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THE MINIMUM MATCHING ENERGY OF BICYCLIC GRAPHS WITH GIVEN GIRTH

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ABSTRACT. The matching energy of a graph was introduced by Gutman and Wagner in 2012 and defined as the sum of the absolute values of zeros of its matching polynomial. Let $\theta(r,s,t)$ be the graph obtained by fusing two triples of pendant vertices of three paths P_{r+2}, P_{s+2} and P_{t+2} to two vertices. The graph obtained by identifying the center of the star S_{n-g} with the degree 3 vertex u of $\theta(1,g-3,1)$ is denoted by $S_{n-g}(u)\theta(1,g-3,1)$. In this paper, we show that, $S_{n-g}(u)\theta(1,g-3,1)$ has minimum matching energy among all bicyclic graphs with order n and girth g.

1. Introduction. All graphs in this paper are finite, simple and nondirected. Let $G=(V,E)$ be such a graph with order $|V|=n$ and size $|E|=m$. In a graph a *matching* is a set of pairwise nonadjacent edges, and we denote the number of k-matchings of G by $m_k(G)$. Note that $m_1(G)=m$ and $m_k(G)=0$ for $k>\lfloor n/2\rfloor$. It is both consistent and convenient to define $m_0(G)=1$. The matching polynomial of G is defined as

$$\alpha(G,x)=\sum_{k\geq 0}(-1)^k m_k(G)x^{n-2k}.$$

All the zeros of $\alpha(G,x)$ are real-valued and the theory of matching polynomials is well elaborated in **[3, 5]**.

Recently, Gutman and Wagner **[7]** introduced the matching energy of a graph G, denoted by $\mathrm{ME}(G)$ and defined as

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$$\text{ME}(G) = \frac{2}{\pi}\int_0^{\infty} \frac{1}{x^2} \ln\left[\sum_{k\geq 0} m_k(G)x^{2k}\right] dx, \tag{1.1}$$

which coincides with the Coulson-type integral formula for the energy which has been studied extensively (see an excellent monograph **[13]** and **[14]** and the references therein for recent advances), when the graph under consideration is a tree. As mentioned in **[7]**, matching energy can also be defined in another form as follows.

Let G be a simple graph, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the zeros of its matching polynomial. Then

$$\text{ME}(G) = \sum_{i=1}^{n} |\mu_i|.$$

The integral on the right side of equation (1.1) is increasing in all the coefficients $m_k(G)$. This means that if two graphs G and G' satisfy $m_k(G) \leq m_k(G')$ for all $k \geq 1$, then $\text{ME}(G) \leq \text{ME}(G')$. If, in addition, $m_k(G) < m_k(G')$ for at least one k, then $\text{ME}(G) < \text{ME}(G')$. This then motivates the introduction of a *quasi-order* $\succeq$, defined by

$$G \succeq H \Longleftrightarrow m_k(G) \geq m_k(H),$$

for all nonnegative integers k. If $G \succeq H$ and there exists some k such that $m_k(G) > m_k(H)$, then we write $G \succ H$. It is said that G is *m-greater than* H if $G \succeq H$ and *strictly m-greater than* H if $G \succ H$. It is easy to see that

$$G \succeq H \Longrightarrow \text{ME}(G) \geq \text{ME}(H)$$

and

$$G \succ H \Longrightarrow \text{ME}(G) > \text{ME}(H).$$

Initial work on matching energy of graphs is attributed to **[7]** and then followed by Li and Yan **[16]** who characterized the maximal connected graphs with given connectivity and chromatic numbers. The extremal graphs in connected bicyclic graphs were determined by Ji, Li and Shi **[10]** and further by Chen, Liu and Shi **[1, 2]** for unicyclic, bicyclic and tricyclic graphs. More generally, the minimal matching energy of (m, n)-graphs with a given matching number was obtained by Xu, Das and Zheng [**19**]. Huang, Kuang and Deng [**9**] characterized the extremal graph for a random polyphenyl chain. For more results, see [**11, 12, 17**] and there may be other results which are unknown to the authors.

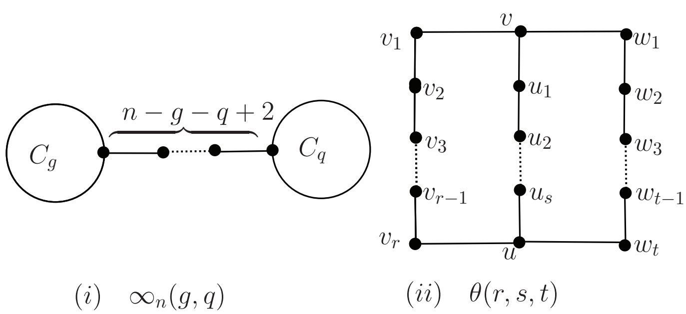


FIGURE 1. Two types of braces: $\infty_n(g, q)$ and $\theta(r, s, t)$.

Denote the set of all connected bicyclic graphs with order n and girth g by $\mathscr{B}_{n,g}$. We now define two special classes of bicyclic graphs. Let $\infty_n(g, q)$ denote the graph obtained by the coalescence of two end vertices of a path $P_{n-g-q+2}$ with one vertex of two cycles C_g and C_q, respectively, and $\theta(r, s, t)$ the graph obtained by fusing two triples of pendant vertices of three paths P_{r+2}, P_{s+2} and P_{t+2} to two vertices, as given in Figure 1. The distance of two cycles C_g and C_q in G is defined as

$$d_G(C_g, C_q) = \min\{d_G(x, y) \mid x \in V(C_g),\ y \in V(C_q)\},$$

sometimes written as d_G for short. Note that $d_G(C_g, C_q) = 0$ if C_g and C_q have a common vertex, e.g., for $G = \infty_n(g, q)$ such that $q = n-g+1$, and in this case, $\infty_n(g, q)$ with $q = n-g+1$ is simply written as $\infty(g, q)$ for convenience. Clearly, any bicyclic graph must contain either graph (i) or (ii) in Figure 1 as an induced subgraph, called its *brace*. Then the set $\mathscr{B}_{n,g}$ can be partitioned into two subsets $\mathscr{B}^1_{n,g}$ and $\mathscr{B}^2_{n,g}$, where $\mathscr{B}^1_{n,g}$ is the set of all bicyclic graphs which contain a brace of the form $\infty_n(g, q)$, and $\mathscr{B}^2_{n,g}$ is the set of all bicyclic graphs which contain a brace of the form $\theta(r, s, t)$.

There have been some papers on characterizing the minimal Hosoya index of graphs, see **[4, 18]**. In this paper, minimal graphs in $\mathscr{B}^i_{n,g}$ for $i = 1, 2$, are determined respectively and, by comparing them, we show that $S_{n-g}(u)\theta(1, g-3, 1)$, obtained by identifying the center of the star S_{n-g} with a vertex u of degree 3 in $\theta(1, g-3, 1)$, has minimum matching energy among all bicyclic graphs in $\mathscr{B}_{n,g}$.

2. Preliminaries. In this section, we shall present some basic results which will be used in the proof of our main results.

Given a graph G and an edge uv of G, we denote by $G - uv$ (respectively $G - v$) the graph obtained from G by deleting the edge uv (respectively the vertex v and edges incident to it).

Lemma 2.1 ([**10**]). *If u, v are adjacent vertices of G, then*

$$m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v),$$

for all nonnegative integers k.

Let $G(v)S_{t+1}$ (or $S_{t+1}(v)G$) denote the graph obtained by identifying the vertex v of a graph G with the center of the star S_{t+1}, as given in Figure 2.

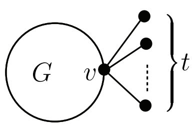


FIGURE 2. $G(v)S_{t+1}$.

Note. Consider the graph in Figure 2.

- $m_k(G)$ is the number of k-matchings that do not contain any of the edges of S_{t+1}.
- $tm_{k-1}(G - v)$ is the number of k-matchings that contain one of these edges, as there are t choices for these edges in S_{t+1}.

Consequently, we have what follows without giving a formal proof.

Lemma 2.2. *Let G be a graph, and let v be a vertex of G. Then,* $m_k(G(v)S_{t+1}) = m_k(G) + tm_{k-1}(G - v)$.

Recall a result in [**13**], which establishes the order of the union of two paths with a given number of vertices according to the quasi-order as stated in the introduction.

Lemma 2.3 ([**13**])**.** *Let $n = 4k$, $4k+1$, $4k+2$ or $4k+3$. Then,*

$$\begin{aligned} P_n &\succ P_2 \cup P_{n-2} \succ P_4 \cup P_{n-4} \succ \cdots \succ P_{2k} \cup P_{n-2k} \succ P_{2k+1} \cup P_{n-2k-1} \\ &\succ P_{2k-1} \cup P_{n-2k+1} \succ \cdots \succ P_3 \cup P_{n-3} \succ P_1 \cup P_{n-1}. \end{aligned}$$

Lemma 2.4 ([**6**])**.** *If $G_1 \succ G_2$, then $G_1 \cup H \succ G_2 \cup H$, where H is an arbitrary graph.*

Applying Lemma 2.4, we can generalize Lemma 2.3 to the following form, the union of three paths.

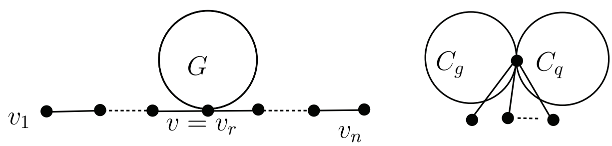
Lemma 2.5. *Let r, s, t be nonnegative integers with $r \leq s - 2$. If r is even, then*

$$\begin{aligned} P_{r-2} \cup P_{s+2} \cup P_t \succ P_r \cup P_s \cup P_t &\succ P_{r+1} \cup P_{s-1} \cup P_t \\ &\succ P_{r-1} \cup P_{s+1} \cup P_t. \end{aligned}$$

Lemma 2.6 ([**7**])**.** *Suppose that G is a connected graph and T an induced subgraph of G such that T is a tree and is connected to the rest of G only by a cut vertex v. If T is replaced by a star of the same order and centered at v, then the matching energy decreases* (*unless T is already such a star*)*. If T is replaced by a path with one end at v, then the matching energy increases* (*unless T is already such a path*)*.*

Recall the definition of a generalized π-transform in [**15**]. We say Q is a *branch* of a connected graph G with root u if Q is a connected induced subgraph of G for which u is the only vertex in Q that has a neighbor not in Q. Let P and Q be branches of a component of a graph G with a common root u_0, which is also their only common vertex. Assume that P is a path and u_0 has at least one neighbor in G that does not lie on P or Q. Form a graph from G by relocating the branch Q from u_0 to v where v is the other end vertex of the path P (by deleting edges u_0w and adding new edges vw for every vertex w in Q adjacent to u_0). We refer to the resulting graph as *a generalized π-transform of G.*

Theorem 2.7 ([**15**]). *If G' is a generalized π-transform of G, then $G' \succ G$ and so* $\mathrm{ME}(G') > \mathrm{ME}(G)$.



(i) $P_n(v_r, v)G$ (ii) $S_n(g, q)$

FIGURE 3.

Let $P_n(v_r, v)G$ denote the graph obtained by identifying the vertex v_r of P_n with the vertex v of G (see Figure 3 (i)). For convenience, we use $P_n(v)G$ (or $G(v)P_n$) to stand for $P_n(v_1, v)G$. Note that $P_n(v_r, v)G$ and $P_r(v)G(v)P_{n-r+1}$ are isomorphic.

Theorem 2.8 ([**8**]). *If v is an arbitrary vertex of the graph G, then for $n = 4k + i$, $i \in \{-1, 0, 1, 2\}$, $k \geq 1$,*

$$P_n(v_1, v)G \succ P_n(v_3, v)G \succ \cdots \succ P_n(v_{2k+1}, v)G \succ P_n(v_{2k}, v)G \succ P_n(v_{2k-2}, v)G \succ \cdots \succ P_n(v_2, v)G.$$

Let G be an arbitrary graph with a specified vertex v. The graph obtained from G is denoted by $\widehat{\mathcal{G}}_i$ for $i = 1, 2, \ldots, n-1$ (as given in Figure 4), by adding $n-1$ new vertices to G in the following way. Attach $i-1$ pendant edges and a pendant path of length $n-i$ at v. By Theorem 2.8, we easily obtain the following.

Lemma 2.9. $\widehat{\mathcal{G}}_1 \succ \widehat{\mathcal{G}}_2 \succ \cdots \succ \widehat{\mathcal{G}}_{n-1}$.

Proof. $\widehat{\mathcal{G}}_1 \succ \widehat{\mathcal{G}}_2$ follows immediately from Theorem 2.8, as $\widehat{\mathcal{G}}_1 \cong P_n(v_1, v)G$ and $\widehat{\mathcal{G}}_2 \cong P_n(v_2, v)G$. In fact, other cases can be verified

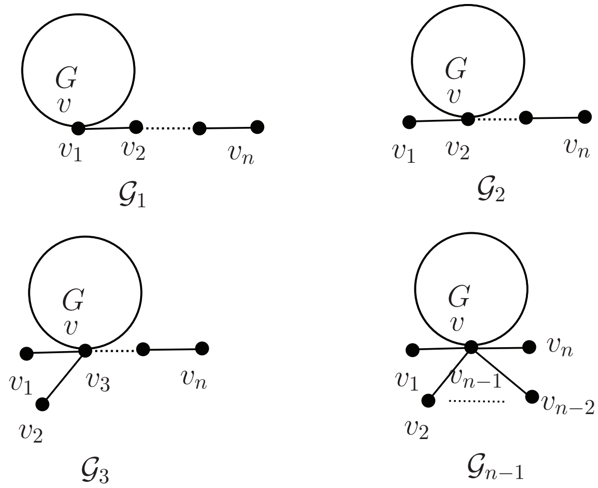


FIGURE 4.

in the same way. Note that if we denote the graph $G(v)S_i$ by H, then $\widehat{\mathcal{G}}_i \cong P_{n-i+1}(v_1, v)H$ and $\widehat{\mathcal{G}}_{i+1} \cong P_{n-i+1}(v_2, v)H$. □

Let $S_n(g, q)$ be the graph in $\mathscr{B}^1_{n,g}$ with $n+1-(g+q)$ pendant edges attached at the common vertex of C_g and C_q (see Figure 3 (ii)).

Theorem 2.10. *$S_n(g, q) \succeq S_n(g, g)$ with equality if and only if $g = q$.*

Proof. Let u (u', respectively) be the common vertex of C_g and C_q (C_g, respectively) in $S_n(g, q)$ ($S_n(g, g)$, respectively), and u_1u_2 ($u'_1u'_2$, respectively) an edge of C_q (C_g, respectively) such that u_1 (u'_1, respectively) is adjacent to u (u', respectively). By Lemma 2.1, we have

$$m_k(S_n(g, q)) = m_k(S_n(g, q) - u_1u_2) + m_{k-1}(S_n(g, q) - u_1 - u_2)$$

and

$$m_k(S_n(g, g)) = m_k(S_n(g, g) - u'_1u'_2) + m_{k-1}(S_n(g, g) - u'_1 - u'_2).$$

Note that

$$S_n(g,q) - u_1u_2 \cong P_{q-1}(u)C_g(u)S_{n+3-g-q}$$
$$S_n(g,g) - u_1'u_2' \cong P_{g-1}(u)C_g(u)S_{n+3-2g}.$$

If we denote the graph $P_{i-1}(u)C_g(u)S_{n+3-g-i}$ by G_i, then $S_n(g,q) - u_1u_2 \cong G_q$ and $S_n(g,g) - u_1'u_2' \cong G_g$. By Lemma 2.9, we have

$$G_q \succ G_{q-1} \succ \cdots \succ G_g.$$

Thus, $S_n(g,q) - u_1u_2 \succeq S_n(g,g) - u_1'u_2'$ with equality if and only if $g = q$.

In the same way, we also get $S_n(g,g) - u_1 - u_2 \succeq S_n(g,g) - u_1' - u_2'$ with equality if and only if $g = q$. Therefore, $S_n(g,q) \succeq S_n(g,g)$ with equality if and only if $g = q$. □

Theorem 2.11. *If $t \geq 2$ and r is even, then*

$$\theta(r-2, s+2, t) \succ \theta(r,s,t) \succ \theta(r+1, s-1, t) \succ \theta(r-1, s+1, t), \tag{2.1}$$

where $r \leq s-2$ and $r+s+2 = g$.

Proof. Let $G = \theta(r,s,t)$, which can be obtained by merging two triples of pendant vertices u_0, v_0, w_0 and u_{r+1}, v_{s+1}, w_{t+1} of three paths

$$P_{r+2} = u_0u_1\cdots u_ru_{r+1}, \qquad P_{s+2} = v_0v_1\cdots v_sv_{s+1},$$

and

$$P_{t+2} = w_0w_1\cdots w_tw_{t+1},$$

to two vertices, say u and v, respectively. By Lemma 2.1, we have

$$\begin{aligned} m_k(G) &= m_k(G - uw_1) + m_{k-1}(G - u - w_1) \\ &= m_k(G - uw_1 - vw_t) + m_{k-1}(G - uw_1 - v - w_t) \\ &\quad + m_{k-1}(G - u - w_1 - vw_t) \\ &\quad + m_{k-2}(G - u - w_1 - v - w_t) \\ &= m_k(P_t \cup C_g) + 2m_{k-1}(P_{g-1} \cup P_{t-1}) \\ &\quad + m_{k-2}(P_r \cup P_s \cup P_{t-2}). \end{aligned}$$

For convenience, let $G_1 = \theta(r-1, s+1, t)$, $G_2 = \theta(r-2, s+2, t)$ and $G_3 = \theta(r+1, s-1, t)$. Applying the same method to the graphs G_1, G_2 and G_3, we get

$$\begin{aligned} m_k(G_1) &= m_k(P_t \cup C_g) + 2m_{k-1}(P_{g-1} \cup P_{t-1}) \\ &\quad + m_{k-2}(P_{r-1} \cup P_{s+1} \cup P_{t-2}), \\ m_k(G_2) &= m_k(P_t \cup C_g) + 2m_{k-1}(P_{g-1} \cup P_{t-1}) \\ &\quad + m_{k-2}(P_{r-2} \cup P_{s+2} \cup P_{t-2}), \\ m_k(G_3) &= m_k(P_t \cup C_g) + 2m_{k-1}(P_{g-1} \cup P_{t-1}) \\ &\quad + m_{k-2}(P_{r+1} \cup P_{s-1} \cup P_{t-2}). \end{aligned}$$

Thus,

$$\begin{aligned} m_k(G_1) - m_k(G) &= m_{k-2}(P_{r-1} \cup P_{s+1} \cup P_{t-2}) \\ &\quad - m_{k-2}(P_r \cup P_s \cup P_{t-2}), \\ m_k(G_2) - m_k(G) &= m_{k-2}(P_{r-2} \cup P_{s+2} \cup P_{t-2}) \\ &\quad - m_{k-2}(P_r \cup P_s \cup P_{t-2}), \\ m_k(G_3) - m_k(G_1) &= m_{k-2}(P_{r+1} \cup P_{s-1} \cup P_{t-2}) \\ &\quad - m_{k-2}(P_{r-1} \cup P_{s+1} \cup P_{t-2}). \end{aligned}$$

By Lemma 2.5, it follows directly that if $r \leq s-2$ and r is even, then

$$\begin{aligned} P_{r-2} \cup P_{s+2} \cup P_t \succ P_r \cup P_s \cup P_t &\succ P_{r+1} \cup P_{s-1} \cup P_t \\ &\succ P_{r-1} \cup P_{s+1} \cup P_t, \end{aligned}$$

and so assertion (2.1) holds. □

As an immediate consequence, we have the following result.

Corollary 2.12. *If $t \geq 2$, then $\theta(r, s, t) \succeq \theta(1, r+s-1, t)$ with equality if and only if $r = 1$ or $s = 1$.*

Proof. Without loss of generality, assume that $r \leq s$. If $r = 0$, by Theorem 2.11, we have either

$$\theta(0, s, t) = \theta(r, s, t) \succ \theta(r+1, s-1, t) = \theta(1, s-1, t)$$

when $s \geq 2$ or $s = 1$ and, in this case, $\theta(0, s, t)$ is already of the form $\theta(1, r + s - 1, t)$.

Now assume that $r \geq 1$. If r is even, then

$$\theta(r, s, t) \succ \theta(r - 1, s + 1, t) \succ \theta(r - 3, s + 3, t) \succ \cdots \succ \theta(1, s + r - 1, t).$$

If r is odd, then

$$\theta(r, s, t) \succ \theta(r-2, s+2, t) \succ \theta(r-4, s+4, t) \succ \cdots \succ \theta(1, s+r-1, t). \quad \square$$

3. Main results. In this section, we first show that $S_n(g, g)$ has minimum matching energy in $\mathscr{B}^1_{n,g}$ and $S_{n-g}(u)\theta(1, g - 3, 1)$ is the minimal graph in $\mathscr{B}^2_{n,g}$. Further, by comparing these two graphs, we conclude that $S_{n-g}(u)\theta(1, g - 3, 1)$ is the unique graph with minimum matching energy in $\mathscr{B}_{n,g}$.

Theorem 3.1. *For any graph $G \in \mathscr{B}^1_{n,g}$, we have $G \succeq S_n(g, g)$ with equality if and only if $G \cong S_n(g, g)$.*

Proof. For any graph $G \in \mathscr{B}^1_{n,g}$, its brace must be of the form $\infty_n(g, q)$ for some $q \geq g$. By Lemma 2.6, if any tree branch is replaced by a star of equal order centered at the root, its matching energy strictly decreases unless the branch is already such a star. To show that $G \succeq S_n(g, g)$ for any $G \in \mathscr{B}^1_{n,g}$, it suffices to show it holds for such a graph G, all of whose tree branches are stars. We distinguish two cases according to the value of d_G, the distance between C_g and C_q in its brace.

Case 1. $d_G = 0$. As above, if G has l tree branches, we can assume that these l branches are stars. Without loss of generality, for $i = 1, \ldots, l$, we will assume G is the coalescence of the vertex u_i in its brace $\infty(g, q)$ and the center of S_{r_i+1}. For convenience, we use the notation G_i to denote such graphs recursively defined as follows. Let $G_0 = \infty(g, q)$, and if G_{i-1} is already defined, then G_i is defined to be $G_{i-1}(u_i)S_{r_i+1}$. Note that $G_l = G$.

Applying Lemma 2.2 to $S_n(g, q)$, we have

$$m_k(S_n(g, q)) = m_k(\infty(g, q)) + (n+1-g-q)m_{k-1}(P_{g-1} \cup P_{q-1}). \tag{3.1}$$

Similarly applying Lemma 2.2 to G, we have

$$\begin{aligned} m_k(G) &= m_k(G_{l-1}) + r_l m_{k-1}(G_{l-1} - u_l) \\ &= m_k(G_{l-2}) + r_{l-1} m_{k-1}(G_{l-2} - u_{l-1}) + r_l m_{k-1}(G_{l-1} - u_l) \\ &= \cdots \\ &= m_k(G_0) + \sum_{i=1}^{l} r_i m_{k-1}(G_{i-1} - u_i), \end{aligned} \tag{3.2}$$

where

$$\sum_{i=1}^{l} r_i = n + 1 - g - q.$$

First, $G_0 = \infty(g, q)$. Second,

$$G_{i-1} - u_i \succeq \infty(g, q) - u_i \succeq P_{g-1} \cup P_{q-1} \quad \text{for } i = 1, \ldots, l,$$

because each graph is a subgraph of the former. If $G \ncong S_n(g, q)$, then for some i, $G_{i-1} - u_i$ has $P_{g-1} \cup P_{q-1}$ as a proper subgraph. Therefore, $G \succeq S_n(g, q)$ with equality if and only if $G \cong S_n(g, q)$. By Theorem 2.10, the assertion holds.

Case 2. $d_G \geq 1$. By Lemma 2.6, we can assume that all tree branches of G are stars. Suppose $P = w_1 w_2 \cdots w_t$ is the unique path connecting two cycles C_g and C_q in G. We proceed by induction on the number of pendant vertices along the path P. If there are no pendant vertices along the path, by applying the inverse generalized π-transform to G, i.e., deleting all edges $w_t u$, where $u \in N(w_t) \setminus \{w_{t-1}\}$ and adding new edges $w_1 u$, it becomes a graph in $\mathscr{B}^1_{n,g}$ with $d_G(C_g, C_q) = 0$ and so, by Case 1, $G \succeq S_n(g, g)$.

Now assume that there is at least a pendant edge uv on the path P. By Lemma 2.1, we have $m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v)$. By the induction hypothesis, $G - uv \succ S_{n-1}(g, q)$. Also, it is easy to see that $G - u - v$ has $P_{g-1} \cup P_{q-1}$ as its subgraph. Thus,

$$\begin{aligned} m_k(G) &= m_k(G - uv) + m_{k-1}(G - u - v) \\ &\geq m_k(S_{n-1}(g, q)) + m_{k-1}(P_{g-1} \cup P_{q-1}) \\ &= m_k(S_n(g, q)), \end{aligned}$$

and strict inequality holds for at least one k. Therefore, $G \succ S_n(g, q)$ and then $G \succ S_n(g, g)$ by Theorem 2.10. $\square$

Theorem 3.2. *For any positive integers* r, s, t *with* $r + s + 2 = g$*, let* $G_1, G_1', G_1'' \in \mathscr{B}^2_{n,g}$ *be defined as* $G_1 = G_0(u)S_{n-r-s-t-1}$, $G_1' = G_0'(u)S_{n-g-t+1}$ *and* $G_1'' = G_0''(u)S_{n-g}$*, where* $G_0 = \theta(r, s, t)$, $G_0' = \theta(1, g-3, t)$, $G_0'' = \theta(1, g-3, 1)$*, as given in Figure* 5*. Then:*

- (i) G_0 *with a vertex of degree* 2 *deleted is strictly* m*-greater than* G_0 *with a vertex of degree* 3 *deleted;*
- (ii) *for any* $G \in \mathscr{B}^2_{n,g}$ *with* $\theta(r, s, t)$ *as its brace,* $G \succeq G_1$ *with equality if and only if* $G \cong G_1$*;*
- (iii) $G_1 \succeq G_1'$ *with equality if and only if* $G_1 \cong G_1'$*;*
- (iv) $G_1' \succeq G_1''$ *with equality if and only if* $G_1' \cong G_1''$*.*

Consequently, for any $G \in \mathscr{B}^2_{n,g}$*,* $G \succeq G_1''$ *with equality if and only if* $G \cong G_1''$*.*

Proof.

(i) Choose two vertices of degrees 2 and 3 respectively in G_0, say w_m and u. Consider $G_0 - w_m$, which can be viewed as a cycle C_g together with two pendant paths, namely, $P = uw_1 \cdots w_{m-1}$ at u and $Q = vw_t \cdots w_{m+1}$ at v, possibly of length 0. If one of the two paths P or Q is of length 0, then we can choose an appropriate edge e such that $G_0 - w_m - e \cong P_{n-1}$. Since P_{n-1} is m-greater than any tree T of order $n-1$, we have $G_0 - w_m \succ P_{n-1} \succeq G_0 - u$ as $G_0 - u$ is a tree.

Now assume that both P and Q are not of length 0. By Lemma 2.1, we have

$$\begin{aligned} m_k(G_0 - w_m) - m_k(G_0 - u) = {} & m_k(G_0 - w_m - uw_1) \\ & + m_{k-1}(G_0 - w_m - u - w_1) \\ & - m_k(G_0 - u - w_{m-1}w_m) \\ & - m_{k-1}(G_0 - u - w_{m-1} - w_m). \end{aligned}$$

Note that

$$G_0 - w_m - u - w_1 \quad \text{and} \quad G_0 - u - w_{m-1} - w_m$$

are the union of the same graph $T_{g+t-m} - w_m$ (see Figure 5) and the path P_{m-2} and so are isomorphic. Thus, $m_{k-1}(G_0 - w_m - u - w_1) = m_{k-1}(G_0 - u - w_{m-1} - w_m)$ for any k. It is clear that $G_0 - w_m - uw_1$ is the union of a graph $P_{t-m+1}(v)C_g$ and a path P_{m-1}, and $G_0 - u - w_{m-1}w_m$ is the union of a graph T_{g+t-m} and a path

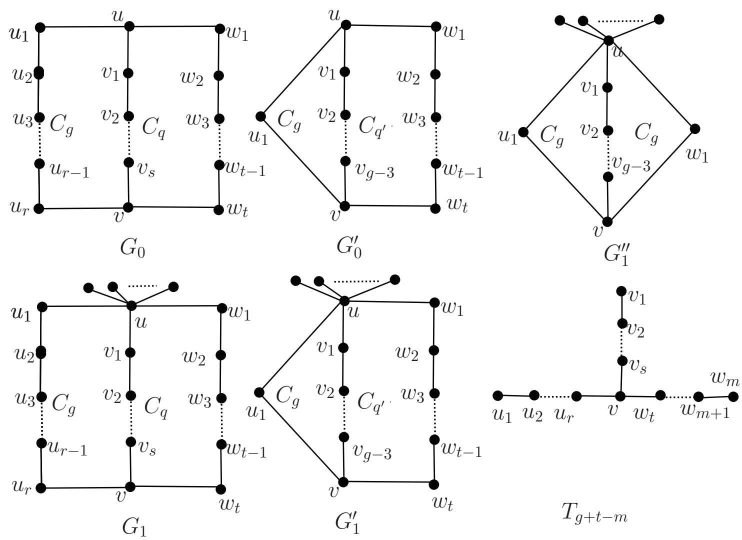


FIGURE 5.

P_{m-1}. Due to $P_{t-m+1}(v)C_g \succ P_{r+s+t-m+2} \succ T_{g+t-m}$ and Lemma 2.4, we have $G_0 - w_m - uw_1 \succ P_{r+s+t-m+2} \cup P_{m-1} \succ G_0 - u - w_{m-1}w_m$. Therefore, $G_0 - w_m \succ G_0 - u$.

(ii) As in the proof of Theorem 3.1, by Lemma 2.6, we can assume that all tree branches at the cycles of G are stars. Without loss of generality, suppose that G is the coalescence of the vertex t_i (here we use new notation for these vertices) in G_0 and the center of S_{r_i+1} for $i = 1, \dots, l$, and $\sum_{i=1}^{l} r_i = a$. For convenience, we use H_i to denote graphs defined recursively as follows. Let $H_0 = G_0$, and if H_{i-1} is already defined, then H_i is defined to be $H_{i-1}(t_i)S_{r_i+1}$. Note that $H_l = G$.

Applying Lemma 2.2, we have

$$m_k(G_1) = m_k(G_0) + am_{k-1}(G_0 - u) \tag{3.3}$$

and

$$\begin{aligned} m_k(G) &= m_k(H_{l-1}) + r_l m_{k-1}(H_{l-1} - t_l) \\ &= m_k(H_{l-2}) + r_{l-1} m_{k-1}(H_{l-2} - t_{l-1}) + r_l m_{k-1}(H_{l-1} - t_l) \\ &\quad + \cdots \\ &= m_k(G_0) + \sum_{i=1}^{l} r_i m_{k-1}(H_{i-1} - t_i). \end{aligned}$$

First note that $G_0 - t_i$ is a (proper) subgraph of $H_i - t_i$, and so $H_i - t_i \succeq G_0 - t_i$. Further, by (i) above, we have $G_0 - t_i \succ G_0 - u$. Therefore, $H_i - t_i \succ G_0 - t_i \succ G_0 - u$ and then $G \succ G_1$ unless G is already such a graph G_1.

(iii) By Corollary 2.12, we have $G_0 \succeq G_0'$. Note that $G_0 - u \cong P_{r+s+1}(u_{r+1}, v)P_t$ and $G_0' - u \cong P_{r+s+1}(u_2, v)P_t$. By Lemma 2.9, we get $G_0 - u \succeq G_0' - u$. So we have

$$\begin{aligned} m_k(G_1) &= m_k(G_0) + a m_{k-1}(G_0 - u) \\ &\geq m_k(G_0') + a m_{k-1}(G_0' - u) \\ &= m_k(G_1'). \end{aligned}$$

Therefore, $G_1 \succeq G_1'$. From the process above, equality holds only if $G_1 \cong G_1'$.

(iv) By Lemma 2.1,

$$m_k(G_1') = m_k(G_1' - vw_t) + m_{k-1}(G_1' - v - w_t)$$

and

$$m_k(G_1'') = m_k(G_1'' - vw_1) + m_{k-1}(G_1'' - v - w_1).$$

By Lemma 2.9, we get $G_1' - vw_t \succeq G_1'' - vw_1$ and $G_1' - v - w_t \succeq G_1'' - v - w_1$. Hence, $G_1' \succ G_1''$ with equality if and only if $G_1' \cong G_1''$. □

Next, we shall compare the minimal graphs from $\mathscr{B}_{n,g}^1$ and $\mathscr{B}_{n,g}^2$.

Theorem 3.3.

$$S_n(g, g) \succ S_{n-g}(u)\theta(1, g-3, 1) \; (= G_1'')$$

(*see Figure* 5).

Proof. Let u_0 be the common vertex of the two copies of C_g in $S_n(g,g)$, where $G_1'' = S_{n-g}(u)\theta(1, g-3, 1)$, and let u_1u_2 be an edge with u_1 adjacent to u_0. Note that $S_n(g,g) - u_1u_2$ is the coalescence of a vertex in C_g with the center of a star and an end vertex of a path, i.e.,

$$S_n(g,g) - u_1u_2 \cong P_{g-1}(u_0)C_g(u_0)S_{n+3-2g}.$$

Similarly,

$$S_n(g,g) - u_1 - u_2 \cong P_{g-2}(u_0)C_g(u_0)S_{n+2-2g},$$

and let $u_1'u_0$ be an edge of C_g in $S_n(g,g) - u_1 - u_2$,

$$S_n(g,g) - u_1 - u_2 - u_1'u_0 \cong P_g(u_0)S_{n+2-2g}(u_0)P_{g-2}.$$

It is obvious that $S_n(g,g) - u_1 - u_2 \succ S_n(g,g) - u_1 - u_2 - u_1'u_0$. In the same way, we have

$$G_1'' - vw_1 \cong C_g(u)S_{n-g+1} \quad \text{and} \quad G_1'' - v - w_1 \cong P_{g-2}(u)S_{n-g+1}.$$

By Lemma 2.1, we have

$$\begin{aligned} m_k(G_1'') &= m_k(G_1'' - vw_1) + m_{k-1}(G_1'' - v - w_1) \\ &= m_k(C_g(u)S_{n-g+1}) + m_{k-1}(P_{g-2}(u)S_{n-g+1}), \end{aligned}$$

and

$$\begin{aligned} m_k(S_n(g,g)) &= m_k(S_n(g,g) - u_1u_2) + m_{k-1}(S_n(g,g) - u_1 - u_2) \\ &= m_k(P_{g-1}(u_0)C_g(u_0)S_{n+3-2g}) \\ &\quad + m_{k-1}(P_{g-2}(u_0)C_g(u_0)S_{n+2-2g}). \end{aligned}$$

By Lemma 2.9, we get

$$P_{g-1}(u_0)C_g(u_0)S_{n+3-2g} \succeq C_g(u)S_{n-g+1}.$$

Choosing an appropriate edge e, we have

$$P_{g-2}(u_0)C_g(u_0)S_{n+2-2g} - e \cong P_g(u_0)S_{n+2-2g}(u_0)P_{g-2},$$

and again, by Lemma 2.9,

$$P_g(u_0)S_{n+2-2g}(u_0)P_{g-2} \succeq P_{g-2}(u)S_{n-g+1}.$$

Thus, $S_n(g,g) \succ G_1''$. □

From Theorems 3.1, 3.2 and 3.3, we obtain the following main result.

Theorem 3.4. *For any graph* $G \in \mathcal{B}_{n,g}$*, we have*

$$G \succeq S_{n-g}(u)\theta(1, g-3, 1),$$

and therefore,

$$\mathrm{ME}(G) \geq \mathrm{ME}(S_{n-g}(u)\theta(1, g-3, 1)),$$

where equality holds if and only if $G \cong S_{n-g}(u)\theta(1, g-3, 1)$*.*

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