the Tate cohomology of modules in $\mathcal{B}_C$ with finite $C$-Gorenstein flat dimension.

The first step in studying Tate cohomology of modules in $\mathcal{B}_C$ with finite $C$-Gorenstein flat dimensions is to give an appropriate resolution. Let

$$\mathcal{H}_C = \{M \mid M \cong C \otimes G \text{ with } G \in \mathcal{A}_C \cap \mathcal{C}\},$$

where $\mathcal{A}_C$ is the Auslander class and $\mathcal{C}$ is the class of cotorsion $R$-modules. It is shown that every $R$-module $M$ in $\mathcal{B}_C$ has a monic $\mathcal{H}_C$-envelope $\varepsilon : M \to B$ with $\mathrm{coker}(\varepsilon) \in \mathcal{F}_C$ (see Proposition 3.4). This fact lets us give the definition of Tate $\mathcal{F}_C$-resolutions of $M$ (see Definition 3.5).

The next result characterizes the modules which admit Tate $\mathcal{F}_C$-resolutions. It is contained in Theorem 3.10.

**Theorem 1.1.** *Let $R$ be a ring. Then an $R$-module $M$ has a Tate $\mathcal{F}_C$-resolution if and only if $M \in \mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) < \infty$.*

We have an $R$-module $M$ with a Tate $\mathcal{F}_C$-resolution

$$T \longrightarrow W \longrightarrow B \longleftarrow M.$$

We use the complex $W$ to define the *relative cohomology* functors $\mathrm{Ext}^n_{\mathcal{GF}_C}(B,-)$ and $\mathrm{Ext}^n_{\mathcal{F}_C}(B,-)$. The complex $T$ is used to define the *Tate cohomology* functors $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,-)$ (see Definition 4.2). These cohomology functors are connected by the next result. See Theorem 4.6.

**Theorem 1.2.** *Let $M$ be any $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) \leqslant n < \infty$. For each $R$-module $N$, there is a long exact sequence*

$$\begin{array}{l} 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^1(B,N)} \mathrm{Ext}^1_{\mathcal{F}_C}(B,N) \longrightarrow \widehat{\mathrm{Ext}}^1_{\mathcal{F}_C}(M,N) \\ \longrightarrow \mathrm{Ext}^2_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^2(B,N)} \mathrm{Ext}^2_{\mathcal{F}_C}(M,N) \longrightarrow \widehat{\mathrm{Ext}}^2_{\mathcal{F}_C}(M,N) \\ \qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \end{array}$$

$$\longrightarrow \operatorname{Ext}^n_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^n(B,N)} \operatorname{Ext}^n_{\mathcal{F}_C}(M,N) \longrightarrow \widehat{\operatorname{Ext}}^n_{\mathcal{F}_C}(M,N)$$
$$\longrightarrow 0,$$

*where $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope of $M$.*

The next result shows that vanishing of the Tate cohomology functor $\widehat{\operatorname{Ext}}^n_{\mathcal{F}_C}(-,-)$ characterizes the finiteness of $\mathcal{F}_C$-projective dimension. See subsection 4.1 for the proof.

**Theorem 1.3.** *Let $M$ be an $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) < \infty$ and $\varepsilon : M \to B$ an $\mathcal{H}_C$-envelope $M$. Then the following are equivalent*:

(i) $\mathcal{F}_C$-pd$(M) < \infty$;
(ii) $\widehat{\operatorname{Ext}}^n_{\mathcal{F}_C}(M,N) = 0$ *for each* (*or some*) $n \in \mathbb{Z}$ *and each $R$-module* $N$;
(iii) $\widehat{\operatorname{Ext}}^n_{\mathcal{F}_C}(M,N) = 0$ *for some $n \in \mathbb{Z}$ and any $N \in \mathcal{H}_C$*;
(iv) $\widehat{\operatorname{Ext}}^n_{\mathcal{F}_C}(N,B) = 0$ *for each* (*or some*) $n \in \mathbb{Z}$ *and each* $N \in \mathcal{B}_C \cap \widehat{\mathcal{GF}_C}$;
(v) $\widehat{\operatorname{Ext}}^0_{\mathcal{F}_C}(M,B) = 0$.

As a consequence of Theorem 1.3, we give a necessary and sufficient condition for an Artinian ring to be semisimple; moreover, we prove that a local Gorenstein ring $(R,m,k)$ is regular if and only if $\widehat{\operatorname{Ext}}^0_{\mathcal{F}}(k,B) = 0$, where $\varepsilon : k \to B$ is a cotorsion envelope of $k$, see Corollaries 4.13 and 4.15.

We conclude this section by summarizing the contents of this paper. Section 2 contains notation and definitions for use throughout this paper. Section 3 focuses on the construction of Tate $\mathcal{F}_C$-resolutions. In Section 4, we consider Tate $\mathcal{F}_C$-cohomology of modules in $\mathcal{B}_C$ with finite $C$-Gorenstein flat dimension and prove Theorem 1.3.

**2. Preliminaries.** Throughout this paper, $R$ is a commutative Noetherian ring and $C$ is a semidualizing $R$-module. We write Mod$(R)$ for the class of $R$-modules and $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{GF}_C$ for the classes of flat, cotorsion and $C$-Gorenstein flat $R$-modules, respectively. For any $R$-module $M$, pd$(M)$, fd$(M)$ and id$(M)$ stand for projective, flat and injective dimensions of $M$, respectively. $\mathrm{Hom}(M,N)$ ($M \otimes N$) means $\mathrm{Hom}_R(M,N)$ ($M \otimes_R N$) for all $R$-modules $M$ and $N$.

Next, we recall the basic definitions and properties needed in the sequel. For more details, the reader may consult **[2, 5, 8, 15]**.

In the following, we let $\mathcal{X}$ be a class of $R$-modules.

**2.1. Covers and envelopes.** Let $M$ be an $R$-module. A homomorphism

$$\phi : M \longrightarrow C$$

with $C \in \mathcal{X}$ is called an *$\mathcal{X}$-preenvelope* of $M$ if, for any homomorphism $f : M \to C'$ with $C' \in \mathcal{X}$, there is a homomorphism $g : C \to C'$ such that $g\phi = f$. Moreover, if the only such $g$ are automorphisms of $C$ when $C' = C$ and $f = \phi$, the $\mathcal{X}$-preenvelope $\phi$ is called an *$\mathcal{X}$-envelope* of $M$. The class $\mathcal{X}$ is called *(pre)enveloping* if every $R$-module has an $\mathcal{X}$-(pre)envelope. Dually, we have the definitions of an $\mathcal{X}$-precover and an $\mathcal{X}$-cover.

**2.2. Complexes.** We let $\mathrm{C}(R)$ be the category of chain complexes of $R$-modules, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote boundedness conditions. For example, $\mathrm{C}_{\sqsupset}(R)$ is the full subcategory of $\mathrm{C}(R)$ of right-bounded complexes. If $X_i = 0$ for $i \neq 0$, we identify $X$ with the module in degree 0, and an $R$-module $M$ is said to be a complex $0 \to M \to 0$, with $M$ in degree 0. To every complex

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \longrightarrow \cdots$$

in $\mathrm{C}(R)$, the *nth homology module* of $X$ is the module

$$\mathrm{H}_n(X) = \frac{\ker(\partial^X_n)}{\mathrm{im}(\partial^X_{n+1})}.$$

We also set $\mathrm{Z}_n(X) = \ker(\partial^X_n)$, $\mathrm{B}_n(X) = \mathrm{im}(\partial^X_{n+1})$ and $\mathrm{C}_n(X) = \mathrm{coker}(\partial^X_{n+1})$. We write $X_{\geqslant n}$ for the subcomplex of $X$ with $i$th component equal to $X_i$ for $i \geqslant n$ and to 0 for $i < n$. We set $X_{\leqslant n} = X/X_{\geqslant n+1}$ and $X_{<n} = X_{\leqslant n-1}$. The *ith shift* of $X$ is the complex $\Sigma^i X$ with $n$th component $X_{n-i}$ and differential $\partial^{\Sigma^i X}_n = (-1)^i \partial^X_{n-i}$; we write $\Sigma X$ instead of $\Sigma^1 X$.

A *homomorphism* $\varphi : X \to Y$ of degree $n$ is a family of $(\varphi_i)_{i\in\mathbb{Z}}$ of homomorphisms of $R$-modules $\varphi_i : X_i \to Y_{i+n}$. In this case, we set $|\varphi| = n$. All such homomorphisms form an abelian group, denoted by $\mathrm{Hom}(X,Y)_n$; it is clearly isomorphic to

$$\prod_{i\in\mathbb{Z}} \mathrm{Hom}(X_i, Y_{i+n}).$$

We let $\mathrm{Hom}(X,Y)$ be the complex of $\mathbb{Z}$-modules with $n$th component $\mathrm{Hom}(X,Y)_n$ and differential

$$\partial(\varphi) = \partial^Y \varphi - (-1)^{|\varphi|}\varphi\partial^X.$$

For any $i \in \mathbb{Z}$, the cycles in $\mathrm{Hom}(X,Y)_i$ are the *chain maps* $X \to Y$ of degree $i$. A chain map of degree 0 is called a *morphism*. Two morphisms $\beta$ and $\beta'$ in $\mathrm{Hom}(X,Y)_0$ are called *homotopic*, denoted by $\beta \sim \beta'$, if there exists a degree 1 homomorphism $\nu$ such that $\partial(\nu) = \beta - \beta'$. A *homotopy equivalence* is a morphism $\varphi : X \to Y$ for which there exists a morphism $\psi : Y \to X$ such that $\varphi\psi \sim \mathrm{id}_Y$ and $\psi\varphi \sim \mathrm{id}_X$.

A *quasi-isomorphism* $\varphi : X \to Y$ is a morphism such that the induced map

$$\mathrm{H}_n(\varphi) : \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(Y)$$

is an isomorphism for all $n \in \mathbb{Z}$. The complexes $X$ and $Y$ are *equivalent* [**5**, page 164, A.1.11] and denoted by $X \simeq Y$, if they can be linked by a sequence of quasi-isomorphisms with arrows in the alternating directions.

A complex $M$ is $\mathrm{Hom}(\mathcal{X}, -)$ *exact* if the complex $\mathrm{Hom}(X, M)$ is exact for each $X \in \mathcal{X}$. Dually, the complex $M$ is $\mathrm{Hom}(-, \mathcal{X})$ exact if $\mathrm{Hom}(M, X)$ is exact for each $X \in \mathcal{X}$.

**Definition 2.1** ([**24**])**.** Let $X$ be a complex. When $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$, the natural morphism $X \to \mathrm{H}_0(X) = M$ is a quasi-isomorphism. In this event, $X$ is an $\mathcal{X}$*-resolution* of $M$ if each $X_n \in \mathcal{X}$, and the associated exact sequence

$$X^+ = \cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

is the *augmented* $\mathcal{X}$*-resolution* of $M$ associated to $X$. An $\mathcal{X}$-resolution is *proper* if $X^+$ is $\mathrm{Hom}(\mathcal{X}, -)$ exact.

The $\mathcal{X}$*-projective dimension* of $M$ is the quantity

$$\mathcal{X}\text{-}\operatorname{pd}(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

We let $\widehat{\mathcal{X}}$ be the subcategory of module $M$ with $\mathcal{X}$-pd$(M) < \infty$. For a non-negative integer $n$, we set res $\widehat{\mathcal{X}}^{\leqslant n}$ as the class of $R$-modules $M$ with $\mathcal{X}$-pd$(M) \leqslant n$. Note that the modules of $\mathcal{X}$-projective dimension 0 are exactly the modules of $\mathcal{X}$.

**Definition 2.2** ([**14**])**.** Let $M$ be an $R$-module. If $M$ has a proper $\mathcal{X}$-resolution $X \to M$, then, for each integer $n$ and each $R$-module $N$, the $n$th *relative cohomology group* $\operatorname{Ext}^n_{\mathcal{X}}(M,N)$ is

$$\operatorname{Ext}^n_{\mathcal{X}}(M,N) = \mathrm{H}_{-n}(\operatorname{Hom}(X,N)).$$

If we choose $\mathcal{X}$ to be the class $\mathcal{P}$ of projective $R$-modules, then $\operatorname{Ext}^n_{\mathcal{P}}(M,N)$ defined here is exactly the classical cohomology group $\operatorname{Ext}^n_R(M,N)$.

For an object $M \in \operatorname{Mod}(R)$, write $M \in {}^{\perp}\mathcal{X}$ if $\operatorname{Ext}^{\geqslant 1}_R(M,X) = 0$ for each $X \in \mathcal{X}$. Dually, we can define $M \in \mathcal{X}^{\perp}$.

**Definition 2.3** ([**5, 10**])**.** Let $C$ be a semidualizing module.

The *Auslander class* $\mathcal{A}_C$ with respect to $C$ consists of $R$-modules $M$ satisfying:

(i) $\operatorname{Tor}^R_{\geqslant 1}(C,M) = 0 = \operatorname{Ext}^{\geqslant 1}_R(C, C\otimes M)$, and
(ii) the natural map $\mu_M : M \to \operatorname{Hom}(C, C\otimes M)$ is an isomorphism of $R$-modules.

The *Bass class* $\mathcal{B}_C$ with respect to $C$ consists of $R$-modules $N$ satisfying

(i) $\operatorname{Ext}^{\geqslant 1}_R(C,N) = 0 = \operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}(C,N))$,
(ii) the natural map $\nu_N : C\otimes \operatorname{Hom}(C,N) \to N$ is an isomorphism of $R$-modules.

**Fact 2.4.**

(i) The classes $\mathcal{A}_C$ and $\mathcal{B}_C$ are closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and summands, see [**17,** Proposition 4.2, Corollary 6.3].

(ii) An $R$-module $M \in \mathcal{A}_C$ if and only $C \otimes M \in \mathcal{B}_C$, see [**10,** Proposition 2.1].
(iii) An $R$-module $N \in \mathcal{B}_C$ if and only $\mathrm{Hom}(C, N) \in \mathcal{A}_C$, see [**10,** Proposition 2.1].

**Definition 2.5** ([**15, 17, 24**])**.** Let $C$ be a semidualizing module. Then:

(i) the class of *$C$-projective* (respectively, *$C$-injective*, *$C$-flat* and *$C$-flat $C$-cotorsion*) $R$-modules, denoted by $\mathcal{P}_C$ (respectively, $\mathcal{I}_C$, $\mathcal{F}_C$, $\mathcal{F}_C^{\mathrm{cot}}$), consists of those $R$-modules of the form $C \otimes P$ (respectively, $\mathrm{Hom}(C, I)$, $C \otimes F$, $C \otimes G$) for some projective $R$-module $P$ (respectively, injective $R$-module $I$, flat $R$-module $F$, flat and cotorsion $R$-module $G$). According to [**24,** Lemma 4.3],

$$\mathcal{F}_C^{\mathrm{cot}} = \mathcal{F}_C \cap \mathcal{F}_C{}^{\perp}.$$

(ii) A complete $\mathcal{F}\mathcal{F}_C$-resolution is an exact sequence

$$\mathbf{X} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes F^0 \longrightarrow C \otimes F^1 \longrightarrow \cdots$$

of $R$-modules with each $F_i, F^i$ flat, and $\mathrm{Hom}(C, I) \otimes \mathbf{X}$ is an exact sequence for any injective $R$-module $I$. An $R$-module $M$ is *$C$-Gorenstein flat* if there exists a complete $\mathcal{F}\mathcal{F}_C$-resolution as above with $M \cong \mathrm{coker}(F_1 \to F_0)$.

(iii) A complete $\mathcal{P}\mathcal{P}_C$-resolution is an exact sequence

$$\mathbf{X} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes P^0 \longrightarrow C \otimes P^1 \longrightarrow \cdots$$

of $R$-modules with each $P_i, P^i$ projective, and $\mathrm{Hom}(\mathbf{X}, C \otimes P)$ is an exact sequence for any projective $R$-module $P$. An $R$-module $M$ is *$C$-Gorenstein projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution as above with $M \cong \mathrm{coker}(P_1 \to P_0)$.

For convenience, we write $\mathcal{G}\mathcal{P}_C$ and $\mathcal{G}\mathcal{F}_C$ for the classes of $C$-Gorenstein projective and $C$-Gorenstein flat $R$-modules, respectively.

**Remark 2.6.** We note that augmented proper left $\mathcal{G}\mathcal{F}_C$-resolutions are exact since every projective $R$-module is in $\mathcal{G}\mathcal{F}_C$.

## 3. Constructions of Tate $\mathcal{F}_C$-resolutions.

**Lemma 3.1.** *The class*

$$\mathcal{H}_C = \{M \mid M \cong C \otimes G \text{ with } G \in \mathcal{A}_C \cap \mathcal{C}\}$$

*is closed under direct summands, extensions and cokernels of monomorphisms.*

*Proof.* To prove that $\mathcal{H}_C$ is closed under direct summands, we consider any split exact sequence

$$0 \longrightarrow M' \longrightarrow C \otimes G \longrightarrow M'' \longrightarrow 0$$

of $R$-modules with $G \in \mathcal{A}_C \cap \mathcal{C}$. Since $G \in \mathcal{A}_C$, we have $G \cong \mathrm{Hom}(C, C \otimes G)$ and $C \otimes G \in \mathcal{B}_C$ by [**10,** Proposition 2.1]. Thus, both $M'$ and $M''$ are in $\mathcal{B}_C$ by [**17,** Proposition 4.2 (a)], and hence, $M' \cong C \otimes \mathrm{Hom}(C, M')$ and $M'' \cong C \otimes \mathrm{Hom}(C, M'')$. So, $\mathrm{Hom}(C, M')$ and $\mathrm{Hom}(C, M'')$ are in $\mathcal{A}_C$ by [**10,** Proposition 2.1].

Applying $\mathrm{Hom}(C, -)$ to the exact sequence $0 \to M' \to C \otimes G \to M'' \to 0$ above, we have a split exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, M') \longrightarrow G \longrightarrow \mathrm{Hom}(C, M'') \longrightarrow 0.$$

Note that $G$ is cotorsion by hypothesis. Then, $\mathrm{Hom}(C, M')$ and $\mathrm{Hom}(C, M'')$ are cotorsion. Hence, $M'$ and $M''$ are in $\mathcal{H}_C$. Thus, $\mathcal{H}_C$ is closed under direct summands.

It remains to show that $\mathcal{H}_C$ is closed extensions and cokernels of monomorphisms. Let

$$0 \longrightarrow C \otimes L \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$$

be an exact sequence of $R$-modules with $L \in \mathcal{A}_C \cap \mathcal{C}$. Then $L \cong \mathrm{Hom}(C, C \otimes L)$. Assume that $M_2 \in \mathcal{H}_C$. It follows that $M_2 \cong C \otimes L_2$ for some $L_2 \in \mathcal{A}_C \cap \mathcal{C}$. Thus, $L_2 \cong \mathrm{Hom}(C, C \otimes L_2) \cong \mathrm{Hom}(C, M_2)$. Applying $\mathrm{Hom}(C, -)$ to the above exact sequence $0 \to C \otimes L \to M_1 \to M_2 \to 0$, we obtain an exact sequence

$$0 \longrightarrow L \longrightarrow \mathrm{Hom}(C, M_1) \longrightarrow \mathrm{Hom}(C, M_2) \longrightarrow 0.$$

Since $\mathcal{C}$ is closed under extension by [**27,** Proposition 3.1.2], $\mathrm{Hom}(C, M_1)$ is cotorsion. Note that $\mathrm{Hom}(C, M_2) \in \mathcal{A}_C$ by the above proof. Then

$\operatorname{Hom}(C, M_1) \in \mathcal{A}_C$ by [**17,** Theorem 6.2]. Thus, $M_1 \in \mathcal{B}_C$ by [**10,** Proposition 2.1], and hence,

$$M_1 \cong C \otimes \operatorname{Hom}(C, M_1).$$

So $M_1 \in \mathcal{H}_C$ and $\mathcal{H}_C$ is closed under extensions. Similarly, we can prove $\mathcal{H}_C$ is closed under cokernels of monomorphisms by noting that $\mathcal{C}$ is closed under cokernels of monomorphisms. □

**Lemma 3.2.** *Let $R$ be a ring. Then $\mathcal{H}_C = \mathcal{B}_C \cap {\mathcal{F}_C}^{\perp}$.*

*Proof.* Let $M$ be an $R$-module in $\mathcal{H}_C$. Then $M \cong C \otimes G$ for some $G \in \mathcal{A}_C \cap \mathcal{C}$. It follows that $M \in \mathcal{B}_C$ by [**10,** Proposition 2.1]. For any flat $R$-module $F$, we have $\operatorname{Ext}^i(C \otimes F, C \otimes G) \cong \operatorname{Ext}^i(F, G) = 0$ by [**17,** Theorem 6.4 (1)]. Thus, $M \in {\mathcal{F}_C}^{\perp}$. So $M \in \mathcal{B}_C \cap {\mathcal{F}_C}^{\perp}$.

For the reverse containment, let $M \in \mathcal{B}_C \cap {\mathcal{F}_C}^{\perp}$. Then $M \cong C \otimes \operatorname{Hom}(C, M)$. It follows from [**10,** Proposition 2.1] that $\operatorname{Hom}(C, M) \in \mathcal{A}_C$. By [**13,** Theorem 4.1.1 (a)], there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(C, M) \longrightarrow G \longrightarrow L \longrightarrow 0$$

of $R$-modules with $G$ cotorsion and $L$ flat. Hence, $G \in \mathcal{A}_C$ by [**17,** Theorem 6.2]. Applying $C \otimes -$ to the exact sequence above, we have an exact sequence

$$0 \longrightarrow M \longrightarrow C \otimes G \longrightarrow C \otimes L \longrightarrow 0$$

with $C \otimes G \in \mathcal{H}_C$. Note that $M \in {\mathcal{F}_C}^{\perp}$ by hypothesis. Then the previous exact sequence is split. Thus, $M \in \mathcal{H}_C$ by Lemma 3.1. □

**Remark 3.3.** It follows from Lemma 3.2 that $\mathcal{F}_C^{\mathrm{cot}} \subseteq \mathcal{H}_C$. However, $\mathcal{F}_C^{\mathrm{cot}} \neq \mathcal{H}_C$ in general. For instance, let $k$ be a field and $R = k[[x^3, x^4, x^5]]$. By [**10,** Example 3.3], the $R$-submodule $C$ of $k[[x]]$ generated by $x$ and $x^2$ is semidualizing with $\operatorname{id}(C) = 1$. We claim that $\mathcal{F}_C^{\mathrm{cot}} \neq \mathcal{H}_C$. Indeed, if $\mathcal{F}_C^{\mathrm{cot}} = \mathcal{H}_C$, then $R^+ \in \mathcal{F}_C^{\mathrm{cot}}$ by noting that every injective $R$-module is in $\mathcal{H}_C$. Thus, $R$ is $C$-injective by [**24,** Lemma 4.1 (a)], and hence, $C$ is injective by [**25,** Lemma 2.11 (b)]. This yields the desired contradiction.

**Proposition 3.4.** *Let $M$ be an $R$-module in $\mathcal{B}_C$. Then $M$ has a monic $\mathcal{H}_C$-envelope*

$$\varepsilon : M \longrightarrow B$$

*with* $\operatorname{coker}(\varepsilon) \in \mathcal{F}_C$.

*Proof.* The proof is modeled on that of [**17,** Proposition 5.3 (a)]. Note that $M \in \mathcal{B}_C$. Then

$$\nu_M : C \otimes \operatorname{Hom}(C, M) \longrightarrow M$$

is an isomorphism. It follows from [**10,** Proposition 2.1] that $\operatorname{Hom}(C, M) \in \mathcal{A}_C$. By [**13,** Theorem 4.1.1 (a)], $\operatorname{Hom}(C, M)$ has a cotorsion envelope

$$\beta : \operatorname{Hom}(C, M) \longrightarrow G$$

with $\operatorname{coker}(\beta)$ flat. Thus, $G \in \mathcal{A}_C$ by [**17,** Theorem 6.2]. Set $B = C \otimes G$. Then $B \in \mathcal{H}_C$. Define $\varepsilon$ to be the composition homomorphism

$$M \xrightarrow{\nu_M^{-1}} C \otimes \operatorname{Hom}(C, M) \xrightarrow{C \otimes \beta} B.$$

Note that

$$0 \longrightarrow \operatorname{Hom}(C, M) \xrightarrow{\beta} G \longrightarrow \operatorname{coker}(\beta) \longrightarrow 0$$

is an exact sequence of $R$-modules with $\operatorname{coker}(\beta)$ flat. Then $C \otimes \beta : C \otimes \operatorname{Hom}(C, M) \to B$ is monic with $\operatorname{coker}(C \otimes \beta) \in \mathcal{F}_C$. Thus, $\varepsilon : M \to B$ is a monomorphism with $\operatorname{coker}(\varepsilon) \in \mathcal{F}_C$. It remains to show that $\varepsilon$ is an $\mathcal{H}_C$-envelope.

First, we show that $\varepsilon$ is an $\mathcal{H}_C$-preenvelope. Let $f : M \to U$ be a homomorphism with $U \in \mathcal{H}_C$. Note that $\operatorname{Ext}^1(\operatorname{coker}(\varepsilon), U) = 0$ by Lemma 3.2. Then the sequence,

$$0 \longrightarrow \operatorname{Hom}(\operatorname{coker}(\varepsilon), U) \longrightarrow \operatorname{Hom}(B, U) \longrightarrow \operatorname{Hom}(M, U) \longrightarrow 0,$$

is exact. Thus,

$$\operatorname{Hom}(\varepsilon, U) : \operatorname{Hom}(B, U) \longrightarrow \operatorname{Hom}(M, U)$$

is epic. Therefore, there is a homomorphism $\varphi : B \to U$ such that $f = \varphi \circ \varepsilon$.

Next we let $U = B$, $f = \varepsilon$ and $\varepsilon = \varphi \circ \varepsilon$. By [**17,** Observation 4.1], we have the following identity:

$$\mathrm{Hom}(C, \nu_M) \circ \ \mu_{\mathrm{Hom}(C,M)} = \mathrm{id}_{\mathrm{Hom}(C,M)}.$$

Then we obtain the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(C, M) \arrow[rr, "\beta"] \arrow[d, "\mathrm{Hom}(C,\nu_M^{-1})"'] & & G \arrow[d, "\cong"', "\mu_G"] \\
\mathrm{Hom}(C, C \otimes \mathrm{Hom}(C, M)) \arrow[rr, "\mathrm{Hom}(C,C\otimes\beta)"] & & \mathrm{Hom}(C, C \otimes G).
\end{tikzcd}$$

Note that $\mu_G$ is an isomorphism. Thus, we have:

$$\begin{aligned}
\beta &= \mu_G^{-1} \circ \mathrm{Hom}(C, C \otimes \beta) \circ \mathrm{Hom}(C, \nu_M^{-1}) \\
&= \mu_G^{-1} \circ \mathrm{Hom}(C, (C \otimes \beta) \circ \nu_M^{-1}) \\
&= \mu_G^{-1} \circ \mathrm{Hom}(C, \varepsilon) \\
&= \mu_G^{-1} \circ \mathrm{Hom}(C, \varphi) \circ \mathrm{Hom}(C, \varepsilon) \\
&= \mu_G^{-1} \circ \mathrm{Hom}(C, \varphi) \circ \mathrm{Hom}(C, C \otimes \beta) \circ \mathrm{Hom}(C, \nu_M^{-1}) \\
&= \mu_G^{-1} \circ \mathrm{Hom}(C, \varphi) \circ \mu_G \circ \beta.
\end{aligned}$$

Since $\beta$ is a cotorsion envelope, $\mu_G^{-1} \circ \mathrm{Hom}(C, \varphi) \circ \mu_G$ is an automorphism. Hence, $\mathrm{Hom}(C, \varphi)$ is an isomorphism. Applying $\mathrm{Hom}(C, -)$ to the exact sequence $0 \to \ker(\varphi) \to B \xrightarrow{\varphi} B$ of $R$-modules, we obtain an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(C, \ker(\varphi)) \longrightarrow \mathrm{Hom}(C, B) \xrightarrow{\mathrm{Hom}(C,\varphi)} \mathrm{Hom}(C, B).$$

Note that $\mathrm{Hom}(C, \ker(\varphi)) = 0$ since $\mathrm{Hom}(C, \varphi)$ is an isomorphism. It follows from [**17,** Proposition 3.1] that $\ker(\varphi) = 0$. Thus, $\varphi : B \to B$ is a monomorphism. Applying $\mathrm{Hom}(C, -)$ to the exact sequence,

$$0 \longrightarrow B \xrightarrow{\varphi} B \longrightarrow \mathrm{coker}(\varphi) \longrightarrow 0$$

of $R$-modules, we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(C, B) \xrightarrow{\mathrm{Hom}(C,\varphi)} \mathrm{Hom}(C, B) \longrightarrow \mathrm{Hom}(C, \mathrm{coker}(\varphi)) \longrightarrow 0.$$

Since $\mathrm{Hom}(C, \varphi)$ is an isomorphism, $\mathrm{Hom}(C, \mathrm{coker}(\varphi)) = 0$. Thus, $\mathrm{coker}(\varphi) = 0$ by [**17,** Proposition 3.1], and so $\varphi$ is an isomorphism. This completes the proof. $\square$

**Definition 3.5.** Let $M$ be an $R$-module. A *Tate $\mathcal{F}_C$-resolution* of $M$ is a diagram

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

of morphisms of complexes satisfying:

(i) $\varepsilon : M \to B$ is a monic $\mathcal{H}_C$-envelope of $M$ such that $\mathrm{coker}(\varepsilon) \in \mathcal{F}_C$;
(ii) $\alpha : W \to B$ is an $\mathcal{F}_C^{\mathrm{cot}}$-resolution of $B$ such that $\mathrm{C}_i(W) \in \mathcal{H}_C$ for every $i = 0, 1, \ldots$;
(iii) $T$ is an exact complex with each entry in $\mathcal{F}_C^{\mathrm{cot}}$ and $\mathrm{Z}_i(T) \in \mathcal{GF}_C$ for all $i \in \mathbb{Z}$;
(iv) $\tau : T \to W$ is a morphism such that $\tau_i$ is bijective for all $i \gg 0$. A Tate $\mathcal{F}_C$-resolution is *split* if $\tau_i$ is a split epimorphism for all $i \in \mathbb{Z}$.

Recall that an $R$-module $M$ is called *Gorenstein flat* [**9**] if there is an exact sequence

$$\mathbb{F} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat $R$-modules with $M \cong \mathrm{im}(F_0 \to F^0)$ such that $I \otimes_R \mathbb{F}$ is exact for every injective $R$-module $I$. Denote by $\mathcal{GF}$ the class of Gorenstein flat $R$-modules.

**Remark 3.6.** Let $C = R$ in Definition 3.5. We have the notion of a Tate $\mathcal{F}$-resolution of an $R$-module $M$, that is, a Tate $\mathcal{F}$-resolution of $M$ is a diagram

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

of morphisms of complexes, where $\varepsilon : M \to B$ is a cotorsion envelope of $M$, $\alpha : W \to B$ is an $\mathcal{F} \cap \mathcal{C}$-resolution of $B$ such that $\mathrm{C}_i(W)$ is cotorsion for every $i = 0, 1, \ldots$, $T$ is an exact complex with each entry in $\mathcal{F} \cap \mathcal{C}$ and $\mathrm{Z}_i(T) \in \mathcal{GF}$ for all $i \in \mathbb{Z}$ and $\tau : T \to W$ is a morphism such that $\tau_i$ is bijective for all $i \gg 0$.

**Lemma 3.7.** *Let $R$ be an $R$-module. Then the following are equivalent*:

(i) *$M \in \mathcal{B}_C \cap \mathcal{GF}_C$;*
(ii) *There exists a $\mathrm{Hom}_R(\mathcal{P}_C, -)$ exact and $\mathrm{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$ exact exact sequence*

$$\cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow T_{-1} \longrightarrow T_{-2} \longrightarrow \cdots$$

*with $T_i \in \mathcal{F}_C$ for all $i \in \mathbb{Z}$ such that $M \cong \mathrm{im}(T_0 \to T_{-1})$;*
(iii) *$M \in \mathcal{B}_C \cap {}^{\perp}\mathcal{F}_C^{\mathrm{cot}}$, and there exists a $\mathrm{Hom}(-, \mathcal{F}_C^{\mathrm{cot}})$ exact exact sequence*

$$0 \longrightarrow M \longrightarrow A_0 \longrightarrow A_{-1} \longrightarrow \cdots$$

*with $A_i \in \mathcal{F}_C^{\mathrm{cot}}$ for all $i \leqslant 0$.*

*Proof.*

(i) $\Leftrightarrow$ (ii) holds by [**24,** Definition 3.12, Proposition 6.4].

(i) $\Rightarrow$ (iii) follows from [**24,** Lemmas 5.1 and 5.6].

(iii) $\Rightarrow$ (i). By (iii), there is a $\mathrm{Hom}(-, \mathcal{F}_C^{\mathrm{cot}})$ exact exact sequence

$$\mathbf{X} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes F^0 \longrightarrow C \otimes F^1 \longrightarrow \cdots$$

of $R$-modules with each $F_i$, $F^i$ flat and $M \cong \mathrm{coker}(F_1 \to F_0)$. Let $E$ be any $C$-injective $R$-module. Then $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \in \mathcal{F}_C^{\mathrm{cot}}$ by [**24,** Lemma 4.1(d)]. Note that

$$\mathrm{Hom}_{\mathbb{Z}}(E \otimes \mathbf{X}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbf{X}, \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})).$$

Since $\mathrm{Hom}(\mathbf{X}, \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}))$ is exact by hypothesis, $E \otimes \mathbf{X}$ is exact. Thus, $M$ is $C$-Gorenstein flat. This completes the proof. $\square$

**Remark 3.8.** We note that the class of $R$-modules satisfying Lemma 3.7 (ii) was denoted by $\mathcal{H}_C(\mathcal{F}_C)$ [**24,** Definition 3.12].

**Proposition 3.9.** *Let $M$ be an $R$-module. Then the following are equivalent for any non-negative integer $n$:*

(i) *$M \in \mathcal{B}_C$ and $\mathcal{GF}_C$-pd$(M) \leqslant n$;*
(ii) *for any exact sequence*

$$\cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0$$

*of $R$-modules with each $A_i$ in $\mathcal{F}_C$, the cokernel*

$$L_n = \mathrm{coker}(A_{n+1} \longrightarrow A_n)$$

*belongs to* $\mathcal{GF}_C \cap \mathcal{B}_C$;

(iii) $M \in \mathcal{B}_C$ *and* $\mathcal{GF}_C$-pd$(\varepsilon(M)) \leqslant n$, *where* $\varepsilon : M \to \varepsilon(M)$ *is an* $\mathcal{H}_C$*-envelope of* $M$.

*Proof.*

(i) $\Rightarrow$ (ii). Let $\cdots \to A_1 \to A_0 \to M \to 0$ be an exact sequence of $R$-modules with each $A_i$ in $\mathcal{F}_C$. Since $M$ is in $\mathcal{B}_C$, $L_i = \mathrm{coker}(A_{i+1} \to A_i)$ is in $\mathcal{F}_C$ for any $i = 0, 1, 2, \ldots$. Thus,

$$\cdots \longrightarrow \mathrm{Hom}(C, A_1) \longrightarrow \mathrm{Hom}(C, A_0) \longrightarrow \mathrm{Hom}(C, M) \longrightarrow 0$$

is an exact sequence with each $\mathrm{Hom}(C, A_i)$ flat. Since $\mathcal{GF}$-pd$(\mathrm{Hom}(C, M)) \leqslant n$ by (i) and [**15,** Theorem 4.4], $\mathrm{coker}(\mathrm{Hom}(C, A_{n+1}) \to \mathrm{Hom}(C, A_n))$ is Gorenstein flat by [**6,** Theorem 3.5]. Thus, $\mathrm{Hom}(C, L_n)$ is Gorenstein flat by noting that

$$\mathrm{Hom}(C, L_n) \cong \mathrm{coker}(\mathrm{Hom}(C, A_{n+1}) \longrightarrow \mathrm{Hom}(C, A_n)),$$

and so $L_n$ is $C$-Gorenstein flat by [**15,** Theorem 4.4], as desired.

(ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii). Let $\varepsilon : M \to \varepsilon(M)$ be an $\mathcal{H}_C$-envelope of $M$. By Proposition 3.4, the sequence

$$0 \longrightarrow M \longrightarrow \varepsilon(M) \longrightarrow \mathrm{coker}(\varepsilon) \longrightarrow 0$$

of $R$-modules is exact with $\mathrm{coker}(\varepsilon) \in \mathcal{F}_C$. Thus, (iii) holds by [**3,** Theorem 2.11] and [**15,** Theorem 2.16], as desired.

(iii) $\Rightarrow$ (i). The proof is similar to that of (i) $\Rightarrow$ (iii). $\square$

We are now in a position to state and prove the next result which contains Theorem 1.1 from the introduction.

**Theorem 3.10.** *Let* $M$ *be an* $R$*-module. Then the following are equivalent for any non-negative integer* $n$:

(i) $M \in \mathcal{B}_C$ *and* $\mathcal{GF}_C$-pd$(M) \leqslant n$;

(ii) $M$ *has a Tate* $\mathcal{F}_C$*-resolution*

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

*such that $\tau_i$ is bijective for all $i \geqslant n$;*

(iii) *$M$ has a split Tate $\mathcal{F}_C$-resolution*

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

*such that $\tau_i$ is bijective for all $i \geqslant n$.*

*Proof.*

(i) $\Rightarrow$ (ii). Since $M \in \mathcal{B}_C$, $M$ has a monic $\mathcal{H}_C$-envelope $\varepsilon : M \to B$ such that $\operatorname{coker}(\varepsilon) \in \mathcal{F}_C$ by Proposition 3.4. By Lemma 3.2 and [**24,** Lemma 4.5], $B$ has an $\mathcal{F}_C^{\mathrm{cot}}$-resolution $\alpha : W \to B$ such that $\mathrm{C}_i(W) \in \mathcal{H}_C$ for every $i = 0, 1, \ldots$. Thus, $\mathrm{C}_n(W) \in \mathcal{B}_C \cap \mathcal{GF}_C$ by Proposition 3.9, and hence, there exists a $\operatorname{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$ exact exact sequence

$$0 \longrightarrow \mathrm{C}_n(W) \longrightarrow A_0 \longrightarrow A_{-1} \longrightarrow \cdots$$

with $A_i \in \mathcal{F}_C^{\mathrm{cot}}$ for all $i \leqslant 0$. Let $\widehat{X} = \Sigma^{n-1} X$, where $X$ is the complex

$$0 \longrightarrow A_0 \longrightarrow A_{-1} \longrightarrow \cdots .$$

Thus, there exists a morphism $\gamma : \widehat{X} \to W_{<n}$ such that the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}_n(W) \arrow[r] \arrow[d, equal] & \widehat{X}_{n-1} \arrow[r] \arrow[d, "\gamma_{n-1}"] & \widehat{X}_{n-2} \arrow[r] \arrow[d, "\gamma_{n-2}"] & \cdots \\
0 \arrow[r] & \mathrm{C}_n(W) \arrow[r] & W_{n-1} \arrow[r] & W_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Let $T$ be the complex obtained by splicing $W_{\geqslant n}$ and $\widehat{X}$ along $\mathrm{C}_n(W)$. It may easily be checked that $T$ is an exact complex with each entry in $\mathcal{F}_C^{\mathrm{cot}}$ and $\mathrm{Z}_i(T) \in \mathcal{GF}_C$ for each $i \in \mathbb{Z}$. Set

$$\tau_i = \begin{cases} \gamma_i & \text{for } i < n, \\ \mathrm{id}_{W_i} & \text{for } i \geqslant n. \end{cases}$$

Thus, $\tau : T \to A$ is a morphism, and so the diagram

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

is a Tate $\mathcal{F}_C$-resolution.

(ii) $\Rightarrow$ (iii). The proof is modeled on that of [**23,** Lemma 3.4 (1)]. By (ii), there exists a Tate $\mathcal{F}_C$-resolution

$$T^{(1)} \xrightarrow{\tau^{(1)}} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

such that $\tau_i^{(1)}$ is bijective for all $i \geqslant n$. Let $T^{(2)} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W<n})$. Then $T^{(2)}$ is contractible and $T_i^{(2)} = 0$ for each $i \geqslant n$. Thus, $T^{(2)}$ is an exact complex with each entry in $\mathcal{F}_C^{\mathrm{cot}}$ and $\mathrm{Z}_i(T^{(2)}) \in \mathcal{F}_C^{\mathrm{cot}}$ for all $i \in \mathbb{Z}$ by [**24,** Lemma 4.4]. Let $f : T^{(2)} \to W$ denote the composition of the natural morphisms

$$T^{(2)} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W<n}) \longrightarrow W_{<n} \longrightarrow W.$$

Note that $f_i$ is a split epimorphism for each $i < n$, and $f_i = 0$ for each $i \geqslant n$. Let

$$T = T^{(1)} \oplus T^{(2)}$$

and

$$\tau_i = (\tau_i^{(1)}\ \ f_i).$$

One easily checks that $\tau$ is a morphism such that each $\tau_i$ is a split epimorphism and $\tau_i$ is bijective for all $i \geqslant n$. Therefore, $T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$ is a split Tate $\mathcal{F}_C$-resolution, as desired.

(iii) $\Rightarrow$ (i). By (iii), $M$ has a split Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

such that $\tau_i$ is bijective for all $i \geqslant n$. Then $\mathcal{GF}_C\text{-pd}(B) \leqslant n$ by Proposition 3.9. Note that $\varepsilon : M \to B$ is a monic $\mathcal{H}_C$-envelope of $M$ such that $\operatorname{coker}(\varepsilon) \in \mathcal{F}_C$. Then $M \in \mathcal{B}_C$. So $\mathcal{GF}_C\text{-pd}(M) \leqslant n$ by Proposition 3.9. This completes the proof. □

**Corollary 3.11.** *Let $R$ be a ring. Then an $R$-module $M$ has finite Gorenstein flat dimension if and only if $M$ has a Tate $\mathcal{F}$-resolution*

$$T \longrightarrow W \longrightarrow B \longleftarrow M.$$

*Proof.* The result holds by Remark 3.6 and Theorem 3.10. □

We end this section with the next remark.

**Remark 3.12.** It would be interesting to compare the results of Corollary 3.11, [**1,** Theorem 3.7] and [**2,** Theorem 3.1]. More specifically, an $R$-module $M$ has finite Gorenstein flat dimension if and only if $M$ has a Tate $\mathcal{F}$-resolution

$$T \longrightarrow W \longrightarrow B \longleftarrow M$$

by Corollary 3.11; $M$ has finite Gorenstein projective dimension if and only if $M$ has a Tate projective resolution (or complete resolution) $T \to P \to M$ by [**2,** Theorem 3.1]; $M$ has finite Gorenstein injective dimension if and only if $M$ has a Tate injective resolution (or complete coresolution) $M \to E \to T$ by [**1,** Theorem 3.7]. Motivated by the fact that Tate projective resolutions can be used to define Tate cohomology based on projective modules, one can define Tate cohomology based on flat modules by using Tate $\mathcal{F}$-resolutions. This observation may be viewed as an illustration of the usefulness of Tate $\mathcal{F}$-resolutions.

## 4. Tate $\mathcal{F}_C$-cohomology.

We begin with the next lemma.

**Lemma 4.1.** *Let*

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

*and*

$$T' \xrightarrow{\tau'} W' \xrightarrow{\alpha'} B' \xleftarrow{\varepsilon'} M'$$

*be Tate $\mathcal{F}_C$-resolutions of $M$ and $M'$, respectively. For each morphism of modules $\mu : M \to M'$, there exists a morphism $\widetilde{\mu}$ making the right-hand square of the diagram*

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, "\widehat{\mu}"] & W \arrow[r, "\alpha"] \arrow[d, "\overline{\mu}"] & B \arrow[d, "\widetilde{\mu}"] & M \arrow[l, "\varepsilon"'] \arrow[d, "\mu"] \\ T' \arrow[r, "\tau'"] & W' \arrow[r, "\alpha'"] & B' & M' \arrow[l, "\varepsilon'"'] \end{tikzcd}$$

*commute; for each choice of $\widetilde{\mu}$, there exists a morphism $\overline{\mu}$, unique up to homotopy, making the middle square commute; for each choice of $\overline{\mu}$ there exists a morphism $\widehat{\mu}$, unique up to homotopy, making the left-hand square commute up to homotopy.*

*If $\mu = \mathrm{id}_M$, then $\overline{\mu}$ and $\widehat{\mu}$ are homotopy equivalences.*

*Proof.* Since $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope and $B' \in \mathcal{H}_C$, there is a homomorphism $\widetilde{\mu} : B \to B'$ such that $\varepsilon' \circ \mu = \widetilde{\mu} \circ \varepsilon$. Note that $\alpha' : W' \to B'$ is an $\mathcal{F}_C^{\mathrm{cot}}$-resolution of $B'$ such that $\mathrm{C}_i(W') \in \mathcal{H}_C$ for every $i = 0, 1, 2, \ldots$. Then

$$0 \longrightarrow \ker(\alpha') \longrightarrow W' \xrightarrow{\alpha'} B' \longrightarrow 0$$

is exact in $\mathrm{C}(R)$ with $\ker(\alpha') \in \mathrm{C}_{\sqsupset}(R)$ such that $\ker(\alpha')$ is $\mathrm{Hom}(\mathcal{F}_C, -)$ exact. Thus, $\mathrm{Hom}(W, \ker(\alpha'))$ is exact by [**6,** Lemma 2.4], and hence,

$$\mathrm{Hom}(W, \alpha') : \mathrm{Hom}(W, W') \longrightarrow \mathrm{Hom}(W, B')$$

is an epic quasi-isomorphism. It follows from [**2,** 1.1 (1)] that there is a unique, up to homotopy, morphism $\overline{\mu}$ such that $\alpha' \circ \overline{\mu} = \widetilde{\mu} \circ \alpha$.

First, we assume that each $\tau_i'$ is a split epimorphism. Then there exists an exact sequence

$$0 \longrightarrow \ker(\tau') \longrightarrow T' \xrightarrow{\tau'} W' \longrightarrow 0$$

in $\mathrm{C}(R)$ with $\ker(\tau') \in \mathrm{C}_{\sqsubset}(R)$ such that each entry of $\ker(\tau')$ belongs to $\mathcal{F}_C^{\mathrm{cot}}$. Applying $\mathrm{Hom}(T, -)$ to the above exact sequence, we get that the sequence

$$0 \longrightarrow \mathrm{Hom}(T, \ker(\tau')) \longrightarrow \mathrm{Hom}(T, T') \longrightarrow \mathrm{Hom}(T, W') \longrightarrow 0$$

of complexes is exact in $\mathrm{C}(R)$. Note that $\mathrm{Hom}(T, \ker(\tau'))$ is an exact complex by [**6,** Lemma 2.5]. It follows that

$$\mathrm{Hom}(T, \tau') : \mathrm{Hom}(T, T') \longrightarrow \mathrm{Hom}(T, W')$$

is an epic quasi-isomorphism. By [**2,** 1.1 (1)], there exists a unique, up to homotopy, morphism $\widehat{\mu}$ such that $\tau' \circ \widehat{\mu} = \overline{\mu} \circ \tau$.

In general, factor $\tau'$ as $T' \xrightarrow{\beta} T'' \xrightarrow{\tau''} W'$, with a homotopy equivalence $\beta$ and a morphism $\tau''$ with each $\tau_i''$ a split epimorphism by the proof of Theorem 3.10 (ii) $\Rightarrow$ (iii). Thus, $\mathrm{Hom}(T, \beta)$ is a quasi-isomorphism. Note that $\mathrm{Hom}(T, \tau'')$ is a quasi-isomorphism by the proof above, it follows that

$$\mathrm{Hom}(T, \tau') = \mathrm{Hom}(T, \tau'') \circ \mathrm{Hom}(T, \beta)$$

is also a quasi-isomorphism. Applying [**2,** 1.1 (1)] again, there exists a unique, up to homotopy, morphism $\widehat{\mu}$ such that $\tau' \circ \widehat{\mu} \sim \overline{\mu} \circ \tau$.

If $\mu = \mathrm{id}_M$, then $\widetilde{\mu}$ is an isomorphism by noting that $\varepsilon$ is an $\mathcal{H}_C$-envelope. Thus, reversing the roles of $M$ and $M'$, we get a morphism $\overline{\mu}' : W' \to W$ such that $\alpha \circ \overline{\mu}' = \widetilde{\mu}^{-1} \circ \alpha'$. Thus, $\overline{\mu}' \circ \overline{\mu} \sim \mathrm{id}_W$. By symmetry, $\overline{\mu} \circ \overline{\mu}' \sim \mathrm{id}_{W'}$. Similarly, we have that $\widehat{\mu}$ is a homotopy equivalence. □

**Definition 4.2.** Let $M$ be an $R$-module. If $M$ admits a Tate $\mathcal{F}_C$-resolution

$$T \longrightarrow W \longrightarrow B \longleftarrow M,$$

define the *nth Tate $\mathcal{F}_C$-cohomology group* $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,N)$ as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,N) = \mathrm{H}_{-n}(\mathrm{Hom}(T,N))$$

for each integer $n$.

**Remark 4.3.**

(i) According to Lemma 4.1, one can see that $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,-)$ is a cohomological functor for each integer $n$, independent of the choice of the Tate $\mathcal{F}_C$-resolution of $M$.

(ii) If $M$ has a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M,$$

then

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\mathrm{id}} B$$

is a Tate $\mathcal{F}_C$-resolution of $B$. So $\widehat{\mathrm{Ext}}^i_{\mathcal{F}_C}(B,N) \cong \widehat{\mathrm{Ext}}^i_{\mathcal{F}_C}(M,N)$ for each $i \in \mathbb{Z}$ and any $R$-module $N$.

**Lemma 4.4.** *Let*

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

*be a split Tate $\mathcal{F}_C$-resolution of an $R$-module $M$. Then there exists a degreewise split exact sequence of complexes*

$$0 \longrightarrow \Sigma^{-1} X \longrightarrow \widetilde{T} \longrightarrow W \longrightarrow 0$$

*with $\widetilde{T} = (T_{\geqslant 0})^+$ such that $X$ is a proper $\mathcal{GF}_C$-resolution of $B$.*

*Proof.* By hypothesis, there is a non-negative integer $n$ such that $\tau_i$ is bijective for all $i \geqslant n$. We set $\widetilde{T} = (T_{\geqslant 0})^+$, that is,

$$\widetilde{T}_i = \begin{cases} T_i & \text{if } i \geqslant 0; \\ \mathrm{C}_0(T) & \text{if } i = -1; \\ 0 & \text{if } n < -1, \end{cases}$$

and

$$\partial_i^{\widetilde{T}} = \begin{cases} \partial_i^T & \text{if } i > 0; \\ \pi & \text{if } i = 0; \\ 0 & \text{if } n < -t, \end{cases}$$

where $\pi : T_0 \to \mathrm{C}_0(T)$ is the natural map. Let $\widetilde{\beta} : \widetilde{T} \to W$ be a morphism such that $\widetilde{\beta}_i = \tau_i$ for all $i \geqslant 0$ and $\widetilde{\beta}_i = 0$ for all $i < 0$, and let $X = \Sigma \ker(\widetilde{\beta})$. It follows from [**24,** Lemma 4.4] that $\ker(\tau)$ is a complex with each entry in $\mathcal{F}_C^{\mathrm{cot}}$. Thus, $X_0 = \mathrm{C}_0(T) \in \mathcal{GF}_C$, $X_i \in \mathcal{F}_C^{\mathrm{cot}}$ for $1 \leqslant i \leqslant n-1$ and $X_i = 0$ for $i \geqslant n+1$ and $i \leqslant -1$. Thus, we have the following exact sequence of complexes:

$$0 \longrightarrow \Sigma^{-1} X \longrightarrow \widetilde{T} \longrightarrow W \longrightarrow 0$$

with $\widetilde{T} = (T_{\geqslant 0})^+$ such that $X$ is a proper $\mathcal{GF}_C$-resolution of $B$. This completes the proof. $\square$

**Fact 4.5.** Note that both $\mathcal{GF}_C$ and $\mathcal{F}_C$ are covered by [**16,** Theorem 3.3(a)] and [**17,** Proposition 5.3(a)]. Let $M$ be an $R$-module and $X \to M$ a proper $\mathcal{GF}_C$-resolution. Choose a proper $\mathcal{F}_C$-resolution $W \to M$ and a morphism $\gamma : W \to X$ lifting the identity on $M$. For each $R$-module $N$, the morphism of complexes

$$\mathrm{Hom}(\gamma, N) : \mathrm{Hom}(X, N) \longrightarrow \mathrm{Hom}(W, N)$$

induces a natural homomorphism of abelian groups

$$\varepsilon^n(M, N) : \mathrm{Ext}^n_{\mathcal{GF}_C}(M, N) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}_C}(M, N)$$

for every $n \in \mathbb{Z}$. The groups and maps defined above do not depend on the choices of resolutions and liftings by [**14,** Proposition 2.2].

Now we can prove the next theorem, which is Theorem 1.2 from the introduction.

**Theorem 4.6.** *Let $M$ be any $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) \leqslant n < \infty$. For each $R$-module $N$, there is a long exact sequence*

$$
\begin{array}{l}
0 \longrightarrow \mathrm{Ext}^1_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^1(B,N)} \mathrm{Ext}^1_{\mathcal{F}_C}(B,N) \longrightarrow \widehat{\mathrm{Ext}}^1_{\mathcal{F}_C}(M,N) \\
\longrightarrow \mathrm{Ext}^2_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^2(B,N)} \mathrm{Ext}^2_{\mathcal{F}_C}(M,N) \longrightarrow \widehat{\mathrm{Ext}}^2_{\mathcal{F}_C}(M,N) \\
\qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \\
\longrightarrow \mathrm{Ext}^n_{\mathcal{GF}_C}(B,N) \xrightarrow{\varepsilon^n(B,N)} \mathrm{Ext}^n_{\mathcal{F}_C}(M,N) \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,N) \\
\longrightarrow 0,
\end{array}
$$

*where $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope of $M$.*

*Proof.* Let $M$ be an $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) \leqslant n < \infty$. By Theorem 3.10 and Lemma 4.4, there exists a degreewise split exact sequence of complexes

$$0 \longrightarrow \Sigma^{-1}X \longrightarrow \widetilde{T} \longrightarrow W \longrightarrow 0$$

with $\widetilde{T} = (T_{\geqslant 0})^+$ such that $X$ is a proper $\mathcal{GF}_C$-resolution of $B$. Let $N$ be an $R$-module. Applying $\mathrm{Hom}(-,N)$ to the exact sequence $0 \to \Sigma^{-1}X \to \widetilde{T} \to W \to 0$ of complexes above, we obtain an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(W,N) \longrightarrow \mathrm{Hom}(\widetilde{T},N) \longrightarrow \mathrm{Hom}(\Sigma^{-1}X,N) \longrightarrow 0.$$

Its long cohomology sequence induces the following exact sequence

$$\cdots \longrightarrow \mathrm{H}_i(\mathrm{Hom}(W,N)) \longrightarrow \mathrm{H}_i(\mathrm{Hom}(\widetilde{T},N)) \longrightarrow \mathrm{H}_i(\mathrm{Hom}(\Sigma^{-1}X,N)) \longrightarrow \cdots.$$

Since $X$ is a proper $\mathcal{GF}_C$-resolution of $B$, we have

$$\mathrm{H}_{-i}(\mathrm{Hom}(X,N)) \cong \mathrm{Ext}^i_{\mathcal{GF}_C}(B,N)$$

for all $i \geqslant 1$ and $\mathrm{H}_{-i}(\mathrm{Hom}(\Sigma^{-1}X, N)) = 0$ for all $i < n$. Moreover,

$$\mathrm{H}_{-i}(\mathrm{Hom}(\widetilde{T}, N)) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(M, N) \quad \text{for all } i \geqslant 1,$$

and

$$\mathrm{H}_0(\mathrm{Hom}(\widetilde{T}, N)) = 0.$$

By Fact 4.5, we get the desired long exact sequence. □

To prove Theorem 1.3, we need some preparation.

**Lemma 4.7.** *Let*

$$0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$$

*be an exact sequence of $R$-modules with $B$ and $B''$ in $\mathcal{H}_C$. If*

$$\sup\{\mathcal{GF}_C\text{-}\mathrm{pd}(B), \mathcal{GF}_C\text{-}\mathrm{pd}(B'')\} < \infty,$$

*then there exists a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "\widehat{\mu}"] \arrow[d, "\tau"'] & T' \arrow[r, "\widehat{\mu}'"] \arrow[d, "\tau'"'] & T'' \arrow[r] \arrow[d, "\tau''"'] & 0 \\
0 \arrow[r] & W \arrow[r, "\overline{\mu}"] \arrow[d, "\alpha"'] & W' \arrow[r, "\overline{\mu}'"] \arrow[d, "\alpha'"'] & W'' \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
0 \arrow[r] & B \arrow[r, "\mu"] & B' \arrow[r, "\mu'"] & B'' \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r, "\mu"] \arrow[u, "\mathrm{id}"] & B' \arrow[r, "\mu'"] \arrow[u, "\mathrm{id}"] & B'' \arrow[r] \arrow[u, "\mathrm{id}"] & 0
\end{tikzcd}$$

*whose columns are Tate $\mathcal{F}_C$-resolutions. Moreover, for any $R$-module $N$, there is a long exact sequence*

$$\begin{aligned}\cdots \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B'', N) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B', N) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B, N) \\ \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(B'', N) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(B', N) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(B, N) \longrightarrow \cdots .\end{aligned}$$

*Proof.* Note that $\sup\{\mathcal{GF}_C\text{-}\mathrm{pd}(B), \mathcal{GF}_C\text{-}\mathrm{pd}(B'')\} < \infty$. Then $\mathcal{GF}_C\text{-}\mathrm{pd}(B') < \infty$ by [**3,** Theorem 2.11] and [**15,** Theorem 2.16]. Moreover, $B'$ is in $\mathcal{H}_C$ by hypothesis and Lemma 3.1. Hence, $B'$ has a Tate $\mathcal{F}_C$-resolution by Theorem 3.10. As in the proof of [**2,** Lemma 5.5] or [**23,** Lemma 3.9], we obtain the desired commutative diagram.

Let $N$ be an $R$-module. Applying $\mathrm{Hom}(-, N)$ to the exact sequence

$$0 \longrightarrow T \xrightarrow{\widehat{\mu}} T' \xrightarrow{\widehat{\mu}'} T'' \longrightarrow 0,$$

we have the following exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(T'', N) \longrightarrow \mathrm{Hom}(T', N) \longrightarrow \mathrm{Hom}(T, N) \longrightarrow 0.$$

Its long cohomology sequence induces the desired long exact sequence of the lemma. $\square$

**Lemma 4.8.** *The following are true for any $R$-module $M$:*

(i) $\mathcal{F}_C$-$\mathrm{pd}(M) = \mathcal{F}_C - \mathrm{pd}(B)$, *where $M \in \mathcal{B}_C$ and $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope.*

(ii) *There is an inequality*:

$$\mathcal{GF}_C\text{-}\mathrm{pd}(M) \leqslant \mathcal{F}_C - \mathrm{pd}(M),$$

*and the equality holds if $\mathcal{F}_C$-$\mathrm{pd}(M) < \infty$.*

*Proof.*

(i) Note that $M \in \mathcal{B}_C$ and $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope. Then the sequence

$$0 \longrightarrow M \longrightarrow B \longrightarrow \mathrm{coker}(\varepsilon) \longrightarrow 0$$

of $R$-modules is exact with $B \in \mathcal{H}_C$ and $\mathrm{coker}(\varepsilon) \in \mathcal{F}_C$ by Proposition 3.4. So $\mathcal{F}_C$-$\mathrm{pd}(M) = \mathcal{F}_C - \mathrm{pd}(B)$ by [**22,** Corollary 5.7].

(ii) Let $\mathcal{F}_C$-$\mathrm{pd}(M) = n$. There is nothing to prove if $n = \infty$ or $n = 0$. We may assume that $n \geqslant 1$ is an integer. It follows from [**22,** Lemma 5.1] that $M \in \mathcal{B}_C$. Hence, $\mathcal{GF}_C$-$\mathrm{pd}(M) \leqslant n$ by Proposition 3.9.

Suppose $\mathcal{GF}_C$-$\mathrm{pd}(M) < n$. Since $\mathcal{F}_C$-$\mathrm{pd}(M) = n$, we have an exact sequence

$$0 \longrightarrow W_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

of $R$-modules with each $W_i \in \mathcal{F}_C$ such that $K_{n-1}$ is not in $\mathcal{F}_C$, where $K_i = \ker(W_{i-1} \to W_{i-2})$ for $i \geqslant 2$, $K_0 = M$ and $K_1 = \ker(W_0 \to M)$. Note that $\mathcal{GF}_C$-$\mathrm{pd}(M) < n$. Then $K_{n-1} \in \mathcal{GF}_C$ by Proposition 3.9.

Since

$$0 \longrightarrow W_n \longrightarrow W_{n-1} \longrightarrow K_{n-1} \longrightarrow 0$$

is exact, $K_{n-1} \in \mathcal{GF}_C \cap \operatorname{res} \widehat{\mathcal{F}_C}^{\leqslant 1}$. Hence, $K_{n-1} \in \mathcal{F}_C$ by [**24,** Lemma 5.12], a contradiction. So $\mathcal{GF}_C$-pd$(M) = n$, as desired. □

**Lemma 4.9.** *If $M$ is an $R$-module with $\mathcal{F}_C$-pd$(M) = n < \infty$, then $\widehat{\mathrm{Ext}}^i_{\mathcal{F}_C}(M, N) = 0$ for any $i \in \mathbb{Z}$ and any $R$-module $N$.*

*Proof.* Let $M$ be an $R$-module with $\mathcal{F}_C$-pd$(M) = n < \infty$. Then $M \in \mathcal{B}_C$ by [**22,** Lemma 5.1]. It follows from Proposition 3.4 that $M$ has an $\mathcal{H}_C$-envelope $\varepsilon : M \to B$. Thus, $\mathcal{GF}_C$-pd$(B) = \mathcal{F}_C$-pd$(B) < \infty$ by Lemma 4.8 (ii). By the proof of (i) $\Rightarrow$ (ii) in Theorem 3.10, $M$ has a Tate $\mathcal{F}_C$-resolution

$$0 \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M\ .$$

So $\widehat{\mathrm{Ext}}^i_{\mathcal{F}_C}(M, N) = 0$ for any $i \in \mathbb{Z}$ and any $R$-module $N$. This completes the proof. □

The next result parallels [**2,** Propositions 5.4, 5.6] and [**23,** Lemmas 4.6, 4.7].

**Proposition 4.10.** *Let $M$ be an $R$-module. Consider an exact sequence of complexes*

$$\mathbb{X} : 0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0.$$

(i) *If $\mathbb{X}$ is $\mathrm{Hom}(\mathcal{F}_C^{\mathrm{cot}}, -)$ exact and $M$ is in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) < \infty$, then there is a long exact sequence*

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M, N) \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M, N') \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M, N'') \\ \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(M, N) \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(M, N') \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(M, N'') \longrightarrow \cdots .$$

(ii) *If $\mathbb{X}$ is an exact sequence of modules of finite $\mathcal{GF}_C$-projective dimensions with $N'$ and $N''$ in $\mathcal{H}_C$, then there is a long exact* *sequence*

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(N'', M) \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(N', M) \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(N, M)$$
$$\longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(N'', M) \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(N', M) \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{F}_C}(N, M) \longrightarrow \cdots .$$

*Proof.*

(i) Since $M$ is an $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-pd$(M) < \infty$, it has a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

by Theorem 3.10. Thus, we get the following exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(T, N) \longrightarrow \mathrm{Hom}(T, N') \longrightarrow \mathrm{Hom}(T, N'') \longrightarrow 0.$$

Its long cohomology sequence induces the desired long exact sequence, as desired.

(ii) Note that $N'$ and $N''$ are in $\mathcal{H}_C$. It follows that $N \in \mathcal{B}_C$ by [**17,** Theorem 6.2]. Thus, $N$ has a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} N$$

by Theorem 3.10. Hence, there is an exact sequence

$$0 \longrightarrow N \xrightarrow{\varepsilon} B \longrightarrow L \longrightarrow 0$$

with $L \in \mathcal{F}_C$ such that $\varepsilon$ is an $\mathcal{H}_C$-envelope of $N$ by Proposition 3.4. By [**3,** Theorem 2.11] and [**15,** Theorem 2.16], it follows that $\mathcal{GF}_C$-pd$(B) < \infty$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & N'' \arrow[r] & 0 \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Note that $B$ and $N''$ are in $\mathcal{H}_C$. It follows from Lemma 3.1 that $U$ and $L$ are in $\mathcal{H}_C$. Applying Lemma 4.7 to the middle row in the above diagram yields the following long exact sequence

$$\begin{aligned}\cdots \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N'', M) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(U, M) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B, M)\\ \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(N'', M) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(U, M) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(B, M) \longrightarrow \cdots .\end{aligned}$$

To complete the proof, it suffices to show that

$$\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B, M) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N, M)$$

and

$$\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(U, M) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N', M)$$

for each $i \in \mathbb{Z}$. Note that

$$\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(B, M) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N, M)$$

holds by Remark 4.3 (ii). It remains to show that $\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(U, M) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N', M)$ for each $i \in \mathbb{Z}$.

Note that $L \in \mathcal{F}_C \cap \mathcal{H}_C$ by the above proof. Then $L \in \mathcal{F}_C^{\mathrm{cot}}$ by Lemma 3.2 and [**24,** Lemma 4.3]. Applying Lemma 4.7 again to the middle column in the above diagram, we obtain the following long exact sequence

$$\begin{aligned}\cdots \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(L, M) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(U, M) \longrightarrow \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N', M)\\ \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(L, M) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(U, M) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{\mathcal{F}_C}(N', M) \longrightarrow \cdots .\end{aligned}$$

Note that $\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(L, M) = 0$ for each $i \in \mathbb{Z}$ by Lemma 4.9. Then

$$\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(U, M) \cong \widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(N', M) \quad \text{for each } i \in \mathbb{Z}.$$

This completes the proof. $\square$

**Remark 4.11.** We do not know whether the long exact sequence in Proposition 4.10 (ii) holds when we drop the conditions "$N'$ and $N''$ in $\mathcal{H}_C$."

Now we can give the proof of Theorem 1.3.

**4.1. Proof of Theorem 1.3.**

(i) $\Rightarrow$ (ii) holds by Lemma 4.9.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Assume that $n$ is an integer such that $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_C}(M,N)=0$ for any $N \in \mathcal{H}_C$. Choose a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

of $M$. It follows that $\mathrm{C}_i(T) \cong \mathrm{C}_i(W)$ for $i \gg 0$. Since $\mathrm{C}_i(W) \in \mathcal{H}_C$ by Definition 3.5, $\mathrm{C}_i(T) \in \mathcal{B}_C$ for $i \gg 0$. Hence, $\mathrm{C}_i(T) \in \mathcal{B}_C$ for each $i \in \mathbb{Z}$ by [**17,** Corollary 6.3]. Set $G = \mathrm{C}_n(T)$, and let $l : G \to T_{n-1}$ be the canonical injection. Let $N$ be any $R$-module in $\mathcal{H}_C$. Then, $\mathrm{H}_{-n}(\mathrm{Hom}(T,N)) = 0$ by hypothesis, and so we have the following exact sequence:

$$\mathrm{Hom}(T_{n-1},N) \xrightarrow{\mathrm{Hom}(\partial_n^T,N)} \mathrm{Hom}(T_n,N) \xrightarrow{\mathrm{Hom}(\partial_{n+1}^T,N)} \mathrm{Hom}(T_{n+1},N).$$

It is easy to check that

$$\mathrm{Hom}(l,N) : \mathrm{Hom}(T_{n-1},N) \longrightarrow \mathrm{Hom}(G,N)$$

is epic. Note that

$$0 \longrightarrow G \xrightarrow{l} T_{n-1} \longrightarrow \mathrm{C}_{n-1}(T) \longrightarrow 0$$

is an exact sequence of $R$-modules. Then we have an exact sequence

$$\mathrm{Hom}(T_{n-1},N) \longrightarrow \mathrm{Hom}(G,N) \longrightarrow \mathrm{Ext}^1(\mathrm{C}_{n-1}(T),N) \longrightarrow \mathrm{Ext}^1(T_{n-1},N) = 0.$$

Thus, $\mathrm{Ext}^1(\mathrm{C}_{n-1}(T),N) = 0$ for any $N \in \mathcal{H}_C$. Note that $\mathrm{C}_{n-1}(T) \in \mathcal{B}_C$ by the above proof. As in the proof of [**17,** Proposition 5.3 (a)], there exists an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow \mathrm{C}_{n-1}(T) \longrightarrow 0$$

of $R$-modules with $U \in \mathcal{F}_C{}^{\perp}$ and $V \in \mathcal{F}_C$. It follows from Lemma 3.2 that $U \in \mathcal{H}_C$. Thus, the exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow \mathrm{C}_{n-1}(T) \longrightarrow 0$$

of $R$-modules is split, and hence, $\mathrm{C}_{n-1}(T) \in \mathcal{F}_C$ by [**17,** Proposition 5.1 (a)]. Consequently, $\mathrm{C}_s(T) \in \mathcal{F}_C$ for all $s \geqslant n$. Let $s$ be an integer such that $s \geqslant \max\{n, \mathcal{GF}_C\text{-pd}(M)\}$. Then $\mathrm{C}_s(T) \in \mathcal{F}_C$. So, $\mathcal{F}_C$-pd$(M) < \infty$, as desired.

(i) $\Rightarrow$ (iv). Let $N$ be an $R$-module in $\mathcal{B}_C \cap \widehat{\mathcal{GF}_C}$. Then $N$ has a Tate $\mathcal{F}_C$-resolution

$$T \longrightarrow W \longrightarrow B \longleftarrow N$$

by Theorem 3.10. Note that $\mathcal{F}_C$-pd$(B) < \infty$ by (i) and Lemma 4.8 (i). Then, $B \in \text{res}\ \widehat{\mathcal{F}_C^{\text{cot}}}^{\leqslant s}$ for some non-negative integer $s$ by [**24,** Proposition 4.6]. Hence, there exists an exact sequence

$$0 \longrightarrow W_s \longrightarrow W_{s-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow B \longrightarrow 0$$

with $\text{coker}(W_i \to W_{i-1}) \in \mathcal{H}_C$ for $1 \leqslant i \leqslant s$ and $W_j \in \mathcal{F}_C^{\text{cot}}$ for $0 \leqslant j \leqslant s$. Thus, we obtain the following exact sequence of complexes:

$$0 \longrightarrow \text{Hom}(T, W_s) \longrightarrow \text{Hom}(T, W_{s-1}) \longrightarrow \cdots \longrightarrow \text{Hom}(T, W_0) \longrightarrow \text{Hom}(T, B) \longrightarrow 0.$$

Since $\text{Hom}(T, W_i)$ is exact for $0 \leqslant i \leqslant s$, $\text{Hom}(T, B)$ is exact. Thus, $\widehat{\text{Ext}}^n_{\mathcal{F}_C}(N, B) = 0$ for each $n \in \mathbb{Z}$, as desired.

(iv) $\Rightarrow$ (v). Assume that $\widehat{\text{Ext}}^n_{\mathcal{F}_C}(N, B) = 0$ for some $n \in \mathbb{Z}$ and each $N \in \mathcal{B}_C \cap \widehat{\mathcal{GF}_C}$. By Theorem 3.10, there exists a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

of $M$. Note that $\varepsilon : M \to B$ is an $\mathcal{H}_C$-envelope of $M$. If follows that $\mathcal{GF}_C$-pd$(B) < \infty$ by Proposition 3.9.

*Case* 1. $n = 0$. Condition (v) holds immediately.

*Case* 2. $n > 0$. Note that there exists an exact sequence

$$0 \longrightarrow B \longrightarrow A_0 \longrightarrow L_{-1} \longrightarrow 0$$

of $R$-modules with $\mathcal{F}_C^{\text{cot}}$-pd$(A_0) < \infty$ and $L_{-1} \in \mathcal{GF}_C$ by [**24,** Corollary 5.10 (c)]. It follows that $A_0$ and $L_{-1}$ belong to $\mathcal{H}_C$ by Lemma 3.1. According to Proposition 4.10 (ii), there exists a long exact sequence

$$\cdots \longrightarrow \widehat{\text{Ext}}^i_{\mathcal{F}_C}(A_0, B) \longrightarrow \widehat{\text{Ext}}^i_{\mathcal{F}_C}(B, B) \longrightarrow \widehat{\text{Ext}}^{i+1}_{\mathcal{F}_C}(L_{-1}, B) \longrightarrow \widehat{\text{Ext}}^{i+1}_{\mathcal{F}_C}(A_0, B) \longrightarrow \cdots.$$

It follows from Lemma 4.9 that $\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(A_0, B) = 0$ for each $i \in \mathbb{Z}$. Hence, $\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(B, B) \cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{F}_C}(L_{-1}, B)$. According to Lemma 3.7, there exist exact sequences

$$0 \longrightarrow L_{-i} \longrightarrow A_{-i} \longrightarrow L_{-i-1} \longrightarrow 0$$

of $R$-modules with $A_{-i} \in \mathcal{F}_C^{\mathrm{cot}}$ and $L_{-i-1} \in \mathcal{GF}_C$ for $i \geqslant 1$. Repeated application of Proposition 4.10 (ii) gives rise to the following isomorphisms

$$\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(B, B) \cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{F}_C}(L_{-1}, B) \cong \cdots \cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{F}_C}(L_{-n}, B).$$

Note that $\widehat{\mathrm{Ext}}^{n}_{\mathcal{F}_C}(L_{-n}, B) = 0$ by hypothesis, and so $\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(B, B) = 0$. Thus, $\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(M, B) = 0$ by Remark 4.3 (ii), as desired.

*Case* 3. $n < 0$. Note that $\alpha : W \to B$ is an $\mathcal{F}_C^{\mathrm{cot}}$-resolution of $B$ such that $\mathrm{C}_i(W) \in \mathcal{H}_C$ for every $i = 0, 1, \ldots,$ by hypothesis. Then we have exact sequences $0 \to \mathrm{C}_1(W) \to W_0 \to B \to 0$ and $0 \to \mathrm{C}_i(W) \to W_{i-1} \to \mathrm{C}_{i-1}(W) \to 0$ for $i \geqslant 2$. It follows from Lemma 4.9 that $\widehat{\mathrm{Ext}}^{i}_{\mathcal{F}_C}(W_i, B) = 0$ for each $i \in \mathbb{Z}$. Repeated application of Proposition 4.10 (ii) yields the isomorphisms:

$$\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(B, B) \cong \widehat{\mathrm{Ext}}^{-1}_{\mathcal{F}_C}(\mathrm{C}_1(W), B) \cong \cdots \cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{F}_C}(\mathrm{C}_{-n}(W), B).$$

Note that $\widehat{\mathrm{Ext}}^{n}_{\mathcal{F}_C}(\mathrm{C}_{-n}(W), B) = 0$ by hypothesis. It follows that $\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(B, B) = 0$. So $\widehat{\mathrm{Ext}}^{0}_{\mathcal{F}_C}(M, B) = 0$ by Remark 4.3 (ii), as desired.

(v) $\Rightarrow$ (i). Note that there exists a Tate $\mathcal{F}_C$-resolution

$$T \xrightarrow{\tau} W \xrightarrow{\alpha} B \xleftarrow{\varepsilon} M$$

of $M$ by Theorem 3.10. Then

$$0 \longrightarrow \ker(\alpha) \longrightarrow W \xrightarrow{\alpha} B \longrightarrow 0$$

is an exact sequence of complexes such that $\ker(\alpha)$ is exact in $\mathrm{C}_{\sqsupset}(R)$ with $\mathrm{Z}_i(\ker(\alpha)) \in \mathcal{H}_C$ for $i \geqslant 1$. Thus $\mathrm{Hom}(X, \ker(\alpha))$ is exact for any $X \in \mathcal{F}_C$. It follows from **[6,** Lemma 2.4] that $\mathrm{Hom}(T, \ker(\alpha))$ is exact. Hence, we obtain the exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(T, \ker(\alpha)) \longrightarrow \mathrm{Hom}(T, W) \longrightarrow \mathrm{Hom}(T, B) \longrightarrow 0.$$

Thus, we have an exact sequence

$$0 = \mathrm{H}_0(\mathrm{Hom}(T, \ker(\alpha))) \longrightarrow \mathrm{H}_0(\mathrm{Hom}(T, W)) \longrightarrow \mathrm{H}_0(\mathrm{Hom}(T, B)).$$

Since $\mathrm{H}_0(\mathrm{Hom}(T, B)) = 0$, by hypothesis, it follows that $\mathrm{H}_0(\mathrm{Hom}(T, W)) = 0$. Thus, $\tau \in \mathrm{B}_0(\mathrm{Hom}(T, W))$, and hence, there exists a $\varphi \in \mathrm{Hom}(T, W)_1$ such that $\partial(\varphi) = \tau$. Since $\tau_i$ is bijective for $i \gg 0$, we have that

$$\varphi_{i-1}\partial_i^T + \partial_{i+1}^T\varphi_i = \mathrm{id}_{T_i} \quad \text{for } i \gg 0.$$

Note that $T$ is an exact complex. Let $i$ be an integer such that $\varphi_{i-1}\partial_i^T + \partial_{i+1}^T\varphi_i = \mathrm{id}_{T_i}$. Choose $x \in \mathrm{im}(\partial_{i+1}^T)$. Then $x = \partial_{i+1}^T\varphi_i(x)$. Thus, the map $T_{i+1} \to \mathrm{im}(\partial_{i+1}^T)$ is split, and hence, $\mathrm{im}(\partial_{i+1}^T) \in \mathcal{F}_C$. So $\mathcal{F}_C$-pd$(M) < \infty$. This completes the proof.

We end this paper with the following corollaries.

**Corollary 4.12.** *Let $M$ be an $R$-module in $\mathcal{B}_C$ with $\mathcal{GF}_C$-*pd$(M) < \infty$ *and $\varepsilon : M \to B$ an $\mathcal{H}_C$-envelope of $M$. Then the following are equivalent*:

(i) *$\mathcal{F}_C$-pd$(M) < \infty$;*
(ii) *$\varepsilon^n(B, N) : \mathrm{Ext}^n_{\mathcal{GF}_C}(B, N) \to \mathrm{Ext}^n_{\mathcal{F}_C}(B, N)$ is an isomorphism for each $n \in \mathbb{Z}$ and any $R$-module $N \in \mathcal{H}_C$;*
(iii) *$\varepsilon^n(B, B) : \mathrm{Ext}^n_{\mathcal{GF}_C}(B, B) \to \mathrm{Ext}^n_{\mathcal{F}_C}(B, B)$ is an isomorphism for each $n \in \mathbb{Z}$.*

*Proof.*

(i) $\Rightarrow$ (ii). Note that $\mathcal{F}_C$-pd$(B) = n < \infty$ by (i) and Lemma 4.8 (i). There is nothing to prove if $n \leqslant 0$. For $n > 0$, it follows from [**24,** Proposition 4.6] that there exists an exact sequence

$$\mathbf{X} : 0 \longrightarrow W_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow B \longrightarrow 0$$

of $R$-modules with each $W_i \in \mathcal{F}_C^{\mathrm{cot}}$ and $\mathrm{Z}_i(\mathbf{X}) \in \mathcal{H}_C \cap \mathrm{res}\,\widehat{\mathcal{F}_C^{\mathrm{cot}}}^{<n}$. Thus, $\mathbf{X}$ is $\mathrm{Hom}(\mathcal{F}_C, -)$ exact. According to [**24,** Lemma 5.1], $\mathbf{X}$ is $\mathrm{Hom}(\mathcal{GF}_C, -)$ exact. Thus, (ii) holds by Fact 4.5.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) holds by Theorems 1.2 and 1.3. $\square$

Let $R$ be an Artinian ring. Then $R$ has a dualizing module $D$ by [**18,** proof of Lemma 3.8]. Note that $\mathrm{Mod}(R) = \mathcal{C}$ by [**27,** Proposition 3.3.1]. It follows that $\mathcal{H}_D = \mathcal{B}_D$. Thus, we have the following.

**Corollary 4.13.** *Let $R$ be an Artinian ring with a dualizing module $D$ and $\mathrm{Max}(R)$ the set of maximal ideals of $R$. Then the following are equivalent*:

(i) *$R$ is semisimple*;

(ii) *$\mathrm{Max}(R) \subseteq \mathcal{A}_D$, and there exists an integer $n$ such that*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{F}_D}(C \otimes m, N) = 0$$

*for any $m \in \mathrm{Max}(R)$ and any $N \in \mathcal{B}_D$*;

(iii) *$\mathrm{Max}(R) \subseteq \mathcal{A}_D$ and $\widehat{\mathrm{Ext}}^0_{\mathcal{F}_D}(C \otimes m, C \otimes m) = 0$ for any $m \in \mathrm{Max}(R)$*;

(iv) *$\mathrm{Max}(R) \subseteq \mathcal{A}_D$ and $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_D}(N, C \otimes m) = 0$ for each (or some) $n \in \mathbb{Z}$, each $N \in \mathcal{B}_D$ and each $m \in \mathrm{Max}(R)$.*

*Proof.* Note that $D$ is a dualizing module over an Artinian ring $R$. It follows from [**18,** Theorem 3.11, Proposition 4.6] that $\mathrm{Mod}(R) = \mathcal{GF}_D$.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from Theorem 1.3 and Fact 2.4.

(iv) $\Rightarrow$ (i). Note that $\mathcal{F}_D$-$\mathrm{pd}(C \otimes m) < \infty$ for any $m \in \mathrm{Max}(R)$ by (iv) and Theorem 1.3. Then $\mathrm{fd}(m) < \infty$ for any $m \in \mathrm{Max}(R)$ by [**26,** Lemma 2.3 (1)]. Hence, $\mathrm{pd}(m) < \infty$ for any $m \in \mathrm{Max}(R)$. Note that $\dim(R) = 0$ by the above proof. Thus, $R$ is semisimple by [**21,** Theorem 5.84]. $\square$

Let $(R, m, k)$ be a local Cohen-Macaulay ring. It follows from [**11**] that $R$ has a dualizing module if and only if $R$ is a homomorphic image of a local Gorenstein ring $Q$. Hence, every regular local ring has a dualizing module $D$ by [**4,** Proposition 3.1.20]. However, we have the following:

**Corollary 4.14.** *Let $(R, m, k)$ be a local Cohen-Macaulay ring with a dualizing module $D$. Then, the following are equivalent*:

(i) *$R$ is regular*;

(ii) *$k \in \mathcal{B}_D$ and $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_D}(k, N) = 0$ for some $n \in \mathbb{Z}$ and any $N \in \mathcal{H}_D$*;

(iii) *$k \in \mathcal{B}_D$ and $\widehat{\mathrm{Ext}}^0_{\mathcal{F}_D}(k, B) = 0$, where $\varepsilon : k \to B$ is an $\mathcal{H}_D$-envelope of $k$;*

(iv) *$k \in \mathcal{B}_D$ and $\widehat{\mathrm{Ext}}^n_{\mathcal{F}_D}(N, B) = 0$ for each (or some) $n \in \mathbb{Z}$ and each $N \in \mathcal{B}_D$, where $\varepsilon : k \to B$ is an $\mathcal{H}_D$-envelope of $k$.*

*Proof.* Note that $D$ is a dualizing module by hypothesis. It follows from [**19,** Proposition 2.6] that $\mathrm{Mod}(R) = \widehat{\mathcal{GF}_D}$.

(i) $\Rightarrow$ (ii). Note that $\mathcal{F}_D$-pd$(k) \leqslant \mathcal{P}_D$-pd$(k) < \infty$ by (i) and [**25,** Proposition 5.1]. So $k \in \mathcal{B}_D$ and (ii) follows from Theorem 1.3.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold by Theorem 1.3.

(iv) $\Rightarrow$ (i). Note that $\mathcal{F}_D$-pd$(k) < \infty$ by (iv) and Theorem 1.3. It follows that $\mathrm{fd}(\mathrm{Hom}(D, k)) < \infty$ by [**26,** Lemma 2.3 (2)]. Thus, $\mathrm{pd}(\mathrm{Hom}(D, k)) < \infty$ by [**6,** page 237, 1.4] and [**20,** Proposition 6], and hence, $\mathcal{P}_D$-pd$(k) < \infty$ by [**25,** Theorem 2.11 (c)]. Thus, (i) holds by [**25,** Proposition 5.1]. $\square$

Recall that a ring $R$ is called *Gorenstein* [**8,** Definition 9.1.1] if the injective dimension of $R$ is finite. Note that $\mathcal{A}_R = \mathcal{B}_R = \mathrm{Mod}(R)$ and $\mathcal{H}_R = \mathcal{C}$ provided that $R$ is a Gorenstein ring. Specializing Corollary 4.14 to the case $C = R$ gives the following new criteria for a local Gorenstein ring to be regular.

**Corollary 4.15.** *Let $(R, m, k)$ be a local Gorenstein ring and $\varepsilon : k \to B$ a cotorsion envelope of $k$. Then the following are equivalent:*

(i) *$R$ is regular;*

(ii) *$\widehat{\mathrm{Ext}}^n_{\mathcal{F}}(k, N) = 0$ for some $n \in \mathbb{Z}$ and any $N \in \mathcal{C}$;*

(iii) *$\widehat{\mathrm{Ext}}^0_{\mathcal{F}}(k, B) = 0$;*

(iv) *$\widehat{\mathrm{Ext}}^n_{\mathcal{F}}(N, B) = 0$ for each (or some) $n \in \mathbb{Z}$ and each $N \in \mathcal{C}$.*

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