τ-REGULAR FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO-DIVISORS

CHRISTOPHER PARK MOONEY

ABSTRACT. Recently there has been a flurry of research on generalized factorization techniques in both integral domains and rings with zero-divisors, namely, τ-factorization. There are several ways that authors have studied factorization in rings with zero-divisors. This paper focuses on the method of regular factorizations introduced by Anderson and Valdes-Leon. We investigate how one can extend the notion of τ-factorization to commutative rings with zero-divisors by using the regular factorization approach. The study of regular factorization is particularly effective because the distinct notions of associate and irreducible coincide for regular elements. We also note that the popular U-factorization developed by Fletcher also coincides since every regular divisor is essential. This will greatly simplify many of the cumbersome finite factorization definitions that exist in the literature when studying factorization in rings with zero-divisors.

1. Introduction. There has been a considerable amount of research done on the factorization properties of commutative rings, especially domains. For various aspects of the theory see [**3, 13, 22, 27, 28, 32**]. Unique factorization domains (UFDs) are well understood and have been studied extensively over the years. More recently, many authors have studied rings which satisfy various weakenings of the UFD conditions. These factorization properties of domains have been extended, in several distinct ways to rings with zero-divisors. Traditionally, in the domain case, authors have studied prime or irreducible factorizations. More recently, research has been done on generalizing the types of factorizations that have been studied to include things like co-maximal factorizations or using $\star$-operations to generalize factorization.

DOI:10.1216/RMJ-2016-46-4-1309

Of particular interest to the current article is the work of Anderson and Frazier. This is a survey article, [**7**], on the study of factorization in domains in which the authors introduce τ-factorization. The use of τ-factorization yields a beautiful synthesis of many of these generalizations of factorizations studied in the integral domain case. In many ways, this article was able to consolidate all of the factorization research in integral domains into a single method of studying factorization. Recently, the author has begun to study methods of extending this powerful approach of τ-factorization to the case of a commutative ring with zero-divisors. Because of the numerous approaches that have been taken to study factorization in rings with zero-divisors, this has led to many approaches to extending τ-factorization.

The theory of factorization in commutative rings with zero-divisors was initiated by Anderson, et al., [**1, 6, 8, 9, 10**] and has been continued by many authors. Recently, there has been an explosion of different approaches, by many different authors, of studying the arithmetic of rings with zero-divisors. For instance, see the work of Chang, Frei, Frisch, Geroldinger, Smertnig, et al., in [**21, 26, 29**].

In [**42**], the author used the methods established by Anderson and Valdes-Leon in [**9**] to extend many of the τ-factorization definitions to also work in rings with zero-divisors. In [**43**], the author investigated extending τ-factorization using the notion of U-factorizations first developed by Fletcher in [**24, 25**] and then studied extensively by Axtell, Forman, Roersma and Stickles in [**11, 12**]. In [**44**], the author studied yet another approach to extending τ-factorization, by using complete factorizations which was touched on in [**7**] in the case of integral domains.

In the present article, we concentrate on the approach studied in [**10, Section 5**] in which Anderson and Valdes-Leon study what was called regular factorization. This approach takes advantage of the fact that, for regular elements, all of the traditionally distinct associate relations and irreducible elements behave as they do in integral domains, where they are all equivalent once again. We see that this approach will greatly simplify matters and in fact unifies many of the previous methods in [**42, 43**].

In Section 2, we provide some necessary background definitions and theorems. In Section 3, we develop many of the definitions of τ-regular-

factorization, τ-regular irreducible elements as well as τ-regular finite factorization properties that rings may have. This is done by using the approach of Anderson and Valdez-Leon in [**10**, Section 5], where they restrict their study of τ-factorization to the regular elements of a commutative ring with 1. In subsection 3.2, we prove several theorems which describe the relationships between the various τ-regular finite factorization properties that rings may possess. In Section 4, we compare this new method of extending τ-factorization with the previous work in [**42**] and the relation $\tau_r := \tau \cap \operatorname{Reg}(R) \times \operatorname{Reg}(R)$. In Section 5, we demonstrate how these τ-regular finite factorization properties are related to other finite factorization properties defined in other works, especially [**42, 43**]. Finally, in Section 6, we study the τ-regular factorization properties of two particular rings that have been of much interest recently. These rings are the self-idealization of a PID, $R(D)$ in [**21**] and $\mathbb{Z}_{p^n}[X]$ studied in [**26**]. We see which of the τ-factorization, τ-U-factorization and τ-regular factorization properties studied in [**42, 43**] and the present paper are satisfied by these rings. We hope this motivates some future questions to be answered about an even larger class of rings, such as Mori, v-Marot, Krull and C-rings, that have been of interest in the literature on commutative rings with zero-divisors.

2. Preliminary definitions and results. We will assume R is a commutative ring with $1 \neq 0$. Let $R^* = R - \{0\}$, let $U(R)$ be the set of units of R and let $R^{\#} = R^* - U(R)$ be the non-zero, non-units of R. As in [**9**], we let $a \sim b$ if $(a) = (b)$, $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$, and $a \cong b$ if

(1) $a \sim b$

and

(2) $a = b = 0$ or, if $a = rb$ for some $r \in R$, then $r \in U(R)$.

We say a and b are *associates* (respectively *strong associates, very strong associates*) if $a \sim b$ (respectively $a \approx b$, $a \cong b$). As in [**5**], a ring R is said to be *strongly associate* (respectively *very strongly associate*) ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (respectively $a \cong b$).

2.1. τ-factorization in rings with zero-divisors. Let τ be a relation on $R^{\#}$, that is, $\tau \subseteq R^{\#} \times R^{\#}$. We will always assume further that

τ is symmetric. For non-units $a, a_i \in R$, and $\lambda \in U(R)$, $a = \lambda a_1 \cdots a_n$ is said to be a τ-*factorization* if $a_i \tau a_j$ for all $i \neq j$. If $n = 1$, then this is said to be a trivial τ-*factorization*. Given the above τ-factorization, we would say that a_i *is a* τ-*factor of* a or write $a_i \mid_\tau a$.

As in [**42**], we say τ is *multiplicative* (respectively *divisive*) if for $a, b, c \in R^{\#}$ (respectively $a, b, b' \in R^{\#}$), $a\tau b$ and $a\tau c$ imply $a\tau bc$ (respectively $a\tau b$ and $b' \mid b$ imply $a\tau b'$). We say τ is *associate* (respectively *strongly associate*, *very strongly associate*) *preserving* if, for $a, b, b' \in R^{\#}$ with $b \sim b'$ (respectively $b \approx b'$, $b \cong b'$) $a\tau b$ implies $a\tau b'$. A τ-*refinement* of a τ-factorization $\lambda a_1 \cdots a_n$ is a factorization of the form,

$$(\lambda \lambda_1 \cdots \lambda_n) b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n},$$

where $a_i = \lambda_i b_{i_1} \cdots b_{i_{m_i}}$ is a τ-factorization for each i. We say that τ is *refinable* if every τ-refinement of a τ-factorization is a τ-factorization. We say τ is *combinable* if whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$.

We now pause to supply the reader with a few examples of particularly useful or interesting τ-relations to give an idea of the power of τ-factorization.

**Example 2.1.** Let R be a commutative ring with 1.

(i) $\tau = R^{\#} \times R^{\#}$. This yields the usual factorizations in R and $\mid_\tau$ is the same as the usual divides. τ is multiplicative and divisive (hence associate preserving as we shall soon see).

(ii) $\tau = \emptyset$. For every $a \in R^{\#}$, there is only the trivial factorization and $a \mid_\tau b \Leftrightarrow a = \lambda b$ for $\lambda \in U(R) \Leftrightarrow a \approx b$. Again, τ is both multiplicative and divisive (vacuously).

(iii) Let S be a non-empty subset of $R^{\#}$, and let $\tau = S \times S$, $a\tau b \Leftrightarrow a, b \in S$. So τ is multiplicative (respectively divisive) if and only if S is multiplicatively closed (respectively closed under non-unit factors). A non-trivial τ-factorization is up to unit factors a factorization into elements from S.

(iv) Let $\star$ be a star-operation on R and define $a\tau b \Leftrightarrow (a, b)^{\star} = R$, that is a and b are $\star$-coprime or $\star$-comaximal. This particular operation has been studied more in depth by Juett [**33**]. When $\star = d$, the identity star operation, we get the co-maximal factorizations of McAdam and Swan [**41**].

(v) Let $a\tau_z b \Leftrightarrow ab = 0$. Then every $a \in R^\#$ is a τ-atom. The only nontrivial τ-factorizations are $0 = \lambda a_1 \cdot \ldots \cdot a_n$ where $a_i \cdot a_j = 0$ for all $i \neq j$. This example was studied extensively in [**42**] and has a close relationship with zero-divisor graphs.

(vi) Let $a\tau b \Leftrightarrow a, b \in \text{Reg}\,(R)$. Then this gives us the regular factorization studied in [**1**]. This is the inspiration for Section 3.

(vii) Letting $\tau \subseteq R^\# \times R^\#$, then we define $\tau_{\text{reg}} := \tau \cap (\text{Reg}\,(R) \times \text{Reg}\,(R))$. Because the collection of regular elements is a saturated, multiplicatively closed set, this has the effect of only allowing trivial factorizations of the zero-divisors. This is the type of τ-factorization we would like to use to compare with the notion of τ-factorizations by way of the regular factorizations studied in [**10**]. This will be studied more in depth in Section 4.

We now summarize several of the definitions given in [**42, 44**]. Let $a \in R$ be a non-unit. Then a is said to be τ-*irreducible* or τ-*atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some i. We will say a is τ-*strongly irreducible* or τ-*strongly atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some a_i. We will say that a is τ-*m-irreducible* or τ-*m-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all i. Note that the m is for "maximal" since such an a is maximal among principal ideals generated by elements which occur as τ-factors of a. As in [**44**], $a \in R$ is said to be a τ-*unrefinable atom* if a admits only trivial τ-factorizations. We will say that a is τ-*very strongly irreducible* or τ-*very strongly atomic* if $a \cong a$ and a has no non-trivial τ-factorizations. See [**42, 44**] for more equivalent definitions of these various forms of τ-irreducibility.

We have the following relationship between the various types of τ-irreducibles which is proved in [**42,** Theorem 3.9] as well as [**44**].

Theorem 2.2. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^\#$. Let $a \in R$ be a non-unit. The following diagram illustrates the relationship between the various types of τ-irreducibles and might satisfy where $\approx$ represents R being a strongly associate ring.*

τ-very strongly irreducible $\implies$ τ-unrefinably irred. $\implies$ τ-strongly irred. $\implies$ τ-irred.

τ-m-irred.
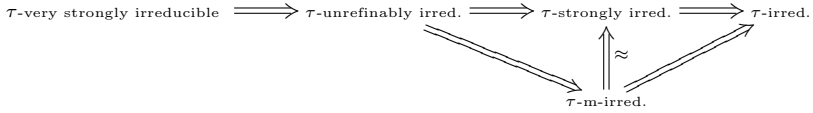
Following Bouvier, a ring R is said to be *présimplifiable* if $x = xy$ implies $x = 0$ or $y \in U(R)$ as in [**14, 15, 16, 17**]. When R is présimplifiable, the various associate relations coincide. As seen in [**42**], for non-zero elements, if R is présimplifiable, then τ-irreducible will imply τ-very strongly irreducible and the various types of irreducible elements will also coincide. Any integral domain or quasi-local ring is présimplifiable. Examples are given in [**9**] and abound in the literature which show that in a general commutative ring setting, each of these types of irreducible elements are distinct. For further discussion of the different τ-irreducible elements, the reader is directed to [**42**].

This leads to the following τ-finite factorization properties that a commutative ring may possess given a particular choice for τ, defined in [**42, 44**]. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$, $\beta \in \{$associate, strongly associate, very strongly associate$\}$ and τ a symmetric relation on $R^{\#}$. Then R is said to be τ-α if every non-unit $a \in R$ has a τ-factorization $a = \lambda a_1 \cdots a_n$ with a_i being τ-α for all $1 \leq i \leq n$. We will call such a factorization a τ-α-*factorization*. We say R satisfies the τ-*ascending chain condition on principal ideals* (ACCP) if for every chain $(a_0) \subseteq (a_1) \subseteq \cdots \subseteq (a_i) \subseteq \cdots$ with $a_{i+1} \mid_\tau a_i$, there exists an $N \in \mathbb{N}$ such that $(a_i) = (a_N)$ for all $i > N$.

A ring R is said to be a τ-α-β-*unique factorization ring* (UFR) if

(1) R is τ-α and
(2) for every non-unit $a \in R$ any two τ-α factorizations $a = \lambda_1 a_1 \cdots a_n = \lambda_2 b_1 \cdots b_m$ have $m = n$ and there is a rearrangement so that a_i and b_i are β.

A ring R is said to be a τ-α-*half factorization ring or half factorial ring* (HFR) if

(1) R is τ-α and

(2) for every non-unit $a \in R$ any two τ-α-factorizations have the same length.

A ring R is said to be a τ-*bounded factorization ring* (BFR) if for every non-unit $a \in R$, there exists a natural number $N(a)$ such that for any τ-factorization $a = \lambda a_1 \cdots a_n$, $n \le N(a)$. A ring R is said to be a τ-β-*finite factorization ring* (FFR) if for every non-unit $a \in R$ there are only a finite number of non-trivial τ-factorizations up to rearrangement and β. A ring R is said to be a τ-β-*weak finite factorization ring* (WFFR) if for every non-unit $a \in R$, there are only finitely many $b \in R$ such that b is a non-trivial τ-divisor of a up to β. A ring R is said to be a τ-α-β-*divisor finite ring* (df ring) if for every non-unit $a \in R$, there are only finitely many τ-α τ-divisors of a up to β.

These result in the following diagram accompanying [**42**, Theorem 4.1] illustrating the relationship between the various τ-finite factorization properties in rings with zero-divisors, where ∇ represents τ refinable.
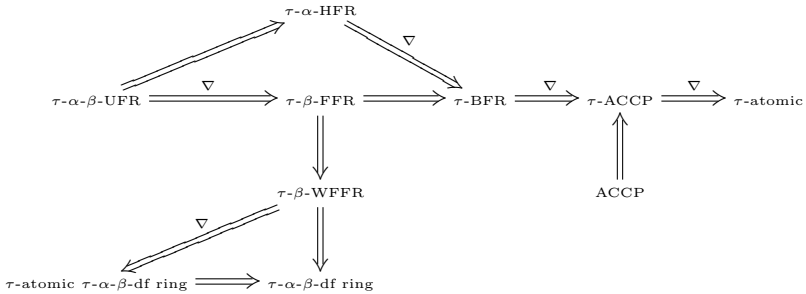
2.2. τ-U-factorization definitions. In this section we briefly present the requisite τ-U-factorization definitions and results from [**43**]. As in [**12**], we define U-factorization as follows. Let $a \in R$ be a non-unit. If $a = \lambda a_1 \cdots a_n b_1 \cdots b_m$ is a factorization with $\lambda \in U(R)$, $a_i, b_i \in R^{\#}$, then we will call

$$a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$$

a U-*factorization* of a if

(1) $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for all $1 \le i \le n$ and
(2) $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \ne (b_1 \cdots \widehat{b_j} \cdots b_m)$ for $1 \le j \le m$ where $\widehat{b_j}$ means b_j is omitted from the product.

Here $(b_1 \cdots b_m)$ is the principal ideal generated by $b_1 \cdots b_m$. The b_i's in this particular U-factorization above will be referred to as *essential divisors*. The a_i's in this particular U-factorization above will be referred to as *inessential divisors*. A U-factorization is said to be *trivial* if there is only one essential divisor.

A τ-U-*factorization* of a non-unit $a \in R$ is a U-factorization

$$a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$$

for which $\lambda a_1 \cdots a_n b_1 \cdots b_m$ is also a τ-factorization.

Given a symmetric relation τ on $R^{\#}$, we say R is τ-U-*refinable* if for every τ-U-factorization of any non-unit $a \in U(R)$,

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil,$$

any τ-U-factorization of an essential divisors,

$$b_i = \lambda' c_1 \cdots c_{n'} \lceil d_1 \cdots d_{m'} \rceil$$

satisfies

$$a = \lambda \lambda' a_1 \cdots a_n c_1 \cdots c_{n'} \lceil b_1 \cdots b_{i-1} d_1 \cdots d_{m'} b_{i+1} \cdots b_m \rceil$$

is a τ-U-factorization.

Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible$\}$. Let a be a non-unit. If

$$a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$$

is a τ-U-factorization, then this factorization is said to be a τ-U-α-*factorization* if it is a τ-U-factorization and the essential divisors b_i are τ-α for $1 \leq i \leq m$.

We now define the finite factorization properties using the τ-U-factorization approach. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible$\}$ and let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. R is said to be τ-*U*-α if for all non-units $a \in R$, there is a τ-U-α-factorization of a. R is said to satisfy τ-U-ACCP (ascending chain condition on principal ideals) if every properly ascending chain of principal ideals $(a_1) \subsetneq (a_2) \subsetneq \cdots$ such that a_{i+1} is an essential divisor in some τ-U-factorization of a_i, for each i terminates after finitely many principal ideals. R is said to be a τ-U-BFR if for all non-units $a \in R$, there is

a bound on the number of essential divisors in any τ-U-factorization of a.

R is said to be a τ-U-β-FFR if for all non-units $a \in R$, there are only finitely many τ-U-factorizations up to rearrangement of the essential divisors and β. R is said to be a τ-U-β-WFFR if for all non-units $a \in R$, there are only finitely many essential divisors among all τ-U-factorizations of a up to β. R is said to be a τ-U-α-β-*divisor finite* (df) *ring* if for all non-units $a \in R$, there are only finitely many essential τ-α divisors up to β in the τ-U-factorizations of a.

R is said to be a τ-U-α-HFR if R is τ-U-α and for all non-units $a \in R$, the number of essential divisors in any τ-U-α-factorization of a is the same. R is said to be a τ-U-α-β-UFR if R is a τ-U-α-HFR and the essential divisors of any two τ-U-α-factorizations can be rearranged to match up to β.

The following diagram summarizes the main results from [**43,** Theorem 4.3, Theorem 4.4] where $\approx$ represents R being strongly associate, $\dagger$ represents R is τ-U-refinable, and ∇ represents τ is associate preserving and refinable:
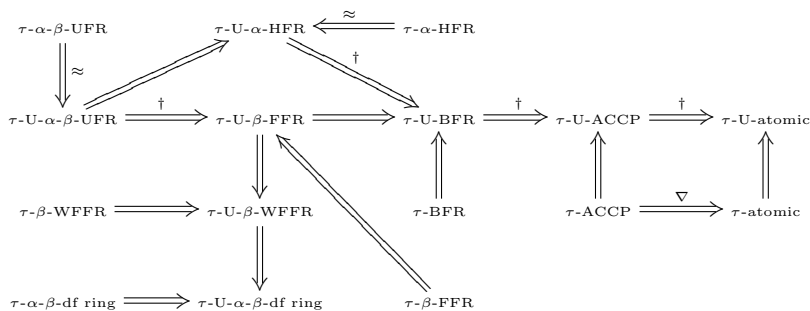
3. τ-regular factorization. We will denote the set of regular elements in a ring R by $\mathrm{Reg}\,(R)$, so $R - Z(R) = \mathrm{Reg}\,(R)$. We will use $\mathrm{Reg}\,(R)^{\#}$ to denote the regular elements which are not units. The primary benefit of looking at the factorization of the regular elements is that for regular elements, all of the associate relations coincide. That is, let $a, b \in \mathrm{Reg}\,(R)$, then $a \sim b$ implies $a \cong b$. Suppose $a = rb$. Neither a nor b can be zero, or else they could not be regular elements since we assume R has an identity which is not zero. But $a \sim b$ implies there

is an $s \in R$ such that $b = sa$. Thus $a = rb = r(sa) = (rs)a$, but a is regular, so $a(1 - rs) = 0$ implies $rs - 1 = 0$ or $rs = 1$ and $r \in U(R)$ as desired. Another important consequence is that, for a regular element, we always have $a \cong a$. This means that for a regular, non-unit element $a \in \text{Reg}\,(R)$, if a is irreducible, then a is very strongly irreducible. As a consequence, for a regular, non-unit $a \in R$ we can simply refer to it as *irreducible* without any ambiguity. We will soon see that this simplifies matters considerably.

3.1. τ-regular factorization definitions. Let τ be a symmetric relation on $R^{\#}$. A τ-factorization, $a = \lambda a_1 \cdots a_n$ with $\lambda \in U(R)$, and $a_i \tau a_j$ for all $i \neq j$ is said to be a *τ-regular-factorization* or *τ-r-factorization* if $a \in \text{Reg}\,(R)$. Note that a is regular if and only if a_i is regular for each $1 \leq i \leq n$.

Proposition 3.1. *Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Given $a \in \text{Reg}\,(R)$, the following are equivalent.*

(i) *For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some $1 \leq i \leq n$.*

(ii) *For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some $1 \leq i \leq n$.*

(iii) *For any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all $1 \leq i \leq n$.*

(iv) *The only τ-regular factorizations of a are of the form $a = \lambda(\lambda^{-1}a)$.*

(v) *$a \cong a$ and for any τ-regular-factorization, $a = \lambda a_1 \cdots a_n$, we have $a \cong a_i$ for some $1 \leq i \leq n$.*

Proof. (v) $\Rightarrow$ (iv). Suppose $a = \lambda a_1 \cdots a_n$ is a τ-regular factorization with $n \geq 2$. Then by hypothesis $a \cong a_i$ for some $1 \leq i \leq n$. Then

$$a = (\lambda a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n) a_i$$

implies that $(\lambda a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n)$ is a unit. Hence the factorization was a trivial factorization to begin with.

(iv) $\Rightarrow$ (iii) is immediate. After noting that any divisor of a regular element must be regular and hence $\sim$, $\approx$ and $\cong$ coincide, it is clear that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (v). Since a is regular by hypothesis, $a \cong a$ and again $\sim$, $\approx$ and $\cong$ coincide on any divisors of a regular element, completing the proof. $\square$

We say that a non-unit, $a \in \mathrm{Reg}\,(R)$ is τ-r-*irreducible* or a τ-r-*atom* if a satisfies any of the above equivalent conditions. We say R is τ-r-*atomic* if for all $a \in \mathrm{Reg}\,(R)^{\#}$, there is a τ-r-factorization into τ-r-irreducible elements. R satisfies τ-r-ACCP if for every chain of principal ideals generated by regular elements $(a_1) \subsetneq (a_2) \subsetneq \cdots (a_i) \subsetneq \cdots$ with a_{i+1} occurring as a τ-divisor in some τ-r-factorization of a_i for all i becomes stationary.

R is a τ-r-*half factorization ring* (HFR) if

(1) R is τ-r-atomic and
(2) if $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ are two τ-r-atomic τ-factorizations implies that $m = n$.

R is said to be a τ-r-*unique factorization ring* (UFR) if R is a τ-r-HFR and there is a rearrangement of any two τ-r-atomic factorizations as above such that $a_i \sim b_i$ for all $1 \le i \le n = m$. We define the τ-*regular-elasticity* as τ-r-$\rho(R) = \sup\{\rho(a) \mid a \in \mathrm{Reg}\,(R)^{\#}\}$ where

$$\rho(a) = \sup\left\{\frac{m}{n} \mid \lambda a = a_1 \cdots a_m = \mu b_1 \cdots b_n \text{are } \tau\text{-atomic-factorizations}\right\}.$$

Then it is clear that R is a τ-r-HFR if and only if R is τ-atomic and τ-r-$\rho(R)$=1.

R is said to be a τ-r-*bounded factorization domain* (BFR) if for every $a \in \mathrm{Reg}\,(R)$ there exists a natural number $N_r(a)$ such that for all τ-r-factorizations $a = \lambda a_1 \cdots a_n$, we have $n \le N_r(a)$. R is said to be a τ-r-*irreducible-divisor-finite ring* (idf ring) if each $a \in \mathrm{Reg}\,(R)^{\#}$ has at most a finite number of τ-r-irreducible τ-divisors (up to order and associates). R is said to be a τ-r-*finite factorization ring* (FFR) if for every $a \in \mathrm{Reg}\,(R)^{\#}$, a has only a finite number (up to order and associates) of τ-factorizations. R is said to be a τ-r-*weak finite factorization ring* (WFFR) if for every $a \in \mathrm{Reg}\,(R)^{\#}$ there are only a finite number of τ-divisors (up to order and associates).

3.2. τ-regular factorization results. We begin with a useful lemma by Dickson [**23**].

Lemma 3.2 (Dickson's theorem [**23**])**.** *Let $S = \mathbb{N} \times \cdots \times \mathbb{N}$ be the direct Cartesian product of n copies of $\mathbb{N}$. Let $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ in S if and only if $a_i \leq b_i$ for $1 \leq i \leq n$. This is a partial ordering on S. If $\emptyset \neq X \subseteq S$, then X has only finitely many minimal elements.*

Proposition 3.3. *Let R be a commutative ring with 1. Let τ be a symmetric relation on $R^{\#}$ with τ refinable, then the following are equivalent.*

- (i) *R is a τ-r-FFR.*
- (ii) *R is a τ-r-WFFR.*
- (iii) *R is a τ-r-atomic τ-r-idf ring.*
- (iv) *R is τ-r-atomic and each $a \in \operatorname{Reg}(R)^{\#}$, a has only finitely many τ-r-atomic τ-factorizations up to order and associates.*
- (v) *For all $a \in \operatorname{Reg}(R)^{\#}$, there are only finitely many $b \in \operatorname{Reg}(R)^{\#}$ up to associate such that b occurs as a τ-factor in a τ-r-factorization of a.*
- (vi) *For all $a \in \operatorname{Reg}(R)^{\#}$, (a) is contained in only finitely many principal ideals (b) where $b \in \operatorname{Reg}(R)^{\#}$ such that b occurs as a τ-factor in a τ-r-factorization of a.*
- (vii) *For all $a \in \operatorname{Reg}(R)^{\#}$, there are only finitely many $b \in \operatorname{Reg}(R)^{\#}$ up to associate such that $b \mid_{\tau} a$.*
- (viii) *For all $a \in \operatorname{Reg}(R)^{\#}$, (a) is contained in only finitely many principal ideals (b) where $b \in \operatorname{Reg}(R)^{\#}$ such that $b \mid_{\tau} a$.*

Proof. (i) $\Rightarrow$ (ii). Letting R be a τ-r-FFR and $a \in \operatorname{Reg}(R)^{\#}$, then there are only a finite number of τ-factorizations (up to order and associate), each of finite length. Hence, since every τ-divisor of a must be among these up to associate, R is a τ-r-WFFR.

(ii) $\Rightarrow$ (iii). Let R be a τ-r-WFFR and $a \in \operatorname{Reg}(R)^{\#}$. If a has a finite number of τ-divisors, then certainly it has a finite number of irreducible τ-divisors, so it suffices to show a has a τ-r-atomic factorization. We instead show the stronger condition, that R satisfies τ-r-ACCP, that is, any chain of principal ideals generated by regular elements

$$(a_0) \subsetneq (a_1) \subsetneq \cdots \subsetneq (a_i) \subsetneq \cdots$$

with a_{i+1} occurring as a τ-factor in a τ-r-factorization of a_i and $a_i \in \operatorname{Reg}(R)^{\#}$ for all i comes to a halt. Suppose there is an infinite chain, but then each a_i is a τ-divisor of a_0 and none of them is associate

since each containment is proper. Therefore we would have an infinite number of non-associate τ-r-divisors contradicting the fact that R is a τ-r-WFFR (note, here we emphasize that τ is refinable to ensure that at each step we retain a τ-factorization).

(iii) $\Rightarrow$ (i). We use Dickson's theorem and modify the proof of [**2,** Theorem 4**]. Let R be a τ-r-atomic τ-r-idf ring and $a \in \text{Reg}(R)^{\#}$. Let $a_1, \ldots, a_n$ be the τ-r-irreducible τ-factors of a representatives up to associate. In particular, they are all regular elements of R since they divide a regular element. Because τ is refinable and R is τ-r-atomic, any τ-r-factorization of $a = \lambda b_1 \cdots b_n$ can be τ-refined into a τ-r-atomic factorization, it suffices to show that there are a finite number of τ-r-atomic factorizations of a up to rearrangement and associate.

Given any τ-r-atomic factorization of a, since τ is refinable, we can replace each τ-atomic factor with one from the list $a_1, \ldots, a_n$ and collect all the unit multiples in the front with the $\lambda \in U(R)$. Thus, given any τ-r-atomic factorization of a, we let e_i represent the number of not necessarily distinct τ-r-factors in the τ-factorization which are associates of a_i. We then could group the terms together based on the associates of the n given τ-r-atomic divisors of a as follows

$$a = \lambda a_1^{e_1} \cdot a_2^{e_2} \cdots a_n^{e_n}.$$

Then e_i is a non-negative integer for each $1 \leq i \leq n$ representing the number of times an associate of a_i appears as a factor in the given τ-r-atomic factorization of a. Consider the set $X = \{(e_1, \ldots, e_n)\}$, which stem from the τ-r-atomic factorizations in this way. Since R is τ-r-atomic, $X \neq \emptyset$. We use the partial ordering provided by Lemma 3.2. Let $(e_i), (f_i) \in X$ be such that $(e_i) \leq (f_i)$. If $(e_i) \neq (f_i)$, then there is some $e_i < f_i$, $1 \leq i \leq n$. Without loss of generality, we suppose it is $e_1 < f_1$. This would imply that

$$(a) = (a_1^{f_1} \cdot a_2^{f_2} \cdots a_n^{f_n}) \subsetneq (a_1^{e_1} \cdot a_2^{f_2} \cdots a_n^{f_n}) = (a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n}) = (a).$$

This is a contradiction, so each element of X is in fact minimal in X. By Dickson's theorem, the set of minimal elements is finite. This means X itself must be finite. Hence there are only a finite number of τ-r-atomic factorizations of a up to rearrangement and associates. Thus R is a τ-r-FFR as desired.

(i) $\Rightarrow$ (iv). This is clear as we have already seen that a τ-r-FFR is τ-r-atomic and a τ-r-atomic factorization is certainly a τ-r-factorization,

so there must be a finite number of τ-r-atomic factorizations up to order and associate for every $a \in \operatorname{Reg}(R)^{\#}$.

(iv) $\Rightarrow$ (iii). Letting $a \in \operatorname{Reg}(R)^{\#}$, then there are a finite number of τ-r-atomic factorizations, each of which has a finite number of τ-r-atomic factors. Therefore the collection of τ-r-atomic divisors is finite, so R is a τ-r-atomic τ-r-idf ring.

(v) and (vi) are restatements of (ii) and their equivalence is immediate. Furthermore, (v) and (vii) (respectively (vi) and (viii)) are seen to be equivalent after noting that for $b \in \operatorname{Reg}(R)$, $a \mid_{\tau} b$ implies there is some τ-factorization $b = \lambda a a_1 \cdots a_n$, but since b is regular and the set of regular elements is saturated, every τ-factor must be regular so this is really a τ-factorization. $\square$

Theorem 3.4. *Let R be a commutative ring with 1, with τ a symmetric relation on $R^{\#}$. We have the following*:

(i) *R is a τ-r-UFR implies R is a τ-r-HFR.*
(ii) *For τ refinable, R is a τ-r-HFR implies R is a τ-r-BFR.*
(iii) *For τ refinable, R is a τ-r-UFR implies R is a τ-r-FFR.*
(iv) *R is a τ-r-FFR implies R is a τ-r-BFR.*
(v) *For τ refinable, R is a τ-r-BFR implies R satisfies τ-r-ACCP.*
(vi) *For τ refinable, R satisfies τ-r-ACCP implies R is τ-r-atomic.*

Proof.

(i) This is immediate from the definition.

(ii) Let R be a τ-r-HFR. Suppose $a = \lambda a_1 \cdots a_n$ is a τ-r-atomic factorization. We claim $N_r(a) = n$. Let $a = \mu b_1 \cdots b_m$ be a τ-r-factorization of a. Since R is τ-r-atomic, we can find τ-r-atomic factorizations for b_i, $1 \leq i \leq m$. We have assumed τ to be refinable, so we can replace each b_i with the corresponding τ-r-atomic factorization and collect the units in front of the factorization and retain a τ-r-factorization which is τ-atomic and thus must have length n. The refinement process can only increase the length of the factorization, so the length of the original factorization is no longer than n, proving the claim.

(iii) We show for τ-refinable, R a τ-r-UFR, R is a τ-r-atomic τ-r-idf-ring which has been shown in Theorem 3.3 to be equivalent to being a

τ-r-FFR. Since R is a τ-r-UFR, we get τ-r-atomic for free. Furthermore, any τ-atomic factorization of $a \in \text{Reg}\,(R)^{\#}$ has the same length, say n, and can be reordered so that the associates match up. This tells us there are precisely n τ-irreducible divisors of a up to associate; hence, R is a τ-r-idf-ring.

(iv) Suppose R is a τ-r-FFR; by definition, we know R is τ-r-atomic. Now, let $a \in \text{Reg}\,(R)^{\#}$ and let S be the finite set of all τ-atomic factors of a. Set $N(a) = |S|$. Letting $a = \lambda a_1 \cdots a_n$ be a τ-atomic factorization of a, then $a_i \in S$ for all i, but then $\{a_i\}_{i=1}^{n} \subseteq S$ and hence is finite and $n \leq N(a) = |S|$ as desired. So R is a τ-r-BFR.

(v) Let R be a τ-r-BFR, and suppose for a moment that R does not satisfy τ-r-ACCP. There must exist an infinite sequence $\{a_i\}_{i=1}^{\infty} \subseteq \text{Reg}\,(R)^{\#}$ such that $a_{n+1} \mid_{\tau} a_n$, but $a_{n+1} \not\sim a_n$ for all $n \geq 1$. Let $a_n = \lambda_{n+1} r_{n+11} \cdots r_{n+1 s_{n+1}} a_{n+1}$ be a τ-factorization of a_n for all $n \geq 1$. But then we have

$$a_1 = \lambda_2 r_{2_1} \cdots r_{2_{s_2}} a_2 = \lambda_2 r_{2_1} \cdots r_{2_{s_2}} \lambda_3 r_{3_1} \cdots r_{3_{s_3}} a_3 = \cdots$$

is a τ-factorization (note we use τ refinable here). This shows that we can find arbitrarily large τ-factorizations of a_1, which contradicts the fact the R is a τ-r-BFR.

(vi) Let R satisfy τ-r-ACCP, but suppose that R is not τ-r-atomic. Then there exists $a \in \text{Reg}\,(R)^{\#}$ with no τ-factorization into τ-atoms. a itself cannot be a τ-atom, so say $a = \lambda a_1 \cdots a_n$ is a τ-factorization with $n > 1$. Now again, some a_i must not be a product of τ-atoms, or with τ refinable, we could find a τ-atomic factorization, say a_1. Then $a_1 \mid_{\tau} a$ and $a_1 \not\sim a$ produce $b_1 = a_1$. Then a_1 must have a τ-factorization $a_1 = \lambda_2 a_{2_1} \cdots a_{2_{n_2}}$ where $n_2 > 1$. Again, one of the τ-factors, say a_{2_1} cannot be a τ-product of τ-atoms. Here $a_{2_1} \mid_{\tau} a_1$ and $a_{2_1} \not\sim a_1$. Put $b_2 = a_{2_1}$. Continuing in this fashion, we obtain a sequence $\{b_i\}_{i=1}^{\infty}$ of elements of $\text{Reg}\,(R)^{\#}$ such that $b_{n+1} \mid_{\tau} b_n$ but $b_{n+1} \not\sim b_n$ for every $n \geq 1$. This contradicts R satisfying τ-r-ACCP. $\square$

The following diagram summarizes our result where ∇ represents τ refinable.

4. τ_{reg}-factorizations. In this section, we study another approach which could have been used to extend τ-factorization to commutative rings with zero-divisors using regular factorizations. In Section 3, we
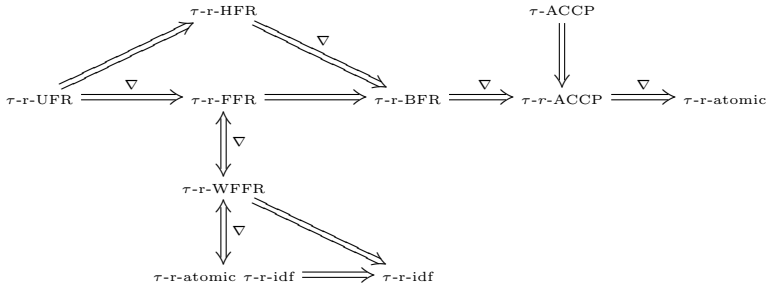
decided to consider only factorizations of the regular elements. In other words, we chose to restrict the elements we attempt to factor to the regular elements of a commutative ring R. We could have instead chosen to restrict the relation τ itself. This gives us the benefit of not completely ignoring a possible large number of zero-divisors in the ring R, but at the cost of choosing a less natural relation τ. Moreover, it allows us to use much of the work done previously in [**42**] by just picking a different τ and keeping all of the original definitions the same. It turns out that, in many ways, either choice is fine and we end up at the same place anyway. Studying this will be the motivation of this section.

Let R be a commutative ring with 1 and τ a symmetric relation on $R^{\#}$. Then we define a new relation

$$\tau_{\mathrm{reg}} := \tau \cap \left(\mathrm{Reg}\,(R) \times \mathrm{Reg}\,(R) \right).$$

We may now pursue the τ-factorizations using the approach from [**42**] and look at factoring all the non-units in R instead of just the regular elements. There is certainly a very close relationship between τ_{reg}-factorizations and τ-regular factorizations; however, there are a few subtle differences that cause some problems, especially with the definition of τ_{reg}-very strongly atomic elements. In, [**42**], the author insisted that part of a being τ-very strongly atomic was that $a \cong a$.

The fact that the very strongly associate relation need not be reflexive is the main reason there is not a perfect correspondence between the two approaches. We will see that τ_{reg}-factorizations are simply very poorly behaved when it comes to τ_{reg}-very strong atoms and rearrangement up to very strong associates. On the bright side, the τ-unrefinably irreducible element introduced in [**44**] will also behave quite nicely here.

Of course, any non-trivial idempotent element, e, is a zero-divisor since $e(e - 1) = 0$. Furthermore, since $e = e^2 = e \cdot e$, with e not a unit, we see that $e \not\approx e$. This means that e is not very strongly atomic for any non-trivial idempotent element. On the other hand, since every non-trivial τ_{reg}-factorization consists of a product of regular elements, we can have no non-trivial τ_{reg}-factorizations of e. This means the only τ_{reg}-factorizations of any zero-divisor, in particular e, are the trivial factorizations. Unfortunately, in the case of a non-trivial idempotent, e, this means e is not a τ-very strong atom and will never have a τ_{reg}-very strongly atomic factorization. We demonstrate this in the following example.

Example 4.1. Let K be an infinite field,

$$R = K \times K \quad \text{with } \tau = R^{\#} \times R^{\#}.$$

We consider the element $(1, 0) \in Z(R)$. This ring has only elements which are strongly associate to idempotent elements and units. So the set of non-unit regular elements is empty and our ring is vacuously a τ-r-UFR. On the other hand, we have $(1, 0) = (\mu^{-1}, 1)(\mu, 0)$, for any unit $\mu \in K^*$, which is the only type of τ_{reg}-factorization of $(1, 0)$. Yet, none of these are τ_{reg}-very strongly atomic factorizations. The problem is that $(\mu, 0) \not\approx (\mu, 0)$ since we have $(\mu, 0) = (1, 0)(\mu, 0)$ and $(1, 0)$ is not a unit. This shows we can have a τ-r-UFR which is not even τ_{reg}-atomic. Moreover, each of these factorizations is non-very strongly associate.

Let $\mu, \lambda \in K^*$. Then $(1, 0) = (\mu^{-1}, 1)(\mu, 0) = (\lambda^{-1}, 1)(\lambda, 0)$ are two τ_{reg}-factorizations of $(1, 0)$, but $(\mu, 0) = (\mu\lambda^{-1}, 0)(\lambda, 0)$ with $(\mu\lambda^{-1}, 0)$ not a unit shows $(\mu, 0) \not\approx (\lambda, 0)$. Since K is infinite, there are infinitely many τ_{reg}-factorizations of $(1, 0)$, none of which are very strongly associate.

This leads us to the following results.

Lemma 4.2. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\text{reg}} := \tau \cap (\text{Reg}\,(R) \times \text{Reg}\,(R)).$$

The collection of non-trivial τ-regular-factorizations and non-trivial τ_{reg}-factorizations coincide.

Proof. Let $a = \lambda a_1 \cdots a_n$ be a non-trivial τ-regular factorization. Then $a \in \text{Reg}(R)$ by definition of τ-regular factorization, and $a_i \tau a_j$ for all $i \neq j$. Since a is regular, and the set of regular elements is saturated, we have $a_i \mid a \in \text{Reg}(R)$ for each $1 \leq i \leq n$, and we know that $a_i \in \text{Reg}(R)$ for each $1 \leq i \leq n$. This means $a_i \tau_{\text{reg}} a_j$ for each $i \neq j$. Thus, $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-factorization.

Conversely, suppose $a = \lambda a_1 \cdots a_n$ is a non-trivial τ_{reg}-factorization. Then $a_i \tau_{\text{reg}} a_j$ for each $i \neq j$. This means $a_i \tau a_j$ and $a_i, a_j \in \text{Reg}(R)$. In particular, since $n \geq 2$, we can conclude that $a_1 a_2 \cdots a_n$ is a product of regular elements, so $a \in \text{Reg}(R)$. This means $a = \lambda a_1 \cdots a_n$ is a τ-regular-factorization. $\qquad\square$

Theorem 4.3. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\text{reg}} := \tau \cap \left(\text{Reg}(R) \times \text{Reg}(R) \right).$$

For $a \in \text{Reg}(R)$, the following are equivalent:

  (i) *a is a τ-regular-atom.*
 (ii) *a is a τ_{reg}-atom.*
(iii) *a is a τ_{reg}-strong atom.*
(iv) *a is a τ_{reg}-m-atom.*
 (v) *a is a τ_{reg}-unrefinable atom.*
(vi) *a is a τ_{reg}-very strong atom.*

Proof. When we consider Theorem 2.2, it suffices to show that (ii) $\Rightarrow$ (vi) and then we show that (i) $\Leftrightarrow$ (v). Let $a \in \text{Reg}(R)$ be a τ_{reg}-atom. Since $a \in \text{Reg}(R)$, we have $a \cong a$ since $a = ra$ implies $r = 1$. Furthermore, if $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-factorization of a, then $a \sim a_i$ for some $1 \leq i \leq n$. Since $a \in \text{Reg}(R)$, $a \cong a_i$ and we have shown that a is a τ_{reg}-very strongly atom.

(i) $\Leftrightarrow$ (v). In light of Lemma 4.2, a has a non-trivial τ-regular factorization if and only if a has a non-trivial τ_{reg}-factorization. $\qquad\square$

Corollary 4.4. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\text{reg}} := \tau \cap \left(\text{Reg}(R) \times \text{Reg}(R) \right).$$

Let

$\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic$\}$.

Letting $a \in \mathrm{Reg}\,(R)$ be a non-unit, then $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-α-factorization if and only if $a = \lambda a_1 \cdots a_n$ is a τ-regular-atomic factorization.

Proof. This is immediate from what we have shown in Theorem 4.3. $\square$

Theorem 4.5. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\mathrm{reg}} := \tau \cap (\mathrm{Reg}\,(R) \times \mathrm{Reg}\,(R)).$$

If $a \in Z(R)$, then the following hold:

  (i) *a is a τ_{reg}-atom.*
 (ii) *a is a τ_{reg}-strong atom.*
(iii) *a is a τ_{reg}-m-atom.*
 (iv) *a is a τ_{reg}-unrefinable atom.*

Proof. By Theorem 2.2, it suffices to show that, for $a \in Z(R)$, a is τ_{reg}-unrefinable. Suppose $a = \lambda a_1 \cdots a_n$ is a non-trivial τ_{reg}-factorization. This implies $n \geq 2$, and therefore $a_i \tau_{\mathrm{reg}} a_j$ for each $i \neq j$. In particular, $a_i \in \mathrm{Reg}\,(R)$ for all $1 \leq i \leq n$. This means a is a product of regular elements and is therefore regular, and hence not a zero-divisor, a contradiction. Thus there are only non-trivial τ_{reg}-factorizations of a making it a τ_{reg}-unrefinable atom as desired. $\square$

Theorem 4.6. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\mathrm{reg}} := \tau \cap (\mathrm{Reg}\,(R) \times \mathrm{Reg}\,(R)).$$

The following are equivalent:

  (i) *R is τ-regular-atomic.*
 (ii) *R is a τ_{reg}-atomic.*
(iii) *R is a τ_{reg}-strongly atomic.*
 (iv) *R is a τ_{reg}-m-atomic.*
  (v) *R is τ_{reg}-unrefinably atomic.*

Proof. Let a be a non-unit in R. Then $a \in Z(R)$ or $a \in \operatorname{Reg}(R)$. If $a \in Z(R)$, we apply Theorem 4.5 to see that a itself is τ_{reg}-atomic, τ_{reg}-strongly atomic, τ_{reg}-m-atomic, and τ_{reg}-unrefinably atomic; and $a = 1 \cdot a$ is a τ_{reg}-atomic, τ_{reg}-strongly atomic, τ_{reg}-m-atomic, and τ_{reg}-unrefinably atomic factorization of a. For R to be a τ-regular-atomic ring, we need only check the regular elements for τ-regular atomic factorizations. If $a \in \operatorname{Reg}(R)$, we apply Corollary 4.4 to see that a has a τ-regular-atomic factorization if and only if a has a τ_{reg}-atomic (respectively τ_{reg}-strongly atomic, τ_{reg}-m-atomic, τ_{reg}-unrefinably atomic) factorization. This completes the equivalence since we have checked both the zero-divisors as well as the regular elements. □

Lemma 4.7. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $a = \lambda(\lambda^{-1}a) = \mu(\mu^{-1}a)$ be two trivial factorizations of a. Then we have the following*:

  (i) $\lambda^{-1}a$ *and* $\mu^{-1}a$ *are associates.*
  (ii) $\lambda^{-1}a$ *and* $\mu^{-1}a$ *are strong associates.*

Proof. $(\mu^{-1}\lambda)(\lambda^{-1}a) = \mu^{-1}a$ with $(\mu^{-1}\lambda) \in U(R)$ proves $\lambda^{-1}a \approx \mu^{-1}a$. If $\lambda^{-1}a \approx \mu^{-1}a$, then $\lambda^{-1}a \sim \mu^{-1}a$. This proves both (ii) and (i). □

Remark 1. Given the above situation, $\lambda^{-1}a$ and $\mu^{-1}a$ need not be very strong associates. For instance $R = \mathbb{R} \times \mathbb{R}$,

$$(1,0) = (1,1)(1,0) = (-1,-1)(-1,0),$$

yet $(1,0) \not\cong (-1,0)$.

Theorem 4.8. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\tau_{\mathrm{reg}} := \tau \cap (\operatorname{Reg}(R) \times \operatorname{Reg}(R)).$$

Let

   $\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic}\}$

and

> $\beta \in \{\text{associate, strongly associate}\}.$

Then we have the following:

(i) *R satisfies τ-regular-ACCP if and only if R satisfies τ_{reg}-ACCP.*
(ii) *R is a τ-regular-UFR if and only if R is a τ_{reg}-α-β-UFR.*
(iii) *R is a τ-regular-HFR if and only if R is a τ_{reg}-α-HFR.*
(iv) *R is a τ-regular-BFR if and only if R is a τ_{reg}-BFR.*
(v) *R is a τ-regular-idf ring if and only if R is a τ_{reg}-α-β-df ring.*
(vi) *R is a τ-regular-atomic τ-regular-idf ring if and only if R is a τ_{reg}-α, τ_{reg}-α-β-df ring.*
(vii) *R is a τ-regular-WFFR if and only if R is a τ_{reg}-β-WFFR.*
(viii) *R is a τ-regular-FFR if and only if R is a τ_{reg}-β-FFR.*

If τ is refinable, then (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii).

Proof.

(i) The statement that $(a) \subsetneqq (a_1)$ with $a_1 \mid_\tau a$ implies that $a = \lambda a_1 a_2 \cdots a_n$. We notice here that $n \geq 2$ or else we would have $a = \lambda a_1$ or $a \approx a_1$ which implies $(a) = (a_1)$, a contradiction. So these properly ascending chains yield non-trivial factorizations at each step. Thus any properly ascending chain of principal ideals

$$(4.1) \qquad\qquad (a_1) \subsetneqq (a_2) \subsetneqq (a_3) \subsetneqq \cdots$$

such that $a_{i+1} \mid_{\tau_{\text{reg}}} a_i$ yields a τ-regular factorization of a_i with a_{i+1} as a τ-regular factor. Conversely, any ascending chain as in (4.1) with a_i regular for all i and a_{i+1} occurring as a τ-factor in some τ-regular factorization of a_i yields a τ_{reg}-factorization of a_i as well. Hence, R fails to satisfy τ-regular ACCP if and only if R fails to satisfy τ_{reg}-ACCP, and the proof is complete.

(ii) We know from Theorem 4.6 that R is τ-regular-α if and only if R is τ_{reg}-α. Let $a \in R$ be a non-unit. If $a \in Z(R)$, we know from Theorem 4.5 that a is τ_{reg}-α. Furthermore, any trivial τ_{reg}-factorization of a is unique up to β by Lemma 4.7. For R to be a τ-regular UFR, we need only check the regular elements. Let $a \in \text{Reg}(R)$. We know from Corollary 4.4, for regular elements, τ-atomic and τ_{reg}-α-factorizations of a coincide, so the uniqueness up to rearrangement and β is immediate.

(iii) By Theorem 4.6, R is τ-regular-α if and only if R is τ_{reg}-α. If $a \in Z(R)$, then a is τ_{reg}-α and has only trivial τ_{reg}-factorizations, each of which has length 1. For $a \in \mathrm{Reg}(R)$, τ-atomic and τ_{reg}-α-factorizations of a coincide by Corollary 4.4, and the equivalence is clear.

(iv) For $a \in Z(R)$, all τ_{reg}-factorizations are trivial and have length 1. By Lemma 4.2, the set of non-trivial τ-regular factorizations and τ_{reg}-factorizations coincide, and the equivalence is apparent.

(v) If $a \in Z(R)$, a itself is τ_{reg}-α, and there is precisely one unique τ_{reg}-α-divisor of a up to β since all trivial τ_{reg}-factorizations are β from Lemma 4.7. If $a \in \mathrm{Reg}(R)$, then the set of τ-regular atomic divisors and τ_{reg}-α-divisors of a are all regular and hence coincide by Theorem 4.3. So the equivalence is clear.

(vi) This is simply (v) plus Theorem 4.6.

(vii) For $a \in Z(R)$, the only τ_{reg}-divisors of a are unit multiples of a, so there is only one τ_{reg}-divisor of a up to β. For $a \in \mathrm{Reg}(R)$, since the set of τ-regular factorizations and the set of τ_{reg}-factorizations of a are the same, the set of τ_{reg}-divisors and τ-regular divisors coincide and are regular, so the associate relations also coincide. Thus the equivalence follows.

(viii) For $a \in Z(R)$, the only τ_{reg}-factorizations of a are of the form $a = \lambda(\lambda^{-1}a)$, so there is only one τ_{reg}-factorization of a up to β. For $a \in \mathrm{Reg}(R)$, the set of τ-regular factorizations and the set of τ_{reg}-factorizations of a are the same. Moreover, the set of τ_{reg}-factors and τ-regular factors coincide and are regular; hence, the associate relations also coincide. Thus, the equivalence follows. $\qquad\square$

**5. Relationship with other finite factorization properties.** In this final section, we would like to demonstrate where the rings satisfying the properties in the present article fit in with the various finite factorization properties already existing in the literature, that is, we would like to compare the τ-regular and τ_{reg}-finite factorization properties with the regular factorization from [**10**], the τ-finite factorization properties defined originally in [**42**] as well as the τ-U-finite factorization properties defined in [**43**]. As a note to the reader, many of these terms were defined in Section 2.

The following theorem demonstrates that the τ-finite factorization properties defined in [**42**] are stronger than the ones in the present article.

Theorem 5.1. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$, $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Then we have the following*:

  (i) *If R is a τ-α-β-UFR, then R is a τ-r-UFR.*
 (ii) *If R is a τ-α-HFR, then R is a τ-r-HFR.*
(iii) *If R is a τ-β-FFR, then R is a τ-r-FFR.*
 (iv) *If R is a τ-β-WFFR, then R is a τ-r-WFFR.*
  (v) *If R is a τ-β-α-df ring, then R is a τ-r-idf ring.*
 (vi) *If R is a τ-BFR, then R is a τ-r-BFR.*
(vii) *If R satisfies τ-ACCP, then R satisfies τ-r-ACCP.*
(viii) *If R is τ-α, then R is τ-r-atomic.*

This yields the following diagram where ∇ represents τ refinable.
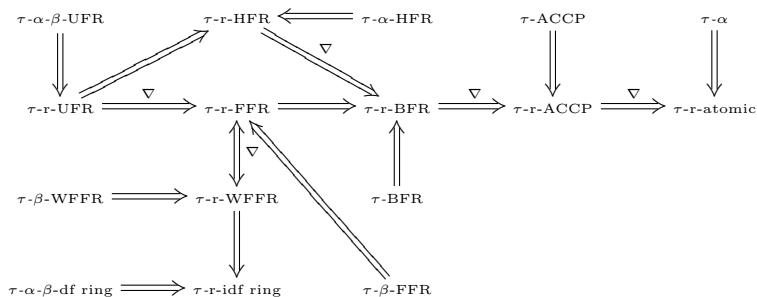
Proof.

(viii) Let $a \in \operatorname{Reg}(R)$. Since R is a τ-α, there is a τ-α-factorization of the form $a = \lambda a_1 \cdots a_n$. Since $a \in \operatorname{Reg}(R)$, $a_i \in \operatorname{Reg}(R)$ for all i, by Proposition 3.1, each of these factorizations is a τ-r-atomic factorization of a, showing R is τ-r-atomic.

(ii) (respectively (i)). Let a be a regular non-unit element. We have just seen that R is τ-r-atomic. Given two τ-r-atomic factorizations, $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$, there are also two τ-α-factorizations. By assumption, we have $m = n$ (respectively and there is a rearrangement

so that $a_i \sim b_i$ for each $1 \leq i \leq n$). This proves R is a τ-r-HFR (respectively τ-r-UFR).

(iii)–(vi). Let $a \in \mathrm{Reg}\,(R)$. For a regular element a, the set of τ-r-factorizations and τ-factorizations are identical, proving (iii) and (vi). Similarly, since every divisor of a regular element is regular, the set of regular τ-divisors is the same as the set of τ-divisors, proving (iv). As in 3.1, we know that the set of τ-α-divisors is the same as the set of τ-r-atoms, proving (v).

(vii) Suppose $(a_1) \subsetneq (a_2) \subsetneq \cdots$ is a chain of regular principal ideals such that $a_{i+1} \mid_\tau a_i$. Then, since R satisfies τ-ACCP, it must become stationary, proving (vii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following gives us a comparison of the regular factorization rings defined in [**10**] with the rings defined in the current article.

Theorem 5.2. *Let R be a commutative ring with 1 and $\tau \subset \mathrm{Reg}\,(R)^{\#} \times \mathrm{Reg}\,(R)^{\#}$:*

  (i) *R an r-BFR implies R is a τ-r-BFR,*
 (ii) *R an r-FFR implies R is a τ-r-FFR,*
(iii) *R an r-WFFR implies R is a τ-r-WFFR,*
(iv) *R satisfies r-ACCP implies R satisfies τ-r-ACCP.*

  Proof.

  (i) Let R be an r-BFR, but suppose R is not a τ-r-BFR. Then there exists a regular element $a \in \mathrm{Reg}\,(R)^{\#}$ with τ-factorizations of arbitrarily long length, but any τ-factorization is certainly a factorization into regular elements, so this would contradict the fact that R is an r-BFR.
 (ii) Let R be an r-FFR, but suppose that R is not a τ-r-FFR. We then have a regular element $a \in \mathrm{Reg}\,(R)^{\#}$ that has an infinite number of τ-r-factorizations up to rearrangement and associate, but again each of these is also an r-factorization and are still unique up to rearrangement and associates, which contradicts the fact that R is an r-FFR.
(iii) Let $a \in \mathrm{Reg}\,(R)^{\#}$. Every τ-r-divisor divisor is a regular divisor of a, so there can only be finitely many up to associate.

(iv) Suppose we have an infinite sequence $\{a_i\}_{i=1}^{\infty}$, $a_k \in \mathrm{Reg}\,(R)^{\#}$ for all k with $a_{n+1} \mid_{\tau} a_n$ but $a_{n+1} \not\sim a_n$ for all $n \geq 1$. But then we still have $a_{n+1} \mid_{\tau} a_n$, $a_k \in \mathrm{Reg}\,(R)^{\#}$ for all k but $a_{n+1} \not\sim a_n$, so we contradict r-ACCP, concluding the proof. □

Corollary 5.3. *The* r-UFRs, r-FFRs, r-HFRs *and* r-BFRs *as defined in* [**1**, Section 5] *satisfy* r-ACCP, *and therefore* τ-r-ACCP. *Hence, for* τ *refinable, each is* τ-r-*atomic by Theorems* 3.4 *and* 5.2.

The following diagram summarizes our results (∇ represents τ refinable):
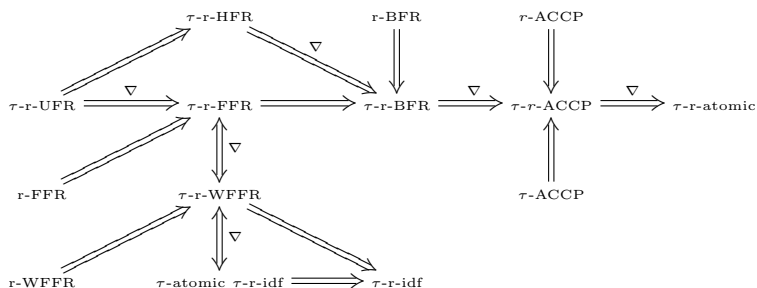
Lemma 5.4. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{\emptyset$, atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$. Every non-unit element in a τ_{reg}-U-α-factorization is an essential divisor. Moreover, given a τ_{reg}-α-factorization, every τ-factor is essential. When $\alpha = \emptyset$, we mean simply a τ_{reg}-U-factorization.*

Proof. Let $a \in R$ be a non-unit, and let

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$$

be a τ_{reg}-U-α-factorization. Then

$$a = \lambda a_1 \cdots a_n b_1 \cdots b_m$$

is a τ_{reg}-factorization. If there is only one τ_{reg}-factor in the factorization, i.e., $m + n = 1$, then this factor is certainly essential. If it were removed, then it would imply that a were a unit, a contradiction. We

now may assume that $m + n \geq 2$, and therefore

$$a = \lambda a_1 \cdots a_n b_1 \cdots b_m$$

is a τ_{reg}-factorization implying that a is a product of regular elements and hence is regular. Moreover, we have $(a) = (b_1 \cdots b_m)$ so $ar = b_1 \cdots b_m$ for some $r \in R$. Hence,

$$a = \lambda a_1 \cdots a_n \cdot a \cdot r$$

and a is regular so cancellation implies that

$$1 = \lambda a_1 \cdots a_n \cdot r,$$

and in particular $a_i \in U(R)$ for all $1 \leq i \leq n$. Hence, there can be no non-unit inessential τ_{reg}-divisors as desired.

Given a τ_{reg}-α-factorization of a non-unit $a \in R$, say $a = \lambda a_1 \cdots a_n$, we show that a_i is essential for each $1 \leq i \leq n$. If $n = 1$, this is immediate as above. Thus, $n \geq 2$, and therefore, a_i is regular for each $1 \leq i \leq n$. Suppose for a moment that a_i were not essential. Then

$$(a) = (a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n) = (a_1 \cdots a_n).$$

But this means that there is an $r \in R$ such that

$$a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n = r \cdot a_1 \cdots a_n.$$

After canceling common factors, since each element on the left is regular, we see that $1 = r \cdot a_i$, which means $a_i \in U(R)$, a contradiction since each $a_j \in R^{\#}$ for all $1 \leq j \leq n$. Thus, a_i is essential for each $1 \leq i \leq n$ and $\lambda \lceil a_1 \cdots a_n \rceil$ is indeed a τ-U-α factorization. $\qquad\square$

The consequence of this lemma is that we see that τ_{reg}-α-factorizations and τ_{reg}-U-α-factorizations coincide and we see there is a correspondence between the sets, given by the map

$$\phi : \{\tau_{\mathrm{reg}}\text{-U-}\alpha\text{-factorizations}\} \longrightarrow \{\tau_{\mathrm{reg}}\text{-}\alpha\text{-factorizations}\}$$

is given by

$$\lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil \longmapsto (\lambda a_1 \cdots a_n) b_1 \cdots b_m,$$

and the inverse

$$\phi^{-1} : \{\tau_{\mathrm{reg}}\text{-}\alpha\text{-factorizations}\} \longrightarrow \{\tau_{\mathrm{reg}}\text{-U-}\alpha\text{-factorizations}\}$$

is given by

$$\lambda a_1 \cdots a_n \longmapsto \lambda \lceil a_1 \cdots a_n \rceil .$$

This observation allows us to further consolidate many of our finite factorization properties when it comes to regular factorization. In particular, we formalize this by way of the following result.

Theorem 5.5. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{\mathrm{reg}} := \tau \cap (\mathrm{Reg}\,(R) \times \mathrm{Reg}\,(R))$. Let*

$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic,}$

$\text{very strongly atomic}\}$

and

$\beta \in \{\text{associate, strongly associate, very strongly associate}\}.$

Then, for any choice of α and β, we have the following:

(i) *R is τ_{reg}-U-α if and only if R is τ_{reg}-α.*
(ii) *R satisfies τ_{reg}-U-ACCP if and only if R satisfies τ_{reg}-ACCP.*
(iii) *R is a τ_{reg}-U-α-β-UFR if and only if R is a τ_{reg}-α-β-UFR.*
(iv) *R is a τ_{reg}-U-α-HFR if and only if R is a τ_{reg}-α-HFR.*
(v) *R is a τ_{reg}-U-BFR if and only if R is a τ_{reg}-BFR.*
(vi) *R is a τ_{reg}-U-α-β-df ring if and only if R is a τ_{reg}-α-β-df ring.*
(vii) *R is a τ_{reg}-U-α, τ_{reg}-U-α-β-df ring if and only if R is a τ_{reg}-α, τ_{reg}-α-β-df ring.*
(viii) *R is a τ_{reg}-U-β-WFFR if and only if R is a τ_{reg}-β-WFFR.*
(ix) *R is a τ_{reg}-U-β-FFR if and only if R is a τ_{reg}-β-FFR.*

If τ is refinable, then (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix).

Proof. (i) ($\Rightarrow$). Let $a \in R$ be a non-unit. Then there is a τ_{reg}-U-α factorization of a, by Lemma 5.4. This factorization is of the form $a = \lambda \lceil a_1 \cdots a_n \rceil$. By definition, $a = \lambda a_1 \cdots a_n$ is a τ_{reg}-factorization and a_i is τ_{reg}-α for each $1 \leq i \leq n$, and therefore, this is a τ_{reg}-α-factorization of a.

($\Leftarrow$). This is shown in [**43,** Theorem 4.3].

(ii) ($\Rightarrow$). Let $a \in R$ be a non-unit. Suppose there was an ascending chain of principal ideals of the form $(a) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \cdots$, such that

$a_{i+1} \mid_{\tau_{\text{reg}}} a_i$ for each i. Say the τ_{reg}-factorization for each i is given by

$$a_i = \lambda a_{i+1} a_{i1} \cdots a_{in_i},$$

because $(a_i) \subsetneq (a_{i+1})$, we know that this τ_{reg}-factorization is non-trivial, and therefore, each τ_{reg}-factor is regular. In particular, a_i is regular, and therefore by Lemma 5.4, is essential. This would contradict the fact that R satisfies τ_{reg}-U-ACCP.

($\Leftarrow$). This is shown in [**43,** Theorem 4.3].

(iii) (respectively (iv)). Let $a \in R$ be a non-unit. Then, by Lemma 5.4, a has a τ_{reg}-U-α factorization if and only if a has a τ_{reg}-α-factorization. Furthermore, since the τ_{reg}-U-factorizations have no inessential divisors, it is clear that the equivalence of the uniqueness (respectively constant length) of these factorizations follows as well.

(v) and (ix). Let $a \in R$ be a non-unit. By Lemma 5.4, the correspondence shows that we may apply ϕ^{-1} to any τ_{reg}-factorization of a of length n and get a τ_{reg}-U-factorization with the same n τ_{reg}-factors all occurring as the τ_{reg} essential divisors in the corresponding τ_{reg}-U-factorization. Similarly, given a τ_{reg}-U-factorization with n essential divisors, we may apply ϕ to this factorization and get a τ_{reg}-factorization of length n with the same τ_{reg}-factors as the essential τ_{reg}-divisors. Hence, there is a bound on the length of the number of essential divisors in any τ_{reg}-U-factorization of a if and only if there is a bound on the length of any τ_{reg}-factorization of a. Moreover, this same correspondence shows that there are the same number of τ_{reg}-factorizations of a up to β as there are τ_{reg}-U-factorizations of a up to β.

(vi) (respectively (viii)). Let $a \in R$ be a non-unit. As in the proof of (v) and (ix), it is clear that the set of τ_{reg}-divisors and essential τ_{reg}-divisors of a are the same by the correspondence given in Lemma 5.4 and map ϕ. This means the set of τ_{reg}-divisors of a and essential τ_{reg}-divisors of a up to β are the same. Moreover, this also means that the set of τ_{reg}-α divisors and the set of τ_{reg}-α-essential divisors are the same up to β as well.

(vii). This follows immediately by combining the results of parts (i) and (vi). $\square$

We can further relate the various properties by removing the very strongly atomic choice for α and the very strongly associate choice for β in Theorem 5.5. The results are combined below.

Corollary 5.6. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Let $\tau_{\text{reg}} := \tau \cap (\text{Reg}\,(R) \times \text{Reg}\,(R))$. Let*

$$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic}\}$$

and

$$\beta \in \{\text{associate, strongly associate}\}.$$

Then, for any choice of α and β, we have the following:

(i) *R is τ_{reg}-U-α if and only if R is τ_{reg}-α if and only if R is τ-regular-atomic.*
(ii) *R satisfies τ_{reg}-U-ACCP if and only if R satisfies τ_{reg}-ACCP if and only if R satisfies τ-regular-ACCP.*
(iii) *R is a τ_{reg}-U-α-β-UFR if and only if R is a τ_{reg}-α-β-UFR if and only if R is τ-regular-UFR.*
(iv) *R is a τ_{reg}-U-α-HFR if and only if R is a τ_{reg}-α-HFR if and only if R is τ-regular-HFR.*
(v) *R is a τ_{reg}-U-BFR if and only if R is a τ_{reg}-BFR if and only if R is τ-regular-BFR.*
(vi) *R is a τ_{reg}-U-α-β-df ring if and only if R is a τ_{reg}-α-β-df ring if and only if R is a τ-regular-idf ring.*
(vii) *R is a τ_{reg}-U-α, τ_{reg}-U-α-β-df ring if and only if R is a τ_{reg}-α, τ_{reg}-α-β-df ring if and only if R is τ-regular-atomic, τ-regular-idf ring.*
(viii) *R is a τ_{reg}-U-β-WFFR if and only if R is a τ_{reg}-β-WFFR if and only if R is τ-regular-WFFR.*
(ix) *R is a τ_{reg}-U-β-FFR if and only if R is a τ_{reg}-β-FFR if and only if R is τ-regular-FFR.*

If τ is refinable, then (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix).

Proof. The first equivalence in each statement (i) for $1 \leq i \leq 9$ follows directly from Theorem 5.5. Similarly, the second equivalence in each statement (i) for $1 \leq i \leq 9$ follows from Theorem 4.8. $\square$

We conclude the article with a diagram which summarizes many of the equivalences and relationships demonstrated thus far where τ_{reg} is defined as above,

$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic}\}$,

$\beta \in \{\text{associate, strongly associate}\}$,

and ∇ represents τ refinable.
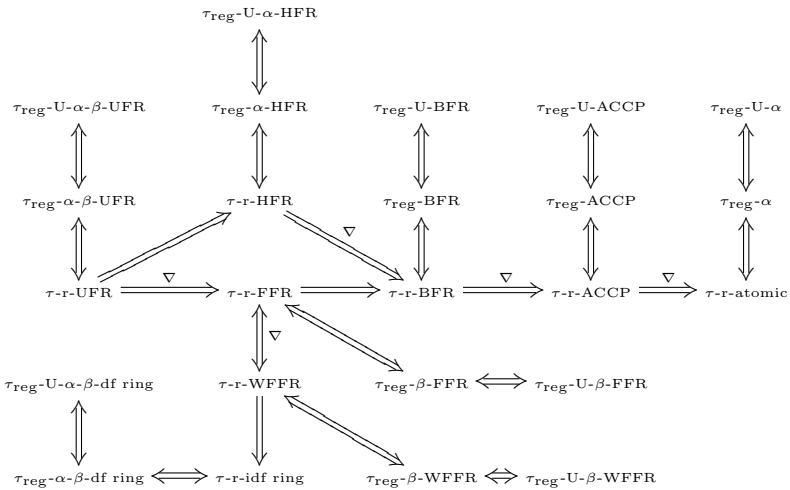
6. Analysis of two explicit rings and further questions. In this section, we explore some rings which have been studied extensively in the literature to see which of the τ-r-finite factorization properties these rings satisfy. Many authors have pursued the theory of factorization in integral domains by way of ideal theory. In the case of integral domains studying factorization from an element perspective is the same as studying the principal ideal structure. When dealing with commutative rings with zero-divisors, these two approaches diverge. There is a substantial amount of research done on ideal theoretic properties of commutative rings with zero-divisors. For instance, the monograph of Huckaba [**32**] uses this approach. The theory of Krull, Mori, Prüfer, Marot, v-Marot, and C-rings has been studied extensively in [**18, 19, 20, 30, 31, 34, 35, 36, 37, 38, 39, 40, 45**].

We supply some results regarding the τ-regular factorization properties of the two very explicit rings recently studied in [**21, 26**]. We hope that this will serve as an example of the kinds of questions we hope to eventually be able to answer about a larger class of rings that have been of interest over the years by many authors studying commutative rings with zero-divisors.

We begin with the self-idealization of a principal ideal domain (PID) studied in [**21**]. Let D be a PID, and let

$$R(D) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \ \middle| \ a, b \in D \right\}.$$

Then, it is well known that $R(D)$ is a Noetherian commutative ring with identity. We include definitions in the theorem for convenience for the reader. An *irreducible* or *atom* is a non-zero non-unit such that $a = bc$ implies $(a) = (b)$ or $(a) = (c)$. Moreover, $a, b \in R$ are said to be *associates* if $(a) = (b)$. R is said to be *atomic* if every non-zero non-unit has a factorization into irreducible elements. Chang and Smertnig further investigate factorization properties of these rings, which we summarize below.

Theorem 6.1. ([**21,** Corollary 9]). *Let D be a* PID. *Then we have the following*:

(i) $R(D)$ *is Noetherian.*
(ii) $R(D)$ *is a* BFR (*R is atomic and there is a bound on the length of irreducible factorizations of every non-zero non-unit of R*).
(iii) $R(D)$ *is an* FFR (*R is atomic and each nonzero non-unit has only nitely many factorizations into irreducibles, up to order and associates*) *if and only if D/pD is finite for all prime elements $p \in D$.*
(iv) *If D is a field, then every non-zero non-unit of $R(D)$ is a prime, and hence $R(D)$ is a* UFR (*R is atomic and every non-zero principal ideal of R can be written as a product of principal prime ideals*) *with a unique nonzero* (*prime*) *ideal.*

We will use these observations to give examples of rings which will satisfy various τ-regular finite factorization properties studied in

the present article as well as other τ-finite factorization properties previously studied in the literature.

Corollary 6.2. *Let D be a PID, and let τ be any symmetric relation on $R(D)^{\#}$. Let $\tau_{\mathrm{reg}} := \tau \cap (\mathrm{Reg}\,(R) \times \mathrm{Reg}\,(R))$. Let*

$$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic}\}$$

and

$$\beta \in \{\text{associate, strongly associate}\}.$$

Then, for any choice of α and β, we have the following:

(i) *$R(D)$ satisfies τ-ACCP, τ-U-ACCP and τ-r-ACCP (equivalently, τ_{reg}-U-ACCP or τ_{reg}-ACCP).*

(ii) *If τ is refinable, then $R(D)$ is τ-atomic, τ-U-atomic, and τ-r-atomic (equivalently, τ_{reg}-U-α or τ_{reg}-α).*

(iii) *$R(D)$ is a τ-BFR, τ-U-BFR, τ-r-BFR (equivalently, τ_{reg}-U-BFR or τ_{reg}-BFR).*

(iv) *If D/pD is finite for all prime elements $p \in D$, then $R(D)$ is a τ-associate-FFR, τ-U-associate-FFR and τ-r-FFR (equivalently, τ_{reg}-U-β-FFR or τ_{reg}-β-FFR).*

(v) *If D is a field, the $R(D)$ is a τ-atomic-associate-UFR, τ-U-atomic-associate-UFR, τ-r-UFR (equivalently, τ_{reg}-U-α-β-UFR or τ_{reg}-α-β-UFR).*

Proof. We begin by noting the equivalent part of each of the above follows directly from Corollary 5.6.

(i) Since $R(D)$ is Noetherian by Theorem 6.1, $R(D)$ satisfies ACCP. It was shown in [**42,** Theorem 4.1] that ACCP implies τ-ACCP for any τ. By [**43,** Theorem 4.3], τ-ACCP implies τ-U-ACCP. Finally, by Theorem 5.1, we see that τ-ACCP implies τ-r-ACCP.

(ii) By [**42,** Theorem 4.1], since τ is assumed to be refinable, τ-ACCP implies τ-atomic. By [**43,** Theorem 4.3], we have τ-atomic implies τ-U-atomic. Finally, by Theorem 5.1, τ-atomic implies τ-r-atomic.

(iii) $R(D)$ is a BFR which satisfies ACCP. $R(D)$ satisfies ACCP, so certainly $R(D)$ is atomic. An atomic ring which has a bound on the length of atomic factorizations of every non-zero non-unit certainly has

a bound on the length of all factorizations. We can simply use the same bound since given any factorization, we could replace the factors with atomic factorizations and get a possibly even longer factorization which is atomic. Hence, we can be sure $R(D)$ satisfies the notion of BFR used in [**42**], where there is a bound on the length of all factorizations of each non-zero non-unit. Let $a \in R(D)$ be a non-zero, non-unit. Any τ-factorization of a is certainly a factorization of a, so we may again use the same bound and this proves $R(D)$ is a τ-BFR. By [**43,** Theorem 4.3], τ-BFR implies τ-U-BFR and, by Theorem 5.1, we see that τ-BFR implies τ-r-BFR.

(iv) We again point out that an atomic FFR using the definition from [**21**] coincides with the definition used in [**42**] since any factorization can be refined into an atomic factorization. Thus, given any non-zero non-unit $a \in R$, the set of atomic factorizations of a up to rearrangement and associate is finite if and only if the set of factorizations of a is finite. If D/pD is finite for all prime elements $p \in D$, then $R(D)$ is an FFR. For every non-zero non-unit $a \in R(D)$, there are a finite number of factorizations of a up to rearrangement and associate. Every τ-factorization of a is certainly among the factorizations of a, and hence the set of τ-factorizations up to rearrangement and associate is finite proving $R(D)$ is a τ-associate-FFR. By [**43,** Theorem 4.3], we have τ-associate-FFR implies τ-U-associate-FFR. By Theorem 5.1, we see that τ-associate-FFR implies τ-r-FFR.

(v) If D is a field, then $R(D)$ is a UFR. In general, this is not necessarily enough to conclude that R is a τ-atomic-associate-UFR; however, in this case, we actually have a stronger condition that every non-zero, non-unit is prime. Prime implies irreducible, but not necessarily any of the stronger forms of irreducible. We show that irreducible implies τ-irreducible. Let $a = \lambda a_1 \cdots a_n$ be a τ-factorization. Then this is a factorization, so $a \sim a_i$ for some $1 \leq i \leq n$. Hence a is τ-atomic. This shows that every non-zero non-unit of $a \in R(D)$ is a τ-irreducible element. This means $a = 1 \cdot a$ is a τ-atomic factorization of a. This is also an atomic factorization. $R(D)$ is known to be a UFR. Suppose there were another τ-irreducible factorization of a, say $a = \lambda b_1 \cdots b_m$. Every b_i is a non-zero, non-unit of $R(D)$, and therefore prime and irreducible. This would yield a distinct irreducible factorization of a, contradicting the fact that $R(D)$ is an atomic-associate-UFR, a contradiction. This proves $R(D)$ is a τ-

atomic-associate-UFR. Since every τ-factorization has length 1, we can also be sure that every τ-factor is essential, and therefore a τ-atomic-U-factorization. This proves $R(D)$ is a τ-atomic-associate-UFR. By Theorem 5.1, we see that τ-atomic-associate-UFR implies τ-r-UFR, completing the proof. $\qquad\square$

Corollary 6.3. *Let D be a field, and let τ be a symmetric relation on $R(D)^{\#}$. Let*

$$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic}\}$$

and

$$\beta \in \{\text{associate, strongly associate}\}.$$

Then, for any choice of α and β, we have the following:

(i) $R(D)$ *is a τ-r-UFR (respectively τ_{reg}-U -α-β-UFR, τ_{reg}-α-β-UFR, τ-atomic-associate-UFR, τ-U-atomic-associate-UF).*

(ii) $R(D)$ *is a τ-r-HFR (respectively τ_{reg}-U-α-HFR, τ_{reg}-α-HFR, τ-atomic-HFR, τ-U-atomic-HFR).*

(iii) $R(D)$ *is a τ-r-BFR (respectively τ_{reg}-U-BFR, τ_{reg}-BFR, τ-BFR, τ-U-BFR).*

(iv) $R(D)$ *is a τ-r-FFR (respectively τ_{reg}-U-β-FFR, τ_{reg}-β-FFR, τ-associate-FFR, τ-U-associate-FFR).*

(v) $R(D)$ *is a τ-r-WFFR (respectively τ_{reg}-U-β-WFFR, τ_{reg}-β-WFFR, τ-associate-WFFR, τ-U-associate-WFFR).*

(vi) $R(D)$ *is a τ-r-atomic (respectively τ_{reg}-U-α, τ_{reg}-α, τ-atomic, τ-U-atomic) τ-r-idf ring (respectively τ_{reg}-U-α-β-df ring, τ_{reg}-α-β-df ring, τ-atomic-associate-idf ring, τ-U-atomic-associate-idf ring).*

(vii) $R(D)$ *satisfies τ-r-ACCP (respectively τ_{reg}-U-ACCP, τ_{reg}-ACCP, τ-ACCP, τ-U-ACCP).*

(viii) $R(D)$ *is τ-r-atomic (respectively τ_{reg}-U-α, τ_{reg}-α, τ-atomic, τ-U-atomic).*

Proof. These are immediate from the observation in Corollary 6.2 (v) as well as Corollary 5.6. $\qquad\square$

We now turn our attention to the ring $\mathbb{Z}_{p^n}[X]$ which was studied extensively in [**26**]. In this article the authors choose the definitions

weakly irreducible and weakly associate for what we have called *irreducible* and *associate*, respectively. They use the terms irreducible and associate for what we have called *strongly irreducible* and *strongly associate*, respectively. We will soon see that the ring in question is présimplifiable and therefore certainly strongly associate, and our definitions will be equivalent (in fact, very strongly atomic, unrefinably atomic and m-atomic are also equivalent here), so there will be no confusion for $\mathbb{Z}_{p^n}[X]$.

Many of the key factorization properties of this ring derive from the following fact [**26,** Fact 1.5], where $\mathrm{Nil}\,(R)$ is the nilradical, $J(R)$ is the Jacobson radical and $Z(R)$ is the set of zero-divisors.

$$\mathrm{Nil}\,(\mathbb{Z}_{p^n}[X]) = J(\mathbb{Z}_{p^n}[X]) = (p) = Z(\mathbb{Z}_{p^n}[X]).$$

If the zero-divisors are contained in the Jacobson radical, then the ring is présimplifiable. This is discussed in many sources, but for τ-factorization, the author suggests [**42,** Lemma 2.2] and [**44**]. As mentioned following Theorem 2.2, the présimplifiable property has the effect of making all of the associate relations coincide, which in turn makes irreducible and very strongly irreducible (hence everything in between) as well as τ-irreducible and τ-very strongly irreducible equivalent. As in [**43,** Theorem 3.5], this also has the added benefit of making every non-unit τ-factor essential, so τ-U-factorizations will also coincide with the usual factorizations.

We now summarize several of the results about $\mathbb{Z}_{p^n}[X]$ from [**26**] which will be of interest in order to draw conclusions about which τ-regular finite factorization properties this ring will satisfy.

Theorem 6.4 ([**26**])**.** *Let* $R = \mathbb{Z}_{p^n}[X]$. *Then we have the following*:

   (i) *R is Noetherian,*
  (ii) *R is atomic,*
 (iii) *R has infinite elasticity,*
 (iv) *each regular element of R has a finite number of factorizations into irreducibles.*

These results will allow us to answer several questions about the factorization properties discussed in this article.

Theorem 6.5. *Let $R = \mathbb{Z}_{p^n}[X]$, and let τ be a symmetric relation on $R^{\#}$. Let*

$$\alpha \in \{\text{atomic, strongly atomic, m-atomic, unrefinably atomic,}$$
$$\text{very strongly atomic}\},$$

and let

$$\beta \in \{\text{associate, strongly associate, very strongly associate}\}.$$

(i) *R satisfies τ-ACCP, τ-U-ACCP, τ-r-ACCP and τ_{reg}-ACCP.*

(ii) *R is r-atomic, and if τ is refinable, then R is τ-α, τ-U-α, τ-r-atomic, τ_{reg}-α and τ_{reg}-U-α.*

(iii) *R is an r-FFR, τ-r-FFR, τ_{reg}-β-FFR and τ_{reg}-U-β-FFR.*

(iv) *R is an r-WFFR, τ-r-WFFR, τ_{reg}-β-WFFR and τ_{reg}-U-β-WFFR.*

(v) *R is an r-idf ring, τ-r-idf ring, τ_{reg}-U-α-β-df ring and a τ_{reg}-α-β-df ring.*

(vi) *R is a r-atomic r-idf ring and, if τ is refinable, then R is a τ-r-atomic τ-r-idf ring, a τ_{reg}-U-α τ_{reg}-U-α-β-df ring, and a τ_{reg}-α τ_{reg}-α-β-df ring.*

Proof. We begin by noting that, since R is présimplifiable, all the associate relations and irreducibles coincide. This allows us to add very strongly associate and very strongly atomic to the choices of α and β regarding τ_{reg}-factorizations and to use the equivalences laid forth in Corollary 5.6.

The proof of (i) and (ii) follows from the same argument as for (i) and (ii) from Corollary 6.2 and from the fact that R is Noetherian.

(iii) Let $a \in \text{Reg}\,(R)^{\#}$. Then, since we know R is atomic and there are finitely many irreducible factorizations of a up to rearrangement and associate from Corollary 6.4, we can be sure that there are only finitely many factorizations of a up to rearrangement and associate. This makes R an r-FFR. If there are finitely many regular factorizations of a, then we can be sure that R is also a τ-r-FFR since every τ-r-factorization is among the finitely many r-factorizations (up to rearrangement and associate).

(iv) Since we have established that R is an r-FFR, there are only finitely many factorizations of any $a \in \text{Reg}\,(R)^{\#}$ up to rearrangement

and associates. This means there are only finitely many divisors of a up to associate, making R an r-WFFR. If there are only finitely many divisors of a up to associate, then there are certainly only finitely many τ-divisors of a up to associate, making R a τ-r-WFFR.

(v) Let $a \in \text{Reg}(R)^{\#}$. Then a has a finite number of factors and τ-factors up to associate. Then a certainly has a finite number of irreducible divisors and τ-irreducible τ-divisors up to associate. Thus, R is both an r-idf ring and a τ-r-idf ring.

(vi) This follows immediately from (ii) and (v). $\square$

As a consequence of Theorem 6.4 (iii), we have the following.

Corollary 6.6. *Let* $R = \mathbb{Z}_{p^n}[X]$, *and let* τ *be a symmetric relation on* $R^{\#}$. *Then*:

  (i) *R is neither an* HFR *nor a* UFR.
 (ii) *R need not be a τ-HFR, τ-U-HFR, τ-UFR nor a τ-U-UFR.*

  Proof.

  (i) By Theorem 6.4 (iii), we know that R has infinite elasticity. As in
       [**26**], this means that there are non-zero, non-unit elements $a \in R$
       such that a can be factored into a product of irreducible elements
       of different lengths. This forces R to fail to be an HFR or a UFR.
 (ii) Because R is présimplifiable, τ-factorizations and τ-U-factoriza-
       tions coincide since every factor is essential. If we were to set
       $\tau_d := R^{\#} \times R^{\#}$, then we would get the usual factorizations. This
       would yield an example of a τ for which R is neither τ-HFR,
       τ-U-HFR, τ-UFR nor τ-U-UFR. $\square$

Our hope is that this type of analysis could be performed on the enormous class of rings that have been studied by many authors mentioned in the introduction to this section. It would be fascinating to know which of the various classes of rings satisfy the equivalences given in the present paper in a similar fashion as above. As one can see by only looking at these two very explicit rings in this section, there were a number of results regarding their regular factorization and τ-regular factorization properties.

Many of the major breakthroughs in factorization in integral domains have come through studying ideal theory. In the integral domain case, studying factorization through principal ideals and elements coincide; however, this is not the case in rings with zero-divisors. Many authors have sought to study factorization in rings with zero-divisors by way of generalizing various types of rings defined through their ideal theory. To name a few, we have Krull, Mori, Prüfer, Marot, v-Marot, and C-rings, which been studied extensively in [**18, 19, 20, 30, 31, 34, 35, 36, 37, 38, 39, 40, 45**].

Many of these rings are well studied in the case of integral domains; however, as is often the case when extending definitions to rings with zero-divisors, many formerly equivalent definitions diverge. When dealing with regular elements as in the present article, there is hope that many of these definitions may again return to being equivalent, as we have seen thus far. It would be interesting for future research to answer many of the questions which arise about which of the various equivalences laid forth in this article will hold for the different types of rings mentioned above. Moreover, as suggested by the referee, much of this work could be reframed in the context of cancellative semigroups where research abounds in the literature. This may help to connect the present research with much of the research done on Krull rings. Unfortunately, the author fears the scope of the aforementioned project is too large to take on in the present article.

REFERENCES

1. Ahmet G. Ağargün, D.D. Anderson and Silvia Valdez-Leon, *Factorization in commutative rings with zero divisors*, III, Rocky Mountain J. Math. **31** (2001), 1–21.

2. M. Alan and A.G. Ağargün, *Some factorization properties in commutative rings*, Int. J. Pure Appl. Math. **18** (2005), 349–358.

3. D.D. Anderson, ed., *Factorization in integral domains*, Lect. Notes Pure Appl. Math. **189**, Marcel Dekker, New York, 1997.

4. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Alg. **69** (1990), 1–19.

5. D.D. Anderson, M. Axtell, S.J. Forman and Joe Stickles, *When are associates unit multiples?*, Rocky Mountain J. Math. **34** (2004), 811–828.

6. D.D. Anderson and S. Chun, *Irreducible elements in commutative rings with zero-divisors*, Rocky Mountain J. Math. **37** (2011), 741–744.

7. D.D. Anderson and Andrea M. Frazier, *On a general theory of factorization in integral domains*, Rocky Mountain J. Math. **41** (2011), 663–705.

8. D.D. Anderson and R. Markanda, *Unique factorization in rings with zero divisors*, Houston J. Math. **11** (1985), 15–30.

9. D.D. Anderson and Silvia Valdes-Leon, *Factorization in commutative rings with zero divisors*, Rocky Mountain J. Math. **26** (1996), 439–480.

10. _____, *Factorization in commutative rings with zero divisors*, II, in *Factorization in integral domains*, Lect. Notes Pure Appl. Math. **189**, Dekker, New York, 1997.

11. M. Axtell, *U-factorizations in commutative rings with zero divisors*, Comm. Alg. **30** (2002), 1241–1255.

12. M. Axtell, S. Forman, N. Roersma and J. Stickles, *Properties of U-factorizations*, Inter. J. Comm. Rings **2** (2003), 83–99.

13. N.R. Baeth and R. Wiegand, *Factorization theory and decomposition of modules*, Amer. Math. Month. **120** (2013), 3–34.

14. A. Bouvier, *Sur les anneaux de fractions des anneaux atomiques présimplifables*, Bull. Sci. Math. **95** (1971), 371–376.

15. _____, *Anneaux présimplifables*, C.R. Acad. Sci. Paris **274** (1972), A1605–A1607.

16. _____, *Résultats nouveaux sur les anneaux présimplifables*, C.R. Acad. Sci. Paris **275** (1972), A955–A957.

17. _____, *Anneaux présimplifables*, Rev. Roum. Math. Pures Appl. **19** (1974), 713–724.

18. Gyu Whan Chang, *Characterizations of a Krull ring $R[X]$*, Bull. Korean Math. Soc. **28** (2001), 543–549.

19. _____, *Eakin-Nagata theorem for commutative rings whose regular ideals are finitely generated*, Korean J. Math. **18** (2010), 271–275.

20. Gyu Whan Chang and Byung Gyun Kang, *Integral closure of a ring whose regular ideals are finitely generated*, J. Algebra **251** (2002), 529–537.

21. Gyu Whan Chang and D. Smertnig, *Factorization in the self-idealization of a PID*, Boll. Union. Mat. Ital. **9** (2013), 363–377.

22. S.T. Chapman, ed., *Arithmetical properties of commutative rings and monoids*, Lect. Notes Pure Appl. Math. **241**, Chapman & Hall/CRC, Boca Raton, 2005.

23. L.E. Dickson, *Finiteness of the odd perfect and primitive abundant numbers with distinct factors*, Amer. J. Math. **35** (1913), 413–442.

24. C.R. Fletcher, *Unique factorization rings*, Math. Proc. Cambr. Philos. Soc. **65** (1969), 579–583.

25. _____ , *The structure of unique factorization rings*, Math. Proc. Cambr. Philos. Soc. **673** (1970), 535–540.

26. C. Frei and S. Frisch, *Non-unique factorization of polynomials over residue class rings of the integers*, Comm. Alg. **39** (2011), 1482–1490.

27. A. Geroldinger, *Additive group theory and non-unique factorizations*, in *Combinatorial number theory and additive group theory*, A. Geroldinger and I. Ruzsa, eds., Birkhäuser, Berlin, 2009.

28. A. Geroldinger and F. Halter-Koch, *Non-unique factorizations. algebraic, combinatorial and analytic theory*, Pure Appl. Math. **278**, Chapman & Hall/CRC, Boca Raton, 2006.

29. A. Geroldinger, S. Ramacher, and A. Reinhart, *On v-marot mori rings and c-rings*, http://arxiv.org/abs/1401.2761.

30. S. Glaz, *Controlling the zero divisors of a commutative ring*, in *Commutative ring theory and applications*, Lect. Notes Pure Appl. Math. **231**, Marcel Dekker, New York, 2003.

31. F. Halter-Koch, *Weak module systems and applications: a multiplicative theory of integral elements and the Marot property*, in *Commutative Ring Theory and Applications*, Lect. Notes Pure Appl. Math. **231**, Marcel Dekker, New York, 2003.

32. J.A. Huckaba, *Commutative rings with zero-divisors*, Pure Appl. Math. **117**, Marcel Dekker, New York, 1988.

33. J.R. Juett, *Generalized comaximal factorization of ideals*, J. Algebra **352** (2012), 141–166.

34. Byung Gyun Kang, *A characterization of Krull rings with zero divisors*, J. Pure Appl. Alg. **72** (1991), 33–38.

35. _____ , *Integral closure of rings with zero-divisors*, J. Alg. **162** (1993), 317–323.

36. _____ , *Characterizations of Krull rings with zero divisors*, J. Pure Appl. Alg. **146** (2003), 283–290.

37. R.E. Kennedy, *Krull rings*, Pacific J. Math. **89** (1980), 131–136.

38. T.G. Lucas, *The Mori property in rings with zero divisors*, in *Rings, modules, algebras, and Abelian groups*, Lect. Notes Pure Appl. Math. **236**, Marcel Dekker, New York, 2004.

39. Thomas G. Lucas, *Krull rings, Prüfer v-multiplication rings and the ring of finite fractions*, Rocky Mountain J. Math. **35** (2005), 1251–1325.

40. Thomas G. Lucas, *The Mori property in rings with zero divisors* II, Rocky Mountain J. Math. **37** (2007), 1195–1228.

41. S. McAdam and R. Swan, *Unique comaximal factorization*, J. Alg. **276** (2004), 180–192.

42. C.P. Mooney, *Generalized factorization in commutative rings with zero-divisors*, Houston J. Math. **41** (2015), 15–32.

43. Christopher Park Mooney, *Generalized u-factorization in commutative rings with zero-divisors*, Rocky Mountain J. Math. **45** (2015), 637–660.

44. ———, *τ-complete factorization in commutative rings with zero-divisors*, Houston J. Math. **42** (2016), 23–44.

45. E. Osmanagić, *On an approximation theorem for Krull rings with zero-divisors*, Comm. Alg. **27** (1999), 3647–3657.

Department of Mathematics and Physics, Westminster College, Fulton, MO 65251
Email address: **christopher.mooney@westminster-mo.edu**