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QUASICONFORMAL EXTENDABILITY OF INTEGRAL TRANSFORMS OF NOSHIRO-WARSCHAWSKI FUNCTIONS

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ABSTRACT. Since the nonlinear integral transforms

$$J_\alpha[f](z) = \int_0^z (f'(u))^\alpha du$$

and

$$I_\alpha[f](z) = \int_0^z (f(u)/u)^\alpha du$$

with a complex number α were introduced, a great number of studies have been dedicated to deriving sufficient conditions for univalence on the unit disk. However, little is known about the conditions where $J_\alpha[f]$ or $I_\alpha[f]$ produce a holomorphic univalent function in the unit disk which extends to a quasiconformal map on the complex plane. In this paper, we discuss quasiconformal extendability of the integral transforms $J_\alpha[f]$ and $I_\alpha[f]$ for holomorphic functions which satisfy the Noshiro-Warschawski criterion. Various approaches using pre-Schwarzian derivatives, differential subordination and Loewner theory are applied to this problem.

1. Introduction.

1.1. Integral transforms. Let $\mathcal{A}$ be the family of analytic functions defined in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{LU}$ and $\mathcal{ZF}$ be the subclasses of $\mathcal{A}$ defined by

$$\mathcal{LU} := \{f \in \mathcal{A} : f'(z) \neq 0 \text{ for all } z \in \mathbb{D}\}$$

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and

$$\mathcal{ZF} := \{f \in \mathcal{A} : f(z)/z \neq 0 \text{ for all } z \in \mathbb{D}\}.$$

In 1915, Alexander [1] first observed the integral transform defined by

$$J[f](z) = \int_0^z \frac{f(u)}{u}\, du$$

on the class $\mathcal{ZF}$ maps the class of starlike functions onto the class of convex functions. Thus, one might expect that $J[f]$ always produces a univalent function for all $f \in \mathcal{S}$, where $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions on $\mathbb{D}$. However, in 1963, Krzyż and Lewandowski [23] gave the counterexample

$$f(z) = \left(\frac{z}{1-iz}\right)^{1-i},$$

which is $\pi/4$-spirallike but transformed into a non-univalent function. In 1972, Kim and Merkes [20] extended this type of transform by introducing a complex parameter $\alpha \in \mathbb{C}$:

$$J_\alpha[f](z) := \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du$$

for $f \in \mathcal{ZF}$, where the branch is chosen so that $(f(z)/z)^\alpha = 1$ for $z = 0$. In their investigation, it was shown that $J_\alpha[\mathcal{S}] \subset \mathcal{S}$ when $|\alpha| \leq 1/4$, while $J_\alpha[\mathcal{S}] \not\subset \mathcal{S}$ if $|\alpha| > 1/2$. Consider $J_\alpha[K](z)$ and Royster's example [31], where $K(z) := z/(1-z)^2$ is the Koebe function.

Another object of investigation in the study of integral transforms is $I_\alpha[f]$, defined by

$$I_\alpha[f](z) := \int_0^z (f'(u))^\alpha du \tag{1.1}$$

on $\mathcal{LU}$, where the branch of $(f')^\alpha = \exp(\alpha \log f')$ is chosen so that $(f')^\alpha(0) = 1$. Then, $J_\alpha[f]$ is represented by $J_\alpha[f] = I_\alpha[J[f]]$. In 1975, Pfaltzgraff [30] proved that $I_\alpha[\mathcal{S}] \subset \mathcal{S}$ if $|\alpha| \leq 1/4$. Additionally, Royster's example again shows that there exists a function $f \in \mathcal{S}$ such that $I_\alpha[f] \notin \mathcal{S}$ if $|\alpha| > 1/3$ or $\alpha \neq 1$.

Currently, no better estimates of the range of $|\alpha|$ have been obtained in the problems of univalence of $I_\alpha[f]$ and $J_\alpha[f]$. The reader may refer to **[10]** for basic terminology in the theory of univalent functions and **[13,** Chapter 15] for basic information about integral transforms on $\mathcal{S}$.

1.2. The Noshiro-Warschawski criterion. It is known that, for a function $f \in \mathcal{A}$, the condition that $f'(\mathbb{D})$ lies in the right half-plane ensures univalence of f on $\mathbb{D}$. This is referred to as the *Noshiro-Warschawski criterion* due independently to Noshiro **[29]** and Warschawski **[34]**. We shall provide the original form of the theorem next, see also **[3,** Theorem 8].

Theorem 1.1 (The Noshiro-Warschawski criterion)**.** *A non constant function f that is analytic in a convex domain D is univalent in D if*

$$\operatorname{Re}\{e^{-ic} f'(z)\} \geq 0 \tag{1.2}$$

for all $z \in D$, where c is a fixed real number.

As special cases, Alexander **[1]** showed the case when D is the unit disk and $f'(\mathbb{D})$ is contained in a half-plane bounded by a straight line through the origin, and Wolff **[35]** showed when D is the right half-plane. On the other hand, Tims **[15]** and Herzog and Piranian **[33]** showed that convexity of D is essential in the theorem, that is, equation (1.2) implies univalence of f on D if and only if D is convex.

In what follows, we will treat the family of functions $f \in \mathcal{A}$ satisfying the hypothesis of the theorem in which D is the unit disk $\mathbb{D}$ and $c = 0$. It is denoted by $\mathcal{R}$, i.e.,

$$\mathcal{R} := \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0 \quad \text{for all } z \in \mathbb{D}\}.$$

Then, Theorem 1.1 states that $\mathcal{R} \subset \mathcal{S}$. Compared with the other typical subclasses of $\mathcal{S}$, a geometric characterization of $\mathcal{R}$ is unknown. Several geometric properties of $f(\mathbb{D})$ by means of Loewner chains are observed in **[17]**. Roughly speaking, $f(\mathbb{D})$ is the complement of the union of the rays $\{f(e^{i\theta}) + te^{i\theta} : t \in [0, \infty)\}$, where $f(e^{i\theta})$ is understood as the impression of the prime end at $e^{i\theta} \in \partial\mathbb{D}$.

A more general problem has been posed for finding a domain $R \subset \mathbb{C}$ such that, for a given simply-connected domain $D \subset \mathbb{C}$, the condition

$$f'(D) \subset R$$

implies univalence of f on D. It is studied as a first-order criterion and can be expressed more generally as

$$\log f'(D) \subset R^*,$$

which means that $f'(z) = \exp g(z)$ where $g(D) \in R^*$. It is particularly concerned with the special case in which $R^* = \alpha I$, where $\alpha \in \mathbb{C}$ and I is an infinite strip parallel to the real axis with width π, i.e., $I := \{z : a - \pi/2 < \operatorname{Im} z < a + \pi/2\}$, $a \in \mathbb{R}$. Theorem 1.1 gives a criterion of the case when $\alpha = 1$ and $a = c$. For further information about first-order univalence criteria, see e.g., **[11, 12]** as well as the more recent work **[2]**.

1.3. Direction of this paper. Until now, a great number of studies were dedicated to deriving sufficient conditions for the univalence of $J_\alpha[f]$ and $I_\alpha[f]$ on $\mathbb{D}$. However, little seems to be known about the conditions where $J_\alpha[f]$ or $I_\alpha[f]$ produces a univalent function in $\mathbb{D}$ which extends to a quasiconformal map on $\mathbb{C}$, except that a straightforward application of the λ-lemma $i(\lambda, z) := J_{\lambda/4}[f](z)$ forms a holomorphic motion on $(\lambda, z) \in \mathbb{D} \times \mathbb{D}$.

In this paper, we discuss quasiconformal extendability of the integral transforms $J_\alpha[f]$ and $I_\alpha[f]$ for holomorphic functions which satisfy the Noshiro-Warschawski criterion. Various approaches using pre-Schwarzian and Schwarzian derivatives, differential subordinations and Loewner theory are applied to this problem.

In the last section, our study contributes to constructing explicit quasiconformal extensions which are formed by inverse counterparts of Loewner chains introduced by Betker **[8]**.

2. Preliminaries.

2.1. Schwarzian and pre-Schwarzian derivatives. As important quantities for investigating properties of functions f in $\mathcal{LU}$, we introduce T_f and S_f defined by

$$T_f := \frac{f''}{f'}, \qquad S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Quantities T_f and S_f are called the *pre-Schwarzian derivative* and *Schwarzian derivative*, respectively. These are considered to be elements of the Banach space of functions $f \in \mathcal{LU}$, for which the norms

$$||T_f|| := \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f|,$$
$$||S_f|| := \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f|,$$

are finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism f on a domain G is said to be *k-quasiconformal*, $0 \le k < 1$, if $\partial_{\bar z} f$ and $\partial_z f$, the partial derivatives of f in z and $\bar z$ in the distributional sense, are locally integrable on G and satisfy $|\partial_{\bar z} f| \le k|\partial_z f|$ almost everywhere in G. If, for a given $f \in \mathcal{S}$, there exists a k-quasiconformal F of $\mathbb{C}$ such that its restriction on $\mathbb{D}$ is equivalent to f, then f is said to have a *k-quasiconformal extension* to $\mathbb{C}$.

Theorem 2.1. *Let $f \in \mathcal{LU}$.*

(i) *If $||T_f|| \le 1$, then f is univalent in $\mathbb{D}$;*
(ii) *if $||T_f|| \le k < 1$, then f has a quasiconformal extension to $\mathbb{C}$;*
(iii) *if $f \in \mathcal{S}$, then $||T_f|| \le 6$;*
(iv) *if $||S_f|| \le 2$, then f is univalent in $\mathbb{D}$;*
(v) *if $f \in \mathcal{S}$, then $||S_f|| \le 6$.*

Becker proved (i) and (ii) [**4, 5**]. The sharpness of constant 1 in (i) is due to Becker and Pommerenke [**7**]. Assertion (iii) is an easy consequence of the well-known inequality $|(1-|z|^2)f''(z)/f'(z) - 2\bar z| \le 4$ for $f \in \mathcal{S}$. Assertion (iv) was first shown by Kraus [**22**] and subsequently rediscovered by Nehari [**28**]. Hille [**16**] showed that constant 2 is the best possible with the function

$$f(z) = \left(\frac{1+z}{1-z}\right)^{i\varepsilon}, \quad \varepsilon > 0,$$

for it is not univalent for all $\varepsilon > 0$, but $||S_f|| = 2(1+\varepsilon^2)$ can approach 2. Nehari [**28**] also verified assertion (v) and the sharpness follows from $||S_K|| = 6$ for the Koebe function K.

2.2. Subordination properties. For analytic functions f and g, it is said that f is *weakly subordinate* to g if there exists an analytic function ω which maps $\mathbb{D}$ into $\mathbb{D}$ such that $f(z) = (g \circ \omega)(z)$. Further, if w can be taken so as to fulfill $\omega(0)=0$, then f is said to be *subordinate* to g, whose relation is denoted by $f(z) \prec g(z)$. Below, we will state two subordination properties which will play central roles in Section 3. The first is a result on differential subordinations due to Hallenbeck and Ruscheweyh.

Theorem 2.2 (Hallenbeck and Ruscheweyh [**14**]). *Let p be an analytic function in $\mathbb{D}$ with $p(0)=1$. Let q be convex univalent in $\mathbb{D}$ with $q(0)=1$, and suppose that $p(z) \prec q(z)$. Then, for all $\gamma \neq 0$ with* $\operatorname{Re}\gamma > 0$, *we have*

$$\gamma z^{-\gamma} \int_0^z u^{\gamma-1} p(u)\, du \prec \gamma z^{-\gamma} \int_0^z u^{\gamma-1} q(u)\, du.$$

For example, if f satisfies $\operatorname{Re} f'(z)(z/f(z))^{1-\gamma} > 0$, then

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\gamma} \prec \frac{1+z}{1-z},$$

and Theorem 2.2 gives

$$\left(\frac{f(z)}{z}\right)^{\gamma} \prec 1 + \frac{2\gamma}{z^\gamma} \int_0^z \frac{u^\gamma}{1-u}\, du.$$

In particular, setting $\gamma = 1$, we have

$$\frac{f(z)}{z} \prec \frac{-z - 2\log(1-z)}{z} \tag{2.1}$$

for all $f \in \mathcal{R}$. This gives the best dominant for $\mathcal{R}$ because, if

$$\phi(z) := -z - 2\log(1-z),$$

then

$$\phi'(z) = \frac{1+z}{1-z},$$

and therefore, $\phi \in \mathcal{R}$.

The second is a fundamental subordination principle in geometric function theory. The original idea is due to Littlewood.

Theorem 2.3 (Kim and Sugawa [**21,** page 195]). *Let g be locally univalent in $\mathbb{D}$. For an analytic function f in $\mathbb{D}$, if f' is weakly* *subordinate to g', then we have $||T_f|| \leq ||T_g||$. In particular, f is uniformly locally univalent on $\mathbb{D}$.*

Theorem 2.3 has a wide range of applications so that we might hope also to obtain the inequality $||S_f|| \leq ||S_g||$ for functions f and g such that f' is weakly subordinate to g'. However, it is shown that the inequality does not always hold under this assumption. Here, we note that the Schwarz-Pick lemma shows that all analytic self-mappings ω of the unit disk satisfy

$$\frac{|\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{1}{1-|z|^2} \tag{2.2}$$

for all $z \in \mathbb{D}$.

Proposition 2.4. *Let g be locally univalent in $\mathbb{D}$. For an analytic function f in $\mathbb{D}$, if f' is weakly subordinate to g', then we have*

$$||S_f|| \leq ||S_g|| + ||T_\omega|| \cdot ||T_g||, \tag{2.3}$$

where ω is an analytic function which appears in the definition of subordination. In particular, f is uniformly locally univalent on $\mathbb{D}$.

Proof. By assumption, we have $T_f = T_g \circ \omega \cdot \omega'$, and hence, equation (2.2) implies that

$$\begin{aligned}
&(1-|z|^2)^2\left|\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right| \\
&\qquad = (1-|z|^2)^2\left|\left(\frac{g''}{g'}\right)'\omega'^2 + \frac{g''}{g'}\cdot\omega'' - \frac{1}{2}\left(\frac{g''}{g'}\cdot\omega'\right)^2\right| \\
&\qquad \leq \frac{(1-|\omega|^2)^2}{|\omega'|^2}\left|\left(\frac{g''}{g'}\right)'\omega'^2 - \frac{1}{2}\left(\frac{g''}{g'}\cdot\omega'\right)^2\right| \\
&\qquad\quad + (1-|z|^2)\frac{1-|\omega|^2}{|\omega'|}\left|\frac{g''}{g'}\cdot\omega''\right| \\
&\qquad \leq ||S_g|| + ||T_\omega|| \cdot ||T_g||.
\end{aligned}$$

□

The term $||T_\omega|| \cdot ||T_g||$ in equation (2.3) is eliminated in only a few cases. So, $||T_g|| = 0$ if and only if g is an affine transform, and then $||S_g||$ also vanishes. Therefore, $||S_f|| = 0$, which implies that f is a Möbius transformation. $||T_\omega|| = 0$ if and only if ω is an affine transform, which is equivalent to the case $f(z) = ag(z) + b$, where $a, b \in \mathbb{C}$ are complex constants.

3. Pre-Schwarzian derivatives and differential subordinations for $J_\alpha[f]$.

3.1. Evaluation of $||T_{J_\alpha[f]}||$ on $\mathcal{R}$. First, we give a sharp estimation of the norm of $T_{J_\alpha[f]}$ for a function $f \in \mathcal{R}$ and make use of Theorem 2.1 to obtain the range of $|\alpha|$ which ensures univalence and quasiconformal extensibility of $J_\alpha[f]$.

Theorem 3.1. *Let $f \in \mathcal{R}$. Then, we have the sharp estimate*

$$||T_{J_\alpha[f]}|| \leq |\alpha| \cdot h(r_0),$$

where $h(r_0) \approx 1.055681$. Here, h is the function defined by

$$h(r) := \frac{-(1+r)^2}{r + 2\log(1-r)} - \frac{1-r^2}{r}, \tag{3.1}$$

and $r_0 \approx 0.329423$ is the unique root of the equation

$$2(r^2+1)(r-1)[\log(1-r)]^2 - 2r(r-1)^2\log(1-r) + r^3(r+3) = 0 \tag{3.2}$$

in $r \in (0, 1)$.

Proof. Using logarithmic differentiation, we have

$$||T_{J_\alpha[f]}|| = |\alpha| \, ||T_{J[f]}||.$$

Then, it suffices to estimate $||T_{J[f]}||$. Suppose that $f \in \mathcal{R}$. By equation (2.1) and Theorem 2.3, we have

$$||T_{J[f]}|| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{\phi'(z)}{\phi(z)} - \frac{1}{z} \right|$$

for all $f \in \mathcal{R}$, where $\phi(z) = -z - 2\log(1-z)$ as defined in subsection 2.2. Then, computation shows that

$$\begin{aligned} \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{\phi'(z)}{\phi(z)} - \frac{1}{z} \right| &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{2(z + (1-z)\log(1-z))}{(1-z)z(z + 2\log(1-z))} \right| \\ &= \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{|z|} \left| 1 + \frac{1+z}{1-z} \cdot \frac{z}{z + 2\log(1-z)} \right|. \end{aligned}$$

Let

$$g(z) := 1 + \frac{1+z}{1-z} \cdot \frac{z}{z + 2\log(1-z)}.$$

It is obvious that g is symmetric with respect to the real axis. Next, we show that all the coefficients of g are negative. g is written as:

$$\begin{aligned} g(z) &= 1 + \frac{1+z}{1-z} \cdot \frac{z}{z + 2\log(1-z)} \\ &= 1 - \frac{1+z}{1-z} \cdot \frac{1}{1 + 2\sum_{n=1}^{\infty} z^n/(n+1)} \\ &= 1 - \frac{1+z}{1 - z + 2\sum_{n=1}^{\infty} z^{n+1}/(n+2) - 2\sum_{n=1}^{\infty} z^{n+1}/(n+1)} \\ &= 1 - \frac{1+z}{1 - 2\sum_{n=1}^{\infty} z^{n+1}/[(n+1)(n+2)]}. \end{aligned}$$

Thus, g has negative coefficients. This fact implies that

$$\sup_{z\in\mathbb{D}} |g(z)| = -\sup_{r\in(0,1)} g(r).$$

Therefore,

$$||T_{J[f]}|| \leq \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{|z|} \left| 1 + \frac{1+z}{1-z} \cdot \frac{z}{z + 2\log(1-z)} \right| = \sup_{r\in(0,1)} h(r).$$

Simple calculation shows that $h'(r)$ has only one critical point r_0 in $r \in (0,1)$ which is the root of equation (3.2). By numerical experiments, we obtain that $r_0 \approx 0.329423$ and $h(r_0) \approx 1.055681$. □

Applying Theorem 2.1 to the above estimate, we can deduce the range of $|\alpha|$ of which $J_\alpha[f]$ is univalent in $\mathbb{D}$ and has a quasiconformal extension to $\mathbb{C}$.

Corollary 3.2. *Let $f \in \mathcal{R}$ and $k \in [0,1)$. Then,*

(i) *if $|\alpha| \leq 1/h(r_0) \approx 0.947255$, then $J_\alpha[f] \in \mathcal{S}$.*
(ii) *If $|\alpha| < k/h(r_0)$, then $J_\alpha[f]$ can be extended to a k-quasiconformal mapping of $\mathbb{C}$.*

3.2. Univalence of $J_\alpha[f]$ when $\alpha \in \mathbb{R}$. In the previous subsection, we dealt with $J_\alpha[f]$ in the case where α is a complex number. However, some geometric properties of $J_\alpha[f]$ on typical subclasses of $\mathcal{S}$ under the restriction of $\alpha \in \mathbb{R}$ have already been investigated. Next, is a list of some fundamental results. Here, we denote by $\mathcal{K}, \mathcal{S}^*$ and $\mathcal{C}$ the well-known classes of convex, starlike and close-to-convex functions in $\mathcal{A}$, respectively.

Theorem 3.3 (Merkes and Wright **[27]**). *Let $\alpha \in \mathbb{R}$. Then, the next assumptions are true*:

(i) *Let $f \in \mathcal{K}$. If $\alpha \in [-1, 3]$, then $J_\alpha[f] \in \mathcal{C}$; otherwise, there exists a function $g \in \mathcal{K}$ such that $J_\alpha[g] \notin \mathcal{S}$.*
(ii) *Let $f \in \mathcal{S}^*$. If $\alpha \in [-1/2, 3/2]$, then $J_\alpha[f] \in \mathcal{C}$; otherwise, there exists a function $g \in \mathcal{S}^*$ such that $J_\alpha[g] \notin \mathcal{S}$.*
(iii) *Let $f \in \mathcal{C}$. If $\alpha \in [-1/2, 1]$, then $J_\alpha[f] \in \mathcal{C}$; otherwise, there exists a function $g \in \mathcal{C}$ such that $J_\alpha[g] \notin \mathcal{C}$.*
(iv) *Let $f \in \mathcal{K}$. If $\alpha \in [-1/2, 3/2]$ then $I_\alpha[f] \in \mathcal{C}$; otherwise, there exists a function $g \in \mathcal{K}$ such that $J_\alpha[g] \notin \mathcal{S}$.*
(v) *Let $f \in \mathcal{C}$. If $\alpha \in [-1/3, 1]$ then $I_\alpha[f] \in \mathcal{C}$; otherwise, there exists a function $g \in \mathcal{C}$ such that $J_\alpha[g] \notin \mathcal{S}$.*

Next, the following is easily proved.

Theorem 3.4. *Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{R}$. If $\alpha \in [-\alpha_0, \alpha_0]$, then $J_\alpha[f] \in \mathcal{R}$; otherwise, there exists a function $g \in \mathcal{R}$ such that $J_\alpha[g] \notin \mathcal{R}$. Here, $\alpha_0 \approx 1.723078$ is defined by $\alpha_0 := \pi/2q(e^{i\theta_0})$, where*

$$q(z) := \frac{-z - 2\log(1-z)}{z},$$

and θ_0 is the unique root of the equation:

$$\varsigma'(\theta) - \varsigma(\theta)^2 - 1 = 0,$$

in $\theta \in (0, \pi/2)$, where ς is defined by

$$\varsigma(\theta) := \frac{\sin\theta + \theta - \pi}{\cos\theta + 2\log\left(2\sin\dfrac{\theta}{2}\right)}. \tag{3.3}$$

Proof. Suppose that $f \in \mathcal{R}$. Again, we will use relation (2.1), namely,

$$J[f]'(z) \prec q(z) \prec \frac{1+z}{1-z}$$

for all $f \in \mathcal{R}$. Here, q is a convex function, see **[25,** Theorem 2]. Since $f \prec g$ implies that $f^\alpha \prec g^\alpha$ for any $\alpha \in \mathbb{R}$ and $(J[f]')^\alpha = J_\alpha[f]'$, our problem reduces to finding the largest $\alpha_0 \in \mathbb{R}$ such that

$$\mathrm{Re}\,[q(z)^{\alpha_0}] > 0 \quad \text{for all } z \in \mathbb{D}.$$

This is equivalent to finding the smallest $\beta_0 \in \mathbb{R}$ such that the sector domain

$$\Delta_{\beta_0} := \left\{ w : |\arg w| < \pi \frac{\beta_0}{2} \right\}$$

contains $q(\mathbb{D})$. Then, $\alpha_0 = 1/\beta_0$. Note that $z \in \Delta_{\beta_0}$, and $1/z \in \Delta_{\beta_0}$.

We obtain

$$\begin{aligned} \arg q(e^{i\theta}) &= \arg\left[-e^{i\theta} - 2\log\left(2\sin\frac{\theta}{2}\right) - i(\theta-\pi)\right] - \theta \\ &= \arctan\varsigma(\theta) - \theta, \end{aligned}$$

by using

$$1 - e^{i\theta} = -2i\sin\frac{\theta}{2}e^{i\theta/2}.$$

Since

$$\frac{\partial \arg q(e^{i\theta})}{\partial\theta} = \frac{\varsigma'(\theta)}{1+\varsigma(\theta)^2} - 1,$$

β_0 is one of the zeros of $\varsigma'(\theta) - \varsigma(\theta)^2 - 1$. With the aid of `Mathematica`, the maximum of $\arg q(e^{i\theta})$ is attained at $\theta_0 \approx 1.141377$. Then,

$$\beta_0 = \frac{2q(\theta_0)}{\pi} \approx 0.580356,$$

and we conclude that $\alpha_0 = 1/\beta_0 \approx 1.723078$. $\square$

Remark 3.5. The Alexander transformation $J[f]$ preserves $\mathcal{R}$.

Theorem 3.4 will be refined to a quasiconformal extension criterion by using the Loewner chains in Section 5.

4. Results for $I_\alpha[f]$ on $\mathcal{R}$. We shall derive some further properties of $I_\alpha[f]$ on $\mathcal{R}$. In particular, Theorem 4.2 will be used in the next section.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{R}$. If $\alpha \in [-1,1]$, then $I_\alpha[f] \in \mathcal{R}$.*

Proof. Since $I_\alpha[f]' = (f')^\alpha$, it is clear that $I_\alpha[f] \in \mathcal{R}$ when $\alpha \in [-1,1]$. $\square$

Theorem 4.2. *Let $\alpha \in \mathbb{C}$. If $|\alpha| > 1$, then there exists a function $g \in \mathcal{R}$ such that $I_\alpha[g] \notin \mathcal{S}$.*

Proof. A counterexample is given by the function

$$\phi(z) = -z - 2\log(1-z),$$

which belongs to $\mathcal{R}$. In fact, it follows from calculations that

$$||S_{I_\alpha[\phi]}|| = 2|\alpha|(|\alpha|+2).$$

Then, Theorem 2.1(v) shows that $I_\alpha[\phi]$ is not univalent if $|\alpha| > 1$. $\square$

Theorem 4.3. *Let $\alpha \in \mathbb{C}$ and $f \in \mathcal{R}$. If $|\alpha| \leq 1/2$, then $I_\alpha[f] \in \mathcal{S}$.*

Proof. Let $f \in \mathcal{R}$. Then,

$$f'(z) \prec \frac{1+z}{1-z},$$

and hence, by Theorem 2.3 we obtain the sharp bound $||T_f|| \leq 2$ for an $f \in \mathcal{R}$, see also **[26,** Lemma 1]. Since $||T_{I_\alpha[f]}|| = |\alpha| \cdot ||T_f||$, it follows from Theorem 2.1 (i) that $I_\alpha[f] \in \mathcal{S}$ if $|\alpha| \leq 1/2$. $\square$

5. Quasiconformal extension of $J_\alpha[f]$ with Loewner chains. In this section, we will use the theory of Loewner chains and its applications to derive quasiconformal extension conditions for $J_\alpha[f]$ and $I_\alpha[f]$ under the class $\mathcal{R}$.

5.1. Loewner chains and inverse Loewner chains. Before starting our argument, we describe the theory of Loewner chains and results of quasiconformal extensions due to Becker and Betker with some notation and terminology we will use.

Let

$$f_t(z) = \sum_{n=1}^{\infty} a_n(t) z^n, \quad a_1(t) \neq 0,$$

be a function defined on $\mathbb{D} \times [0, \infty)$, where $a_1(t)$ is a complex-valued, locally absolutely continuous function on $[0, \infty)$. Then, f_t is called a *Loewner chain* if f_t satisfies the following conditions:

(i) f_t is univalent in $\mathbb{D}$ for each $t \geq 0$,
(ii) $|a_1(t)|$ increases strictly as t increases, and $\lim_{t\to\infty} |a_1(t)| \longrightarrow \infty$,
(iii) $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for $0 \leq s < t < \infty$.

It should be noted that strict monotonicity of $|a_1(t)|$ implies that $f_s(\mathbb{D}) \neq f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$.

The key properties of Loewner chains are that f_t is absolutely continuous on $t \geq 0$ for each $z \in \mathbb{D}$, which implies that $\partial_t f_t$ ($\partial_t := \partial/\partial t$) exists almost everywhere on $[0, \infty)$ and satisfies the partial differential equation

$$\partial_t f_t(z) = z \partial_z f_t(z) p(z,t), \quad z \in \mathbb{D}, \text{ almost everywhere } t \geq 0, \tag{5.1}$$

where $p(z,t)$ is analytic for all $z \in \mathbb{D}$, for each $t \geq 0$, measurable for all $t \geq 0$ for each $z \in \mathbb{D}$ and satisfies $\mathrm{Re}\, p(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \geq 0$. We call the function p a *Herglotz function.* Further, Becker [**4, 6**] showed that, if p satisfies

$$\left| \frac{1 - p(z,t)}{1 + p(z,t)} \right| \leq k, \quad z \in \mathbb{D}, \text{ almost everywhere } t \geq 0;$$

then, f_0 has a k-quasiconformal extension to $\mathbb{C}$. This enables us to derive various kinds of sufficient conditions under which a function $f \in \mathcal{S}$ has a quasiconformal extension, see e.g., [**18, 19**].

Betker introduced the following notion of inverse counterparts of Loewner chains. Let

$$\omega_t(z) = \sum_{n=1}^{\infty} b_n(t) z^n, \quad b_1(t) \neq 0,$$

be a function defined on $\mathbb{D} \times [0, \infty)$, where $b_1(t)$ is a complex-valued, locally absolutely continuous function on $[0, \infty)$. Then, ω_t is said to be an *inverse Loewner chain* if

(i) ω_t is univalent in $\mathbb{D}$ for each $t \geq 0$;
(ii) $|b_1(t)|$ decreases strictly as t increases; and $\lim_{t\to\infty} |b_1(t)| \to 0$;
(iii) $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$ for $0 \leq s < t < \infty$;
(iv) $\omega_0(z) = z$ and $\omega_s(0) = \omega_t(0)$ for $0 \leq s \leq t < \infty$.

The next partial differential equation is also satisfied by ω:

$$\partial_t \omega_t(z) = -z\partial_z \omega_t(z) q(z,t), \quad z \in \mathbb{D}, \text{ almost everywhere } t \geq 0, \tag{5.2}$$

where q is a Herglotz function. Conversely, we can construct an inverse Loewner chain by means of equation (5.2) according to the next lemma.

Lemma 5.1 (Betker [**8**]). *Let $q(z,t)$ be a Herglotz function. Suppose that $q(0,t)$ is locally integrable in $[0,\infty)$ with*

$$\int_0^\infty \operatorname{Re} q(0,t)\, dt = \infty.$$

Then, there exists an inverse Loewner chain w_t satisfying equation (5.2)*.*

Applying the notion of an inverse Loewner chain, we obtain a generalization of Becker's result.

Theorem 5.2 (Betker [**8**]). *Let $k \in (0,1]$. Let f_t be a Loewner chain satisfying equation* (5.1) *with*

$$\left| \frac{p(z,t) - \overline{q(z,t)}}{p(z,t) + q(z,t)} \right| \leq k < 1, \quad z \in \mathbb{D}, \text{ almost everywhere } t \geq 0,$$

where $q(z,t)$ is a Herglotz function. Let ω_t be the inverse Loewner chain which is generated by q with equation (5.2)*. Then, f_t and ω_t are continuous and injective on $\overline{\mathbb{D}}$ for each $t \geq 0$, and f_0 has a k-quasiconformal extension $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which is defined by*

$$\Phi\left(\frac{1}{\overline{\omega_t(e^{i\theta})}}\right) = f_t(e^{i\theta}), \quad \theta \in [0,2\pi),\ t \geq 0. \tag{5.3}$$

Case $q(z,t) = 1$ reflects Becker's theorem. In this case, $\omega_t(z) = e^{-t}z$. Further, if ω is obtained from the choice $q = p$, then we have the next corollary.

Corollary 5.3 (Betker [**8**]). *Let $\gamma \in (0,1]$. Suppose that f_t is a Loewner chain for which p in equation* (5.1) *satisfies the condition*

$$|\arg p(z,t)| \leq \frac{\gamma\pi}{2}, \quad z \in \mathbb{D}, \ \textit{almost everywhere } t \geq 0.$$

Then, f_t admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the map defined by equation (5.3) *is a* $\sin(\gamma\pi/2)$*-quasiconformal extension of f_0 to $\mathbb{C}$.*

In contrast to Becker's quasiconformal extension theorem mentioned above, the theorem due to Betker does not explicitly give a quasiconformal extension in all cases. In general, this is due to the fact that it is difficult to express an inverse Loewner chain ω_t which has the same Herglotz function as a Loewner chain f_t given in an explicit form.

Specifically, let f_t be a given Loewner chain, and let $p(z,t)$ be a Herglotz function associated with f_t by equation (5.1). Fix an arbitrary $T > 0$, and define a Herglotz function $q(z,t)$ by

$$q(z,t) := \begin{cases} p(z, T-t), & t \in [0,T], \\ 1, & t \in (T,\infty). \end{cases} \tag{5.4}$$

It is known that there exists a Loewner chain h_t with the equation $\partial_t h_t(z) = z\partial_z h_t(z) q(z,t)$. One can see that $g_t(z)$, defined by

$$g_t(z) := \begin{cases} (h_T^{-1} \circ h_t)(z), & t \in [0,T] \\ e^{T-t}, & t \in (T,\infty), \end{cases} \tag{5.5}$$

is also a Loewner chain whose Herglotz function is q. Such g_t is uniquely determined by the condition $g_T(z) = z$. Therefore, g_t is the unique solution of the differential equation

$$\partial_t g_t(z) = z\partial_z g_t(z) p(z, T-t)$$

for all $z \in \mathbb{D}$ and $t \in [0,T]$. Hence, $\omega_t := g_{T-t}$ is defined on $z \in \mathbb{D}$, $t \in [0,T]$, and satisfies

$$\partial_t \omega_t(z) = -z\partial_z \omega_t(z) p(z,t).$$

It is also easily seen that $\omega_0(z) = z$, $\omega_t(0) = 0$, $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$, and $b_1(t)$ is monotonically decreasing with $|b_1| \to 0$ as $t \to \infty$. Since T is arbitrary, we obtain our desired inverse Loewner chain.

The above argument indicates that, in order to obtain the concrete expression of ω_t, we need to write h_t and h_t^{-1} by a given f_t, and this is not always possible. Loewner chains for spirallike functions are one of the few known cases in which this method works well. Here, $f \in \mathcal{A}$ is said to be λ-*spirallike*, $\lambda \in (-\pi/2, \pi/2)$, if f satisfies

$$\operatorname{Re}\left\{ e^{-i\lambda} \frac{zf'(z)}{f(z)} \right\} > 0$$

for all $z \in \mathbb{D}$. We know that $f_t(z) = e^{e^{i\lambda}t} f(z)$ describes an expanding flow for λ-spirallike domains. In this case, the corresponding inverse Loewner chain ω_t can be written explicitly:

$$\omega_t(z) := f^{-1}(e^{-e^{i\lambda}t} f(z)).$$

Let $\alpha \in (-\pi/2, \pi/2)$ be given. Suppose that

$$\left| \arg \frac{zf'(z)}{f(z)} - \lambda \right| < \frac{\pi\alpha}{2}.$$

Then, by Corollary 5.3, f has a continuous extension to $\overline{\mathbb{D}}$, and the function $\Phi : \mathbb{C} \to \mathbb{C}$,

$$\begin{cases} \Phi(z) = f(z), & z \in \overline{\mathbb{D}}, \\ \Phi\left(\dfrac{1}{\overline{f^{-1}(e^{-e^{i\lambda}t} f(e^{i\theta}))}} \right) = e^{e^{i\lambda}t} f(e^{i\theta}), & \theta \in [0, 2\pi),\ t \geq 0, \end{cases} \tag{5.6}$$

defines a $\sin(\pi\alpha/2)$-quasiconformal extension of f. If

$$z = 1/\overline{f^{-1}(e^{-e^{i\lambda}t} f(e^{i\theta}))},$$

then we have

$$f\left(\frac{1}{\overline{z}}\right) = e^{-e^{i\lambda}t} f(e^{i\theta}), \tag{5.7}$$

and hence, equation (5.6) is expressed by

$$\Phi(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}}, \\ \dfrac{(f(e^{i\theta}))^2}{f(1/\bar{z})}, & z \in \mathbb{C} \backslash \overline{\mathbb{D}}, \end{cases} \tag{5.8}$$

where $f(e^{i\theta})$ is uniquely determined by the equation

$$\arg_\lambda f(1/\overline{z}) = \arg_\lambda f(e^{i\theta}),$$

which is deduced by equation (5.7), where $\arg_\lambda$ represents the λ-argument, for details, see **[24]**. Function (5.8) is the same as that given in **[32]**.

5.2. Results. Several conditions are known under which $f \in \mathcal{R}$ has a quasiconformal extension to the complex plane. One remarkable result is due to Chuaqui and Gevirtz **[9]**, who gave necessary and sufficient conditions under which $f(\mathbb{D})$ can be a quasidisk by introducing the notion of *property M*. Comparing the two, our results provide quantitative estimates for the dilatations of quasiconformal extensions.

A Loewner chain for class $\mathcal{R}$ is simply given by

$$f_t(z) := f(z) + tz.$$

In fact, a straightforward calculation shows that

$$\frac{1}{p(z,t)} = \frac{\partial_t f_t(z)}{z\partial_z f_t(z)} = f'(z) + t.$$

If we assume that $|\arg f'(z)| \leq \gamma\pi/2$ for a fixed constant $\gamma \in [0,1)$, then it follows from Corollary 5.3 that f has a $\sin(\gamma\pi/2)$-quasiconformal extension to $\mathbb{C}$. Consequently, we obtain the following.

Theorem 5.4. *Let $f \in \mathcal{A}$ and $\gamma \in [0,1)$. If $|\arg f'(z)| \leq \gamma\pi/2$ for all $z \in \mathbb{D}$, then f belongs to $\mathcal{R}$ and has a $\sin(\gamma\pi/2)$-quasiconformal extension to $\mathbb{C}$.*

As was shown above, it does not seem possible to obtain an explicit quasiconformal extension by equation (5.3) in this case because there is no feasible means of finding a Loewner chain h_t whose Herglotz function is given by equation (5.4) with $q(z,t) = f'(z)+t$ and its inverse function $h_t^{-1}(z)$ to define g_t by equation (5.5).

Theorem 5.5. *Let $f \in \mathcal{R}$. Let $\beta_0 \approx 0.580356$ and $\alpha_0 = 1/\beta_0 \approx 1.723078$ be constants which are given in subsection 3.2, and let $\alpha \in (-\alpha_0, \alpha_0)$ be fixed. Then, $J_\alpha[f]$ has a $\sin(|\alpha|\beta_0\pi/2)$-quasiconformal extension to $\mathbb{C}$.*

Proof. In subsection 3.2, we have shown that, if $f \in \mathcal{R}$, then $\{f(z)/z : z \in \mathbb{D}\}$ lies in the sector domain

$$\Delta_{\beta_0} = \left\{ w : |\arg w| < \pi \frac{\beta_0}{2} \right\}.$$

This implies that

$$\left(\frac{f(z)}{z} \right)^{\alpha} = J_\alpha[f]'(z) \in \Delta_{\alpha\beta_0} \quad \text{for all } z \in \mathbb{D},$$

and therefore,

$$|\arg J_\alpha[f]'(z)| \leq \frac{|\alpha|\beta_0\pi}{2}.$$

Hence, Theorem 5.4 yields our assertion. □

Theorem 5.6. *Let $f \in \mathcal{R}$. Then, for a fixed $\alpha \in (-1, 1)$, $I_\alpha[f]$ has a $\sin(|\alpha|\pi/2)$-quasiconformal extension to $\mathbb{C}$. However, if α lies on $(-\infty, -1] \cup [1, \infty)$, then there exists a function $g \in \mathcal{R}$ such that $I_\alpha[g]$ does not have any quasiconformal extension.*

Proof. Let $f \in \mathcal{R}$. Since $|\arg I_\alpha[f]'| < |\alpha|\pi/2$, applying Theorem 5.4, we conclude that f has a $\sin(|\alpha|\pi/2)$-quasiconformal extension to $\mathbb{C}$.

As for the second statement of the theorem, by Theorem 4.2 it suffices to consider the case where either $\alpha = 1$ or $\alpha = -1$. If $\alpha = 1$, then our statement easily follows because

$$I_1[\phi] = \phi(z) = -z - 2\log(1 - z) \in \mathcal{R}$$

maps $\mathbb{D}$ onto a domain which is not a quasicircle. It can be shown similarly in the case when $\alpha = -1$ with the counterexample

$$\psi(z) = -z + 2\log(1 + z).$$

□

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