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ASYMPTOTIC RESEMBLANCE

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ABSTRACT. *Uniformity* and *proximity* are two different ways of defining small scale structures on a set. *Coarse structures* are large scale counterparts of uniform structures. In this paper, motivated by the definition of proximity, we develop the concept of asymptotic resemblance as a relation between subsets of a set to define a large scale structure on it. We use our notion of asymptotic resemblance to generalize some basic concepts of coarse geometry. We introduce a large scale compactification which, in special cases, agrees with the *Higson compactification.* At the end of the paper we show how the *asymptotic dimension* of a metric space can be generalized to a set equipped with an asymptotic resemblance relation.

1. Introduction and preliminaries. There are several ways to define *small scale* structures on a set. In 1937, Weil **[10]** defined the concept of uniformity. A few years later, Tukey **[9]** used the notion of uniform coverings to find another definition for uniform spaces. In 1950, Efremovich **[4, 5]** used proximity relations to define a small scale structure on a set. He axiomatized the relation "A is near B" for subsets A and B of a set. Let us recall the definition of a proximity space.

Definition 1.1. A relation δ on the family of all subsets of a nonempty set X is called a *proximity* on X if, for all $A, B, C \subseteq X$, it satisfies the following properties (by $A\overline{\delta}B$ we mean that $A\delta B$ does not hold.)

(i) If $A\delta B$, then $B\delta A$.
(ii) $\emptyset\overline{\delta}A$.
(iii) If $A\bigcap B \neq \emptyset$, then $A\delta B$.

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(iv) $A\delta(B\bigcup C)$ if and only if $A\delta B$ or $A\delta C$.
(v) If $A\bar{\delta}B$, then there is $E \subseteq X$ such that $A\bar{\delta}E$ and $(X \setminus E)\bar{\delta}B$.

The pair (X, δ) is called a *proximity space*.

There are also some ways of defining large scale structures on a set. In recent contexts, one can find notions of coarse structures **[8]**, large scale structures **[3]** and ball structures **[7]**. A *coarse structure* $\mathcal{E}$ *on a set* X is a family of subsets of $X \times X$ such that all subsets of a member of $\mathcal{E}$ are members of $\mathcal{E}$ and, for all $E, F \in \mathcal{E}$, the sets E^{-1}, $E \circ F$ and $E \bigcup F$ are in $\mathcal{E}$. The pair $(X, \mathcal{E})$ is called a *coarse space*. Let us recall that

$$E \circ F = \{(x, y) \mid (x, z) \in F, (z, y) \in E \quad \text{for some } z \in X\},$$

and

$$E^{-1} = \{(x, y) \mid (y, x) \in E\}, \quad \text{for all } E, F \subseteq X \times X.$$

A member of $\mathcal{E}$ is called an *entourage*. A coarse structure $\mathcal{E}$ is called *unitary* if it contains the diagonal $\Delta = \{(x, x) \mid x \in X\}$. From now on, by *coarse structure* we mean a unitary coarse structure. A coarse structure is known as a large scale counterpart of a uniformity.

In Section 2, we introduce a large scale counterpart of proximity. For this reason, the relation A and B are asymptotically alike for two subsets A and B of a set X is axiomatized and the notion of asymptotic resemblance is introduced. We call a set equipped with an asymptotic resemblance relation, an *asymptotic resemblance* (an AS.R.) *space*. In Section 2, we show how one can generalize basic concepts of coarse geometry (coarse maps, coarse connectedness, coarse subspace, etc.) by our definition. Also in this section, we show that every coarse structure on a set X can induce an asymptotic resemblance relation on X.

In Section 3, we investigate the relation between coarse structures and asymptotic resemblance relations. We give an example of two different coarse structures on a set X such that they induce the same asymptotic resemblance relation on X. We show how asymptotic resemblance relations on a set X can admit an equivalence relation on the family of all coarse structures on X.

A coarse structure $\mathcal{E}$ on a topological space X is said to be *compatible with the topology* of X if each entourage is contained in an open entourage. A compatible coarse structure on a topological space is called *proper* if each bounded subset has compact closure. One can easily check that a unitary coarse structure is compatible with the topology of a space if and only if it contains an open entourage containing the diagonal **[12]**. Let $\mathcal{E}$ be a proper coarse structure on a topological space $(X, \mathcal{T})$. A continuous and bounded map $f : X \to \mathbb{C}$ is called a *Higson function* if for each $E \in \mathcal{E}$ and $\epsilon > 0$ there exists a compact subset K of X such that $|f(x) - f(y)| < \epsilon$ for all $(x, y) \in E \setminus (K \times K)$. The family of all Higson functions is denoted by $C_h(X)$. The Gelfand-Naimark theorem on C^*-algebras shows that there is a compactification hX of X, such that $C(hX)$ (the family of all continuous functions on hX) and $C_h(X)$ are isomorphic **[8,** subsection 2.3]. The compactification hX of X is called the *Higson compactification* of X. The compact set $\nu X = hX \setminus X$ is called the *Higson corona* of X.

In Section 4, we use our notion of asymptotic resemblance to make a compactification of a space (the *asymptotic compactification*) that in some cases agrees with the Higson compactification of coarse spaces. We are going to use the Wallman compactification of a topological space to generate our desired compactification. Let us briefly recall the Wallman compactification of a topological space (**[11]**).

Let $(X, \mathcal{T})$ be a Hausdorff topological space, and let γX be the family of all closed ultrafilters on X. For each open subset U of X, the set $U^* = \{\mathcal{F} \in \gamma X \mid U \notin \mathcal{F}\}$. It is straightforward to show that $\mathcal{F} \in U^*$ if and only if $\mathcal{F}$ contains a subset of U. The family $\mathcal{B} = \{U^* \mid U$ is open in $X\}$ is a basis for a topology on γX and γX is compact by this topology. Let σ_x denote the unique closed ultrafilter that converges to $x \in X$. The map $\sigma : X \to \gamma X$, defined by $\sigma(x) = \sigma_x$, is a topological embedding and γX is called the *Wallman compactification* of X.

A *cluster* $\mathcal{C}$ in a proximity space (X, δ) is a family of subsets of X such that, for all $A, B \in \mathcal{C}$, we have $A\delta B$, if $A, B \subseteq X$ and $A \bigcup B \in \mathcal{C}$, then $A \in \mathcal{C}$ or $B \in \mathcal{C}$; and, if $A\delta B$ for all $B \in \mathcal{C}$, then $A \in \mathcal{C}$. A proximity space (X, δ) is said to be *separated* if $x\delta y$ implies $x = y$, for all $x, y \in X$. A proximity δ on a topological space $(X, \mathcal{T})$ is said to be *compatible with* $\mathcal{T}$ if $a \in \bar{A}$ and $a\delta A$ are equivalent. Let $\mathfrak{X}$ denote the family of all clusters in a separated proximity space (X, δ). For $\mathfrak{M}, \mathfrak{N} \subseteq \mathfrak{X}$ define $\mathfrak{M}\delta^*\mathfrak{N}$ if $A \subseteq X$ absorbs $\mathfrak{M}$ and $B \subseteq X$ absorbs $\mathfrak{N}$, then $A\delta B$. A set D *absorbs* $\mathfrak{M} \subseteq \mathfrak{X}$ means that $A \in \mathcal{C}$ for all $\mathcal{C} \in \mathfrak{M}$.

The relation δ^* is a proximity on $\mathfrak{X}$. The pair $(\mathfrak{X}, \delta^*)$ is a compact proximity space, and it is called the *Smirnov compactification* of (X, δ) [**6**, Section 7].

In Section 5, we introduce a proximity on an AS.R. space such that its Smirnov compactification agrees with the asymptotic compactification.

There are several equivalent definitions for *asymptotic dimension* of a metric space ([**1**]). In this paper, by asymptotic dimension of a metric space (X, d) we mean the following definition.

Definition 1.2. Let X be a metric space. The inequality $\operatorname{asdim} X \leq n$ means that for each uniformly bounded cover $\mathcal{U}$ of X, there exists a uniformly bounded cover $\mathcal{V}$ of X such that $\mathcal{U}$ refines $\mathcal{V}$ and $\mu(\mathcal{V}) \leq n+1$. For a family $\mathcal{M}$ of subsets of a set X, $\mu(\mathcal{M})$ denotes the multiplicity of $\mathcal{M}$, i.e., the greatest number of elements of $\mathcal{M}$ that meets a point of X. By $\operatorname{asdim} X = n$, we mean that $\operatorname{asdim} X \leq n$ and $\operatorname{asdim} X \leq n-1$ do not hold. For a metric space X, $\operatorname{asdim} X$ is called the *asymptotic dimension* of X.

In Section 6, we show how one can generalize the notion of asymptotic dimension to AS.R. spaces.

In this paper, we denote the *Hausdorff distance* between subsets A and B of a metric space (X, d) by $d_H(A, B)$. Let us recall one more thing here. A proper map $f : X \to Y$ between metric spaces (X, d) and (Y, d') is said to be a *coarse map* if, for each $r > 0$, there exists an $s > 0$ such that $d(x, x') < r$ implies $d'(f(x), f(x')) < s$.

2. Asymptotic resemblance.

Definition 2.1. Let X be a metric space. We say that two subsets A and B of X are *asymptotically alike* and we denote it by $A\lambda B$, if $d_H(A, B) < \infty$. We assume that $d_H(\emptyset, \emptyset) = 0$ and $d_H(\emptyset, A) = \infty$ for all $\emptyset \neq A \subseteq X$.

Let us denote the open ball of radius $r > 0$ around $x \in X$ by $\mathbf{B}(x, r)$, and let $\mathbf{B}(A, r) = \bigcup_{a \in A} \mathbf{B}(x, r)$ for each subset A of X. The above definition states that $A\lambda B$ if and only if there is an $r > 0$ such that $A \subseteq \mathbf{B}(B, r)$ and $B \subseteq \mathbf{B}(A, r)$.

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in a metric space (X, d). If there exists a $k > 0$ such that $d(x_n, y_n) < k$ for all $n \in \mathbb{N}$, then we have $\{x_i \mid i \in I\}\lambda\{y_i \mid i \in I\}$ for each $I \subseteq \mathbb{N}$. The converse is also true.

Lemma 2.2. *Let (X, d) be a metric space. Suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in X such that, for each subset I of $\mathbb{N}$, we have $\{x_i \mid i \in I\}\lambda\{y_i \mid i \in I\}$. Then, there exists a $k > 0$ such that $d(x_n, y_n) < k$, for all $n \in \mathbb{N}$.*

Proof. Suppose, contrary to our claim, that, for each $n \in \mathbb{N}$, there is some $i_n \in \mathbb{N}$ such that $d(x_{i_n}, y_{i_n}) > n$. Without loss of generality, we can assume that we have $i_n = n$ for each $n \in \mathbb{N}$. We derive a contradiction by two steps.

Step 1. We claim that, for each $x \in X$ and $s > 0$, the index set $I = \{i \in \mathbb{N} \mid x_i \in \mathbf{B}(x, s)\}$ is finite. Let $C = \{x_i \mid i \in I\}$ and $D = \{y_i \mid i \in I\}$. Then since $C\lambda D$, let $d_H(C, D) = r$. If $j \in I$, we have:

$$i < d(x_i, y_i) \le d(x_i, x_j) + d(x_j, y_i) < 2s + r.$$

This implies that I is finite. Similarly, we can prove that, for each bounded subset D of X, the index set $J = \{j \in \mathbb{N} \mid y_j \in D\}$ is finite.

Step 2. Set $E_k = \{x_n \mid n \ge k\} \bigcup \{y_n \mid n \ge k\}$ for $k \in \mathbb{N}$. By Step 1, for each bounded set D, there is a $k \in \mathbb{N}$ such that $E_k \bigcap D = \emptyset$. Let $k_1 = 1$. For each $i \in \mathbb{N}$, choose $k_{i+1} \in \mathbb{N}$ such that

$$E_{k_{i+1}} \bigcap \mathbf{B}(\{x_{k_i}, y_{k_i}\}, k_i) = \emptyset.$$

Let $A = \{x_{k_i} \mid i \in N\}$ and $B = \{y_{k_i} \mid i \in N\}$. We have $A\lambda B$ so there exists $s > 0$ such that $A \subseteq \mathbf{B}(B, s)$ and $B \subseteq \mathbf{B}(A, s)$. Now, choose $k_i > s$. For $j, l \ge i$, let $\alpha = \min\{j, l\}$. We have $d(y_{k_j}, x_{k_l}) \ge k_\alpha \ge k_i > s$. Therefore, for each $j \ge i$, we have $y_{k_j} \in \mathbf{B}(x_{k_r}, s)$, for some $r = 1, \dots, i-1$, which means the set

$$\Big\{ j \in \mathbb{N} \mid y_j \in \bigcup_{r=1}^{i-1} \mathbf{B}(x_{k_r}, s) \Big\}$$

is infinite, and it contradicts Step 1 of the proof. $\square$

It is well known that a map $f : X \to Y$ between metric spaces X and Y is uniformly continuous if and only if for two subsets A and B of X, $A\delta B$ implies $f(A)\delta f(B)$ [**6,** subsection 4.8], where δ denotes the metric proximity, i.e., $A\delta B$ if and only if $d(A,B) = 0$. The following theorem is the large-scale counterpart of this fact.

Theorem 2.3. *Let X and Y be two metric spaces. A proper map $f : X \to Y$ is coarse if and only if, for each asymptotically alike subset A and B of X, $f(A)$ and $f(B)$ are asymptotically alike too.*

Proof. Suppose that $f : X \to Y$ is a coarse map. Let A and B be two subsets of X such that $A \subseteq \mathbf{B}(B,r)$ and $B \subseteq \mathbf{B}(A,r)$ for some $r > 0$. By hypothesis, there exists an $s > 0$ such that $d(x,x') < r$ yields $d(f(x), f(x')) < s$, so $f(A) \subseteq \mathbf{B}(f(B),s)$ and $f(B) \subseteq \mathbf{B}(f(A),s)$.

To prove the converse, assume that f is not a coarse map. So there are $r > 0$ and sequences x_n and y_n in X such that $d(x_n,y_n) < r$ and $d(f(x_n), f(y_n)) > n$. But the sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ satisfy the hypothesis of Lemma 2.2, a contradiction. $\square$

Proposition 2.4. *Let X be a metric space. The relation λ defined in Definition* 2.1 *is an equivalence relation on the family of all subsets of X, and it has following properties*:

(i) *$A_1\lambda B_1$ and $A_2\lambda B_2$ imply $(A_1 \bigcup A_2)\lambda(B_1 \bigcup B_2)$.*
(ii) *$(B_1 \bigcup B_2)\lambda A$ and $B_1, B_2 \neq \emptyset$ imply that there are nonempty subsets A_1 and A_2 of A such that $A = A_1 \bigcup A_2$, and we have $B_i\lambda A_i$ for $i \in \{1,2\}$.*

Proof. It is straightforward to show that λ is an equivalence relation on the family of all subsets of X, and it satisfies property (i). For property (ii), assume that $B_1 \bigcup B_2 \subseteq \mathbf{B}(A,r)$ and $A \subseteq \mathbf{B}(B_1 \bigcup B_2, r)$ for some $r > 0$ and $B_1, B_2 \neq \emptyset$. For $i \in \{1,2\}$, let $A_i = \mathbf{B}(B_i,r) \bigcap A$. We have

$$A = A_1 \bigcup A_2 \quad \text{and} \quad A_i\lambda B_i \text{ for } i \in \{1,2\}.$$ $\square$

Definition 2.5. Let X be a set. We call a binary relation λ on the power set of X an *asymptotic resemblance* (an AS.R.) if it is an equivalence relation on the family of all subsets of X and satisfies Proposition 2.4 (i) and (ii). For subsets A and B of X, we say that A

and B are *asymptotically alike* if $A\lambda B$. By $A\bar{\lambda}B$, we mean that A and B are not asymptotically alike. We call the pair (X,λ) an AS.R. *space.*

In a metric space (X,d), we call the relation defined in Definition 2.1 the AS.R. *associated to the metric* d on X.

Proposition 2.6. *Let* (X,λ) *be an* AS.R. *space. If* $A\lambda B$ *and* $\emptyset \neq A_1 \subseteq A$*, then there is* $\emptyset \neq B_1 \subseteq B$ *such that* $A_1\lambda B_1$*.*

Proof. It is an immediate consequence of Proposition 2.4 (ii). □

Proposition 2.7. *Let* λ *be an* AS.R. *on a set* X*. Suppose that* $A,B,C \subseteq X$ *and* $A \subseteq B \subseteq C$*. If* $A\lambda C$*, then* $A\lambda B$*.*

Proof. Proposition 2.4 (i) leads to $((B\backslash A)\bigcup A)\lambda((B\backslash A)\bigcup C)$. Thus, $B\lambda C$ and, since λ is an equivalence relation, $A\lambda B$. □

Let us recall that, on a coarse space $(X,\mathcal{E})$,

$$E(A) = \{y \in X \mid (x,y) \in E \text{ for some } x \in A\},$$

for all $E \in \mathcal{E}$ and all $A \subseteq X$.

Example 2.8. Suppose that $\mathcal{E}$ is a coarse structure on a set X. For any two subsets A and B of X, define $A\lambda_{\mathcal{E}}B$ if $A \subseteq E(B)$ and $B \subseteq E(A)$ for some $E \in \mathcal{E}$. The relation $\lambda_{\mathcal{E}}$ is an asymptotic resemblance on X. We call $\lambda_{\mathcal{E}}$ the AS.R. *associated to the coarse structure* $\mathcal{E}$ on X.

In the next section, we will investigate the relation between coarse structures and asymptotic resemblance relations in more details.

Example 2.9. Let X be a set. For any two subsets A and B of X, define $A\lambda B$ if $A\Delta B = (A \setminus B)\bigcup(B \setminus A)$ is finite. The relation λ is an AS.R. on X that we call the *discrete asymptotic resemblance* on a set X.

Definition 2.10. Let λ be an AS.R. on a set X. We say a subset A of X is *bounded* if $A\lambda x$, for some $x \in X$. We assume that the empty set is bounded.

Let λ be the AS.R. associated to a coarse structure $\mathcal{E}$ on a set X. It is easy to verify that $D \subseteq X$ is bounded if and only if it is bounded with respect to $\mathcal{E}$.

Proposition 2.11. *Let λ be an* AS.R. *on a set X, and let $A \subseteq X$. If $A\lambda x$ for some $x \in X$ and $\emptyset \neq B \subseteq A$, then $B\lambda x$. Thus, all subsets of a bounded set are bounded.*

Proof. It is an immediate consequence of Proposition 2.6. □

Example 2.12. Suppose that G is a group. For two subsets A and B of G, define $A\lambda_l B$ if there exists a finite subset K of G such that $A \subset BK$ and $B \subseteq AK$. We call λ_l the *left* AS.R. on G. Similarly, one can define the *right* AS.R. on G. In both cases, a subset D of G is bounded if and only if it is finite. If G is an Abelian group, then λ_r and λ_l obviously coincide. However, they are different in the general case [**3**].

Example 2.13. Suppose that A and B are two subsets of the real line $\mathbb{R}$. Define $A\lambda B$ if there exists $r > 0$ such that

$$A \subseteq \bigcup_{b \in B} (b - r, +\infty)$$

and

$$B \subseteq \bigcup_{a \in A} (a - r, +\infty).$$

It is straightforward to show that λ is an equivalence relation on the family of all subsets of $\mathbb{R}$, and it satisfies Proposition 2.4 (i). Now suppose that $A\lambda(B_1 \bigcup B_2)$ and $B_1, B_2 \neq \emptyset$. So there is an $r > 0$ such that we have

$$A \subseteq \bigcup_{b \in B_1 \bigcup B_2} (b - r, +\infty)$$

and

$$B_1 \bigcup B_2 \subseteq \bigcup_{a \in A} (a - r, +\infty).$$

Let

$$A'_1 = \left(\bigcup_{b\in B_1}(b-r,+\infty)\right)\bigcap A.$$

If $B_1 \subseteq \bigcup_{a\in A'_1}(a-r,\infty)$, then $A'_1\lambda B_1$, and we can let $A_1 = A'_1$.

Now assume that there is a $b_1 \in B_1$ such that $b_1 \le a-r$ for all $a \in A'_1$. Since $A\lambda(B_1 \bigcup B_2)$, there is an $a_1 \in A$ such that $a_1 - r < b_1$. Let

$$A_1 = A'_1 \bigcup \{a_1\}$$

and

$$r_1 = \max\{\mid b_1 - a_1 \mid +1, r\}.$$

Since $a_1 \notin A'_1$, $a_1 \le b-r < b$ for all $b \in B_1$. Thus, $B_1 \subseteq (a_1 - r_1, +\infty)$, which leads to $A_1\lambda B_1$. Similarly, one can find $A_2 \subseteq A$ such that $A_2\lambda B_2$ and $A = A_1 \bigcup A_2$. Let λ_d denote the AS.R. associated to the standard metric on $\mathbb{R}$. It is easy to show that $A\lambda_d B$ yields $A\lambda B$, for all $A, B \subseteq \mathbb{R}$. A set $D \subseteq \mathbb{R}$ is bounded with respect to λ if and only if $D \subseteq (a,+\infty)$ for some $a \in \mathbb{R}$. There is no metric on $\mathbb{R}$ such that we have $A\lambda B$ if and only if $d_H(A,B) < \infty$ for all subsets A and B of $\mathbb{R}$.

Suppose the contrary. For each $n \in \mathbb{N}$, the interval $(-\infty,-n)$ is unbounded. We choose $b_n < -n$ such that $d(-n,b_n) > n$. Therefore, the sequences $(-n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Lemma 2.2, a contradiction.

Definition 2.14. Let (X,λ_1) and (Y,λ_2) be two AS.R. spaces. We call a map $f : X \to Y$ an AS.R. *mapping* if:

(i) (Properness.) $f^{-1}(B)$ is bounded in X for each bounded subset B of Y.
(ii) $A\lambda_1 B$ implies $f(A)\lambda_2 f(B)$, for all subsets A and B of X.

In fact, Theorem 2.3 states that, for metric spaces X and Y, a map $f : X \to Y$ is a coarse map if and only if it is an AS.R. mapping for the AS.R.'s associated to their metrics.

Definition 2.15. Let (Y,λ) be an AS.R. space, and let X be a set. We say that two maps $f : X \to Y$ and $g : X \to Y$ are *close* if we have $f(A)\lambda g(A)$ for each subset A of X.

Proposition 2.16. *Let (Y, d) be a metric space, and let λ be the* AS.R. *associated to d. Two maps $f : X \to Y$ and $g : X \to Y$ are close if and only if there is some $k > 0$ such that $d(f(x), g(x)) < k$ for all $x \in X$.*

Proof. The proof of the *only if* part is straightforward.

Now suppose that f and g are close maps. Assume that, on the contrary, for all $n \in \mathbb{N}$, there exists an $x_n \in X$ such that $d(f(x_n), g(x_n)) > n$. Then the sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(g(x_n))_{n\in\mathbb{N}}$ satisfy the hypothesis of Lemma 2.2, a contradiction. $\square$

Definition 2.17. Let (X, λ_1) and (Y, λ_2) be two AS.R. spaces. We call an AS.R. mapping $f : X \to Y$ an *asymptotic equivalence* if there exists an AS.R. mapping $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps $i_X : X \to X$ and $i_Y : Y \to Y$, respectively. We say AS.R. spaces (X, λ_1) and (Y, λ_2) are *asymptotically equivalent* if there exists an asymptotic equivalence $f : X \to Y$.

Proposition 2.18. *Let (X, λ_1) and (Y, λ_2) be two* AS.R. *spaces. Suppose that $f : X \to Y$ and $g : X \to Y$ are two close maps. If f is an* AS.R. *mapping, then so is g, and if f is an asymptotic equivalence, then so is g.*

Proof. We will prove that, if f is a proper map, then so is g. Other parts of Proposition 2.18 are straightforward results of the property that λ_1 and λ_2 are equivalence relations on the family of all subsets of X and Y.

Let $D \subseteq Y$ be a bounded set. We have $f(g^{-1}(D))\lambda_2 g(g^{-1}(D))$ so $f(g^{-1}(D))$ is bounded. Thus, $f^{-1}(f(g^{-1}(D)))$ is bounded, and Proposition 2.11 leads to $g^{-1}(D)$ being bounded. $\square$

Definition 2.19. Let (X, λ) be an AS.R. space, and let Y be a nonempty subset of X. For the two subsets A and B of Y, define $A\lambda_Y B$ if $A\lambda B$. The pair (Y, λ_Y) is an AS.R. space, and we call λ_Y the *subspace* AS.R. induced by λ on Y.

Lemma 2.20. *Let* (X, λ) *and* (Y, λ') *be two* AS.R. *spaces. Suppose that* $f : X \to Y$ *is an asymptotic equivalence and* $\emptyset \neq C \subseteq X$*. Then* $f|_C : (C, \lambda_C) \to (f(C), \lambda'_{f(C)})$ *is also an asymptotic equivalence.*

Proof. Let $g : Y \to X$ be an AS.R. mapping such that $g \circ f$ and $f \circ g$ are close maps to identity maps $i_X : X \to X$ and $i_Y : Y \to Y$, respectively. Let $q : f(C) \to C$ be a map such that $f \circ q(a) = a$ for each $a \in f(C)$. Suppose that $D \subseteq C$ is bounded. Since $g \circ f(D) \lambda D$, $g \circ f(D)$ is a bounded subset of X. We have $q^{-1}(D) \subseteq f(D) \subseteq g^{-1}(g \circ f(D))$. Proposition 2.11 shows that $q^{-1}(D)$ is bounded. Assume that $A, B \subseteq f(C)$ and $A \lambda'_{f(C)} B$. We have $g \circ f(q(A)) \lambda q(A)$ and, since $f(q(A)) = A$, $q(A) \lambda g(A)$. Similarly, $q(B) \lambda g(B)$ leads to $q(A) \lambda_C q(B)$. Therefore, q is an AS.R. mapping. Now, let $A \subseteq C$. We have $f(q \circ f(A)) = f(A)$ so $g(f(q \circ f(A))) = g \circ f(A) \lambda A$. Also, we have $g \circ f(q \circ f(A)) \lambda q \circ f(A)$ and it leads to $q \circ f(A) \lambda_C A$. Therefore, $f|_C : C \to f(C)$ is an asymptotic equivalence. □

Definition 2.21. We call an AS.R. space (X, λ) *asymptotically connected* if we have $x \lambda y$ for all $x, y \in X$.

It is immediate that the AS.R. associated to a connected coarse structure is asymptotically connected.

Proposition 2.22. *An* AS.R. *space* (X, λ) *is asymptotically connected if and only if, for each nonempty subset* A *and* B *of* X*,* $A \Delta B$ *is finite, which yields* $A \lambda B$*.*

Proof. The *if* part is trivial.

Assume that $A \setminus B = \{x_1, \ldots, x_n\}$ and $B \setminus A = \{y_1, \ldots, y_m\}$. By using Proposition 2.4 (i) and asymptotic connectedness of λ, we can conclude $(A \setminus B) \lambda (B \setminus A)$. By Proposition 2.4 (i), we have

$$\left((A \setminus B) \bigcup \left(A \bigcap B\right)\right) \lambda \left((B \setminus A) \bigcup \left(A \bigcap B\right)\right).$$

Thus, $A \lambda B$. □

3. Coarse structures and asymptotic resemblance relations.

In Example 2.8, we stated that every coarse structure $\mathcal{E}$ on a set X

induces an AS.R. on X. We denoted this AS.R. by $\lambda_{\mathcal{E}}$. The following example shows that two different coarse structures may induce a same AS.R. relation.

Example 3.1. Let $X = \mathbb{N}$. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ denote two families of subsets of $X \times X$ such that:

(i) $E \in \mathcal{E}_1$ if and only if $E(A)$ and $E^{-1}(A)$ are finite for all finite $A \subseteq \mathbb{N}$.
(ii) $E \in \mathcal{E}_2$ if and only if there exists $n_E \in \mathbb{N}$ such that $E(x)$ and $E^{-1}(x)$ have at most n_E members, for all $x \in X$.

Both $\mathcal{E}_1$ and $\mathcal{E}_2$ families are coarse structures on X ([**8,** Examples 2.8 and 2.44]). It is immediate that $\mathcal{E}_2$ is a proper subset of $\mathcal{E}_1$.

For two subsets A, B of X, we claim that $A\lambda_{\mathcal{E}_2}B$ if and only if A and B are both finite or A and B are both infinite. It is straightforward to show that, if $A\lambda_{\mathcal{E}_2}B$ and A is finite, then so is B, and if A and B are both finite, then $A\lambda_{\mathcal{E}_2}B$.

Suppose that A and B are both infinite. Let

$$A = \{a_n \mid n \in \mathbb{N}\} \quad \text{and} \quad B = \{b_n \mid n \in \mathbb{N}\},$$

and assume that $a_n < a_{n+1}$ and $b_n < b_{n+1}$ for all $n \in \mathbb{N}$. Let

$$E = \{(a_n, b_n) \mid n \in \mathbb{N}\} \bigcup \{(b_n, a_n) \mid n \in \mathbb{N}\}.$$

Clearly, $E \in \mathcal{E}_2$ and $n_E = 2$. We have $A \subseteq E(B)$ and $B \subseteq E(A)$, so $A\lambda_{\mathcal{E}_2}B$. Since $\mathcal{E}_2 \subseteq \mathcal{E}_1$, one can easily show that $A\lambda_{\mathcal{E}_1}B$ if and only if A and B are both finite or A and B are both infinite. Thus, $\lambda_{\mathcal{E}_1} = \lambda_{\mathcal{E}_2}$.

Let λ be an AS.R. on a set X. We denote the family of all coarse structures that induce λ by $\mathcal{E}(\lambda)$. Let us recall that, for two coarse structures $\mathcal{E}_1$ and $\mathcal{E}_2$ on a set X, $\mathcal{E}_2$ is said to be *coarser* than $\mathcal{E}_1$ if $\mathcal{E}_1 \subseteq \mathcal{E}_2$ ([**8,** subsection 2.1]).

Proposition 3.2. *Let λ be an* AS.R. *on a set X. If $\mathcal{E}(\lambda) \neq \emptyset$, then there is a coarse structure $\mathcal{E}_\lambda \in \mathcal{E}(\lambda)$ such that $\mathcal{E}_\lambda$ is coarser than each member of $\mathcal{E}(\lambda)$.*

Proof. Let $\mathcal{E}_\lambda$ be the family of all $E \subseteq X \times X$ such that $\pi_1(F)\lambda\pi_2(F)$ for all $F \subseteq E$, where π_1 and π_2 denote projection maps onto first and second factors, respectively. Since λ is an equivalence relation, $\Delta \in \mathcal{E}_\lambda$ and $E^{-1} \in \mathcal{E}_\lambda$ for all $E \in \mathcal{E}_\lambda$. By Proposition 2.4 (i), one easily sees that $E \bigcup F \in \mathcal{E}_\lambda$ for all $E, F \in \mathcal{E}_\lambda$.

Let $E, F \in \mathcal{E}_\lambda$, and suppose that $H \subseteq E \circ F$. Set

$$O_1 = \{(x, y) \in X \times X \mid x \in \pi_1(H), y \in F(x) \bigcap \pi_1(E)\}$$

and

$$O_2 = \{(x, y) \in X \times X \mid x \in \pi_2(O_1), y \in E(x)\}.$$

We have $O_1 \subseteq F$ and $O_2 \subseteq E$, so $\pi_1(H) = \pi_{O_1}\lambda\pi_2(O_1)$ and $\pi_2(O_1) = \pi_1(O_2)\lambda\pi_2(O_2) = \pi_2(H)$. It follows that $\pi_1(H)\lambda\pi_2(H)$, which leads to $E \circ F \in \mathcal{E}_\lambda$. Therefore, $\mathcal{E}_\lambda$ is a coarse structure on X.

Suppose that $\mathcal{E} \in \mathcal{E}(\lambda)$. It is straightforward by the definition to show that, if $E \in \mathcal{E}$ and $F \subseteq E$, then $\pi_1(F)\lambda\pi_2(F)$. It follows that $\mathcal{E} \subseteq \mathcal{E}_\lambda$. Thus, $\mathcal{E}_\lambda$ is coarser than each member of $\mathcal{E}(\lambda)$.

It remains to show $\mathcal{E}_\lambda \in \mathcal{E}(\lambda)$. Suppose that $A, B \subseteq X$ and $A \subseteq E(B)$ and $B \subseteq E(A)$, for some $E \in \mathcal{E}_\lambda$. Let

$$F_1 = \{(a, b) \in E \mid a \in A, b \in B\}$$

and

$$F_2 = \{(b, a) \in E \mid a \in A, b \in B\}.$$

Then, $A = \pi_1(F_1)\lambda\pi_2(F_1)$ and $B = \pi_1(F_2)\lambda\pi_2(F_2)$. We obtain

$$A = \left(\pi_2(F_2) \bigcup (A \setminus \pi_2(F_2))\right) \lambda \left(B \bigcup A \setminus \pi_2(F_2)\right).$$

Since A and $\pi_2(F_1) \subseteq B$ are asymptotically alike, there is a subset L of B such that $(A \setminus \pi_2(F_2))\lambda L$, by Proposition 2.6. Therefore, $A\lambda(B \bigcup L) = B$. Since $\mathcal{E}(\lambda) \neq \emptyset$, and $\mathcal{E}_\lambda$ is greater than each member of $\mathcal{E}(\lambda)$, it is straightforward to show that $A\lambda B$ implies that there is an $E \in \mathcal{E}_\lambda$ such that $A \subseteq E(B)$ and $B \subseteq E(A)$, for all $A, B \subseteq X$. $\square$

In fact, asymptotic resemblance relations on a set X define an equivalence relation on the family of all coarse structures on X. Two coarse structures on X are equivalent if they induce the same asymptotic resemblance relation. Proposition 3.2 shows that these equivalence classes have a biggest member. One can compare this with similar arguments about the relation between uniform structures and proximity in [**6,** Section 12].

4. Asymptotic compactification.

Definition 4.1. Let $(X, \mathcal{T})$ be a topological space, and let λ be an AS.R. on X. We say that an open subset U of X is an *asymptotic neighborhood* of $A \subseteq X$ if $A \subseteq U$ and $A\lambda U$. We call λ a *compatible* AS.R. *with* $\mathcal{T}$ if

(i) Each subset of X has an asymptotic neighborhood.
(ii) $A\lambda\overline{A}$ for all $A \subseteq X$.

Proposition 4.2. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{E}$ be a coarse structure compatible with $\mathcal{T}$. Then the* AS.R. *associated to $\mathcal{E}$ is compatible with $\mathcal{T}$ also.*

Proof. Assume that E is a symmetric open entourage containing the diagonal. For $A \subseteq X$, $E(A)$ is an asymptotic neighborhood of A. Let $a \in \overline{A}$. $E(a)$ is an open neighborhood of a, so $E(a)\bigcap A \neq \emptyset$. Let $a' \in E(a)\bigcap A$. Since $(a, a') \in E^{-1} = E$, $a \in E(a') \subseteq E(A)$. Thus, $\overline{A} \subseteq E(A)$, and this leads to $A\lambda\overline{A}$. $\square$

Definition 4.3. We call two subsets A_1 and A_2 of an AS.R. space (X, λ) *asymptotically disjoint* if, for all unbounded subsets $L_1 \subseteq A_1$ and $L_2 \subseteq A_2$, we have $L_1\overline{\lambda}L_2$. We say that an AS.R. space (X, λ) is *asymptotically normal* if, for asymptotically disjoint subsets A_1 and A_2 of X, there exist $X_1 \subseteq X$ and $X_2 \subseteq X$ such that $X = X_1 \bigcup X_2$, and A_i and X_i are asymptotically disjoint for $i \in \{1, 2\}$.

Let B a bounded subset of an AS.R. space (X, λ). Then B is asymptotically disjoint from all $A \subseteq X$. In [**2**], two subsets A and B of a metric space (X, d) are called asymptotically disjoint if, for some $x_0 \in X$, $\lim_{r\to\infty} d(A \setminus \mathbf{B}(x_0, r), B \setminus \mathbf{B}(x_0, r)) = \infty$.

The next proposition shows that this definition is equivalent to our definition of asymptotical disjointness on metric spaces.

Proposition 4.4. *Let (X,d) be a metric space, and let λ be the associated* AS.R. *to d. Two unbounded subsets A and B of X are asymptotically disjoint if and only if, for some x_0 in X,* $\lim_{r\to\infty} d(A \setminus \boldsymbol{B}(x_0,r), B \setminus \boldsymbol{B}(x_0,r)) = \infty$.

Proof. Let $x_0 \in X$ be a fixed point. Suppose that A and B are two asymptotically disjoint subsets of X. Assume that, on the contrary,

$$\lim_{r\to\infty} d(A \setminus \mathbf{B}(x_0,r), B \setminus \mathbf{B}(x_0,r)) \neq \infty.$$

Thus, there exists $N \in \mathbb{N}$ such that, for each $m \in \mathbb{N}$, we have $d(A \setminus \mathbf{B}(x_0,r_m), B \setminus \mathbf{B}(x_0,r_m)) < N$ for some $r_m \geq m$. We choose $x_m \in A \setminus \mathbf{B}(x_0,r_m)$ and $y_m \in B \setminus \mathbf{B}(x_0,r_m)$ such that $d(x_m,y_m) < N$. Let

$$L_1 = \{x_m \mid m \in \mathbb{N}\}$$

and

$$L_2 = \{y_m \mid m \in \mathbb{N}\}.$$

Thus, $L_1 \subseteq A$ and $L_2 \subseteq B$ are two unbounded subsets and $L_1 \lambda L_2$, a contradiction.

To prove the converse, let $A, B \subseteq X$, and suppose that $\lim_{r\to\infty} d(A \setminus \mathbf{B}(x_0,r), B \setminus \mathbf{B}(x_0,r)) = \infty$. Assume that, on the contrary, there are unbounded subsets $L_1 \subseteq A$ and $L_2 \subseteq B$ such that $d_H(L_1,L_2) < N$, for some $N \in \mathbb{N}$. Since L_1 is unbounded for each $n \in \mathbb{N}$, there exists $x_n \in L_1 \setminus \mathbf{B}(x_0,n)$ such that $d(x_n,b) > N$ for all $b \in \mathbf{B}(x_0,n) \bigcap L_2$. Thus, there is a $y_n \in L_2 \setminus \mathbf{B}(x_0,n)$ such that $d(x_n,y_n) < N$. Then $x_n \in A \setminus \mathbf{B}(x_0,n)$ and $y_n \in B \setminus \mathbf{B}(x_0,n)$ for all $n \in \mathbb{N}$. Thus,

$$\lim_{r\to\infty} d(A \setminus \mathbf{B}(x_0,r), B \setminus \mathbf{B}(x_0,r)) \neq \infty,$$

a contradiction. $\square$

Proposition 4.5. *Let (X,d) be a metric space and let λ be the* AS.R. *associated to d. Then (X,λ) is an asymptotically normal* AS.R. *space.*

Proof. Assume that A and B are asymptotically disjoint subsets of X. For $i \in \mathbb{N} \bigcup \{0\}$, let

$$A_i = \Big\{x \mid d(x, A) \leq i+1\Big\} \bigcap \Big\{x \mid d(x, B) \geq i\Big\}$$

and

$$B_i = \Big\{x \mid d(x, B) \leq i+1\Big\} \bigcap \Big\{x \mid d(x, A) \geq i\Big\}.$$

Suppose that

$$X_1 = \bigcup_{i=0}^{\infty} B_i \quad \text{and} \quad X_2 = \bigcup_{i=0}^{\infty} A_i.$$

For $x \in X$, assume that $i \leq d(x, A) \leq i+1$ and $j \leq d(x, B) \leq j+1$. If $i = j$ then $x \in A_i = B_j$. If $i < j$ then $i+1 \leq j$ so $x \in A_i$. Thus $X = X_1 \bigcup X_2$. We claim that A and X_1 are asymptotically disjoint. Suppose that, contrary to our claim, there are unbounded subsets $L_1 \subseteq A$ and $L_2 \subseteq X_1$ such that $L_1 \lambda L_2$, i.e., $L_1 \subseteq \mathbf{B}(L_2, n)$ and $L_2 \subseteq \mathbf{B}(L_1, n)$ for some $n \in \mathbb{N}$. Thus, $L_2 \subseteq \mathbf{B}(A, n)$, and this leads to

$$L_2 \subseteq \bigcup_{i=0}^{n-1} B_i.$$

Therefore, $L_2 \subseteq \mathbf{B}(B, n)$. Let

$$L_3 = \mathbf{B}(L_2, n) \bigcap B.$$

We obtain $L_3 \lambda L_2$, which leads to $L_3 \lambda L_1$, a contradiction. Therefore, A and X_1 are asymptotically disjoint. Similarly, one can show that B and X_2 are asymptotically disjoint. □

Let X be a Hausdorff and locally compact topological space, and let αX be a compactification of X. Let us recall that the topological coarse structure on X associated to αX is the family of all $E \subseteq X \times X$ such that the closure of E meets $(\alpha X \times \alpha X) \setminus (X \times X)$ only in the diagonal [**8,** Definition 2.28]. It is known that topological coarse structures associated to a second countable compactifications are not metrizable [**8,** Example 2.53].

The next proposition shows that the class of all asymptotic normal AS.R. spaces is much bigger than the family of all metric spaces.

Proposition 4.6. *Let X be a Hausdorff and locally compact metric space, and let αX be a first countable compactification of X. Let $\mathcal{E}$ be the topological coarse structure associated to αX, and let λ be the* AS.R. *associated to $\mathcal{E}$. Then λ is asymptotically normal.*

Proof. First we prove that A and B are asymptotically disjoint subsets of X if and only if

$$\overline{A}\bigcap\overline{B}\bigcap(\alpha X\setminus X)=\emptyset.$$

Let

$$\omega\in\overline{A}\bigcap\overline{B}\bigcap(\alpha X\setminus X)\quad\text{for } A,B\subseteq X.$$

There are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A and B, respectively, such that they converge to ω. Let $E=\{(x_n,y_n)\mid n\in\mathbb{N}\}$. It is straightforward to show that each sequence in E can be assumed to be a subsequence of $((x_n,y_n))_{n\in\mathbb{N}}$, and this shows that

$$\overline{E}\bigcap((\alpha X\times\alpha X)\setminus(X\times X))=\{(\omega,\omega)\}.$$

So $E\in\mathcal{E}$.

Let

$$L_1=\{x_n\mid n\in\mathbb{N}\}\subseteq A$$

and

$$L_2=\{y_n\mid n\in\mathbb{N}\}\subseteq B.$$

We obtain $L_1\lambda L_2$ which shows that A and B are not asymptotically disjoint.

Now assume that A and B are two subsets of X such that they are not asymptotically disjoint. Let L_1 and L_2 be two unbounded and asymptotically alike subsets of A and B, respectively. There is an $E\in\mathcal{E}$ such that $L_1\subseteq E(L_2)$ and $L_2\subseteq E(L_1)$.

Let $\omega\in\overline{L_1}\bigcap(\alpha X\setminus X)$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in L_1 and $x_n\to\omega$. For each $n\in\mathbb{N}$, choose $y_n\in L_2$ such that $(x_n,y_n)\in E$. Since $E\in\mathcal{E}$, $y_n\to\omega$. This shows that

$$\omega\in\overline{L_1}\bigcap\overline{L_2}\bigcap(\alpha X\setminus X).$$

Thus,

$$\overline{A}\bigcap\overline{B}\bigcap(\alpha X\setminus X)\neq\emptyset.$$

Now we give the proof of Proposition 4.6. Let A and B be two asymptotically disjoint subsets of X. Then

$$\overline{A}\bigcap\overline{B}\bigcap(\alpha X\setminus X)=\emptyset.$$

Since αX is a normal topological space, there is a map $f:\alpha X\to[0,1]$ such that

$$f\Big(\overline{A}\bigcap(\alpha X\setminus X)\Big)=0$$

and

$$f\Big(\overline{B}\bigcap(\alpha X\setminus X)\Big)=1.$$

Let

$$X_1=f^{-1}([1/2,1])\bigcap X$$

and

$$X_2=f^{-1}([0,1/2])\bigcap X.$$

By what we first proved, here it is straightforward to show that A and B are asymptotically disjoint from X_1 and X_2, respectively. $\square$

Definition 4.7. Let $(X,\mathcal{T})$ be a topological space and λ be an AS.R. compatible with $\mathcal{T}$. We say that λ is *proper* if each bounded subset of X has a compact closure.

It is straightforward to show that a proper coarse structure admits a proper AS.R. It is an immediate result of the definition that, if there exists a proper AS.R. on a topological space X, then X is a locally compact topological space.

Proposition 4.8. *Suppose that λ is a proper and asymptotically connected* AS.R. *on a topological space X. Then a subset A of X is bounded if and only if $\overline{A}$ is compact.*

Proof. The *only if* part is a part of the definition. Suppose that A is a subset of X with compact closure. We cover $\overline{A}$ with the U_i, $i \in \{1, \ldots, n\}$, such that each U_i is an asymptotic neighborhood of some $a_i \in \overline{A}$. We have

$$\left(\bigcup_{i=1}^{n} U_i\right) \lambda \{a_1, \ldots, a_n\}$$

by Proposition 2.4 (i). Also, we have $\{a_1, \ldots, a_n\} \lambda a_1$ by asymptotic connectedness of λ, so Proposition 2.11 leads to $A \lambda a_1$. □

From now on, we will assume that all AS.R. spaces are asymptotically connected.

Definition 4.9. Let $(X, \mathcal{T})$ be a topological space and λ an AS.R. compatible with $\mathcal{T}$. For two nonempty subsets A and B of X, define $A \sim B$ if $A = B$ or A and B unbounded asymptotically alike subsets of X. The relation $\sim$ is an equivalence relation on the family of all nonempty subsets of X. Let γX denote the family of all closed ultrafilters on X and $\mathcal{F}_1, \mathcal{F}_2 \in \gamma X$. Define $\mathcal{F}_1 \approx \mathcal{F}_2$ if, for any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, there are $L_1 \subseteq A$ and $L_2 \subseteq B$ such that $L_1 \sim L_2$. We denote the equivalence class of $\mathcal{F} \in \gamma X$ by $[\mathcal{F}]$.

Lemma 4.10. *Let (X, λ) be an* AS.R. *space. If A and B are asymptotically disjoint subsets of X and $A \lambda C$ and $B \lambda D$ for some $C, D \subseteq X$, then C and D are asymptotically disjoint too.*

Proof. It is an immediate consequence of Proposition 2.6. □

Proposition 4.11. *Let $(X, \mathcal{T})$ be a topological space, and let λ be an* AS.R. *compatible with $\mathcal{T}$. If (X, λ) is an asymptotically normal* AS.R. *space, then the relation $\approx$ defined in Definition* 4.9 *is an equivalence relation.*

Proof. The relation $\approx$ is obviously symmetric and reflexive. We suppose that $\mathcal{F}_1 \approx \mathcal{F}_2$ and $\mathcal{F}_2 \approx \mathcal{F}_3$ and claim that $\mathcal{F}_1 \approx \mathcal{F}_3$. Suppose that, contrary to our claim, there are disjoint sets $A \in \mathcal{F}_1$ and $C \in \mathcal{F}_3$ such that they are asymptotically disjoint. So $A, C \notin \mathcal{F}_2$. Choose $B \in \mathcal{F}_2$ such that $B \bigcap (A \bigcup C) = \emptyset$. Since (X, λ) is asymptotically normal, there are $X_1 \subseteq X$ and $X_2 \subseteq X$ such that $X_1 \bigcup X_2 = X$, and they are asymptotically disjoint from A and C, respectively. Let

$$B_1 = B \bigcap X_1$$

and

$$B_2 = B \bigcap X_2.$$

By compatibility and Lemma 4.10, $\overline{B_1}$ and $\overline{B_2}$ are asymptotically disjoint from A and C respectively. Since $\mathcal{F}_2$ is a closed ultrafilter and $B = \overline{B_1} \bigcup \overline{B_2}$ then $\overline{B_1} \in \mathcal{F}_2$ or $\overline{B_2} \in \mathcal{F}_2$ which contradicts $\mathcal{F}_1 \approx \mathcal{F}_2$ or $\mathcal{F}_2 \approx \mathcal{F}_3$, respectively. $\square$

Let us recall that, for an open subset U of a topological space X, U^* is the family of all closed ultrafilters on X such that U contains some elements of them.

Proposition 4.12. *Let X be a normal topological space, and let λ be a compatible and asymptotically normal* AS.R. *on X. Then the set*

$$R = \{(\mathcal{F}_1, \mathcal{F}_2) \in \gamma X \times \gamma X \mid \mathcal{F}_1 \approx \mathcal{F}_2\}$$

is closed in $\gamma X \times \gamma X$.

Proof. Suppose that $(\mathcal{F}_1, \mathcal{F}_2) \notin R$. Then there are disjoint sets $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ such that they are also asymptotically disjoint. We choose asymptotic neighborhoods $A \subseteq U$ and $B \subseteq V$ such that $U \bigcap V = \emptyset$. Therefore, $\mathcal{F}_1 \in U^*$ and $\mathcal{F}_2 \in V^*$.

Now assume that $\mathcal{H}_1 \in U^*$ and $\mathcal{H}_2 \in V^*$. Thus, there are $D_1 \in \mathcal{H}_1$ and $D_2 \in \mathcal{H}_2$ such that $D_1 \subseteq U$ and $D_2 \subseteq V$. Then

$$D_1 \bigcup A \in \mathcal{H}_1 \quad \text{and} \quad D_2 \bigcup B \in \mathcal{H}_2.$$

Also, by Proposition 2.7, we have

$$\left(D_1 \bigcup A\right) \lambda A \quad \text{and} \quad \left(D_2 \bigcup B\right) \lambda B.$$

By Lemma 4.10, $D_1 \bigcup A$ and $D_2 \bigcup B$ are asymptotically disjoint. Therefore, the open neighborhood $U^* \times V^*$ of $(\mathcal{F}_1, \mathcal{F}_2)$ is disjoint from R. $\square$

Let λ be a compatible AS.R. on a Hausdorff topological space X. We recall that, for a point $x \in X$, σ_x denotes the family of all closed subsets of X that contains x, and the map $\sigma : X \to \gamma X$ defined by $\sigma(x) = \sigma_x$ is a topological embedding. For two points $x, y \in X$, it is straightforward to show that $\sigma_x \approx \sigma_y$ if and only if $x = y$. Thus, the map

$$\phi : X \to \frac{\gamma X}{\approx}$$

defined by $\phi(x) = [\sigma_x]$ is one-to-one.

Corollary 4.13. *Let X be a normal topological space, and let λ be a proper and asymptotically normal* AS.R. *on X. Then $\gamma X/\approx$ is a Hausdorff compactification of X.*

Proof. Since γX is compact, its quotient $\gamma X/\approx$ is compact too. By Proposition 4.12, $\gamma X/\approx$ is Hausdorff. It suffices to show that $\phi : X \to \gamma X/\approx$ is a topological embedding. Let $\pi : \gamma X \to \gamma X/\approx$ be the quotient map. Since $\phi = \pi \circ \sigma$, ϕ is a continuous map. Suppose that $U \subseteq X$ is an open set and $[\sigma_x] \in \phi(U)$. By Proposition 2.7, we can choose an asymptotic neighborhood W of x such that $W \subseteq U$. It is easy to verify that $\pi^{-1}(\phi(W)) = W^*$. Thus, $\phi(W)$ is open in $\gamma X/\approx$, and we have $[\sigma_x] \in \phi(W) \subseteq \phi(U)$. Therefore, ϕ is a topological embedding and $\phi(X)$ is open in $\gamma X/\approx$. $\square$

Proposition 4.14. *Let $(X, \mathcal{T})$ be a topological space, and let λ be an* AS.R. *compatible with $\mathcal{T}$. Suppose that $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ are two nets in X. Let*

$$T_\beta = \{x_\alpha \mid \alpha \geq \beta\} \quad \text{and} \quad S_\beta = \{y_\alpha \mid \alpha \geq \beta\}.$$

If $T_\beta \lambda S_\beta$ for all $\beta \in I$, $\sigma_{x_\alpha} \to \mathcal{F}_1$ and $\sigma_{y_\alpha} \to \mathcal{F}_2$ for some $\mathcal{F}_1, \mathcal{F}_2 \in \gamma X \setminus \sigma(X)$, then $\mathcal{F}_1 \approx \mathcal{F}_2$.

Proof. Suppose that $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. We choose asymptotic neighborhoods U and V of A and B, respectively. Then $\mathcal{F}_1 \in U^*$ and $\mathcal{F}_2 \in V^*$. Since $\sigma_{x_\alpha} \to \mathcal{F}_1$ and $\sigma_{y_\alpha} \to \mathcal{F}_2$, there are $\alpha, \beta \in I$ such that $T_\alpha \subseteq U$ and $S_\beta \subseteq V$. Let $\alpha, \beta \leq \gamma$ so that $T_\gamma \subseteq U$ and $S_\gamma \subseteq V$. This leads to $T_\gamma \lambda L_1$ and $S_\gamma \lambda L_2$ for some $L_1 \subseteq A$ and $L_2 \subseteq B$ by Proposition 2.6. $\square$

A compactification $\overline{X}$ of a proper coarse space $(X,\mathcal{E})$ is said to be a *coarse compactification* of X, if $E\in\mathcal{E}$ and $(x_\alpha,y_\alpha)_{\alpha\in I}$ is a convergent net in E, then $x_\alpha\to\omega$ for $\omega\in\overline{X}\setminus X$ yields $y_\alpha\to\omega$ [**8**].

Corollary 4.15. *Let X be a normal topological space, and let λ be an* AS.R. *associated to a proper coarse structure $\mathcal{E}$ on X. Suppose that λ is asymptotically normal. Then $\gamma X/\approx$ is a coarse compactification.*

Proof. Let $(x_\alpha,y_\alpha)_{\alpha\in I}$ be a convergent net in $E\in\mathcal{E}$. Assume that $[\sigma_{x_\alpha}]\to[\mathcal{F}_1]$ and $[\sigma_{y_\alpha}]\to[\mathcal{F}_2]$ for $[\mathcal{F}_1],[\mathcal{F}_2]\in\gamma X/\approx\setminus\phi(X)$. Suppose that $\sigma_{x_{\alpha_i}}$ is a convergent subnet of σ_{x_α} and $\sigma_{y_{\alpha_{i_k}}}$ is a convergent subnet of $\sigma_{y_{\alpha_i}}$. If

$$\sigma_{x_{\alpha_{i_k}}}\longrightarrow\mathcal{H}_1\quad\text{and}\quad\sigma_{y_{\alpha_{i_k}}}\longrightarrow\mathcal{H}_2,$$

we have $\mathcal{H}_1\approx\mathcal{F}_1$ and $\mathcal{H}_2\approx\mathcal{F}_2$. Thus, by Proposition 4.14, we have $\mathcal{H}_1\approx\mathcal{H}_2$, and therefore, $\mathcal{F}_1\approx\mathcal{F}_2$. □

Corollary 4.16. *Let X be a normal topological space, and let $\mathcal{E}$ be a proper coarse structure on X. Assume that the* AS.R. *associated to $\mathcal{E}$ is asymptotically normal. Then the identity map $i:X\to X$ extends uniquely to a continuous map of hX into $\gamma X/\approx$.*

Proof. It is an immediate consequence of Corollary 4.16 and [**8,** 2.39]. □

Proposition 4.17. *Assume the hypotheses of Corollary* 4.16 *hold. Each Higson function $f:X\to\mathbb{C}$ has a unique extension $\overline{f}:\gamma X/\approx\,\to\mathbb{C}$.*

Proof. Let $f:X\to\mathbb{R}$ be a Higson function and $\widehat{f}:\gamma X\to\mathbb{C}$ its extension to γX. Suppose that

$$\mathcal{F}_1,\mathcal{F}_2\in\gamma X\setminus X\quad\text{and}\quad\mathcal{F}_1\approx\mathcal{F}_2.$$

Let $\widehat{f}(\mathcal{F}_1)=x_1$ and $\widehat{f}(\mathcal{F}_2)=x_2$. Assume that $x_1\neq x_2$. Let $\delta=|x_1-x_2|/4$. Then $\widehat{f}^{-1}(B(x_1,\delta))$ and $\widehat{f}^{-1}(B(x_2,\delta))$ are open sets containing $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, so there are open sets $U\subseteq X$ and $V\subseteq X$ such that $\mathcal{F}_1\in U^*\subseteq\widehat{f}^{-1}(B(x_1,\delta))$ and $\mathcal{F}_2\in V^*\subseteq\widehat{f}^{-1}(B(x_2,\delta)$. Thus, there are $A\in\mathcal{F}_1$ and $B\in\mathcal{F}_2$ such that $A\subseteq U$ and $B\subseteq V$. Since $\mathcal{F}_1\approx\mathcal{F}_2$, there are unbounded and asymptotically alike subsets $L_1 \subseteq A$ and $L_2 \subseteq B$. Therefore, there is as $E \in \mathcal{E}$ such that $L_1 \subset E(L_2)$ and $L_2 \subseteq E(L_1)$. Since f is a Higson function, there is a compact $K \subseteq X$ such that $|f(x) - f(y)| < \delta$ for all $(x, y) \in E \setminus K \times K$.

Let $x \in L_1 \setminus K$ and $y \in L_2 \setminus K$ so $\sigma_x \in U^*$ and $\sigma_y \in V^*$. This leads to $\widehat{f}(\sigma_x) = f(x) \in B(x_1, \delta)$ and $\widehat{f}(\sigma_y) = f(y) \in B(x_2, \delta)$ so

$$|x_2 - x_1| \leq |x_2 - f(y)| + |f(x) - f(y)| + |x_1 - f(x)| < 3\delta.$$

Thus, $|x_2 - x_1| < 3|x_1 - x_2|/4$, a contradiction. Therefore, $x_1 = x_2$. Define $\overline{f} : \gamma X/\approx \to \mathbb{C}$ by $\overline{f}([\mathcal{F}]) = \widehat{f}(\mathcal{F})$. The map $\overline{f}$ is well defined and, since $\overline{f} \circ \pi = \widehat{f}$, it is continuous. $\square$

Corollary 4.18. *Assume that the hypotheses of Corollary* 4.16 *hold. Then hX and $\gamma X/\approx$ are homeomorphic.*

Proof. Proposition 4.17 shows that the identity map $i : X \to X$ extends uniquely to a map from $\gamma X/\approx$ to hX. Thus, Corollary 4.16 shows that hX and $\gamma X/\approx$ are homeomorphic. $\square$

Suppose that $(X, \mathcal{T})$ is a topological space and λ is a proper and asymptotically normal AS.R. on it. We call $\gamma X/\approx$ the *asymptotic compactification* of X. We also call $\nu X = \gamma X/\approx \setminus \phi(X)$ the *asymptotic corona* of X. For an AS.R. associated to a proper coarse structure $\mathcal{E}$ on X, Corollary 4.18 shows that νX is homeomorphic with the Higson corona.

Example 4.19. Let (X, d) be a metric space. For two subsets A and B of X, define $A \lambda B$ if A and B are both unbounded or A and B are both bounded. The relation λ is a proper AS.R. on (X, d). Two subsets A and B of X are asymptotically disjoint if and only if A is bounded or B is bounded. For a bounded subset $A \subseteq X$, let $X_1 = X \setminus A$ and $X_2 = A$. Then X_1 is asymptotically disjoint from A and X_2 is asymptotically disjoint from B, for all $B \subseteq X$. Thus, (X, λ) is an asymptotically normal AS.R. space. It is straightforward to show that $\mathcal{F}_1 \approx \mathcal{F}_2$, for all $\mathcal{F}_1, \mathcal{F}_2 \in \gamma X \setminus \sigma(X)$. Therefore, the asymptotic compactification of X is the one point compactification of (X, d).

Example 4.20. Suppose that λ is the AS.R. introduced in Example 2.13 on $\mathbb{R}$. Since the two unbounded subsets of $\mathbb{R}$, with respect to λ, are asymptotically alike, the subsets A and B of $\mathbb{R}$ are asymptotically disjoint if and only if A is bounded or B is bounded with respect to λ.

Let A be a bounded subset of $\mathbb{R}$ with respect to λ. Then $A \subseteq (r, +\infty)$ for some $r \in \mathbb{R}$.

Let $X_1 = (-\infty, r)$ and $X_2 = (r, +\infty)$. The sets X_1 and A are asymptotically disjoint and X_2 is asymptotically disjoint from B, for all $B \subseteq X$. Thus, (X, λ) is an asymptotically normal AS.R. space.

At each point $x \in \mathbb{R}$, other than the origin, assume the usual neighborhood basis at x. At the origin, let $\mathcal{B} = \{(-\epsilon, +\epsilon) \bigcup (n, +\infty) \mid n \in \mathbb{N}, \epsilon > 0\}$ be the neighbourhood basis. Let $\mathcal{T}$ be the corresponding topology on $\mathbb{R}$. It is easy to show that λ is a proper AS.R. space on $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T})$ is a normal topological space. For all $\mathcal{F}_1, \mathcal{F}_2 \in \gamma X \setminus \sigma(X)$, we have $\mathcal{F}_1 \approx \mathcal{F}_2$. Therefore, the asymptotic compactification of $(\mathbb{R}, \lambda)$ is the one point compactification of $(\mathbb{R}, \mathcal{T})$.

Proposition 4.21. *Let X and Y be two topological spaces equipped with two proper and asymptotically normal* AS.R.*'s. For every continuous* AS.R. *mapping $f : X \to Y$, there exists a unique continuous extension $\widetilde{f} : \gamma X/\approx \to \gamma Y/\approx$ which sends νX to νY.*

Proof. For $\mathcal{F} \in \gamma X$, define

$$f_*(\mathcal{F}) = \{A \subseteq Y \mid A \text{ is closed and } f^{-1}(A) \in \mathcal{F}\}.$$

Let $\widehat{f}(\mathcal{F})$ be a unique closed ultrafilter that contains $f_*(\mathcal{F})$ ([**11,** 16K]). The map $\widehat{f} : \gamma X \to \gamma Y$ is a continuous extension of f ([**11,** 19K]). Assume that $\mathcal{F} \in \gamma X \setminus \sigma(X)$ and $\widehat{f}(\mathcal{F}) = \sigma_y$ for some $y \in Y$. Therefore, for all $A \in f_*(\mathcal{F})$, we have $y \in A$.

Let $U \subseteq Y$ be an asymptotic neighborhood of y. We have

$$A = (A \setminus U) \bigcup (\overline{A \bigcap U}).$$

Since $y \notin A \setminus U$ and $f_*(\mathcal{F})$ is a prime closed filter, then $\overline{A \bigcap U} \in f_*(\mathcal{F})$. Since f is an AS.R. mapping thus $f^{-1}(\overline{A \bigcap U})$ is bounded and it contradicts $\mathcal{F} \in \gamma X \setminus \sigma(X)$. Thus, $\widehat{f}$ sends $\gamma X \setminus \sigma(X)$ to $\gamma Y \setminus \sigma(Y)$.

Suppose that $\mathcal{F}_1 \approx \mathcal{F}_2$. Let $C \in \widehat{f}(\mathcal{F}_1)$ and $A \in f_*(\mathcal{F}_1)$. Assume that U is an asymptotic neighborhood of C. Since

$$A = (\overline{A \bigcap U}) \bigcup (A \setminus U) \quad \text{and} \quad (A \setminus U) \bigcap U = \emptyset,$$

$\overline{A \bigcap U} \in f_*(\mathcal{F}_1)$. Similarly, one can show that, for $D \in \widehat{f}(\mathcal{F}_2)$ and $B \in f_*(\mathcal{F}_2)$, we have $\overline{B \bigcap V} \in f_*(\mathcal{F}_2)$ for some asymptotic neighborhood V of D. Then $f^{-1}(\overline{A \bigcap U}) \in \mathcal{F}_1$ and $f^{-1}(\overline{B \bigcap V}) \in \mathcal{F}_2$. Thus, there are unbounded and asymptotically alike subsets $L_1 \subseteq f^{-1}(\overline{A \bigcap U})$ and $L_2 \subseteq f^{-1}(\overline{B \bigcap V})$. Since f is an AS.R. mapping, $f(L_1)$ and $f(L_2)$ are unbounded and asymptotically alike subsets of $\overline{A \bigcap U}$ and $\overline{B \bigcap V}$, respectively. Since λ is compatible with the topology, Proposition 2.6 shows that C and D are not asymptotically disjoint. Thus, $\widehat{f}(\mathcal{F}_1) \approx \widehat{f}(\mathcal{F}_2)$. Therefore, $\widetilde{f} : \gamma X/\approx \to \gamma Y/\approx$ defined by $\widetilde{f}([\mathcal{F}]) = [\widehat{f}(\mathcal{F})]$ is well defined. We have $\widetilde{f} \circ \pi = \pi' \circ \widehat{f}$, where $\pi : \gamma X \to \gamma X/\approx$ and $\pi' : \gamma Y \to \gamma Y/\approx$ are quotient maps. So $\widetilde{f}$ is continuous and, since $\widehat{f}$ sends $\gamma X \setminus \sigma(X)$ to $\gamma Y \setminus \sigma(Y)$, it sends νX to νY. $\square$

In the following propositions, $\overline{A}$ denotes the closure of $A \subseteq X$ in $\gamma X/\approx$.

Proposition 4.22. *Let X be a normal topological space, and let $\mathcal{E}$ be a proper coarse structure on X. Assume that the* AS.R. *associated to $\mathcal{E}$ is asymptotically normal. If A and B are two asymptotically alike subsets of X then $\overline{A} \bigcap \nu X = \overline{B} \bigcap \nu X$.*

Proof. Let $[\mathcal{F}] \in \overline{A} \bigcap \nu X$. Let us denote by D' the closure of $D \subseteq X$ in γX. Since $\pi : \gamma X \to \gamma X/\approx$ is a closed map so $(\overline{A} \bigcap \nu X) \subseteq \pi(A') \bigcap \nu X$. Thus, there is an ultrafilter $\mathcal{G} \in A'$ such that $\mathcal{F} \approx \mathcal{G}$.

There is a net $(x_\alpha)_{\alpha \in I}$ in A such that $\sigma_{x_\alpha} \to \mathcal{G}$. Since A and B are asymptotically alike, $A \subseteq E(B)$ and $B \subseteq E(A)$ for some $E \in \mathcal{E}$. For each $\alpha \in I$, we choose $y_\alpha \in B$ such that $(x_\alpha, y_\alpha) \in E$. The net $(\sigma_{y_\alpha})_{\alpha \in I}$ has a convergent subnet $(\sigma_{y_{\alpha_i}})_{i \in J}$. Then $\sigma_{y_{\alpha_i}} \to \mathcal{H}$ for some $\mathcal{H} \in B'$. Two nets $(\sigma_{y_{\alpha_i}})_{i \in J}$ and $(\sigma_{x_{\alpha_i}})_{i \in J}$ satisfy the assumptions of Proposition 4.14. Thus, $\mathcal{G} \approx \mathcal{H}$, and this leads to $[\mathcal{F}] \in \overline{B} \bigcap \nu X$. $\square$

Corollary 4.23. *Assume the hypotheses of Proposition* 4.22 *hold. Two subsets A and B of X are asymptotically disjoint if and only if*

$$\left(\overline{A}\bigcap \nu X\right)\bigcap\left(\overline{B}\bigcap \nu X\right)=\emptyset.$$

Proof. Suppose that $A, B \subseteq X$ and

$$\left(\overline{A}\bigcap \nu X\right)\bigcap\left(\overline{B}\bigcap \nu X\right)=\emptyset.$$

Assume that, on the contrary, there are unbounded and asymptotically alike subsets $L_1 \subseteq A$ and $L_2 \subseteq B$. By Proposition 4.22,

$$\left(\overline{L_1}\bigcap \nu X\right)=\left(\overline{L_2}\bigcap \nu X\right)\neq\emptyset.$$

Since $\overline{L_1} \subseteq \overline{A}$ and $\overline{L_2} \subseteq \overline{B}$ since

$$\left(\overline{A}\bigcap \nu X\right)\bigcap\left(\overline{B}\bigcap \nu X\right)\neq\emptyset,$$

a contradiction.

To prove the converse, assume that A and B are asymptotically disjoint. Let $[\mathcal{F}] \in \overline{A}\bigcap \nu X$. As in the previous proposition, let us denote by D' the closure of $D \subseteq X$ in γX. So there is $\mathcal{G} \in A'$ such that $\mathcal{G} \approx \mathcal{F}$. Let $\mathcal{H} \in B'$. The closures of A and B in topological space X are in $\mathcal{G}$ and $\mathcal{H}$, respectively. Since λ is an AS.R. compatible with the topology, $\mathcal{G}$ and $\mathcal{H}$ contain asymptotically disjoint sets. Therefore, $[\mathcal{F}] \notin \overline{B}\bigcap \nu X$. $\square$

Now we will prove the converse of Proposition 4.22 for metric spaces.

Corollary 4.24. *Assume that (X, d) is a proper metric space. For two subsets A and B of X, if $\overline{A}\bigcap \nu X = \overline{B}\bigcap \nu X$, then A and B are asymptotically alike.*

Proof. Suppose that A and B are not asymptotically alike. We can assume that, without loss of generality, for each $n \in \mathbb{N}$, A is not a subset of $\mathbf{B}(B, n)$. For each $n \in \mathbb{N}$, choose $a_n \in A$ such that $d(a_n, B) \geq n$. Let $L = \{a_n \mid n \in \mathbb{N}\}$. Clearly, L and B are asymptotically disjoint. Thus, by Corollary 4.23, $(\overline{L}\bigcap \nu X)\bigcap(\overline{B}\bigcap \nu X) = \emptyset$. This is a contradiction, since $\overline{L}\bigcap \nu X \subseteq \overline{A}\bigcap \nu X$. $\square$

5. Asymptotic compactification and proximity. Let $(X,\mathcal{T})$ be a topological space, and let λ be an AS.R. compatible with $\mathcal{T}$. Suppose that the relation $\sim$ is as in Definition 4.9. For two subsets A and B of X, define $A\delta_\lambda B$ if there are $L_1 \subseteq \overline{A}$ and $L_2 \subseteq \overline{B}$ such that $L_1 \sim L_2$.

Proposition 5.1. *Let $(X,\mathcal{T})$ be a normal topological space, and let λ be a proper and asymptotically normal* AS.R. *on X. Then δ_λ is a separated proximity on X, and it is compatible with $\mathcal{T}$.*

Proof. The relation δ_λ clearly satisfies Definition 1.1 (i), (ii) and (iii). Assume that $A\delta_\lambda(B\bigcup C)$. Then there are $L_1 \subseteq \overline{A}$ and $L_2 \subseteq \overline{B\bigcup C}$ such that $L_1 \sim L_2$. If $L_1 = L_2$, then

$$\overline{A}\bigcap\left(\overline{B\bigcup C}\right) \neq \emptyset,$$

and this leads to

$$A\delta_\lambda B \quad \text{or} \quad A\delta_\lambda C,$$

clearly. If L_1 and L_2 are two unbounded asymptotically alike subsets of X, then $L_2\bigcap\overline{B}$ or $L_2\bigcap\overline{C}$ should be unbounded. Assume that $L_2\bigcap\overline{B}$ is unbounded. Then there is an unbounded subset $L_3 \subseteq L_1$ such that $(L_2\bigcap\overline{B})\lambda L_3$ by Proposition 2.6. Thus, $A\delta_\lambda B$.

If $A\delta_\lambda B$, it is straightforward to show that $A\delta_\lambda(B\bigcup C)$ for all $C \subseteq X$. Now assume that $A, B \subseteq X$ and $A\overline{\delta_\lambda}B$. Then $\overline{A}$ and $\overline{B}$ are two disjoint and asymptotically disjoint subsets of X. We choose $X_1 \subseteq X$and $X_2 \subseteq X$ such that $X = X_1\bigcup X_2$, and they are asymptotically disjoint from $\overline{A}$ and $\overline{B}$, respectively.

Since $(X,\mathcal{T})$ is a normal topological space and λ is compatible with $\mathcal{T}$, we can find asymptotic neighborhoods

$$\overline{X_1}\bigcap\overline{A} \subseteq U \quad \text{and} \quad \overline{X_2}\bigcap\overline{B} \subseteq V$$

such that

$$\overline{U}\bigcap\overline{B} = \emptyset \quad \text{and} \quad \overline{V}\bigcap\overline{A} = \emptyset.$$

Let $E = (X_1 \setminus U)\bigcup V$. Since X_1 and $\overline{A}$ are asymptotically disjoint, $\overline{X_1}\bigcap\overline{A}$ is bounded, and this shows that U is bounded. Similarly V is bounded. Thus, $\overline{A}$ and $\overline{E}$ are disjoint, and they are asymptotically disjoint as well since V is bounded. Therefore, $A\overline{\delta_\lambda}E$.

Similarly, one can show that

$$\overline{B} \quad \text{and} \quad \overline{X \setminus E} \subseteq \overline{(X_2 \setminus V) \bigcup U}$$

are disjoint and asymptotically disjoint. This leads to $(X \setminus E)\overline{\delta_\lambda}B$. Since λ is proper, one can easily verify that δ_λ is compatible with the topology. $\square$

Let us recall that, on a separated proximity space (X, δ), $\mathfrak{X}$ denotes the family of all clusters in X. For two subsets $\mathfrak{M}$ and $\mathfrak{N}$ of X, $\mathfrak{M}\delta^*\mathfrak{N}$ means that, if $A \subseteq X$ absorbs $\mathfrak{M}$ and $B \subseteq X$ absorbs $\mathfrak{N}$, then $A\delta B$. A subset A of X absorbs $\mathfrak{M} \subseteq \mathcal{X}$ means that $A \in \mathcal{C}$ for all $\mathcal{C} \in \mathfrak{M}$. The proximity space $(\mathfrak{X}, \delta^*)$ is called the *Smirnov compactification* of X.

Proposition 5.2. *Let $(X, \mathcal{T})$ be a normal topological space, and let λ be a proper and asymptotically normal* AS.R. *on X. Then $\gamma X/\approx$ and the Smirnov compactification $(\mathfrak{X}, \delta_\lambda^*)$ are homeomorphic.*

Proof. Let $\mathcal{F} \in \gamma X$, and let

$$\widetilde{\mathcal{F}} = \{A \subseteq X \mid A\delta_\lambda B \text{ for all } B \in \mathcal{F}\}.$$

The family $\widetilde{\mathcal{F}}$ is a cluster in X [**6,** Theorem 5.8]. Define $\psi : \gamma X/\approx \to \mathfrak{X}$ by $\psi([\mathcal{F}]) = \widetilde{\mathcal{F}}$ for all $\mathcal{F} \in \gamma X$. For $\mathcal{F}, \mathcal{G} \in \gamma X$, if $\mathcal{F} \approx \mathcal{G}$, then $A\delta_\lambda B$ for all $A \in \mathcal{F}$ and all $B \in \mathcal{G}$. Therefore, $\widetilde{\mathcal{F}} = \widetilde{\mathcal{G}}$. Thus, the map ψ is well defined.

It is straightforward to show that ψ is one-to-one and, by using [**6,** 5.8], one can easily show that it is also surjective. Suppose that $\mathfrak{M} \subseteq \gamma X$ and $\mathcal{F} \in \overline{\mathfrak{M}}$. Let A be a subset of X such that $A \in \psi(\mathcal{G})$ for all $\mathcal{G} \in \mathfrak{M}$. We claim that $A \in \psi(\mathcal{F})$.

Suppose that, contrary to our claim, $A \notin \psi(\mathcal{F})$. So there exists a $B \in \mathcal{F}$ such that $\overline{A}$ and $\overline{B}$ are disjoint and asymptotically disjoint. We choose an asymptotic neighborhood $\overline{B} \subseteq U$ such that $\overline{A} \bigcap \overline{U} = \emptyset$. The set U^* is an open subset of γX containing $\mathcal{F}$. Thus, there are $\mathcal{G} \in \mathfrak{M}$ and $C \in \mathcal{G}$ such that $C \subseteq U$. This shows that $\overline{C}$ and $\overline{A}$ are disjoint and asymptotically disjoint. Therefore, $A\overline{\delta_\lambda}C$, which contradicts $A \in \psi(\mathcal{G})$. Thus, $\psi(\mathcal{F}) \in \overline{\psi(\mathfrak{M})}$. This shows that $\psi \circ \pi$ is continuous, where $\pi : \gamma X \to \gamma X/\approx$ is the quotient map. So ψ is continuous and, since $\gamma X/\approx$ is compact and Hausdorff, it is a homeomorphism too. $\square$

6. Asymptotic dimension. Let $\mathcal{U}$ be a family of subsets of a set X, and let

$$S_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} U \times U.$$

For two subsets A and B of X, define $A \sim_{\mathcal{U}} B$ if $A \subseteq S_{\mathcal{U}}(B)$ and $B \subseteq S_{\mathcal{U}}(A)$.

Definition 6.1. We call a family $\mathcal{U}$ of subsets of an AS.R. space (X, λ) *uniformly bounded*, if

(i) each $U \in \mathcal{U}$ is bounded.
(ii) $A \sim_{\mathcal{U}} B$ implies $A\lambda B$, for all $A, B \subseteq X$.

The next proposition shows that, if λ is the AS.R. associated to a metric d on a set X, then the above definition coincides with uniformly boundedness with respect to d.

Proposition 6.2. *Let* (X, d) *be a metric space, and let* λ *be the* AS.R. *associated to* d. *A family* $\mathcal{U}$ *of subsets of* X *is uniformly bounded if and only if there is* $k > 0$ *such that* $\operatorname{diam}(U) < k$ *for all* $U \in \mathcal{U}$.

Proof. The *if* part is easy to verify.

To prove the converse, assume that, on the contrary, for each $n \in \mathbb{N}$, there are $U_n \in \mathcal{U}$ and $x_n, y_n \in U_n$ such that $d(x_n, y_n) > n$. For each subset $I \subseteq \mathbb{N}$, we have

$$A_I = \{x_i \mid i \in I\} \sim_{\mathcal{U}} B_I = \{y_i \mid i \in I\},$$

so $A_I \lambda B_I$. Thus, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ satisfy the hypotheses of Lemma 2.2, a contradiction. $\square$

Let us recall that, for a family $\mathcal{M}$ of subsets of a set X, $\mu(\mathcal{M})$ denotes the multiplicity of $\mathcal{M}$, i.e., the greatest number of elements of $\mathcal{M}$ that meets a point of X.

Definition 6.3. Let (X, λ) be an AS.R. space. We say that $\operatorname{asdim}_\lambda X \leq n$ if, for all uniformly bounded covers $\mathcal{U}$ of X, there is a uniformly bounded cover $\mathcal{V}$ for X such that $\mathcal{U}$ refines $\mathcal{V}$ and $\mu(\mathcal{V}) \leq n+1$. We say that $\operatorname{asdim}_\lambda X = n$ if $\operatorname{asdim}_\lambda X \leq n$ and $\operatorname{asdim}_\lambda X \leq n-1$ is not true. We call $\operatorname{asdim}_\lambda X$ the *asymptotic dimension* of an AS.R. space (X, λ).

Proposition 6.2 shows that, on a metric space (X, d), we have $\operatorname{asdim} X = \operatorname{asdim}_\lambda X$, where λ is the AS.R. associated to d.

Proposition 6.4. *Let* (X, λ) *be an* AS.R. *space, and let* $Y \subseteq X$*. Then* $\operatorname{asdim}_{\lambda_Y} Y \leq \operatorname{asdim}_\lambda X$*.*

Proof. Suppose that $\operatorname{asdim}_\lambda X = n$. Let $\mathcal{V}$ be a uniformly bounded cover of Y. Assume that $\mathcal{U} = \mathcal{V} \bigcup_{x \in X \setminus Y} \{\{x\}\}$. If $A, B \subseteq X$ and $A \sim_{\mathcal{U}} B$, then

$$\Big(A \bigcap (X \setminus Y)\Big) = \Big(B \bigcap (X \setminus Y)\Big)$$

and

$$\Big(A \bigcap Y\Big) \sim_{\mathcal{V}} \Big(B \bigcap Y\Big).$$

So $(A \bigcap Y) \lambda (B \bigcap Y)$ and Proposition 2.4 (i) show that $A \lambda B$. Thus, $\mathcal{U}$ is a uniformly bounded cover of X.

Let $\mathcal{W}$ be a uniformly bounded cover of X such that $\mathcal{U}$ refines it and $\mu(\mathcal{W}) \leq n+1$. The family

$$\mathcal{W}_Y = \Big\{W \bigcap Y \mid W \in \mathcal{W}\Big\}$$

is a uniformly bounded cover of Y and $\mathcal{V}$ refines it. Clearly, $\mu(\mathcal{W}_Y) \leq n+1$ so $\operatorname{asdim}_{\lambda_Y} \leq n$. $\square$

Proposition 6.5. *Asymptotic equivalent* AS.R. *spaces have the same asymptotic dimension.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be two AS.R. mappings between the AS.R. spaces (X, λ) and (Y, λ'), such that

$$g \circ f(A) \lambda A \quad \text{and} \quad f \circ g(B) \lambda' B$$

for all subsets $A \subseteq X$ and $B \subseteq Y$. Suppose that $\operatorname{asdim}_\lambda X = n$.

Let $\mathcal{U}$ be a uniformly bounded cover of Y, and let

$$g^*(\mathcal{U}) = \{g(U) \mid U \in \mathcal{U}\}.$$

For all $U \in \mathcal{U}$, we have $f \circ g(U) \lambda U$, so $g(U) \subseteq f^{-1}(f \circ g(U))$ is bounded.

Assume that

$$A, B \subseteq g(Y) \quad \text{and} \quad A \sim_{g^*(\mathcal{U})} B.$$

Let

$$C = g^{-1}(A) \bigcap S_{\mathcal{U}}(g^{-1}(B)) \quad \text{and} \quad D = g^{-1}(B) \bigcap S_{\mathcal{U}}(g^{-1}(A)).$$

Since $A \sim_{g^*(\mathcal{U})} B$, it is straightforward to show that $g(C) = A$ and $g(D) = B$.

We have $C \sim_{\mathcal{U}} D$ so $C\lambda' D$ since g is an AS.R. mapping $A\lambda B$. Thus, $g^*(\mathcal{U})$ is a uniformly bounded cover of $g(Y)$. By Proposition 6.4, $\operatorname{asdim}_{\lambda_{g(Y)}} g(Y) \leq n$. Thus, there is a uniformly bounded cover $\mathcal{V}$ of $g(Y)$ such that $g^*(\mathcal{U})$ refines it and $\mu(\mathcal{V}) \leq n+1$.

Let

$$g_*(\mathcal{V}) = \{g^{-1}(V) \mid V \in \mathcal{V}\}.$$

Since g is an AS.R. mapping, all members of $g_*(\mathcal{V})$ are bounded. Suppose that $M, N \subseteq Y$ and $M \sim_{g_*(\mathcal{V})} N$. It is easy to verify that $g(M) \sim_{\mathcal{V}} g(N)$ so $g(M)\lambda g(N)$. Since $f \circ g(M)\lambda' M$ and $f \circ g(N)\lambda' N$ so $M\lambda' N$. Thus, $g_*(\mathcal{V})$ is a uniformly bounded cover of Y.

It is straightforward to show that $\mathcal{U}$ refines $g_*(\mathcal{V})$ and $\mu(g_*(\mathcal{V})) \leq n+1$. Therefore, $\operatorname{asdim}_{\lambda'} Y \leq \operatorname{asdim}_{\lambda} X$. Similarly, one can show that $\operatorname{asdim}_{\lambda} X \leq \operatorname{asdim}_{\lambda'} Y$. $\square$

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