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GEOMETRY OF BOUNDED FRÉCHET MANIFOLDS

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ABSTRACT. In this paper, we develop the geometry of bounded Fréchet manifolds. We prove that a bounded Fréchet tangent bundle admits a vector bundle structure. But, the second order tangent bundle T^2M of a bounded Fréchet manifold M becomes a vector bundle over M if and only if M is endowed with a linear connection. As an application, we prove the existence and uniqueness of an integral curve of a vector field on M.

1. Introduction. The geometry of Fréchet manifolds has received serious attention in recent years, cf., [**3**] for a survey. In particular, second order tangent bundles have been studied due to their applications in the study of second order ordinary differential equations that arise via geometric objects (such as autoparallel curves and parallel translation) on manifolds (see [**1, 2**]). However, due to intrinsic difficulties with Fréchet spaces, only a certain type of manifolds was considered, namely, those Fréchet manifolds which can be obtained as a projective limit of Banach manifolds (PLB-manifolds). It was proved that the second order tangent bundle T^2M of a PLB-manifold M admits a vector bundle structure if and only if M is endowed with a linear connection (see [**4**]).

Some of the basic issues in the theory of Fréchet spaces are mainly related to the space of continuous linear mappings. Indeed, the space of continuous linear mappings of one Fréchet space to another is not a Fréchet space in general. On the other hand, the general linear group of a Fréchet space does not admit any non-trivial topological group structure. This defect brings into question the method of defining a vector bundle. Another drawback is the lack of a general solvability

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theory for ordinary differential equations. Because of these reasons, an arbitrary connection is hard to handle in the framework of Fréchet bundles.

As mentioned above, there is a solution to these difficulties for Fréchet manifolds which can be obtained as projective limits of Banach manifolds. However, there is another manner of overcoming the aforementioned problems. Recently, in [**17**], Müller introduced the concept of bounded Fréchet manifolds and provided an inverse function theorem in the sense of Nash and Moser in this category. Such spaces arise in geometry and physical field theory and have many desirable properties. For instance, the space of all smooth sections of a fiber bundle (over closed or non-compact manifolds), which is the foremost example of infinite-dimensional manifolds, has the structure of a bounded Fréchet manifold, (see [**17,** Theorem 3.34]). As for the importance of bounded Fréchet manifolds, we refer to [**6**], where Sard's theorem was obtained in this category. The statement of the theorem is as follows. *Let M, respectively N, be bounded Fréchet manifolds with compatible metrics d_M, respectively d_N, modeled on Fréchet spaces E, respectively F, with standard metrics. Let $f : M \to N$ be an MC^k- Lipschitz Fredholm map with $k > \max\{\operatorname{Ind} f, 0\}$. Then, the set of regular values of f is residual in N.*

One of the essential ideas of this setting is to replace the space of all continuous linear maps by the space $\mathcal{L}_{d',d}(E, F)$, for all linear Lipschitz continuous maps. Then, $\mathcal{L}_{d',d}(E, F)$ is a topological group that has satisfactory properties. For example, the composition map,

$$\mathcal{L}_{d,g}(F, G) \times \mathcal{L}_{d',d}(E, F) \longrightarrow \mathcal{L}_{d',g}(E, G),$$

is bilinear continuous. In particular, the evaluation map $\mathcal{L}_{g,d}(E, F) \times E \to F$ is continuous.

Our goal in this paper is to extend the known results of Fréchet geometry to bounded Fréchet manifolds. We define the tangent bundles TM and T^2M of a bounded Fréchet manifold M, modeled on a Fréchet space F, and prove that they too are endowed with bounded Fréchet manifold structures of the same type modeled on F^2 and F^4, respectively. In addition, we show that TM admits a vector bundle structure, which allows us to define a connection on TM via a connection map, cf., [**18, 19**]. We shall interpret linear connections as linear systems of ordinary differential equations on trivial bundles.

Our main result is that T^2M admits a vector bundle structure if and only if M is endowed with a linear connection. Moreover, a linear connection on M determines a vector bundle structure on T^2M and a vector bundle isomorphism $T^2M \to TM \oplus TM$. We conclude by proving the existence and uniqueness of the integral curve of a vector field on M.

It turns out that bounded Fréchet manifolds have some advantages over both PLB-manifolds and infinite-dimensional convenient manifolds. In the case of PLB-manifolds, the difficulty is that to construct a geometric object on manifolds, we need to establish the existence of the projective limit of its Banach corresponding factors. In the case of convenient manifolds, to construct such geometrical structures, we need to define the notion of manifolds by charts. However, this drastically restricts the consequences of Cartesian closedness (see [**13, 16**]). In addition, for convenient manifolds, we have two different kinds of tangent bundles (kinematic and operational) and hence, we have two different types of vector fields. Another drawback is that operational vector fields do not necessarily have integral curves. On the other hand, for a given kinematic vector field, integral curves may not exist locally, and, if they exist, they may not be unique for the same initial condition (see [**13**]).

2. Prerequisites. In this section, we summarize all the necessary preliminary material that we need for a self contained presentation of the paper. For detailed studies on bounded Fréchet manifolds we refer to [**6, 10, 17**].

We denote by (F, d) a Fréchet space whose topology is defined by a complete translational-invariant metric d. We define $\|f\|_d = d(f, 0)$ for $f \in F$, and write $L \cdot f$ instead of $L(f)$ when L is a linear map between Fréchet spaces. A metric with absolutely convex balls will be called a *standard metric*. Note that every Fréchet space admits a standard metric which defines its topology. If α_n is an arbitrary sequence of positive real numbers converging to 0, and if ρ_n is any sequence of continuous semi-norms defining the topology of F, then,

$$d_{\alpha,\rho}(e, f) := \sup_{n\in\mathbb{N}} \alpha_n \frac{\rho_n(e - f)}{1 + \rho_n(e - f)}$$

is a metric on F with the desired properties.

As mentioned in the introduction, we replace the space of all linear continuous maps between Fréchet spaces by the space of all linear Lipschitz continuous maps. Let (E, g) be another Fréchet space, and let $\mathcal{L}_{g,d}(E, F)$ be the set of all globally linear Lipschitz continuous maps, i.e., linear maps $L : E \to F$, such that

$$\|L\|_{g,d} := \sup_{x \in E \setminus \{0\}} \frac{\|L \cdot x\|_d}{\|x\|_g} < \infty.$$

We abbreviate $\mathcal{L}_g(E) := \mathcal{L}_{g,g}(E, E)$ and write $\|L\|_g = \|L\|_{g,g}$ for $L \in \mathcal{L}_g(E)$. If d is a standard metric, then,

$$D_{g,d} : \mathcal{L}_{g,d}(E, F) \times \mathcal{L}_{g,d}(E, F) \longrightarrow [0, \infty),\ (L, H) \longmapsto \|L - H\|_{g,d} \tag{2.1}$$

is a translational-invariant metric on $\mathcal{L}_{d,g}(E, F)$, turning it into an Abelian topological group (see [**10,** Remark 1.9]). The latter is not a topological vector space, in general, but a locally convex vector group with absolutely convex balls. We shall always equip Fréchet spaces with standard metrics and define the topology on $\mathcal{L}_{d,g}(E, F)$ by the metric $D_{g,d}$. The vector groups

$$\mathcal{L}^{(i+1)}_{g,d}(F, E) := (F, \mathcal{L}^{i}_{g,d}(F, E))$$

are defined by induction.

Let E, F be Fréchet spaces, let U be an open subset of E, and let $P : U \to F$ be a continuous map. Let $CL(E, F)$ be the space of all continuous linear maps from E to F, topologized by the compact-open topology. We say that P is *differentiable* at the point $p \in U$, if there exists a linear map:

$$\mathrm{d}\, P(p) : E \longrightarrow F, \text{ with } \mathrm{d}\, P(p)h = \lim_{t \to 0} \tfrac{P(p+th) - P(p)}{t}, \quad \text{for all } h \in E.$$

If P is differentiable at all points $p \in U$, if $\mathrm{d}\, P(p) : U \to CL(E, F)$ is continuous for all $p \in U$, and if the induced map $P' : U \times E \to F$, $(u, h) \mapsto \mathrm{d}\, P(u)h$ is continuous in the product topology, then we say that P is *Keller-differentiable*. We define $P^{(k+1)} : U \times E^{k+1} \to F$ inductively by:

$$P^{(k+1)}(u, f_1, \ldots, f_{k+1}) = \lim_{t \to 0} \tfrac{P^{(k)}(u + t f_{k+1})(f_1, \ldots, f_k) - P^{(k)}(u)(f_1, \ldots, f_k)}{t}.$$

If P is Keller-differentiable, $\mathrm{d}\,P(p) \in \mathcal{L}_{d,g}(E,F)$, for all $p \in U$, and the induced map $\mathrm{d}\,P(p) : U \to \mathcal{L}_{d,g}(E,F)$ is continuous, then P is called *b-differentiable.* We say P is MC^0 and write $P^0 = P$ if it is continuous. We say P is an MC^1 and write $P^{(1)} = P'$ if it is b-differentiable. Let $\mathcal{L}_{d,g}(E,F)_0$ be the connected component of $\mathcal{L}_{d,g}(E,F)$ containing the 0 map. If P is b-differentiable and if $V \subseteq U$ is a connected, open neighborhood of $x_0 \in U$, then $P'(V)$ is connected, and hence contained, in the connected component

$$P'(x_0) + \mathcal{L}_{d,g}(E,F)_0,$$

of $P'(x_0)$, in $\mathcal{L}_{d,g}(E,F)$. Thus,

$$P' \mid_V - P'(x_0) : V \longrightarrow \mathcal{L}_{d,g}(E,F)_0$$

is again a map between subsets of Fréchet spaces. This enables a recursive definition. If P is MC^1 and V can be chosen for each $x_0 \in U$, such that $P' \mid_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$ is MC^{k-1}, then P is called an MC^k-*map.* We give a piecewise definition of $P^{(k)}$ by

$$P^{(k)} \mid_V := \left(P' \mid_V - P'(x_0)\right)^{(k-1)},$$

for x_0 and V, as before. The map P is MC^∞ if it is MC^k for all $k \in \mathbb{N}_0$. We shall denote by D, D^2 the first and the second differential, respectively.

A bounded Fréchet manifold is a Hausdorff second countable topological space, with an atlas of coordinate charts, taking their values in Fréchet spaces such that the coordinate transition functions are all MC^∞-maps.

We will need to consider the space of all globally Lipschitz continuous k-multilinear maps. Let

$$B = \prod_{i=1}^{k} F_i$$

be the topological product of any finite number k of Fréchet spaces $(F_1, d_1), \ldots, (F_k, d_k)$. For $x = (x_1, \ldots, x_k) \in B$ and $y = (y_1, \ldots, y_k) \in B$, we define the maximum metric $d_{\max}$ as follows:

$$d_{\max}(x,y) = \max_{1 \le i \le k} d_i(x_i, y_i).$$

We shall always use this metric on B. Let $(F_1, d_1), \dots, (F_k, d_k)$ and (F, d) be Fréchet spaces. The space of all globally Lipschitz continuous k-multilinear maps is the space of all k-multilinear maps $L : F_1 \times \cdots \times F_k \to F$ such that, for all $f_i \in F_i \setminus \{0\}$, $1 \leq i \leq k$,

$$\|L\|_{d_1,\dots,d_k,d} \quad = \sup_{f_i \in F_i \setminus \{0\}} \frac{\|L(f_1, \dots, f_k)\|_d}{\|f_1\|_{d_1}, \dots, \|f_k\|_{d_k}} < \infty.$$

This space is denoted by $\mathcal{L}_{d_1,\dots d_k,d}(F_1, \dots, F_k;\ F)$. On the latter space, we define a metric

$$D_{d_1,\dots,d_k,d}(L, H) = \ \|L - H\|_{d_1,\dots,d_k,d},$$

which produces an Abelian topological group.

Throughout the paper, we suppose that $d_1, \dots, d_k, d$ are fixed metrics, and for ease of notation, we will not write them when they appear as indices.

Lemma 2.1. *There are canonical topological group isomorphisms:*

$$\begin{aligned}\mathcal{L}(F_1, \mathcal{L}(F_2, \dots, F_k; F)) &\cong \mathcal{L}(F_1, \dots, F_k; F)\\ &\cong \mathcal{L}(F_1, \dots, F_{k-1}; \mathcal{L}(F_k, F))\\ &\cong \mathcal{L}(F_{i_1}, \dots, F_{i_k}; F),\end{aligned}$$

where $(i_1, \dots, i_k)$ *is a permutation of* $(1, \dots, k)$.

Proof. Define

$$\begin{aligned}K : \mathcal{L}(F_1, \mathcal{L}(F_2, \dots, F_k; F)) &\longrightarrow \mathcal{L}(F_1, \dots, F_k; F)\\ K(L(f_1)(f_2, \dots, f_k)) &= \widehat{L}(f_1, \dots, f_k).\end{aligned}$$

The association $L \overset{K}{\mapsto} \widehat{L}$ is linear and a group isomorphism. Since K is linear, we only need to show $\|L\| = \|\widehat{L}\|$, to prove continuity of K and its inverse. By straightforward verification, we have

$$\begin{aligned}\|L\| &= \sup\left\{\|L(f_1)(f_2, \dots, f_k)\| \ \mid \ \|f_1\| = 1, \dots, \|f_k\| = 1\right\}\\ &= \sup\left\{\|\widehat{L}(f_1, \dots, f_k)\| \ \mid \ \|f_1\| = 1, \dots, \|f_k\| = 1\right\}\\ &= \ \|K(L) = \widehat{L}\|.\end{aligned}$$

Likewise, the other isomorphisms are proved. ∎

Convention. The terms *bounded Fréchet tangent bundle* and *bounded Fréchet second order tangent bundle* are too long, so we remove *bounded Fréchet* from the terms.

3. Constructions of TM and T^2M. In this section, we construct TM and T^2M based on the work of Yano and Ishihara [**20**].

3.1. Tangent bundle. Let M be a bounded Fréchet manifold modeled on a Fréchet space F, and let $\mathcal{MC}_p(M)$ be the set of all MC^∞-mappings $f : \mathbb{R} \to M$ that send 0 to $p \in M$. On $\mathcal{MC}_p(M)$, we define an equivalence relation $\sim$ as follows. Let $\Phi = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a compatible atlas for M, $(p \in U_\alpha, \varphi_\alpha)$ an admissible chart, and $f, g \in \mathcal{MC}_p(M)$. Let r be a fixed natural number. We say that f and g are equivalent and write $f \sim g$, if they satisfy:

$$(\varphi_\alpha \circ f)'(0) = (\varphi_\alpha \circ g)'(0), \ldots, (\varphi_\alpha \circ f)^r(0) = (\varphi_\alpha \circ g)^r(0), \tag{3.1}$$

where the orders of the derivatives run between 1 and r. It follows from the chain rule for MC^k-maps (see [**10,** Lemma B.1]) that the equivalency at a point p is well defined. The equivalence class containing a mapping $f \in \mathcal{MC}_p(M)$ is called the r-jet of f at p, and is denoted by $j_p^r f$.

Let TM be the set of all 1-jets of M, and let $\pi_M : TM \to M$ be a natural projection. The fiber $\pi_M^{-1}(p)$ is the tangent space T_pM. The space T_pM has the structure of a Fréchet space, which is isomorphic to F by means of the mapping $\varphi_\alpha \circ \pi_M : T_pF \to F$, given by $j_p^1 f \mapsto \varphi_\alpha(p)$. It is easily verified that this structure of T_pM is independent of the choice of the chart $(U_\alpha, \varphi_\alpha)$. Then, TM is the disjoint union of the tangent spaces T_pM and is called the tangent bundle over M. Let $h : M \to N$ be an MC^k-map of manifolds. The tangent map $Th : TM \to TN$ is defined by

$$Th(j_p^1(f)) = j_{h(p)}^1(h \circ f).$$

The following lemma is fundamental for constructing trivializing atlases and vector bundle structures for TM and T^2M.

Lemma 3.1.

(i) *Let $h : M \to N$ and $g : N \to K$ be MC^k-maps of manifolds. Then, $T(h \circ g) = Tg \circ Th$.*

(ii) *If* $h : M \to N$ *is an* MC^k*-diffeomorphism, then* $Th : TM \to TN$ *is a bijection and* $(Th)^{-1} = T(h^{-1})$.
(iii) *Let* $h : U \subset E \to V \subset F$ *be a diffeomorphism of open sets of Fréchet spaces. The tangent map* $Th : U \times F \to V \times E$ *is a local vector bundle isomorphism.*
(iv) *If* $h : U \subset E \to V \subset F$ *is an* MC^k*-diffeomorphism of open sets of Fréchet spaces, then* Th *is an* MC^{k-1}*-diffeomorphism.*

Proof.

(i) $g \circ h$ is MC^k ([**10,** Lemma B.1]). Furthermore,

$$\begin{aligned} T(g \circ h)(j^1_p f) &= j^1_{(g \circ h)(p)}(g \circ h \circ f) \\ &= Tg(j^1_{h(p)})(h \circ f) \\ &= (Tg \circ Th)(j^1_p f). \end{aligned}$$

(ii) By (i) and the definition of the tangent map, $Th \circ Th^{-1} = T\mathrm{id}_{\mathrm{TN}}$, while $Th^{-1} \circ Th = T\mathrm{id}_{\mathrm{TM}}$.
(iii) Th is a local vector bundle morphism. Since h is a diffeomorphism, it follows that $(Th)^{-1} = T(h^{-1})$ is a local vector bundle morphism; thus, Th is a vector bundle isomorphism.
(iv) Let C be a curve passing through $u \in U$ such that $\mathrm{D}\, C(0) \cdot 1 = e$ for a given $e \in F$. Define the map $\eta(t) : \mathbb{R} \to E$, by $\eta(t) = u + et$, which is tangent to C at $t = 0$. Define $\lambda : U \times F \to TU$ by $\lambda(u, e) = j^1_u(\eta(t))$. We have

$$(Th \circ \lambda)(u, e) = Th \cdot j^1_u(\eta(t)) = j^1_{h(u)}(h \circ \eta(t)).$$

We also have

$$(\lambda \circ h')(u, e) = \lambda(h(u), \mathrm{D}\, h(u) \cdot e) = j^1_{h(u)}(h(u) + (\mathrm{D}\, h(u) \cdot e)t).$$

These are equal because the curves $t \mapsto h(u + et)$ and $t \mapsto h(u) + (\mathrm{D}\, h(u) \cdot e)t$ are tangent at 0 by the definition of the derivative and the previous parts. Therefore, $Th \circ \lambda = \lambda \circ h'$, which means λ identifies $U \times E$ with TU. Correspondingly, we can identify h' with Th, so the results of earlier parts imply statement (iv). □

Proposition 3.2. *Let $\pi_M : TM \to M$ be a tangent bundle. Then, the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$, gives rise to a trivializing atlas*

$$\{(\pi_M^{-1}(U_\alpha), T\varphi_\alpha)\}_{\alpha\in\mathcal{A}}, \quad \text{on } TM,$$

with

$$\begin{aligned} T\varphi_\alpha : \pi_M^{-1}(U_\alpha) &\longrightarrow \varphi_\alpha(U_\alpha) \times F, \\ j_p^1(f) &\longmapsto (\varphi_\alpha(p), (\varphi_\alpha \circ f)'(0)); f \in \mathcal{MC}_p(M). \end{aligned}$$

This makes TM into a bounded Fréchet manifold modeled on $F \times F$.

Proof. The proof follows from Lemma 3.1. □

The definition of vector bundles for Banach manifolds applies to bounded Fréchet manifolds, with evident modifications (see [**14**] for the definition of a Banach vector bundle). Note that the group of automorphisms, $\mathrm{Aut}(F)$, is topological ([**10,** Proposition 1.2]); thus, it can serve as the structure group of a vector bundle. Let X be a topological space, and let $\Pi : X \to M$ be a surjective continuous map. Let (E, g) be a Fréchet space. Consider the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ of M, and for each $\alpha \in \mathcal{A}$, suppose that we are given a mapping

$$\tau_\alpha : \Pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times E,$$

satisfying the following conditions.

(VB1) The map τ_α is an MC^∞-isomorphism commuting with the projection on U_α, i.e., the following diagram is commutative.

$$\begin{array}{ccc} \Pi^{-1}(U_\alpha) & \xrightarrow{\ \tau_\alpha\ } & U_\alpha \times E \\ & {\scriptstyle \Pi} \searrow \quad \swarrow {\scriptstyle (u,e)\longmapsto u} & \\ & U_\alpha & \end{array}$$

In particular, for each $p \in U_\alpha$, the induced map $\tau_{\alpha p} : \Pi^{-1}(p) \to E$ is an isomorphism.

(VB2) If U_α and U_β are two members of the open covering, then the map

$$\tau_{\alpha p} \circ \tau_{\beta p}^{-1} : E \longrightarrow E$$

is an isomorphism in the category of topological vector spaces.

(VB3) Let U_α and U_β be two members of the open covering. Then the map $U_\alpha \cap U_\beta \to \mathrm{Aut}(E)$, given by $p \mapsto (\tau_\alpha \circ \tau_\beta^{-1})_p$, is a morphism.

The collection $\{(U_\alpha, \tau_\alpha)\}_{\alpha\in\mathcal{A}}$ is called a *trivializing covering* for Π, and the maps τ_α are called *trivializing maps*. Two trivializing coverings are said to be VB-*equivalent* if their union satisfies conditions (VB2) and (VB3). An equivalence class of such a trivializing covering is said to determine the structure of a vector bundle on Π. The space M is called the *base space* of the bundle, X the *total space*, and E the *fiber*. The group $\mathrm{Aut}(E)$ is called the *structure group* of the bundle. For each $p \in M$, the fiber $\Pi^{-1}(p)$ over p has the structure a Fréchet space, which is isomorphic to E via $\tau_{\alpha p}$. Condition (VB2) insures that this structure of $\Pi^{-1}(p)$ is independent of the choice of the trivializing map $\tau_{\alpha p}$. Note that a vector bundle, as defined above, leads to a groupless vector bundle defined by Hamilton [**11,** Definition 4.3.1]. Indeed, given $p \in M$, one can choose $\alpha \in \mathcal{A}$ so that $p \in U_\alpha$, and then τ_α is an obvious candidate for the local trivialization around p required in the definition of Hamilton.

Theorem 3.3. *TM admits a vector bundle structure over M, with fiber of type F, and structure group* $\mathrm{Aut}(F)$.

Proof. Consider the above atlas of M and its corresponding trivializing atlas for TM. Let $\pi\mathrm{r}_1$ and $\pi\mathrm{r}_2$ be the projections to the first and second factors, respectively. For all $\alpha \in \mathcal{A}$, we have $\pi\mathrm{r}_1 \circ T\varphi_\alpha = \pi_M$; therefore, TM is a fiber bundle. Suppose $U_\alpha \cap U_\beta \neq 0$. Then, by Lemma 3.1 (iii), the overlap map

$$T\varphi_\alpha \circ T\varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \times F \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta) \times F$$

is a local vector bundle isomorphism. Thereby, the transition maps $\Theta_{\alpha\beta} = T\varphi_\alpha \circ T\varphi_\beta^{-1}$ can be considered as taking values in $\mathrm{Aut}(F)$. The following

$$U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(F), \qquad p \longmapsto (\pi\mathrm{r}_2 \circ T\varphi_\alpha\,|_{T_pM} \circ T\varphi_\beta^{-1}\,|_{T_pM})$$

is a smooth morphism; hence, all the conditions of [**14,** Proposition 1.2] are verified. Thus, TM is a vector bundle over M, with structure group $\mathrm{Aut}(F)$. □

3.2. Second order tangent bundle. Now that TM is a manifold, we can define second order tangents. Assume $r = 2$ in the equivalence relation (3.1). Let $T^2_p M$ be the set of all 2-jets at p, and let

$$T^2 M = \bigcup_{p \in M} T^2_p M.$$

Let $\Pi_{TM} : T^2 M \to M$ be a natural projection, defined by

$$\Pi_{TM}(j^2_p(f)) = p.$$

If we topologize $T^2 M$ in a natural way, then $T^2 M$ is called the *second order tangent bundle* over M.

By virtue of Lemma 3.1, we have a trivializing atlas

$$\{(\Pi^{-1}_{TM}(\pi^{-1}_M(U_\alpha)), \widetilde{\Phi}_\alpha)\}_{\alpha \in \mathcal{A}},$$

for $T^2 M$, with

$$\begin{aligned} \widetilde{\Phi}_\alpha : \Pi^{-1}_{TM}(\pi^{-1}_M(U_\alpha)) &\longrightarrow \varphi_\alpha(U_\alpha) \times F, \\ j^2_p(f) &\longmapsto (\varphi_\alpha(p), (\varphi_\alpha \circ f)''(0)); \quad f \in \mathcal{MC}_p(M). \end{aligned}$$

$T^2_p M$ can be identified with $F \times F$ under the isomorphism:

$$\Psi : T^2_p M \longrightarrow F \times F, \quad j^2_p(f) \longmapsto ((\varphi_\alpha \circ f)'(0), (\varphi_\alpha \circ f)''(0)),$$

but fails to be a vector bundle over M because the trivializing isomorphism does not respect the linear structure of the fibers. The submersion $\pi_{12} : T^2 M \to TM$, defined by $\pi_{12}(j^2_p(f)) = j^1_p(f)$, is a vector bundle. Let

$$\pi_2 : T(TM) \longrightarrow TM$$

be an ordinary tangent bundle over TM. The space $T^2 M$ coincides with

$$\{\Upsilon \in T(TM) \mid \pi_2(\Upsilon) = T\pi_M(\Upsilon)\}, \tag{3.2}$$

and can be identified with a submanifold of $T(TM)$, see [**15,** page 372, Proposition 3.2]. The bundle $T(TM)$ is a fiber bundle over M, with the projection $\pi^2 = \pi_M \circ T\pi_M$. The restriction $\pi^2 |_{T^2 M} : T^2 M \to M$ is again a fiber bundle.

Let $\Pi_i : N_i \to M$, $i = 1, 2$, be fiber bundles with the same group structure $\mathrm{Aut}(F)$. The fiber product is defined as usual. The bundle

$T(N_1 \times N_2)$ is canonically isomorphic to $T(N_1) \times T(N_2)$, with structure group $\mathrm{Aut}(F \times F)$. Furthermore, if TN_i are vector bundles, then their product is the Whitney sum $TN_1 \oplus TN_2$ (see [**8, 12**]).

4. Connections. Here we define connections by using Vilms's [**19**] point of view for connections on infinite-dimensional vector bundles. Also, we show that each linear connection corresponds, in a bijective way, to an ordinary differential equation analogous to the case of Banach manifolds (see [**18**]).

Henceforth, we keep the formalism of Section 3 for tangent bundles and second order tangent bundles.

Definition 4.1. A smooth connection map $\mathcal{K}$ for the tangent bundle $\pi_M : TM \to M$ is a smooth bundle morphism $\mathcal{K} : T(TM) \to TM$ such that there exist smooth maps

$$\tau_\alpha : \varphi_\alpha(U_\alpha) \times F \longrightarrow \mathcal{L}_d(F),$$

which give the local representatives of $\mathcal{K}$ by

$$\begin{gathered}\mathcal{K}_\alpha = \Phi_\alpha \circ \mathcal{K} \circ (\widetilde{\Phi}_\alpha)^{-1} : \varphi_\alpha(U_\alpha) \times F \times F \times F \longrightarrow \varphi_\alpha(U_\alpha) \times F,\\ \mathcal{K}_\alpha(f, g, h, k) = (f, k + \tau_\alpha(f, g) \cdot h)\end{gathered}$$

$\mathcal{L}_d(F)$ is topologized by the metric (2.1).

A connection on M is a connection map on the tangent bundle $\pi_M : TM \to M$. A connection $\mathcal{K}$ is linear if and only if it is linear on the fibers of the tangent map. Locally, $T\pi$ is the map

$$U_\alpha \times F \times F \times F \longrightarrow U_\alpha \times F,$$

defined by $T\pi(f, \xi, h, \gamma) = (f, h)$; hence, locally its fibers are the spaces $\{f\} \times F \times \{h\} \times F$. Therefore, $\mathcal{K}$ is linear on these fibers if and only if the maps $(g, k) \mapsto k + \tau_\alpha(f, g)h$ are linear, and this means that the mappings τ_α need to be linear with respect to the second variable.

Assume that the connection $\mathcal{K}$ is linear and $f \in U_\alpha$. The unique local Christoffel symbol

$$\Gamma_\alpha(p) : \varphi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(F \times F; F),$$

satisfying $\Gamma_\alpha(p)(g,h) = \tau_\alpha(p,g)h$ is associated to

$$\tau_\alpha(f,\cdot) \in \mathcal{L}(F,\mathcal{L}(F,F)) \cong \mathcal{L}(F\times F;F)$$

by the canonical isomorphism of Lemma 2.1.

Christoffel symbols satisfy the following compatibility condition, cf. [**7**],

$$\begin{aligned}\Gamma_\alpha(\Theta_{\alpha\beta}(f))(\mathrm{D}\,\Theta_{\alpha\beta}(f)(g), \mathrm{D}\,\Theta_{\alpha\beta}(f)(h)) + (\mathrm{D}^2\,\Theta_{\alpha\beta}(f)(h))(g)\\ = \mathrm{D}\,\Theta_{\alpha\beta}(f)(\Gamma_\beta(f)(g,h)),\\ \text{for all } (f,g,h)\in\varphi_\alpha(U_\alpha\cap U_\beta)\times F\times F.\end{aligned} \tag{4.1}$$

Here, we denote the diffeomorphisms $\varphi_\alpha\circ\varphi_\beta^{-1}$ of F by $\Theta_{\alpha\beta}$.

Theorem 4.2. *Every linear connection on M induces a vector bundle structure on $\pi^2|_{T^2M} : T^2M \to M$ and gives rise to an isomorphism of this vector bundle with the vector bundle $TM\oplus TM$.*

Proof. If we have a connection, then the connection map $\mathcal{K} : T(TM) \to M$ is defined. The following map,

$$\pi_2\oplus\mathcal{K}\oplus T\pi_M : T(TM) \longrightarrow TM\oplus TM\oplus TM, \tag{4.2}$$

is a diffeomorphism (see [**5**]). The diffeomorphism determines a unique vector bundle structure for $T(TM)$ over M. Let $(U_\alpha,\varphi_\alpha)$ be a chart of M. The induced chart $\{(\pi_M^{-1}(U_\alpha), T\varphi_\alpha)\}$ in TM takes a vector bundle structure by means of the diffeomorphism (4.2). Let

$$\imath : TM\oplus TM \to TM\oplus TM\oplus TM$$

be the natural isomorphism. T^2M is a submanifold of $T(TM)$, consisting of tangent vectors Υ such that $\pi_2(\Upsilon) = T\pi_M(\Upsilon)$. Therefore, the inclusion $\imath$ is the isomorphism onto $(\pi_2\oplus\mathcal{K}\oplus T\pi_M)(T^2M)$; thus,

$$\imath^{-1}\circ(\pi_2\oplus\mathcal{K}\oplus T\pi_M)(T^2M) = \pi_2\oplus\mathcal{K}(T^2M).$$

Hence, the diffeomorphism

$$\pi_2\oplus\mathcal{K} : T^2M \longrightarrow TM\oplus TM, \tag{4.3}$$

gives the structure of a vector bundle to T^2M. Since T^2M is isomorphic to $TM\oplus TM$, it can be considered as a vector bundle with group structure $\mathrm{Aut}(F\times F)$. $\square$

The proof of the following theorem is the same as the usual proof given for Banach manifolds (see [**4,** Theorem 2.4]). We simply provide the scheme of the proof.

Theorem 4.3. *If T^2M admits a vector bundle structure isomorphic to $TM \oplus TM$, then there exists a linear connection on M.*

Proof. Let $\{(\Pi^{-1}(U_\alpha), \Omega_\alpha)\}_{\alpha \in \mathcal{A}}$ be a trivializing atlas of T^2M. By hypothesis $\Omega_{\alpha,p} = \Omega^1_{\alpha,p} \times \Omega^2_{\alpha,p}$, where

$$\Omega^i_{\alpha,p} : \pi_M^{-1}(p) \longrightarrow F, \quad i = 1, 2.$$

Let $(U, \mathit{\Omega})$ be an arbitrary chart such that $U \subseteq U_\alpha$. Define $\mathit{\Omega}_\alpha = \mathit{\Omega} \circ (\Omega^1_{\alpha,p} \circ (\mathrm{D}_x \, \mathit{\Omega})^{-1})$. Then define the Christoffel symbols as follows:

$$\Gamma_\alpha(y)(u, u) = \Omega^2_{\alpha,p}(j^2_p f) - (\mathit{\Omega}_\alpha \circ f)''(0), \qquad y \in \mathit{\Omega}_\alpha(U_\alpha),$$

where f is the representative of the vector u. The remaining values of $\Gamma_\alpha(y)$ on elements of the form (u, v) with $u \neq v$ are automatically defined if we require $\Gamma_\alpha(y)$ to be symmetric and bilinear. They satisfy the compatibility condition since the trivializations $\big\{(\Pi^{-1}(U_\alpha), \Omega_\alpha)\big\}_{\alpha \in \mathcal{A}}$ coincide on all common areas of their domains, and hence, give rise to a linear connection on M. $\square$

Proposition 4.4. *Each linear connection of the trivial vector bundle $(M \times F, M, \mathrm{pr}_1)$ corresponds bijectively to an ordinary differential equation*

$$\frac{dx}{dt} = A(t) \cdot x,$$

where

$$[A(t)](u) = \Gamma_1(t)(u, 1_M),$$

for every $u \in F$ and $t \in M$.

Proof. Keep the above formalism for connections. Suppose that a linear connection $\mathcal{K}$ on $(M \times F, M, \mathrm{pr}_1)$ is given. We want to associate $\mathcal{K}$ with a unique ordinary differential equation on the Fréchet space F. With respect to the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ of M, we consider the Christoffel symbols of $\mathcal{K}$, that is, the smooth maps

$$\Gamma_\alpha(p) : \varphi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(F \times F; F).$$

In particular, we denote the Christoffel symbol defined over the chart (M, id_M) by Γ_1. Let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be charts in M. Then, we define the following map for any $t \in \varphi_\alpha(U_\alpha)$,

$$A_\alpha : \varphi_\alpha(U_\alpha) \longrightarrow \mathcal{L}_d(F), \qquad t \longmapsto \Gamma_\alpha(t)(\cdot, 1_M), \tag{4.4}$$

where 1_M is the unit of M. The map A_β, defined as above, corresponds to (U_β, φ_β). In particular, we set $A \equiv A_1 : M \to \mathcal{L}_d(F)$ when the chart is (M, id_M). If $U_\alpha \cap U_\beta \neq 0$, then the compatibility condition (4.1) imposes the following

$$A_\beta(t) = (\varphi_\alpha \circ \varphi_\beta^{-1})'(t) \cdot A_\alpha((\varphi_\alpha \circ \varphi_\beta^{-1})(t)), \tag{4.5}$$

for every $t \in \varphi_\beta(U_\alpha \cap U_\beta)$. Letting $\alpha = 1$ in (4.5) yields

$$A_\beta(t) = (\varphi_\beta^{-1})'(t) \cdot A(\varphi_\beta^{-1}(t)), \quad \text{for all } t \in \varphi_\beta(U_\beta). \tag{4.6}$$

Now we can define

$$\frac{dx}{dt} = A(t) \cdot x; \qquad [A(t)](u) = \Gamma_1(t)(u, 1_M),$$
$$\text{for all } u \in F, \quad \text{for all } t \in M.$$

Conversely, for a given equation with coefficient $A : M \to \mathcal{L}_d(F)$, we define the smooth maps $A_\beta(t) : \varphi_\beta(U_\beta) \to \mathcal{L}_d(F)$ by (4.6). The same equality proves that A_α and A_β satisfy (4.5). We then define the Christoffel symbols $\{\Gamma_\alpha\}_{\alpha \in \mathcal{A}}$ by $\Gamma_\alpha(t)(u, s) = s \cdot [A_\alpha(t)](u)$, for every $t \in \varphi_\alpha(U_\alpha)$, $s \in M$ and $u \in F$. By virtue of (4.5), the above Christoffel symbols satisfy the compatibility condition over overlapping charts, thereby defining a linear connection $\mathcal{K}$, on $(M \times F, M, \mathrm{pr}_1)$. $\square$

5. Vector fields on TM. Having introduced the tangent bundle over a manifold M, we now consider sections of these bundles. A vector field on M is a section $\xi : M \to TM$ of its tangent bundle, i.e., $\pi_M \circ \xi = \mathrm{id}_M$. For a vector field ξ and a chart $U \subset M \xrightarrow{\varphi} \varphi(U) \subset F$, the principal part $\xi_\varphi : \varphi(U) \to F$ of ξ is defined by $\xi_\varphi(\varphi(p)) = \mathrm{pr}_2 \circ T\varphi(\xi_p)$. Let I be an open interval in $\mathbb{R}$, and let $\ell : I \to M$ be a curve passing through p_0. If ξ is a vector field on M, and if ξ_φ denotes the main part of its local representative in a chart φ, then $\ell(t)$ is called an integral curve of ξ when $(\varphi \circ \ell)'(t) = \xi_\varphi(\varphi \circ \ell(t))$ for each t, where $\varphi \circ \ell$ is the local representative of the curve ℓ. Note that, if the base manifold M is a Fréchet space F with differential structure induced by the chart (F, id_F), then the above condition reduces to $\ell'(t) = \mathrm{D}\, \ell(t)(1_{\mathbb{R}})$, that is, our definition is a natural generalization of the notion of a derivative on a manifold M.

Proposition 5.1. *Let $U \subseteq F$ be open, and let $\xi : U \to F$ be MC^k, $k \geq 1$. Then, for $p_0 \in U$, there is an integral curve $\ell : I \to F$ at p_0. Furthermore, any two such curves are equal on the intersection of their domains.*

Proof. Since ξ is MC^k, it is bounded, say by R. Let L be a positive real number. Pick a positive real number r such that $\overline{B_r(p_0)} \subseteq U$ and $\|\xi(p)\|_d \leq L$ for all $p \in \overline{B_r(p_0)}$. Let $m = \min\{1/R, r/L\}$, and let t_0 be a real number. We shall show that there is a unique MC^1-curve $\ell(t)$, $t \in [t_0 - m, t_0 + m]$, whose image lies in $\overline{B_r(p_0)}$ and that satisfies

$$\ell'(t) = \xi(\ell(t)), \qquad \ell(t_0) = p_0. \tag{5.1}$$

The conditions $\ell'(t) = \xi(\ell(t))$ and $\ell(t_0) = p_0$ are equivalent to the integral equation,

$$\ell(t) = p_0 + \int_{t_0}^{t} \xi(\ell(u))\, du.$$

Now, define $\ell_n(t)$ by induction:

$$\ell_0(t) = p_0, \qquad \ell_{n+1}(t) = p_0 + \int_{t_0}^{t} \xi(\ell_n(u))\, du.$$

The estimation on the size of the integral (see **[10,** Lemma 1.10]) yields $\ell_n(t) \in \overline{B_r(p_0)}$, for all n and $t \in [t_0 - m, t_0 + m]$. Furthermore,

$$\|\ell_{n+1}(t) - \ell_n(t)\|_d \leq \frac{LR^n}{(n+1)!} \mid t - t_0 \mid^{n+1}.$$

Therefore, ℓ_n converges uniformly to a continuous curve $\ell(t)$ satisfying (5.1). Now, let $\jmath(t)$ be another solution. By induction, we obtain

$$\|\ell_n(t) - \jmath(t)\|_d \leq \frac{LR^n}{(n+1)!} \mid t - t_0 \mid^{n+1}.$$

Therefore, letting $n \to \infty$ gives $\ell(t) = \jmath(t)$. □

Corollary 5.2. *Suppose the hypotheses of Proposition* 5.1 *hold. Let $\mathcal{I}_t(p_0)$ be the solution of $\ell'(t) = \xi(\ell(t))$, $\ell(t_0) = p_0$. Then, there is an open neighborhood U_0 of p_0, and a positive real number α such that, for* *every $q \in U_0$, there exists a unique integral curve $\ell(t) = \mathcal{I}_t(q)$ satisfying $\ell(0) = q$ and $\ell'(t) = \xi(\ell(t))$ for all $t \in (-\alpha, \alpha)$.*

Proof. Suppose

$$U_0 = \overline{B_{r/2}(p_0)} \quad \text{and} \quad \alpha = \min\{1/R, r/2L\}.$$

Fix an arbitrary point q_0 in U_0. Then, $\overline{B_{r/2}(q_0)} \subset \overline{B_r(p_0)}$, thereby $\|\xi(z)\|_d < L$, for all $z \in \overline{B_{r/2}(q_0)}$. By Proposition 5.1, with p_0 replaced by q, r replaced by $r/2$ and t_0 by 0, there exists a unique integral curve $\ell(t)$ for all $t \in (-\alpha, \alpha)$ such that $\ell(0) = q$. □

The proof of the following theorem is the same as the standard proof given for Banach manifolds (see [**14,** Theorem 2.1]).

Theorem 5.3. *Let $\xi : M \to TM$ be a vector field. Then, there exists an integral curve for ξ at $p \in M$. Furthermore, any two such curves are equal on the intersection of their domains.*

Proof. The existence follows from Proposition 5.1, by means of local representation. However, that is not applicable for the proof of uniqueness since these curves may lie in different charts. Let $\rho_i(t) : I_i \to M$, $i = 1, 2$, be two integral curves. Let $I = I_1 \cap I_2$ and $J = \{t \in I \mid \rho_1(t) = \rho_2(t)\}$. J is closed, since M is Hausdorff. From Proposition 5.1, J contains some neighborhood of 0. Now define $\delta_1(u) = \rho_1(u + t)$ and $\delta_2(u) = \rho_2(u + t)$, for $t \in J$.

They are integral curves with initial conditions $\rho_1(t)$ and $\rho_2(t)$, respectively. By Proposition 5.1, they coincide on some neighborhood of 0. Therefore, J contains an open neighborhood of t, so J is open. Since I is connected, it follows that $J = I$. □

Remark 5.4. Let ξ be an MC^k-vector field on M, $k \geq 1$. The existence of a flow of class MC^k for ξ depends on the solution of the appropriate time-dependent linear differential equation on the model space F. But, on the Fréchet space F, an ordinary differential equation may admit no, one or multiple solutions for the same initial condition. Therefore, there may not exist an MC^k-flow for ξ in general.

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