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AN EXPLICIT FORMULA FOR BERNOULLI POLYNOMIALS IN TERMS OF r-STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. In this paper, the authors establish an explicit formula for computing Bernoulli polynomials at nonnegative integer points in terms of r-Stirling numbers of the second kind.

1. Introduction. It is well known that the Bernoulli numbers B_k for $k \ge 0$ can be generated by

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}, \quad |t| < 2\pi,$$

and that the Bernoulli polynomials $B_n(x)$ for $n \ge 0$ and $x \in \mathbb{R}$ can be generated by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.1}$$

In combinatorics, Stirling numbers of the second kind $S(n,k)$ are equal to the number of partitions of the set $\{1,2,\dots,n\}$ into k nonempty disjoint sets. Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ can be computed by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n.$$

In [**1**], Stirling numbers $S(n,k)$ were combinatorially generalized as r-Stirling numbers of the second kind, denoted by $S_r(n,k)$ here, for $r \in \mathbb{N}$, which can alternatively be defined as the number of partitions

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of the set $\{1,2,\dots,n\}$ into k nonempty disjoint subsets such that the numbers $1,2,\dots,r$ are in distinct subsets.

Note that

$$S(0,0)=1, \qquad S_0(n,k)=S(n,k),$$
$$S(n,0)=0, \qquad S_1(n,k)=S(n,k)$$

for all $n\ge k\ge 0$.

In [**4,** page 536] and [**5,** page 560], the simple formula

$$B_n=\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S(n,k), \quad n\in\mathbb{N}\cup\{0\} \tag{1.2}$$

for computing the Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n,k)$ was incidentally obtained. Recently, four alternative proofs for formula (1.2) were supplied in [**6, 7, 16**]. For more information on calculation of the Bernoulli numbers B_n, please refer to [**8, 9, 10, 11, 13, 15, 17**], especially to [**3**], and the many references therein.

The aim of this paper is to generalize formula (1.2). Our main result can be formulated as the following theorem.

Theorem 1.1. *For all integers $n,r\ge 0$, the Bernoulli polynomials $B_n(r)$ can be computed in terms of r-Stirling numbers of the second kind $S_r(n+r,k+r)$ by*

$$B_n(r)=\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S_r(n+r,k+r). \tag{1.3}$$

In the final section of this paper, several remarks are listed.

2. Proof of Theorem 1.1. We are now in a position to verify our main result.

For $n,r\ge 0$, let

$$F_{n,r}(x)=\sum_{k=0}^{n}k!S_r(n+r,k+r)x^k.$$

By [**1,** page 250, Theorem 16], we have

$$\sum_{n=0}^{\infty} S_r(n+r,k+r)\frac{t^n}{n!} = \sum_{n=k}^{\infty} S_r(n+r,k+r)\frac{t^n}{n!} = \frac{1}{k!}e^{rt}(e^t-1)^k,$$

where $S_r(n,m) = 0$ for $m > n$, see [**1,** page 243, equation (10)]. Accordingly, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,r}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!x^k S_r(n+r,k+r)\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty} k!x^k \sum_{n=k}^{\infty} S_r(n+r,k+r)\frac{t^n}{n!}\\ &= e^{rt}\sum_{k=0}^{\infty} x^k(e^t-1)^k = \frac{e^{rt}}{1-x(e^t-1)}.\end{aligned}$$

For $s \in \mathbb{R}$, integrating with respect to $x \in [0,s]$ on both sides of the above equation yields

$$\sum_{n=0}^{\infty}\left[\int_0^s F_{n,r}(x)\,\mathrm{d}\,x\right]\frac{t^n}{n!} = -e^{rt}\frac{\ln(1+s-se^t)}{e^t-1}. \tag{2.1}$$

On the other hand,

$$\int_0^s F_{n,r}(x)\,\mathrm{d}\,x = \sum_{k=0}^{n}\frac{k!}{k+1}S_r(n+r,k+r)s^{k+1}.$$

Substituting this into equation (2.1) gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{k!}{k+1}S_r(n+r,k+r)s^{k+1}\frac{t^n}{n!} = -e^{rt}\frac{\ln(1+s-se^t)}{e^t-1}.$$

Taking $s = -1$ in the above equation and using the generating function (1.1) results in

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{k!}{k+1}S_r(n+r,k+r)\right]\frac{t^n}{n!} = -\frac{te^{rt}}{e^t-1} = \sum_{n=0}^{\infty}[-B_n(r)]\frac{t^n}{n!},$$

which implies formula (1.3). The proof of Theorem 1.1 is complete. □

3. Remarks. Finally, we would like to give several remarks on Theorem 1.1 and its proof.

Remark 3.1. Since $B_n(0) = B_n$ and $S_0(n,k) = S(n,k)$, when $r = 0$, formula (1.3) becomes (1.2). Therefore, our Theorem 1.1 generalizes formula (1.2).

Remark 3.2. It is easy to see that

$$F_{n,0}(1) = \sum_{k=0}^{n} k! S(n,k),$$

which are the classical ordered Bell numbers. For more information, please refer to **[2, 14]** and the closely related references therein.

Remark 3.3. In **[12]**, the second author defined a variant of the polynomials $F_{n,r}(x)$. Hence, a simple combinatorial study and interpretation of the polynomials $F_{n,r}(x)$ is available therein.

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