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BLASCHKE'S ROLLING BALL PROPERTY AND CONFORMAL METRIC RATIOS

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ABSTRACT. We characterize the closed sets in Euclidean space that satisfy a two-sided rolling ball property. As an application we show that certain conformal metric ratios have boundary value 1.

1. Introduction. A non-empty closed subset Σ of Euclidean space R^n has the (two-sided) *rolling ball property* if there exists an $R>0$ such that for each point $\xi\in\Sigma$ there are two open balls, each of radius R, that lie in different components of $\mathsf{R}^n\setminus\Sigma$ and whose boundary spheres are tangent at the point ξ. See Figure 3.

Our interest in the rolling ball property arose via an application to certain families of conformal metrics. But, first we establish the following characterization of such sets.

Theorem A. *Let $\Sigma\subset\mathsf{R}^n$ be non-empty and closed, with $n\ge 2$. Then Σ has the* (*two-sided*) *rolling ball property with parameter $R>0$ if and only if each of the following holds*:

(1) *Σ is an orientable $(n-1)$-dimensional $\mathcal{C}^1$ smooth embedded submanifold of R^n.*
(2) *For each pair of distinct components Γ_1 and Γ_2 of Σ,* $\operatorname{dist}(\Gamma_1,\Gamma_2)\ge 2R$.
(3) *There is a globally defined unit normal vector field $\Sigma\xrightarrow{\mathbf{n}}\mathsf{S}^{n-1}$ such that for all points $\xi,\zeta\in\Sigma$,*

$$|\mathbf{n}(\xi)-\mathbf{n}(\zeta)|\le\frac{1}{R}\,|\xi-\zeta|.$$

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This is a modest generalization of a result due to Walther, who used Blaschke's rolling ball property to study various problems from mathematical morphology, image analysis and smoothing; see **[28,** Theorem 1], in particular, for many good references.

Let $\mathcal{O}$ be a translation invariant family of domains (open connected subsets) $\Omega \subsetneq \mathsf{R}^n$. Let $\mathcal{M} = \{\rho_\Omega(x)|dx|\}_{\Omega\in\mathcal{O}}$ be a class of conformal metrics each defined on domain Ω in $\mathcal{O}$. Assume that $\mathcal{M}$ is monotone and translation invariant. See subsection 2.4 for precise definitions of this terminology as well as that stated below.

Theorem B. *Fix $n \geq 2$ and $R > 0$. Let $\mathcal{M} = \{\rho_\Omega(x)|dx|\}_{\Omega\in\mathcal{O}}$ be a monotone and translation invariant class of conformal metrics as described above. Let $\mathcal{M}_R := \{\rho_\Omega(x)|dx|\}_{\Omega\in\mathcal{O}_R}$ where $\mathcal{O}_R$ is the subfamily of all domains Ω in $\mathcal{O}$ that satisfy the* (*two-sided*) *rolling ball property with parameter R.*

Suppose that both B^n_R and A^n_R are domains in $\mathcal{O}$ and that each of the associated conformal metrics $\rho_{\mathsf{B}^n_R}(x)|dx|$ and $\rho_{\mathsf{A}^n_R}(x)|dx|$ is asymptotically quasihyperbolic. Then $\mathcal{M}_R$ is asymptotically quasihyperbolic, that is, for all domains $\Omega \in \mathcal{O}_R$,

$$\lim_{\substack{\operatorname{dist}(x,\partial\Omega)\to 0\\ x\in\Omega}} \operatorname{dist}(x,\partial\Omega)\rho_\Omega(x) = 1,$$

and this uniform limit holds uniformly with respect to all $\Omega \in \mathcal{O}_R$.

A conformal metric $\rho(x)|dx|$ on Ω is asymptotically quasihyperbolic if the above uniform metric ratio limit holds, meaning that, sufficiently near $\partial\Omega$, $\rho(x)|dx|$ is "asymptotic" to the quasihyperbolic metric $\operatorname{dist}(x,\partial\Omega)^{-1}|dx|$. See (2.2).

The rolling ball property is closely related to the notion of the *reach* of a set which is defined, for a non-empty closed set $\Sigma \subset \mathsf{R}^n$, by

$$\begin{aligned}\operatorname{reach}(\Sigma) := \sup\{r > 0 \mid {}&\text{for all } x \in \mathsf{R}^n,\ \operatorname{dist}(x,\Sigma) < r \text{ implies}\\ &\text{there exists } \xi \in \Sigma \text{ such that } |x-\xi| = \operatorname{dist}(x,\Sigma)\};\end{aligned}$$

when there are no such $r > 0$, we set $\operatorname{reach}(\Sigma) := 0$. Federer introduced this terminology and established fundamental properties of sets with positive reach, see **[7]**, and also **[27]** and its many references. For a non-empty closed $\Sigma \subset \mathsf{R}^n$, we define

$$\text{(1.1)} \quad \operatorname{rbp}(\Sigma) := \sup\{R > 0 \mid \Sigma \text{ has the (two-sided) rolling ball property with parameter } R\};$$

when there are no such $R > 0$, we set $\operatorname{rbp}(\Sigma) := 0$. Clearly, there are sets with positive reach which do not enjoy the rolling ball property; indeed, each closed convex set has infinite reach. Also, there are, e.g., compact $\mathcal{C}^{2-\varepsilon}$ curves in R^2 that do not have positive reach, see **[17]**. However, we do have the following folklore result.

Theorem C. *Let $n \geq 2$. Suppose that Σ is a non-empty closed $(n-1)$-dimensional $\mathcal{C}^1$ smooth embedded submanifold of R^n. Then $\operatorname{reach}(\Sigma) = \operatorname{rbp}(\Sigma)$.*

As a corollary of Theorems A and C, we see that, for such Σ the following are equivalent:

(a) $\operatorname{reach}(\Sigma) > 0$.
(b) $\operatorname{rbp}(\Sigma) > 0$.
(c) Σ has a globally defined Lipschitz continuous unit normal vector field.

For compact Σ, the equivalence of (a), (b) and (c) above, in addition to numerous similar conditions, was first observed by Lucas **[22]**, see also **[7]**. It is noteworthy that, in this setting, the unit normal for Σ is a.e. differentiable; hence, the Weingarten map (also known as the *shape operator*) for Σ is defined at a.e. point of Σ, and so we can compute principal curvatures at a.e. point of Σ. In particular, the largest of these principal curvatures is bounded everywhere above by $1/\operatorname{rbp}(\Sigma)$.

There is an analog of Theorem C for non-closed submanifolds Σ, provided we adjust things appropriately; in this case, for the rolling ball property, we drop the requirement that the two balls lie in different components of $\mathsf{R}^n \setminus \Sigma$, and then we have $\operatorname{reach}(\overline{\Sigma}) = \operatorname{rbp}(\Sigma)$.

Theorem A raises the natural question: What is the 'size' of the singular set? Here the *singular set* for a $\mathcal{C}^1$ hypersurface is the set of points where the associated unit normal vector field is *not* differentiable. We provide the following answer to this question for the case where 'size' means Hausdorff dimension.

Example. *For each $n \geq 2$, there is a hypersurface in R^n that has the rolling ball property and whose singular set has Hausdorff dimension $n-1$.*

We prove Theorems A, B, and C in Section 3. Section 2 contains standard information including basic definitions and notation. In subsection 2.4 we list classes of conformal metrics to which Theorem B applies. In subsection 2.5, we give simple examples of sets that enjoy the rolling ball property and we construct the above Example in subsection 2.6. See subsection 3.5 for an intriguing question.

2. Preliminaries. Our notation is relatively standard. We write $|x-y|$ for the Euclidean distance between points x, y in Euclidean space R^n. Then, $\mathsf{B}^n(x;r) := \{y : |x-y| < r\}$ and $\mathsf{S}^{n-1}(x;r) := \{y : |x-y| = r\}$ are the open ball and the sphere of radius r centered at the point x. We set $\mathsf{B}^n_r := \mathsf{B}^n(0;r)$ and $\mathsf{A}^n_r := \mathsf{R}^n \setminus \overline{\mathsf{B}}^n_r$; so, $\mathsf{B}^n := \mathsf{B}^n_1$ is the open unit ball.

The standard unit basis vectors are $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and the Euclidean inner product is written as $x \cdot y$; thus, e.g.,

$$|x \pm y|^2 = |x|^2 \pm 2\, x \cdot y + |y|^2$$

which reduces to $|u \pm v|^2 = 2 \pm 2\, u \cdot v$ when u and v are unit vectors. The *angle* $\theta \in [0, \pi]$ between two non-zero vectors x and y is defined by $\cos\theta = x \cdot y/(|x||y|)$.

2.1. Grassmannians. We let $\mathsf{G}(n,k)$ denote the set of all k-planes in R^n, i.e., $\mathsf{G}(n,k)$ is the set of all k-dimensional vector subspaces of R^n. Then the Grassman space is

$$\mathcal{G}(n) := \bigcup_{k=0}^{n} \mathsf{G}(n,k).$$

By identifying each $V \in \mathcal{G}(n)$ with the orthogonal projection $\mathsf{R}^n \xrightarrow{P_V} \mathsf{R}^n$ onto V, we can define a distance function on $\mathcal{G}(n)$ via

$$\mathrm{d}_{\mathsf{G}}(V,W) := \|P_V - P_W\| := \sup_{x \in \mathsf{S}^{n-1}} |P_V(x) - P_W(x)|\,.$$

In fact, $(\mathcal{G}(n), \mathrm{d}_{\mathsf{G}})$ is a compact metric space. Moreover, for all $V, W \in \mathcal{G}(n)$:

$$\mathrm{d}_{\mathsf{G}}(V, W) = \mathrm{d}_{\mathsf{G}}(V^{\perp}, W^{\perp}) \leq 1 \tag{2.1a}$$

with $\mathrm{d}_{\mathsf{G}}(V, W) = 1$ holding if $\dim(V) \neq \dim(W)$;

$$\mathrm{d}_{\mathsf{G}}(V, W) = \max_{v \in V \cap \mathsf{S}^{n-1}} \mathrm{dist}(v, W) \tag{2.1b}$$

provided $V \neq \{0\} \neq W$; and, if $\dim(V) = 1 = \dim(W)$ or $\dim(V) = n - 1 = \dim(W)$,

$$\mathrm{d}_{\mathsf{G}}(V, W) = \sin \alpha \tag{2.1c}$$

where $\alpha \in [0, \pi/2]$ is the angle between V and W.

In Hilbert space theory, $\mathrm{d}_{\mathsf{G}}(V, W)$ is called the *aperture* of V and W, see [**1,** pages 69–71] and [**16,** pages 56–57, Theorem I-6.34].

From (2.1a), we see that the components of the space $\mathcal{G}(n)$ are precisely the sets $\mathsf{G}(n, k)$; if $\dim(V) \neq \dim(W)$, then the open balls $\mathsf{B}_{\mathsf{G}}(V; 1)$ and $\mathsf{B}_{\mathsf{G}}(W; 1)$ are disjoint, and in fact, $\mathsf{B}_{\mathsf{G}}(V; 1) = \mathsf{G}(n, \dim V)$. Similarly, for any $s \in (0, 1)$,

$$\mathsf{B}_{\mathsf{G}}(V; s) = \{W \in \mathsf{G}(n, \dim V) \mid \mathrm{d}_{\mathsf{G}}(W, V) < s\}.$$

2.2. Cones. It is convenient to introduce the following notation. Given a point $a \in \mathsf{R}^n$, a k-plane $V \in \mathsf{G}(n, k)$ and $s \in (0, 1)$, we set

$$\mathsf{X}(V, s) := \{x \in \mathsf{R}^n \mid \mathrm{dist}(x, V) < s|x|\}$$

and then

$$\mathsf{X}(a, V, s) := a + \mathsf{X}(V, s) = \{x \in \mathsf{R}^n \mid \mathrm{dist}(x - a, V) < s|x - a|\}.$$

Recalling that $\mathrm{dist}(x, V) = |x - P_V(x)| = |P_{V^{\perp}}(x)|$ (where $P_V(x)$ is the orthogonal projection of x onto V), we see that $x \in \mathsf{X}(V, s)$ if and only if the angle α between x and $P_V(x)$ satisfies $\sin \alpha < s$. See Figure 1.

Notice that, when $\dim(V) = 1$, i.e., when V is a line in R^n, $\mathsf{X}(V, s)$ is a doubly infinite right-circular cone with apex at the origin, axis V, and aperture $\alpha := \arcsin(s)$. With this in mind, we call $\mathsf{X}(a, V, s)$ a *cone* at a. Note, too, that we always have

$$\mathsf{X}(V,s)=\mathsf{R}^n\setminus\overline{\mathsf{X}}(V^\perp,\sqrt{1-s^2}),$$

in particular, we can also "see" the shape of the cone $\mathsf{X}(V,s)$ when V is a hyperplane (i.e., when $\dim(V)=n-1$).

In addition, we point out that $\mathsf{X}(V,s)=\bigcup_{W\in\mathsf{B}_{\mathsf{G}}(V;s)}W\setminus\{0\}$.

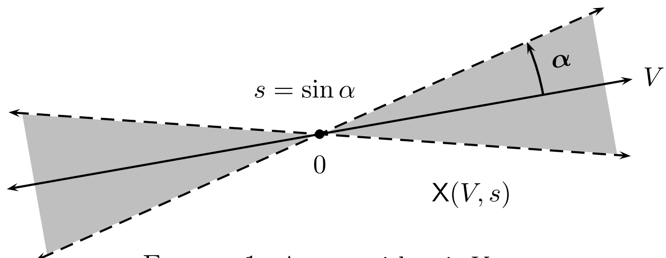

FIGURE 1. A *cone* with axis V.

2.3. Smooth hypersurfaces. We refer to [**19**] for the basic theory of smooth manifolds and embedded submanifolds. For later use, we record the following well-known information, see [**4,** page 48, Theorem 2.1.2 (iii)] or [**19,** Chapter 5].

Fact 2.1. *Suppose that, for each point $\xi\in\Sigma\subset\mathsf{R}^n$, there is an $r>0$ such that, after suitable translation and rotation, $\Sigma\cap\mathsf{B}^n(\xi;r)$ is the graph of a $\mathcal{C}^1$ function $\mathsf{B}^{n-1}_r\to\mathsf{R}$. Then Σ is an $(n-1)$-dimensional $\mathcal{C}^1$ smooth embedded submanifold of R^n.*

We also require the following folklore information concerning *hypersurfaces* in R^n; these are the *connected* submanifolds of dimension $n-1$. The second assertion below is the so called Jordan-Brouwer separation theorem (for smooth hypersurfaces), see [**2, 20, 21, 26, 29**].

Fact 2.2. *Let Σ be a non-empty closed subspace of R^n that is a $\mathcal{C}^1$ smooth embedded hypersurface. Then Σ is orientable and $\mathsf{R}^n\setminus\Sigma$ has exactly two components, each of which has Σ as its topological boundary.*

That such a Σ is orientable means, in particular, that there is a globally defined unit normal vector field along Σ. See, for example, [**19,** Chapter 13].

2.4. Conformal metrics. A conformal metric on a domain $\Omega \subset \mathsf{R}^n$ has the form $\rho(x)|dx|$ where ρ is some positive Borel function defined on Ω (with the property that the line element $\rho(x)|dx|$ integrates to an honest distance function, e.g., this holds if ρ is locally bounded away from 0 and from ∞). The ratio $\rho(x)|dx|/\sigma(x)|dx|$ of two conformal metrics, both defined on some Ω, is a well-defined positive function on Ω. We write $\rho \le C\,\sigma$ to indicate that this metric ratio is bounded above by C.

We recall that when $\Omega \xrightarrow{\tau} \Omega'$ is (the restriction of) a Möbius transformation, so a conformal map, then the *pullback* of a conformal metric $\sigma(y)|dy|$ on $\Omega' = T(\Omega)$ is the conformal metric $\rho(x)|dx|$ on Ω defined by

$$\rho(x)|dx| = \tau^*\left[\sigma(y)|dy|\right] := \sigma(\tau(x))|\tau'(x)||dx|.$$

Here, we consider a family $\mathcal{O}$ of domains $\Omega \subsetneq \mathsf{R}^n$ and a class $\mathcal{M} = \{\rho_\Omega(x)|dx|\}_{\Omega\in\mathcal{O}}$ of conformal metrics, each defined on a domain Ω in $\mathcal{O}$. We call $\mathcal{M}$ *monotone* if, whenever $\Omega_1, \Omega_2 \in \mathcal{O}$ satisfy $\Omega_1 \subseteq \Omega_2$, $\rho_{\Omega_2} \le \rho_{\Omega_1}$. We say that $\mathcal{M}$ is *translation invariant* if $\mathcal{O}$ is translation invariant, so, for each Ω in $\mathcal{O}$, every translate $\tau(\Omega)$ of Ω belongs to $\mathcal{O}$, and, if the pullback metric $\tau^*[\rho_{\tau(\Omega)}(y)|dy|]$ equals $\rho_\Omega(x)|dx|$.

A conformal metric $\rho(x)|dx|$ on Ω is *asymptotically quasihyperbolic*, abbreviated AQH, if the metric ratio $\rho\delta$ has uniform boundary value 1, that is, if and only if

$$\rho(x)\delta(x) := \frac{\rho(x)|dx|}{\delta(x)^{-1}|dx|} \longrightarrow 1 \text{ uniformly as } \delta(x) \to 0. \tag{2.2}$$

Here $\delta(x) = \delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$ is the Euclidean distance from x to the boundary of Ω, and $\delta(x)^{-1}|dx|$ is the *quasihyperbolic metric* in Ω.

A class $\mathcal{M}$ of conformal metrics is *asymptotically quasihyperbolic* if and only if the metrics in $\mathcal{M}$ are *uniformly* AQH, that is, if and only if the limit in equation (2.2) holds uniformly for each metric $\rho(x)|dx|$ in $\mathcal{M}$, and uniformly with respect to $\mathcal{M}$.

As an example, recall that the *hyperbolic metric* on the ball B^n_r is given by $\lambda(x)|dx| = \lambda_{\mathsf{B}^n_r}(x)|dx| = 2r/r^2 - |x|^2\,|dx|$. It is easy to check that the class of all hyperbolic metrics on all balls $\mathsf{B}^n(a;r)$ in R^n is AQH, and similarly for the hyperbolic metrics on the complements of the closures of such balls.

We mention that, in Theorem B, we can relax the two-sided rolling ball property hypothesis and still obtain pointwise limits. For example, as long as there exist an interior ball and an exterior ball whose boundary spheres are tangent at some point $\xi \in \partial\Omega$, then $\lim_{x\to\xi} \delta(x)\rho(x) = 1$, see [**23,** Proposition 4] where Minda obtained a similar result for the Aumann-Carathéodory rigidity constant for plane domains.

Now we list examples of classes of metrics to which Theorem B applies. Trivially, the collection of all quasihyperbolic metrics is monotone, translation invariant and asymptotically quasihyperbolic. The collection of all hyperbolic metrics, on appropriate domains, also has these properties. In R^2, each domain with at least two boundary points supports a maximal constant curvature -1 metric, called its *Poincaré hyperbolic metric*. However, in R^n with $n \geq 3$, we only have the *real hyperbolic metrics* defined on open balls, open half-spaces and complements of closed balls.

Other examples of families of conformal metrics that are monotone, translation invariant and asymptotically quasihyperbolic include the so-called *Ferrand metric* (introduced in [**8**] and studied in [**11, 14**]), the *Kulkarni-Pinkhall-Thurston metric* (introduced in [**18**] and studied in [**5, 11, 12, 13**]), and the *inner Apollonian metric* (see [**3, 9**]). For metrics on plane domains we also have the *Harnack metric* defined via positive harmonic functions, as well as a similar metric defined by bounded harmonic functions, see [**10, 15**], the *Hahn metric*, see [**24**], and the so-called capacity metric, see [**25**].

It is perhaps worthwhile to mention that *all* of the above metric examples are asymptotically quasihyperbolic on balls because in any ball they are equal to the hyperbolic metric.

2.5. Examples. It is easy to see that if Σ is any union of appropriately positioned hyperplanes, spheres of radius R, cylinders of radius R, tori of "radius" R, etc., then Σ will have the rolling ball property with parameter R.

By using suitably rotated and translated copies of the arc $\{(x,x^2) \mid x \in [0,1]\}$, we can construct a "broken-arc-parabola" P that is a $\mathcal{C}^1$ curve in R^2 that (passes through the points $\{(n,n) \mid n \in \mathsf{Z}\}$ and) has the rolling ball property with parameter $R = 1/2$, but that is not $\mathcal{C}^2$. See Figure 2. Then, $\mathsf{P} \times \mathsf{R}^{n-1}$ is a $\mathcal{C}^1$ hypersurface in R^n that has the rolling ball property (in fact, it satisfies a rolling cylinder property) but is not $\mathcal{C}^2$.

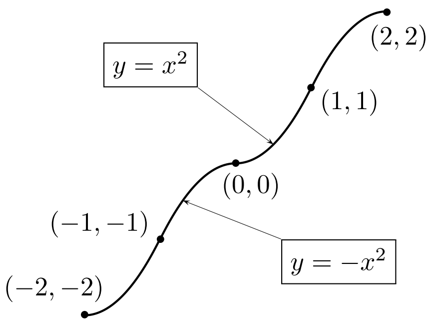

FIGURE 2. A "broken-arc-parabola" with the rolling ball property.

Now we turn to the example stated at the end of the Introduction. Our construction begins with Lemma 2.3; as this is surely folk-lore amongst the experts, we just sketch its proof.

2.6. Proof for the introduction Example. Appealing to Lemma 2.3 to obtain E and g as described there, we proceed as follows. Define

$$\mathsf{R} \xrightarrow{f} \mathsf{R} \quad \text{by} \quad f(x) := \int_0^x g(t)\,dt.$$

Then, f is $\mathcal{C}^1$ with $f' = g$ everywhere; in fact, since g is Lipschitz, f is $\mathcal{C}^{1,1}$. Let Γ be the graph of f,

$$\Gamma := \{(x, f(x)) \mid x \in \mathsf{R}\}.$$

Then, Γ is a curve in R^2 that has the rolling ball property. For $n > 2$, put $\Sigma := \Gamma \times \mathsf{R}^{n-2}$. Then, Σ is a hypersurface in R^n that has the rolling ball property (even the rolling cylinder property).

The singular set for Γ is $\Psi_\Gamma := \{(x, f(x)) \mid x \in E\}$. The projection $\mathsf{R}^2 \xrightarrow{P} \mathsf{R}$ onto the first factor is Lipschitz, so

$$1 = \dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(P(\Psi_\Gamma)) \le \dim_{\mathcal{H}}(\Psi_\Gamma) \le 1.$$

The singular set for Σ is $\Psi_\Sigma := \Psi_\Gamma \times \mathsf{R}^{n-2}$, and therefore,

$$\dim_{\mathcal{H}}(\Psi_\Sigma) = \dim_{\mathcal{H}}(\Psi_\Gamma) + (n-2) = n - 1. \qquad \square$$

Lemma 2.3. *There is an $\mathcal{F}_\sigma$ set $E \subset \mathsf{R}$ and a Lipschitz function $\mathsf{R} \xrightarrow{g} \mathsf{R}$ such that E has Hausdorff dimension $\dim_{\mathcal{H}}(E) = 1$ and such that E is precisely the set of points where g fails to be differentiable.*

There are more precise results known than Lemma 2.3, but an easy proof follows at once from the next lemma.

Lemma 2.4. *Let $\alpha \in (0,1)$ be given. There is a Lipschitz function $[0,1] \xrightarrow{g} [0,1]$ and a 'uniform' Cantor dust $C \subset [0,1]$ such that C has positive finite α-dimensional Hausdorff measure (so, in particular, $\dim_{\mathcal{H}}(C) = \alpha$) and such that the non-differentiability set for g is precisely C, together with the countably many midpoints of each component of $[0,1] \setminus C$.*

To prove Lemma 2.4, we build C in the standard way; e.g., follow the construction given in [**6,** page 57, Example 4.4] using $m = 2$ children at each step. Then $g(x) := \operatorname{dist}(x, C)$ is Lipschitz even with $|g(x) - g(y)| \le |x - y|$ and $\|g\|_\infty \le 1$. Evidently, g is differentiable on each component of $[0,1] \setminus C$, except for the midpoints. It is not too difficult to show that g fails to be differentiable at every point of C.

Now, take any sequence $(\alpha_n)_1^\infty$ in $(0,1)$ with $\alpha_n \to 1$. Let g_n and C_n be the Lipschitz functions and Cantor dusts promised by Lemma 2.4. Let $E_n \subset [n-1, n]$ be the translation of C_n by $n-1$, and define $g(x) := g_n(x - n + 1)$ for $x \in [n-1, n]$ and $g(x) := 0$, elsewhere. Then, $E := \bigcup E_n$ has Hausdorff dimension 1 and is the non-differentiability set for g (up to a countable set).

3. Proofs. Here we prove Theorem A and then Theorem B. First, we establish the necessity of conditions (1), (2), (3), and then their sufficiency.

3.1. Proof of necessity in Theorem A. Let $\Sigma \subset \mathsf{R}^n$ be non-empty and closed, with $n \ge 2$, and assume that Σ has the (two-sided) rolling ball property with parameter $R > 0$. Recall that this means that, for each point $\xi \in \Sigma$, there are two open balls $\mathsf{B}_\xi^\pm$, each of radius R, that lie in different components of $\mathsf{R}^n \setminus \Sigma$ and whose boundary spheres are tangent at the point ξ.

In particular, there is a map $\xi \mapsto T_\xi$ from Σ to $\mathsf{G}(n, n-1)$ with the property that the affine hyperplane $\xi + T_\xi$ is tangent to each of $\mathsf{B}^\pm_\xi$ at ξ. We let $N_\xi := (T_\xi)^\perp$, so $\xi + N_\xi$ is the line normal to both of the $(n-1)$-dimensional spheres $\partial\mathsf{B}^\pm_\xi$ at ξ, that is, $\xi + N_\xi$ is the line that passes through ξ and both of the centers of $\mathsf{B}^\pm_\xi$. (See Figure 3.) We also write $\mathbf{n}(\xi)$ to denote one of the (two) unit vectors that span N_ξ; we make this more precise below. In terms of this notation, the balls $\mathsf{B}^\pm_\xi$ are simply $\mathsf{B}^n(\xi \pm R\,\mathbf{n}(\xi); R)$.

The key steps in our proof are as follows.

(1) We establish continuity of the map $\xi \mapsto T_\xi$.
(2) We verify that, locally, Σ is the graph of a Lipschitz map.
(3) We demonstrate that this Lipschitz map is in fact $\mathcal{C}^1$.
(4) We corroborate all remaining assertions.

Throughout our proof, we assume a "standard setting" obtained as follows. We start with a given fixed point ζ in Σ. Then, by translating and rotating as necessary, i.e., by applying a rigid motion of R^n, we can assume that $\zeta = 0$ and that $\mathbf{n}(\zeta) = \mathbf{e}_n$. The latter assumption means that N_ζ is the x_n-axis and that $T_\zeta = \mathsf{R}^{n-1} \times \{0\} \subset \mathsf{R}^n$. In this setting, the balls $\mathsf{B}^\pm_\zeta$ are just $\mathsf{B}^n(\pm R\,\mathbf{e}_n; R)$.

We begin by establishing the following *ball-cone containment conditions.* (See Figure 4.) For all points $\xi \in \Sigma$ and for each $\varepsilon \in (0, 2)$, we

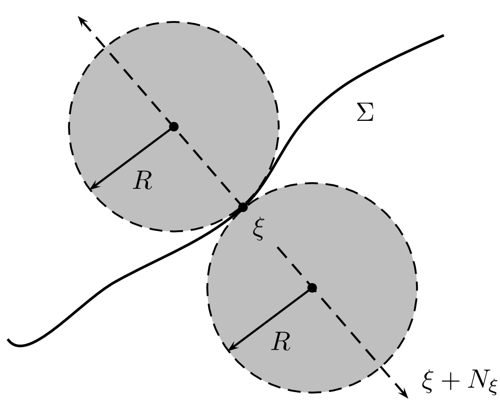

FIGURE 3. Two sided rolling ball property for Σ.

have

$$\Sigma \cap \mathsf{B}^n(\xi;\varepsilon R) \cap \mathsf{X}(\xi, N_\xi, \sqrt{1-(\varepsilon/2)^2}) = \emptyset \tag{3.1a}$$

and

$$\Sigma \cap \mathsf{B}^n(\xi;\varepsilon R) \subset \{\xi\} \cup \mathsf{X}(\xi, T_\xi, \varepsilon/2). \tag{3.1b}$$

To verify the above, let $\zeta \in \Sigma$ and $\varepsilon \in (0,2)$ be given. Assume the "standard setting" where $\zeta = 0$ and $\mathbf{n}(\zeta) = \mathbf{e}_n$.

We use spherical polar coordinates (ρ, φ, θ) for points x of R^n; so here $\rho = |x| \ge 0$, $\varphi \in [0,\pi]$ is the angle between x and $\mathbf{e}_n$, and $\theta \in \mathsf{S}^{n-2}$. As illustrated in Figure 4, the polar equations for the spheres $|x| = \varepsilon R$ and $\partial \mathsf{B}^+_\zeta = \mathsf{S}^{n-1}(R\,\mathbf{e}_n; R)$ are, respectively,

$$\rho = \varepsilon R \quad \text{and} \quad \rho = 2R\cos\varphi.$$

The points of intersection of these two spheres satisfy $\cos\varphi = \varepsilon/2$, so $\sin\varphi = \sqrt{1-(\varepsilon/2)^2}$. It follows that

$$\mathsf{B}^n(\zeta;\varepsilon R) \cap \mathsf{X}(\zeta, N_\zeta, \sqrt{1-(\varepsilon/2)^2}) \subset \mathsf{B}^+_\zeta \cup \mathsf{B}^-_\zeta.$$

Since the balls $\mathsf{B}^\pm_\zeta$ lie in $\mathsf{R}^n \setminus \Sigma$, (3.1a) holds. Since $\sin((\pi/2) - \varphi) = \cos\varphi$, (3.1b) follows.

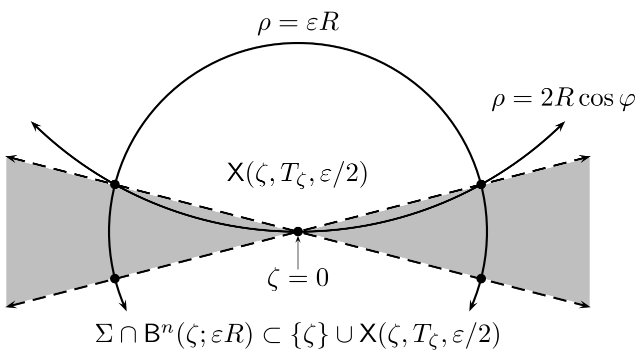

FIGURE 4. The ball-cone containment condition.

3.1.1. The map $\xi \mapsto T_\xi$ is continuous. We show that, for each $\varepsilon \in (0, 2-\sqrt{2})$ and any points $\xi, \zeta \in \Sigma$,

$$|\xi - \zeta| < \varepsilon R \implies \mathrm{d}_{\mathsf{G}}(T_\xi, T_\zeta) < 2\varepsilon. \tag{3.2}$$

To establish this, let $\zeta \in \Sigma$ and $\varepsilon \in (0, 2-\sqrt{2})$ be given. Assume the "standard setting" where $\zeta = 0$ and $\mathbf{n}(\zeta) = \mathbf{e}_n$. Let $\xi \in \Sigma \cap \mathsf{B}^n(\zeta; \varepsilon R)$. By symmetry, we may assume that $\xi = s\,\mathbf{e}_1 + t\,\mathbf{e}_n$, where $s > 0$ and $t \geq 0$.

We claim that the unit normal $\mathbf{n}(\xi)$ must satisfy $\mathbf{n}(\xi) \cdot \mathbf{e}_n \neq 0$. In fact, we show that, if $\mathbf{n}(\xi) \cdot \mathbf{e}_n = 0$, then $\mathsf{B}^+_\zeta \cap \mathsf{B}^+_\xi \neq \emptyset \neq \mathsf{B}^+_\zeta \cap \mathsf{B}^-_\xi$, which would contradict the hypothesis that the open balls $\mathsf{B}^\pm_\xi$ lie in different components of $\mathsf{R}^n \setminus \Sigma$. To this end, let us write down what it means for these balls to overlap. Evidently, two open balls both of radius R have non-empty intersection if and only if their centers are within distance $2R$ of each other. Applying this observation to the balls B^+_ξ and B^+_ζ, we obtain the inequality

$$\begin{aligned} 4R^2 &> |(\xi + R\,\mathbf{n}(\xi)) - (\zeta + R\,\mathbf{n}(\zeta))|^2 = |\xi + R(\mathbf{n}(\xi) - \mathbf{n}(\zeta))|^2 \\ &= |\xi|^2 + 2R\,\xi \cdot (\mathbf{n}(\xi) - \mathbf{e}_n) + R^2|\mathbf{n}(\xi) - \mathbf{e}_n|^2 \\ &= |\xi|^2 + 2R\,\xi \cdot (\mathbf{n}(\xi) - \mathbf{e}_n) + R^2\,(2 - 2\,\mathbf{n}(\xi) \cdot \mathbf{e}_n)\,. \end{aligned}$$

Thus, we deduce that

$$\mathsf{B}^+_\zeta \cap \mathsf{B}^+_\xi \neq \emptyset \iff |\xi|^2 + 2R\,\xi \cdot (\mathbf{n}(\xi) - \mathbf{e}_n) - 2R^2\,\mathbf{n}(\xi) \cdot \mathbf{e}_n < 2R^2, \tag{3.3a}$$

and by similar reasoning,

$$\mathsf{B}^+_\zeta \cap \mathsf{B}^-_\xi \neq \emptyset \iff |\xi|^2 - 2R\,\xi \cdot (\mathbf{n}(\xi) + \mathbf{e}_n) + 2R^2\,\mathbf{n}(\xi) \cdot \mathbf{e}_n < 2R^2. \tag{3.3b}$$

Now suppose that $\mathbf{n}(\xi) \cdot \mathbf{e}_n = 0$. Then, we also have

$$\xi \cdot (\mathbf{n}(\xi) \pm \mathbf{e}_n) = (s\,\mathbf{e}_1 + t\,\mathbf{e}_n) \cdot (\mathbf{n}(\xi) \pm \mathbf{e}_n) = s\,\mathbf{e}_1 \cdot \mathbf{n}(\xi) \pm t.$$

Thus, under the assumption that $\mathbf{n}(\xi) \cdot \mathbf{e}_n = 0$, (3.3a) and (3.3b) become

$$\mathsf{B}^+_\zeta \cap \mathsf{B}^+_\xi \neq \emptyset \iff |\xi|^2 + 2R\,(s\,\mathbf{e}_1 \cdot \mathbf{n}(\xi) - t) < 2R^2, \tag{3.4a}$$

and

$$\text{(3.4b)} \qquad \mathsf{B}_\zeta^+ \cap \mathsf{B}_\xi^- \neq \emptyset \iff |\xi|^2 - 2R\,(s\,\mathbf{e}_1 \cdot \mathbf{n}(\xi) + t) < 2R^2.$$

Since $|\mathbf{e}_1 \cdot \mathbf{n}(\xi)| \le 1$, the left-hand-sides of the inequalities on the right-hand sides of (3.4a) and (3.4b) are both at most

$$|\xi|^2 + 2R(s+t) = (s+R)^2 + (t+R)^2 - 2R^2.$$

Since $(s,t) \in \mathsf{B}^2(0;\varepsilon R) \subset \mathsf{B}^2((-R,-R);2R)$, $(s+R)^2+(t+R)^2 < 4R^2$; thus, both inequalities on the right-hand sides of (3.4a) and (3.4b) hold. Therefore, $\mathsf{B}_\zeta^+ \cap \mathsf{B}_\xi^+ \neq \emptyset \neq \mathsf{B}_\zeta^+ \cap \mathsf{B}_\xi^-$, which contradicts the hypothesis that the open balls $\mathsf{B}_\xi^\pm$ lie in different components of $\mathsf{R}^n \setminus \Sigma$. It follows that $\mathbf{n}(\xi) \cdot \mathbf{e}_n \neq 0$.

Now we take $\mathbf{n}(\xi)$ to be the unit vector that spans N_ξ and satisfies $\mathbf{n}(\xi) \cdot \mathbf{e}_n > 0$. We show that the open ball $\mathsf{B}_\xi^+ := \mathsf{B}^n(\xi + R\,\mathbf{n}(\xi); R)$ has non-empty intersection with B_ζ^+. According to (3.1b), $\xi \in \mathsf{X}(\zeta, T_\zeta, \varepsilon/2)$, so

$$t = \xi \cdot \mathbf{e}_n = \operatorname{dist}(\xi, T_\zeta) < \frac{\varepsilon}{2}\,|\xi|.$$

Similarly, $\zeta \in \mathsf{X}(\xi, T_\xi, \varepsilon/2)$, so

$$|\xi \cdot \mathbf{n}(\xi)| = \operatorname{dist}(-\xi, T_\xi) < \frac{\varepsilon}{2}\,|\xi|.$$

Thus, as $|\xi - \zeta| = |\xi| < \varepsilon R$ with $\varepsilon < \sqrt{2/3}$, we obtain

$$\begin{aligned} \big||\xi|^2 + 2R\,\xi \cdot (\mathbf{n}(\xi) - \mathbf{e}_n)\,\big| &\le |\xi|^2 + 2\varepsilon R|\xi| < 3\,(\varepsilon R)^2 \\ &\le 2R^2 \le 2R^2 + 2R^2\,\mathbf{n}(\xi) \cdot \mathbf{e}_n. \end{aligned}$$

Appealing to (3.3a), we can now assert that $\mathsf{B}_\zeta^+ \cap \mathsf{B}_\xi^+ \neq \emptyset$.

Therefore, $\mathsf{B}_\zeta^+ \cap \mathsf{B}_\xi^- = \emptyset$. Using (3.3b), we see that the angle θ between $\mathbf{n}(\xi)$ and $\mathbf{e}_n$ satisfies

$$\begin{aligned} \cos\theta = \mathbf{n}(\xi) \cdot \mathbf{e}_n &\ge 1 + \frac{1}{R}\,\xi \cdot (\mathbf{n}(\xi) + \mathbf{e}_n) - \frac{|\xi|^2}{2R^2} \\ &\ge 1 - \frac{\varepsilon|\xi|}{R} - \frac{|\xi|^2}{2R^2} > 1 - 2\varepsilon^2, \end{aligned}$$

where we again have used the facts that

$$|\xi \cdot \mathbf{n}(\xi)| < \frac{\varepsilon}{2}\,|\xi|, \qquad |\xi \cdot \mathbf{e}_n| < \frac{\varepsilon}{2}\,|\xi|, \quad \text{and } |\xi| < \varepsilon R.$$

It follows that $\mathrm{d}_{\mathsf{G}}(T_\xi, T_\zeta) = \mathrm{d}_{\mathsf{G}}(N_\xi, N_\zeta) = \sin\theta < 2\varepsilon$.

Thus, the maps $\xi \mapsto T_\xi$ and $\xi \mapsto N_\xi$ are uniformly continuous, as maps from Σ to $\mathsf{G}(n, n-1)$ and to $\mathsf{G}(n, 1)$, respectively. We have also shown that locally there is a continuous unit normal vector field $\mathbf{n}$ on Σ; that is, given $\zeta \in \Sigma$, for each point $\xi \in \Sigma \cap \mathsf{B}^n(\zeta; R/2)$ we can select a unit vector $\mathbf{n}(\xi)$ that spans N_ξ and, by requiring that $\mathbf{n}(\xi)\cdot\mathbf{n}(\zeta) > 0$, we have $\mathbf{n} : \Sigma \cap \mathsf{B}^n(\zeta; R/2) \to \mathsf{S}^{n-1}$ uniformly continuous.

3.1.2. Σ is a Lipschitz graph. We demonstrate that, for each $\xi \in \Sigma$, $\Sigma \cap \mathsf{B}^n(\xi; R/10)$ is the graph of a Lipschitz function. In fact, the desired Lipschitz function is obtained from the (inverse of the) orthogonal projection onto T_ξ. So, let ζ be a given fixed point in Σ. Assume the "standard setting" where $\zeta = 0$ and $\mathbf{n}(\zeta) = \mathbf{e}_n$.

To begin, consider a fixed point $\xi \in \Sigma \cap \mathsf{B}^n(\zeta; R/9)$. Note that $\mathsf{B}^n(\zeta; R/9) \subset \mathsf{B}^n(\xi; R/4)$ and $\mathrm{d}_{\mathsf{G}}(N_\zeta, N_\xi) < 2/9$. If $L \subset \{0\} \cup \mathsf{X}(N_\zeta, 3/4)$ is a one-dimensional vector subspace, then

$$\mathrm{d}_{\mathsf{G}}(L, N_\xi) \le \mathrm{d}_{\mathsf{G}}(L, N_\zeta) + \mathrm{d}_{\mathsf{G}}(N_\zeta, N_\xi) < \frac{3}{4} + \frac{2}{9} = \frac{35}{36} < \frac{\sqrt{63}}{8},$$

and thus $\mathsf{X}(\xi, N_\zeta, 3/4) \subset \mathsf{X}(\xi, N_\xi, \sqrt{63}/8)$. Using this and (3.1a) with $\varepsilon = 1/4$, we obtain

$$\Sigma \cap \mathsf{B}^n(\zeta; R/9) \cap \mathsf{X}(\xi, N_\zeta, 3/4) = \emptyset. \tag{3.5}$$

Now let $\mathsf{R}^n \xrightarrow{P} T_\zeta = \mathsf{R}^{n-1} \times \{0\}$ be the standard orthogonal projection; so, $P(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, 0)$. We use (3.5) to show that $P|_{\Sigma\cap\mathsf{B}^n(\zeta;R/9)}$ is injective with a Lipschitz inverse. Notice that

$$|P(a) - P(b)| = |P(a-b)| < \frac{3}{4}\,|a-b|$$
$$\implies b \in \mathsf{X}(a, N_\zeta, 3/4) \text{ and } a \in \mathsf{X}(b, N_\zeta, 3/4).$$

In light of (3.5), we see that if the above inequality holds then at least one of the points a, b does not belong to $\Sigma \cap \mathsf{B}^n(\zeta; R/9)$. We conclude that if $a, b \in \Sigma \cap \mathsf{B}^n(\zeta; R/9)$ then $|P(a) - P(b)| \ge (3/4)|a-b|$, so $P|_{\Sigma\cap\mathsf{B}^n(\zeta;R/9)}$ is injective with an inverse that is (4/3)-Lipschitz.

Next, we note that $P(\Sigma \cap \mathsf{B}^n(\zeta; R/9)) \supset \mathsf{B}^{n-1}_{R/10} \times \{0\}$. To see this, observe that for any $x \in \mathsf{B}^{n-1}_{R/10}$ the vertical line $L_x := (x, 0) + N_\zeta$ through $(x, 0)$ has the property that the line segment $L_x \cap \mathsf{B}^n(\zeta; R/10)$

joins points of both balls $\mathsf{B}^{\pm}_{\zeta}$ and so must meet Σ. It now follows that the map

$$\Sigma \cap \mathsf{B}^n(\zeta; R/10) \xrightarrow{P_\zeta} \mathsf{B}^{n-1}_{R/10} \times \{0\},$$
$$\text{where } P_\zeta := P|_{\Sigma\cap\mathsf{B}^n(\zeta;R/10)},$$

is a bijection with inverse $F := P_\zeta^{-1}$ a $(4/3)$-Lipschitz map.

Evidently, F has the form $F(x,0) = (x, f(x))$ where $\mathsf{B}^{n-1}_{R/10} \xrightarrow{f} \mathsf{R}$ is $(4/3)$-Lipschitz, and $\Sigma \cap \mathsf{B}^n(\zeta; R/10)$ is the graph of f.

3.1.3. Σ is a $\mathcal{C}^1$ smooth embedded submanifold. We use Fact 2.1 to confirm that Σ is an $(n-1)$-dimensional $\mathcal{C}^1$ smooth embedded submanifold of R^n. Again, let ζ be a given fixed point in Σ, and assume the "standard setting" where $\zeta = 0$ and $\mathbf{n}(\zeta) = \mathbf{e}_n$.

It suffices to show that the map $\mathsf{B}^{n-1}_{R/10} \xrightarrow{f} \mathsf{R}$, defined in the last paragraph of subsection 3.1.2, is $\mathcal{C}^1$ smooth. In fact, we claim that for each $1 \le i < n$ and every $x \in \mathsf{B}^{n-1}_{R/10}$ f has an x_i-partial derivative given by

$$\frac{\partial f}{\partial x_i}(x) = -\frac{\mathbf{n}(\xi)\cdot\mathbf{e}_i}{\mathbf{n}(\xi)\cdot\mathbf{e}_n} \quad \text{where } \xi := (x, f(x)) = F(x,0). \tag{3.6}$$

Since $\xi \mapsto \mathbf{n}(\xi)$ is continuous (as a function of $\xi \in \Sigma \cap \mathsf{B}^n(\zeta; R/10)$), it follows that f is $\mathcal{C}^1$.

Write $\xi := (x, f(x)) = F(x,0)$ and $\eta := (y, f(y)) = F(y,0)$, where $x, y \in \mathsf{B}^{n-1}_{R/10}$ with x fixed. We first show that

$$\lim_{y\to x} \frac{(\eta-\xi)\cdot\mathbf{n}(\xi)}{|y-x|} = 0.$$

Let $0 < \varepsilon \ll 1$ be given, and put $\delta := \varepsilon R$. Since F is $(4/3)$-Lipschitz,

$$|\eta - \xi| = |F(y,0) - F(x,0)| \le \frac{4}{3}\,|y-x|.$$

Thus, $|y-x| < \delta$ implies that $|\eta - \xi| < (4/3)\delta < 2\varepsilon R$, so by (3.1b), we have

$$|y-x| < \delta \implies \eta \in \mathsf{X}(\xi, T_\xi, \varepsilon).$$

Now $\eta \in \mathsf{X}(\xi, T_\xi, \varepsilon)$ means that

$$|(\eta - \xi) \cdot \mathbf{n}(\xi)| = \operatorname{dist}(\eta - \xi, T_\xi) < \varepsilon|\eta - \xi| \leq \frac{4}{3}\,\varepsilon|y - x|.$$

Thus,

$$0 < |y - x| < \delta \implies \frac{(\eta - \xi) \cdot \mathbf{n}(\xi)}{|y - x|} < 2\varepsilon.$$

Fix $1 \leq i < n$. Consider $y = x + h\,\mathbf{e}_i$. (Here $e_i \in R^{n-1}$.) Then, $\eta - \xi = h\,\mathbf{e}_i + (f(y) - f(x))\,\mathbf{e}_n$, (here $e_i, e_n \in R^n$) so

$$(\eta - \xi) \cdot \mathbf{n}(\xi) = h\,\mathbf{n}(\xi) \cdot \mathbf{e}_i + (f(y) - f(x))\,\mathbf{n}(\xi) \cdot \mathbf{e}_n,$$

and therefore,

$$\mathbf{n}(\xi) \cdot \mathbf{e}_i + \frac{f(y) - f(x)}{h}\,\mathbf{n}(\xi) \cdot \mathbf{e}_n = \frac{(\eta - \xi) \cdot \mathbf{n}(\xi)}{h} \longrightarrow 0 \quad \text{as } h \to 0.$$

We conclude that (3.6) holds.

3.1.4. Orientability and Lipschitz normal vector field. Thanks to subsection 3.1.3 and Fact 2.2, we know that each component Γ of Σ is an embedded hypersurface in R^n; hence, Σ is orientable and the vector field $\mathbf{n}$ is globally defined and continuous. It remains to establish (2) and (3).

Let Γ_1 and Γ_2 be two components of Σ. Fix a point $\xi_1 \in \Gamma_1$. Since Γ_2 is closed, there is a point $\xi_2 \in \Gamma_2$ that satisfies $|\xi_1 - \xi_2| = \operatorname{dist}(\Gamma_1, \Gamma_2)$. A routine calculus exercise reveals that the line through the points ξ_1 and ξ_2 is normal to Γ_2 at ξ_2. It follows that the open line segment (ξ_1, ξ_2) contains a diameter of one of the balls $\mathsf{B}_\xi^\pm$, so $\operatorname{dist}(\xi_1, \Gamma_2) = |\xi_1 - \xi_2| \geq 2R$. As ξ_1 was an arbitrary point of Γ_1, $\operatorname{dist}(\Gamma_1, \Gamma_2) \geq 2R$.

Finally, let $\xi, \zeta \in \Sigma$. If these points lie in different components of Σ, then the Lipschitz inequality in (3) follows immediately from (2). Assume that ξ and ζ lie in the same component of Σ. A connectedness argument reveals that the balls B_ξ^+ and B_ζ^- lie in different components of $\mathsf{R}^n \setminus \Sigma$. Thus,

$$2R \leq |(\xi + R\,\mathbf{n}(\xi)) - (\zeta - R\,\mathbf{n}(\zeta))| = |(\xi - \zeta) + R(\mathbf{n}(\xi) + \mathbf{n}(\xi))|$$

so

$$4R^2 \leq |\xi - \zeta|^2 + 2R\,(\xi - \zeta) \cdot (\mathbf{n}(\zeta) + \mathbf{n}(\xi)) + R^2|\mathbf{n}(\zeta) + \mathbf{n}(\xi)|^2.$$

Similarly, B_ξ^- and B_ζ^+ lie in different components, and so

$$4R^2 \le |\xi-\zeta|^2 - 2R\,(\xi-\zeta)\cdot(\mathbf{n}(\zeta)+\mathbf{n}(\xi)) + R^2|\mathbf{n}(\zeta)+\mathbf{n}(\xi)|^2.$$

Therefore,

$$4R^2 \le |\xi-\zeta|^2 + R^2|\mathbf{n}(\zeta)+\mathbf{n}(\xi)|^2 = |\xi-\zeta|^2 + R^2(4-|\mathbf{n}(\zeta)-\mathbf{n}(\xi)|^2),$$

which then gives the asserted Lipschitz condition in (3). $\square$

3.2. Proof of sufficiency in Theorem A. Here we assume that (1), (2), (3) hold for some non-empty closed subset Σ of Euclidean space R^n, and we demonstrate that Σ has the (two-sided) rolling ball property with parameter R. Indeed, it suffices to check that, for each $\xi\in\Sigma$, the balls $\mathsf{B}_\xi^\pm := \mathsf{B}^n(\xi \pm R\,\mathbf{n}(\xi); R)$ have the desired properties. To confirm this, we only need to show that these balls lie in separate components of $\mathsf{R}^n\setminus\Sigma$, since all of the other conditions are known.

So, fix a point $\xi\in\Sigma$, and let Γ be the ξ-component of Σ. Condition (2) ensures that neither of the balls $\mathsf{B}_\xi^\pm$ meets $\Sigma\setminus\Gamma$. Below, we verify that $\mathsf{B}_\xi^+\cap\Gamma=\emptyset=\mathsf{B}_\xi^-\cap\Gamma$. According to the Jordan-Brouwer separation theorem (see Fact 2.2), this means that $\mathsf{B}_\xi^\pm$ lie in different components of $\mathsf{R}^n\setminus\Gamma$, hence in different components of $\mathsf{R}^n\setminus\Sigma$.

To prove that $\mathsf{B}_\xi^+\cap\Gamma=\emptyset=\mathsf{B}_\xi^-\cap\Gamma$, we (slightly) modify Walther's argument; see [**28,** page 313, (v)$\Rightarrow$(iii)]. We show that $\mathsf{B}_\xi^+\cap\Gamma=\emptyset$; a similar argument shows that $\mathsf{B}_\xi^-\cap\Gamma=\emptyset$.

Using the fact that Γ is a $\mathcal{C}^1$ smooth hypersurface with unit normal $\mathbf{n}(\xi)$ at $\xi\in\Gamma$, it is easy to see that, for sufficiently small $\varepsilon>0$, we obtain $\Gamma\cap\mathsf{B}^n(\xi+\varepsilon\mathbf{n}(\xi);\varepsilon/2)=\emptyset$. Let Ω be the component of $\mathsf{R}^n\setminus\Gamma$ that contains the ball $\mathsf{B}^n(\xi+\varepsilon\mathbf{n}(\xi);\varepsilon/2)$. For $0<\rho<r$, we write

$$C(\rho,r)=C(\xi;\rho,r):=\operatorname{cvx}\left[\{\xi\}\cup\mathsf{B}^n(\xi+r\mathbf{n}(\xi);\rho)\right],$$

where $\operatorname{cvx}[A]$ denotes the closed convex hull of A. See Figure 5.

Again using the fact that Γ is a $\mathcal{C}^1$ smooth hypersurface, for sufficiently small $r>0$ and $0<\rho<r$, we have

$$C(\rho,r)\subset\overline{\Omega}. \tag{3.7}$$

Below we verify the crucial fact that, whenever (3.7) holds for some $0<\rho<r\le R$, it also holds with $\rho=r$. In particular, we establish:

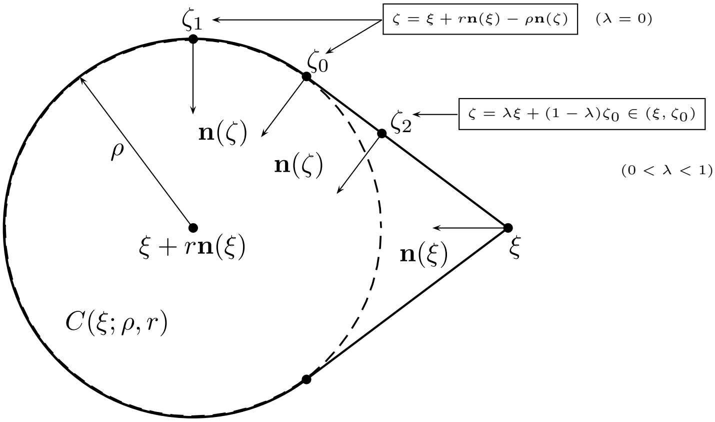

FIGURE 5. Points $\zeta \in \Gamma \cap \partial C(\rho, r)$ where $C(\rho, r) = C(\xi; \rho, r)$.

(3.8) If $0 < \rho < r \leq R$ and (3.7) holds, then $\mathsf{B}^n(\xi + r\mathbf{n}(\xi); r) \subset \Omega$.

Before proving (3.8), we explain how it implies that $\mathsf{B}_\xi^+ \cap \Gamma = \emptyset$.

We already know that (3.7) holds for some $r = r_0 \in (0, R]$ (and some $\rho \in (0, r_0)$). According to (3.8), $\mathsf{B}^n(\xi + r_0\mathbf{n}(\xi); r_0) \subset \Omega$, so, if $r_0 = R$, we are done. Suppose $r_0 < R$, and put $r_1 := \min\{R, (3/2)r_0\}$. A glance at the appropriate picture reveals that (3.7) holds with $r = r_1$ and $\rho = 2r_0 - r_1 \in (0, r_1)$, so by (3.8), $\mathsf{B}^n(\xi + r_1\mathbf{n}(\xi); r_1) \subset \Omega$ and we are done if $r_1 = R$. If $r_1 < R$, then $r_1 = \frac{3}{2}r_0 < R$ and we put $r_2 := \min\{R, \frac{3}{2}r_1\}$ and iterate this process. Since $r_k = (3/2)^k r_0 < R$ for only finitely many $k \in \mathsf{N}$, eventually this process stops with some $r_k = R$.

It remains to establish (3.8). We claim that if (3.7) holds for some $r \in (0, R]$ (and some $\rho \in (0, r)$), then for this r and all $\rho \in (0, r)$,

$$C(\rho, r) \subset \{\xi\} \cup \Omega; \tag{3.9}$$

therefore, $\mathsf{B}^n(\xi + r\mathbf{n}(\xi); r) \cup \{\xi\} = \bigcup_{\rho \in (0,r)} C(\rho, r) \subset \{\xi\} \cup \Omega$, so $\mathsf{B}^n(\xi + r\mathbf{n}(\xi); r) \subset \Omega$ and (3.8) follows. To corroborate this claim, we argue by way of contradiction. To this end, we assume that (3.7) holds for some $r \in (0, R]$ (and some $\rho \in (0, r)$), but there exists some $\rho \in (0, r)$ such that (3.9) is false. Evidently, there is then a smallest $\rho > 0$ such that (3.9) is false, and for this ρ we have

$$C(\rho, r) \subset \overline{\Omega} = \Omega \cup \partial\Omega = \Omega \cup \Gamma, \quad \text{but} \quad C(\rho, r) \cap \Gamma \supset \{\xi, \zeta\}$$

for some $\zeta \neq \xi$. Then $\zeta \in \Gamma \cap \partial C(\rho, r)$, so $\mathbf{n}(\zeta)$ is also normal to $\partial C(\rho, r)$ at ζ. See Figure 5. It is not hard to check that

$$\mathbf{n}(\zeta) \cdot (\zeta - \xi) \leq 0. \tag{3.10}$$

Also, we see (by looking at Figure 5) that there exists a $\lambda \in [0, 1)$ such that

$$\zeta = \lambda\,\xi + (1 - \lambda)(\xi + r\,\mathbf{n}(\xi) - \rho\,\mathbf{n}(\zeta)).$$

Here, $\lambda = 0$ when $\zeta \in \mathsf{S}^{n-1}(\xi + r\,\mathbf{n}(\xi); \rho)$, and $0 < \lambda < 1$ when $\zeta \in \partial C(\rho, r) \backslash \mathsf{S}^{n-1}(\xi + r\,\mathbf{n}(\xi); \rho)$ (and in this latter case, $[\xi, \zeta] \subset \partial C(\rho, r)$ too). Using the above, we get

$$\zeta - \xi = (1 - \lambda)[r\,\mathbf{n}(\xi) - \rho\,\mathbf{n}(\zeta)], \tag{3.11}$$

so with (3.10) we deduce that

$$r\,\mathbf{n}(\xi) \cdot \mathbf{n}(\zeta) = \rho + \frac{\mathbf{n}(\zeta) \cdot (\zeta - \xi)}{1 - \lambda} \leq \rho,$$

and therefore,

$$\mathbf{n}(\xi) \cdot \mathbf{n}(\zeta) \leq \frac{\rho}{r}.$$

However, we now demonstrate that $c := \mathbf{n}(\xi) \cdot \mathbf{n}(\zeta) > \rho/r$. This contradiction reveals that our claim (just above (3.9)) is true, thus completing our proof. To see that $c > \rho/r$, we use the Lipschitz condition (3) on the vector field $\mathbf{n}$ in conjunction with equation (3.11) to obtain

$$R\,|\mathbf{n}(\xi) - \mathbf{n}(\zeta)| \leq |\xi - \zeta| = (1 - \lambda)|r\,\mathbf{n}(\xi) - \rho\,\mathbf{n}(\zeta)|,$$

so

$$R^2\,(2 - 2\,\mathbf{n}(\xi) \cdot \mathbf{n}(\zeta)) \leq (1 - \lambda)^2 \left(r^2 - 2r\rho\,\mathbf{n}(\xi) \cdot \mathbf{n}(\zeta) + \rho^2\right)$$

or

$$2R^2(1 - c) \leq r^2 - 2r\rho c + \rho^2.$$

Since $0 < \rho < r \leq R$, this last inequality implies that $c > \rho/r$. □

3.3. Proof of Theorem B. Let $\varepsilon > 0$ be given. We assume that $\mathsf{B}^n_R, \mathsf{A}^n_R \in \mathcal{O}$ and that the associated conformal metrics $\rho_{\mathsf{B}^n_R}(x)|dx|$ and $\rho_{\mathsf{A}^n_R}(x)|dx|$ are asymptotically quasihyperbolic. Thus, there exists an $r \in (0, R)$ so that for each $z \in D \in \{\mathsf{B}^n_R, \mathsf{A}^n_R\}$

$$\operatorname{dist}(z, \partial D) < r \implies |\rho_D(z) \operatorname{dist}(z, \partial D) - 1| < \varepsilon.$$

Let $\Omega \in \mathcal{O}_R$, let $x \in \Omega$, and suppose $\operatorname{dist}(x, \partial\Omega) < r$. Pick $\xi \in \partial\Omega$ with $|x - \xi| = \operatorname{dist}(x, \partial\Omega)$. Let $\mathsf{B}^\pm_\xi$ be the two promised open balls of radius R, and centers $\xi \pm R\,\mathbf{n}(\xi)$, whose boundary spheres are tangent at ξ, and label these so that $x \in B := \mathsf{B}^+_\xi \subseteq \Omega$. Then since Ω is connected, $\Omega \subseteq A := \mathsf{R}^n \setminus \overline{\mathsf{B}^-_\xi}$.

As $\mathcal{O}$ is translation invariant, $A, B \in \mathcal{O}$. Thus, with $\mathcal{M}$ monotone and $B \subseteq \Omega \subseteq A$, we obtain

$$\rho_A(x) \le \rho_\Omega(x) \le \rho_B(x).$$

Now A and B are translates of $\mathsf{A}^n_R, \mathsf{B}^n_R$, respectively, and the points

$$a := x - \xi + R\,\mathbf{n}(\xi) \in \mathsf{A}^n_R$$

and

$$b := x - \xi - R\,\mathbf{n}(\xi) \in \mathsf{B}^n_R$$

correspond to x. Evidently, $\operatorname{dist}(a, \partial\mathsf{A}^n_R) = \operatorname{dist}(x, \partial A) = \operatorname{dist}(x, \partial B) = \operatorname{dist}(b, \partial\mathsf{B}^n_R) = \operatorname{dist}(x, \partial\Omega) < r$. Also, $\rho_{\mathsf{A}^n_R}(a) = \rho_A(x)$ and $\rho_{\mathsf{B}^n_R}(b) = \rho_B(x)$. Therefore,

$$\begin{aligned}
-\varepsilon < \rho_{\mathsf{A}^n_R}(a) \operatorname{dist}(a, \partial\mathsf{A}^n_R) - 1 &= \rho_A(x) \operatorname{dist}(x, \partial A) - 1 \\
\le \rho_\Omega(x) \operatorname{dist}(x, \partial\Omega) - 1 &\le \rho_B(x) \operatorname{dist}(x, \partial B) - 1 \\
= \rho_{\mathsf{B}^n_R}(b) \operatorname{dist}(b, \partial\mathsf{B}^n_R) - 1 &< \varepsilon.
\end{aligned}$$

□

3.4. Proof of Theorem C. Assume that Σ is a non-empty closed $(n-1)$-dimensional $\mathcal{C}^1$ smooth embedded submanifold of R^n. First, we show that $\operatorname{reach}(\Sigma) \ge \operatorname{rbp}(\Sigma)$.

Suppose that $0 < r < R < \operatorname{rbp}(\Sigma)$. Let $x \in \mathsf{R}^n$ with $d := \operatorname{dist}(x, \Sigma) \in (0, r)$. Choose $\xi \in \Sigma$ with $d = |x - \xi|$. Using notation from the first paragraph of subsection 3.1, we have disjoint open balls

$\mathsf{B}^{\pm}_{\xi} = \mathsf{B}^n(\xi \pm R\,\mathbf{n}(\xi); R)$ whose boundary spheres are tangent at the point ξ.

A standard argument shows that for each tangent vector $v \in T_\xi$, $x - \xi \perp v$ so $x - \xi \in N_\xi$. Since $0 < d < r < R$, we see that x lies in the Euclidean line segment $(\xi - R\,\mathbf{n}(\xi), \xi + R\,\mathbf{n}(\xi))$. Thus, $\mathsf{B}^n(x; d) \subset \mathsf{B}^+ \cup \mathsf{B}^-$, so ξ is the unique point of Σ nearest to x. Therefore, $\mathrm{reach}(\Sigma) \geq \mathrm{rbp}(\Sigma)$.

To establish the reverse inequality, suppose $R > \mathrm{rbp}(\Sigma)$. Then there exists a point $\xi \in \Sigma$ such that any two open balls of radius R whose boundary spheres are tangent at ξ cannot both lie in $\mathsf{R}^n \setminus \Sigma$. By translating and rotating, as necessary, we can assume that $\xi = 0$ and that the tangent hyperplane at ξ is $T_\xi = \mathsf{R}^{n-1} \times \{0\}$. Then,

$$[\overline{B}^n(R\,\mathbf{e}_n; R) \cup \overline{B}^n(-R\,\mathbf{e}_n;)] \cap \Sigma \supsetneq \{\xi\}. \tag{3.12}$$

Since Σ is a $\mathcal{C}^1$ smooth embedded submanifold of R^n, there exists an $\varepsilon \in (0, R)$ such that $\Sigma \cap \mathsf{B}^n(\pm\varepsilon\mathbf{e}_n; \varepsilon) = \emptyset$. Put

$$r := \inf\{t \in [\varepsilon, R] \mid [\overline{B}^n(t\,\mathbf{e}_n; t) \cup \overline{B}^n(-t\,\mathbf{e}_n; t)] \cap \Sigma \supsetneq \{\xi\}\};$$

so, $r \in [\varepsilon, R]$ is the 'first' time that the two closed balls meet Σ in at least two points. Then,

$$\mathsf{B}^n(\pm r\,\mathbf{e}_n; r) \cap \Sigma = \emptyset,$$

and one of the two spheres $\mathsf{S}^{n-1}(\pm r\,\mathbf{e}_n; r)$ meets Σ at both $\xi = 0$ and at some point $\zeta \neq \xi$. It now follows that the center of said sphere lies in $\mathsf{R}^n \setminus \Sigma$ and has two nearest points of Σ, so $\mathrm{reach}(\Sigma) \leq r \leq R$. Therefore, $\mathrm{reach}(\Sigma) \leq \mathrm{rbp}(\Sigma)$. □

3.5. A question. A natural question is whether or not there exists a family of metrics $\mathcal{M} = \{\rho_\Omega(x)|dx|\}_{\Omega \in \mathcal{O}}$ such that the metric ratio $\mathrm{dist}(x, \partial\Omega)\rho_\Omega(x) = \rho_\Omega(x)|dx|/\delta_\Omega^{-1}(x)|dx|$ having "uniform boundary value 1" implies that $\partial\Omega$ has the rolling ball property. It is trivial that this does not hold for the quasihyperbolic metric, and it is not hard to find examples of domains Ω such that $\partial\Omega$ does not have the rolling ball property and such that both the Ferrand and the Kulkarni-Pinkhall metrics in Ω do have this metric ratio property. With this in mind, we raise the following question; here $\lambda_\Omega(z)|dz|$ denotes the Poincaré hyperbolic metric in Ω.

What can be said about a hyperbolic plane domain Ω if we know that

$$\lim_{\substack{\mathrm{dist}(z,\partial\Omega)\to 0\\ z\in\Omega}} \mathrm{dist}(z,\partial\Omega)\lambda_\Omega(z) = 1\,?$$

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