RUAN COHOMOLOGIES OF THE COMPACTIFICATIONS OF RESOLVED ORBIFOLD CONIFOLDS

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ABSTRACT. In this paper, we study the Ruan cohomologies of X^s and X^{sf}, the natural compactifications of V^s and V^{sf}, where V^s and V^{sf} are the two small resolutions of

$$V = \{(x,y,z,w) \mid xy - zw = 0\}/\mu_r(1,-1,0,0), \quad r > 1,$$

the finite group quotient of the singular conifold. There is an additive isomorphism between the Chen-Ruan cohomologies $\phi : H^*_{CR}(X^s) \to H^*_{CR}(X^{sf})$. We study the three-point orbifold Gromov-Witten invariants of the exceptional curves Γ^s on X^s and Γ^{sf} on X^{sf} and show that the corresponding Ruan cohomology ring structures on the Chen-Ruan cohomologies of X^s and X^{sf}, defined by these three-point functions, are isomorphic to each other under the map ϕ and the identification $[\Gamma^s] \leftrightarrow -[\Gamma^{sf}]$.

1. Introduction. In [**18**], Li and Ruan proposed symplectic birational geometry as a generalized algebraic birational geometry. It is suggested that the symplectic birational structure of a symplectic manifold is detected by genus zero Gromov-Witten invariants. In the symplectic category, many obvious properties of algebraic birational geometry are no longer obvious. Notably, the birational invariance of uniruledness in [**10**] is such an example, where the authors have drawn

on new and powerful technology from Gromov-Witten theory. Other progress has also been made in [**11, 18**].

Roughly speaking, symplectic birational geometry mainly concerns the change of Gromov-Witten theory under symplectic surgery. In [**16**], Li and Ruan interpreted flops and extremal transition by symplectic surgery and showed an elegant result that any two smooth minimal models in dimension three have the same quantum cohomology. As mentioned in [**18**], the more appropriate category for symplectic birational geometry is the category of orbifolds. And we should carry symplectic birational geometry to orbifolds.

In [**3, 4**], the authors initiated a program for studying quantum cohomology under birational transformation of orbifolds. In their papers, they considered the singularity

$$W_r = \{(x, y, z, t) \mid xy - z^{2r} + t^2\}/\mu_r(a, -a, 1, 0)$$

with r being a prime number. For this singularity, they defined the orbi-conifold transition and orbi-flop. Their main result was that, for a pair of orbifolds (X, Y) with Y being obtained from X by a sequence of orbi-flops, the quantum cohomologies of X and Y are isomorphic to each other. In [**3**], they first considered a quantum modification of Chen-Ruan cohomology, Ruan cohomology, which is in a sense between Chen-Ruan cohomology and quantum cohomology. They showed that Ruan cohomology is invariant under orbi-flops. Then, by using relative orbifold Gromov-Witten invariants and degeneration formulas, they proved that quantum cohomology is invariant under orbi-flops in [**4**].

In this paper, we study another singularity

$$V = \{(x, y, z, w) \mid xy - zw = 0\}/\mu_r(1, -1, 0, 0),$$

which is a natural replacement for the well-known smooth conifold. There are two small resolutions V^s and V^{sf} of V, which are both orbifolds and orbifold vector bundles over the $\mathbb{P}^1$-orbifolds Γ^s and Γ^{sf}, respectively. There are many symplectic forms on V^s and V^{sf} constructed from the smooth case (see Remark 2.1). We can perform the well-known symplectic cutting (cf., [**8, 15**]) to get two compact orbifolds, X^s and X^{sf}, which are just $\mathbb{P}(V^s \oplus \mathcal{O})$ and $\mathbb{P}(V^{sf} \oplus \mathcal{O})$, hence, orbifold fiber bundles over Γ^s and Γ^{sf} respectively. In addition, X^s is obtained from X^{sf} by an orbi-flop transition.

There is an obvious additive isomorphism between Chen-Ruan cohomologies

$$\phi : H_{CR}^*(X^s) \longrightarrow H_{CR}^*(X^{sf}),$$

which preserves the orbifold Poincaré pairing. The exceptional divisor of X^s (respectively, X^{sf}) is Γ^s (respectively, Γ^{sf}). By using only the moduli spaces of J-holomorphic curves representing multiples of Γ^s's we can define a three-point function on $H_{CR}^*(X^s)$:

$$F^s(\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} + \sum_{d \geq 1} q^d \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d[\Gamma^s]}^{X^s}.$$

Such a function and the orbifold Poincaré pairing define the ring structure of Ruan cohomology $RH_{CR}^*(X^s)$ over the Chen-Ruan cohomology group. Similarly, one can define $F^{sf}(\beta_1, \beta_2, \beta_3)$ on $H_{CR}^*(X^{sf})$ and the Ruan cohomology $RH_{CR}^*(X^{sf})$. Our main theorem is

Theorem 1.1. *Suppose $\alpha_i \in H_{CR}^*(X^s)$ and $\beta_i = \phi(\alpha_i)$, $1 \leq i \leq 3$. Then:*

$$F^s(\alpha_1, \alpha_2, \alpha_3) = F^{sf}(\beta_1, \beta_2, \beta_3),$$

under the identification $[\Gamma^s] \leftrightarrow -[\Gamma^{sf}]$, i.e., $q \leftrightarrow q^{-1}$. Hence, we get the isomorphism of rings:

$$RH_{CR}^*(X^s) \cong RH_{CR}^*(X^{sf}).$$

2. Compactifications of the resolved orbifold conifolds and their Chen-Ruan cohomologies.

2.1. Resolved orbifold conifolds and their compactifications.
The well-known (smooth) conifold singularity is the complex hyperplane given by

$$\widetilde{V} = \{(x, y, z, w) \mid xy - zw = 0\} \subset \mathbb{C}^4.$$

It has an isolated singular point at the origin. Given a prime number r, let $\mu_r = \langle \xi \rangle$, the cyclic group of rth roots of 1 with $\xi = \exp(2\pi i/r)$, act on $\mathbb{C}^4$:

$$\xi \cdot (x, y, z, w) = (\xi x, \xi^{-1} y, z, w).$$

It is clear that this action preserves $\widetilde{V}$. Set

$$V = \frac{\widetilde{V}}{\mu_r}.$$

We call V the *orbifold conifold*.

By blow-ups, we have two small resolutions of $\widetilde{V}$. They are

$$\widetilde{V}^s = \left\{ ((x, y, z, w), [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid xy - zw = 0, \frac{p}{q} = \frac{x}{z} = \frac{w}{y} \right\}$$

$$\widetilde{V}^{sf} = \left\{ ((x, y, z, w), [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid xy - zw = 0, \frac{p}{q} = \frac{x}{w} = \frac{z}{y} \right\}.$$

Let

$$\widetilde{\pi}^s : \widetilde{V}^s \longrightarrow \widetilde{V}, \qquad \widetilde{\pi}^{sf} : \widetilde{V}^{sf} \longrightarrow \widetilde{V}$$

be the projections. Denote the extremal rays $(\widetilde{\pi}^s)^{-1}(0)$ and $(\widetilde{\pi}^{sf})^{-1}(0)$ by $\widetilde{\Gamma}^s$ and $\widetilde{\Gamma}^{sf}$, respectively. Both of them are $\mathbb{P}^1$. It is well known that $\widetilde{V}^s$ and $\widetilde{V}^{sf}$ are both the resolved conifolds, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. The action of μ_r extends naturally to both resolutions by setting

$$\xi \cdot [p, q] = [\xi p, q],$$

for the first model and

$$\xi \cdot [p, q] = [\xi p, q],$$

for the second model.

Set

$$V^s = \frac{\widetilde{V}^s}{\mu_r}, \qquad V^{sf} = \frac{\widetilde{V}^{sf}}{\mu_r}, \qquad \Gamma^s = \frac{\widetilde{\Gamma}^s}{\mu_r}, \qquad \Gamma^{sf} = \frac{\widetilde{\Gamma}^{sf}}{\mu_r}.$$

We call V^s and V^{sf} *small resolutions* of V. We say that V^{sf} is the flop of V^s and vice versa. They are both orbifolds.

Remark 2.1. There is a family of symplectic structures on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (cf., [**16**]). Choose a Hermitian metric on the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then $||z||^2$ for $z \in \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is a smooth function. And $i\partial\bar{\partial}||z||^2$ is a 2-form and nondegenerate on the fiber. Suppose that ω_0 is a symplectic form on $\mathbb{P}^1$. Then,

$$\omega = \pi^* \omega_0 + \epsilon i \partial \bar{\partial} ||z||^2$$

is a symplectic form on the total space in a neighborhood of the zero section, where $\pi : \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ is the projection and ϵ is a small constant. The Hamiltonian function is

$$H(x, z_1, z_2) = |z_1|^2 + |z_2|^2 - \epsilon.$$

The induced S^1-action is given by

$$e^{it} \cdot (x, z_1, z_2) = (x, e^{it} z_1, e^{it} z_2).$$

Then, we can perform the well-known symplectic cutting (cf., [8, 15, 16]) along the hypersurface $H^{-1}(0)$. Therefore, the resulting space $\overline{M}^+$ is just $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$, the projectivization of the bundle.

We can perform this construction on both $\widetilde{V}^s$ and $\widetilde{V}^{sf}$ and get two symplectic manifolds $\widetilde{X}^s$ and $\widetilde{X}^{sf}$, the projectivization of $\widetilde{V}^s$ and $\widetilde{V}^{sf}$, respectively. It is obvious that we can deform the symplectic structures such that they are μ_r-invariant. Then we get two symplectic orbifolds $X^s = \widetilde{X}^s/\mu_r$ and $X^{sf} = \widetilde{X}^{sf}/\mu_r$, the projectivization of V^s and V^{sf}. In this paper, we are going to study the Ruan cohomologies of these two symplectic orbifolds.

2.2. Chen-Ruan cohomologies of X^s and X^{sf}. Let us take X^s. The manifold $\widetilde{X}^s$ is the projectivization of the bundle $\widetilde{E}^s = \widetilde{L}_1^s \oplus \widetilde{L}_2^s \oplus L = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ over $\widetilde{\Gamma}^s = \mathbb{P}^1$. Hence, $X^s = \mathbb{P}(L_1^s \oplus L_2^s \oplus L)$ is an orbifold fiber bundle over Γ^s with fiber $\mathbb{P}^2/\mu_r$. In terms of coordinates $((x, y, z, w, t), [p, q])$, we get a local trivialization of the bundle $\widetilde{E}^s$ with coordinates (u, x, w, t) and (v, z, y, t), where $u = q/p$ and $v = p/q$. The local coordinates of $\widetilde{X}^s$ are $(u, [x, w, t])$ and $(v, [z, y, t])$. The transition map is given by

$$\begin{cases} v = u^{-1}, \\ z = ux, \\ y = uw, \\ t = t. \end{cases}$$

In terms of these local coordinates, the μ_r-action is given by

$$\xi \cdot (u, [x, w, t]) = (\xi^{-1}u, [\xi x, w, t]),$$
$$\xi \cdot (v, [z, y, t]) = (\xi v, [z, \xi^{-1}y, t]).$$

Hence, the set of singular points in X^s is contained in the two fibers of $X^s \to \Gamma^s$ over $0^s = [1,0]$ and $\infty^s = [0,1]$. They are

$$\{(0,[1,0,0])\} \cup \{(0,[0,w,t])\} \quad \text{over } 0^s,$$

and

$$\{(0,[0,1,0])\} \cup \{(0,[z,0,t])\} \quad \text{over } \infty^s.$$

We denote $(0,[1,0,0])$ by p^s and $(0,[0,1,0])$ by q^s. We denote $\{(0,[0,w,t])\}$, the fiber of $\mathbb{P}(L_2^s \oplus L)$ over 0^s, and $\{(0,[z,0,t])\}$, the fiber of $\mathbb{P}(L_1^s \oplus L)$ over ∞^s, by L_0^s and L_∞^s, respectively. We can also write the coordinates of 0^s and ∞^s as

$$0^s = ([1,0],[0,0,1]) \quad \text{and} \quad \infty^s = ([0,1],[0,0,1]).$$

Then the exceptional curve connecting 0^s and ∞^s is $\Gamma^s = ([p,q],[0,0,1])$.

At L_0^s (respectively, L_∞^s, p^s and q^s), for each $\xi^k \in \mu_r$, $1 \le k \le r-1$, there is a corresponding twisted sector (cf., [5]), as a set which is $\{(p,\xi^k)|p \in L_0^s\}$ (respectively, $\{(p,\xi^k)|p \in L_\infty^s\}$, $\{(p^s,\xi^k)\}$ and $\{(q^s,\xi^k)\}$). We denote this twisted sector by $[L_0^s]_k$ (respectively, $[L_\infty^s]_k$, $[p^s]_k$ and $[q^s]_k$). For each twisted sector, a degree shifting number is assigned. Using the description of the μ_r-action in local coordinates given above, we conclude

Lemma 2.2. *For $\xi^k \in \mu_r$, $1 \le k < r$, the degree shifting numbers are*

$$\begin{cases} \iota([L_0^s]_k) = \iota([L_\infty^s]_k) = 1, \\ \iota([p^s]_k) = 3 - 3k/r, \ \iota([q^s]_k) = 3k/r. \end{cases}$$

Topologically, the twisted sectors $[L_0^s]_k$ and $[L_\infty^s]_k$ are both $\mathbb{P}^1$, and $[p^s]_k$ and $[q^s]_k$ are both $\{pt\}$. The Chen-Ruan cohomology of X^s is:

$$H_{CR}^*(X^s) = H^*(X^s) \oplus \bigoplus_{k=1}^{r-1} (H^{*-2}([L_0^s]_k) \oplus H^{*-2}([L_\infty^s]_k))$$

$$\oplus \bigoplus_{k=1}^{r-1} (H^{*-6+6k/r}([p^s]_k) \oplus H^{*-6k/r}([q^s]_k)).$$

A similar structure applies to X^{sf}. The local trivialization of $\widetilde{E}^{sf} = \widetilde{L}_1^{sf} \oplus \widetilde{L}^{sf} \oplus L = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ has coordinates (u, x, z, t) and (v, w, y, t). The induced local trivialization of $\widetilde{X}^{sf}$ has coordinates $(u, [x, z, t])$ and $(v, [w, y, t])$. The transition map is

$$\begin{cases} v = u^{-1}, \\ w = ux, \\ y = uz, \\ t = t. \end{cases}$$

In terms of these local coordinates, the μ_r-action is given by

$$\xi \cdot (u, [x, z, t]) = (\xi^{-1}u, [\xi x, z, t]),$$
$$\xi \cdot (v, [w, y, t]) = (\xi v, [w, \xi^{-1}y, t]).$$

Hence, the set of singular points in $X^{sf} = \mathbb{P}(L_1^{sf} \oplus L_2^{sf} \oplus L)$ consists of

$$\{(0, [1, 0, 0])\} \cup \{(0, [0, z, t])\} \quad \text{over } 0^{sf} = [1, 0]$$

and

$$\{(0, [0, 1, 0])\} \cup \{(0, [w, 0, t])\} \quad \text{over } \infty^{sf} = [0, 1].$$

We denote $(0, [1, 0, 0])$ by p^{sf} and $(0, [0, 1, 0])$ by q^{sf}. We denote $\{(0, [0, z, t])\}$, the fiber of $\mathbb{P}(L_2^{sf} \oplus L)$ over 0^{sf}, and $\{(0, [w, 0, t])\}$, the fiber of $\mathbb{P}(L_1^{sf} \oplus L)$ over ∞^{sf}, by L_0^{sf} and L_∞^{sf}, respectively. We can also write the coordinates of 0^{sf} and ∞^{sf} as

$$0^{sf} = ([1, 0], [0, 0, 1]) \quad \text{and} \quad \infty^{sf} = ([0, 1], [0, 0, 1]).$$

Then the exceptional curve connecting 0^{sf} and ∞^{sf} is

$$\Gamma^{sf} = ([p, q], [0, 0, 1]),$$

and the degree shifting numbers of the corresponding twisted sectors are:

$$\begin{cases} \iota([L_0^{sf}]_k) = \iota([L_\infty^{sf}]_k) = 1, \\ \iota([p^{sf}]_k) = 3 - 3k/r, \ \iota([q^{sf}]_k) = 3k/r. \end{cases}$$

The Chen-Ruan cohomology of X^{sf} is

$$H^*_{CR}(X^{sf}) = H^*(X^{sf}) \oplus \bigoplus_{k=1}^{r-1} \left(H^{*-2}([L^{sf}_0]_k) \oplus H^{*-2}([L^{sf}_\infty]_k) \right)$$

$$\oplus \bigoplus_{k=1}^{r-1} \left(H^{*-6+6k/r}([p^{sf}]_k) \oplus H^{*-6k/r}([q^{sf}]_k) \right).$$

3. Orbifold Gromov-Witten theory and Ruan Cohomology rings of X^s and X^{sf}. In this section, we consider orbifold Gromov-Witten invariants of X^s and X^{sf} and ring structures on the Chen-Ruan cohomology groups of X^s and X^{sf}.

The exceptional curve Γ^s (respectively, Γ^{sf}) generates a subgroup of the degree 2 homology group of X^s (respectively, X^{sf}). Here, we focus on the orbifold Gromov-Witten invariants with degree 2 classes in this subgroup.

3.1. Orbifold Gromov-Witten theory. For a compact symplectic orbifold $\mathcal{X}$ with compatible almost complex structure J, let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli space of n-pointed genus g orbifold stable maps to $\mathcal{X}$ of degree d. That is this space parameterizes the families of representable J-holomorphic morphisms $f : \mathcal{C} \to \mathcal{X}$, from an orbicurve $\mathcal{C}$ to $\mathcal{X}$. Here the condition of *representability* means the induced homomorphism on isotropy groups at every point is injective.

Orbicurve are allowed isotropy only at marked and nodal points, and the orbifold structure near nodal points is required to be *balanced*, which means locally near a nodal point, $\mathcal{C}$ has the form

$$\{xy = 0\}/\mathbb{Z}_\ell$$

with $\mathbb{Z}_\ell = \langle \zeta \rangle$ acting by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$. For the precise definition of orbicurve one can see, for example, [6].

There is an evaluation map for every marked point, x_i:

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) \longrightarrow \Lambda\mathcal{X},$$
$$[\mathcal{C}, f, x_1, \cdots, x_n] \longmapsto (y_i, (g_i)_{G_{y_i}})$$

where $f(x_i) = y_i$ and $g_i = \lambda_f(\sigma)$ with σ the generator of the isotropy group of x_i and λ_f the induced homomorphism. By using the decom-

position of the inertia orbifold $\Lambda\mathcal{X} = \coprod_{(g)\in\mathcal{T}_\mathcal{X}} \mathcal{X}_{(g)}$, where $\mathcal{T}_\mathcal{X}$ is the set of equivalence classes of conjugacy classes in local groups, we can decompose $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ into components

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) = \coprod_{(\mathbf{g})\in\mathcal{T}_\mathcal{X}^n} \overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X}, d),$$

where $(\mathbf{g}) = ((g_1), \ldots, (g_n))$.

Chen and Ruan [**6**] observed that each component $\overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X}, d)$ has a virtual fundamental cycle $[\overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X}, d)]^{\mathrm{vir}}$ of dimension

$$v\dim_\mathbb{C}[\overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X}, d)]^{\mathrm{vir}} = c_1(d) + (\dim_\mathbb{C} X - 3)(1 - g) + n - \iota(\mathbf{g}),$$

where $\iota(\mathbf{g}) = \sum_i \iota(g_i)$ and $\iota(g_i)$ is the degree shifting number of $\mathcal{X}_{(g_i)}$.

Now we can define the orbifold Gromov-Witten invariant for $\alpha_i \in H^*(\mathcal{X}_{(g_i)})$ as

$$\langle\alpha_1, \ldots, \alpha_n\rangle_{g,(\mathbf{g}),d}^\mathcal{X} \triangleq \int_{[\overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X},d)]^{\mathrm{vir}}} \prod_{i=1}^n ev_i^*(\alpha_i).$$

When $\sum_i \deg\alpha_i \neq 2v\dim_\mathbb{C}[\overline{\mathcal{M}}_{g,(\mathbf{g})}(\mathcal{X}, d)]^{\mathrm{vir}}$, the invariant is defined to be zero. We also denote the invariant by $\langle\alpha_1, \ldots, \alpha_n\rangle_{g,n,d}^\mathcal{X}$, since $(\mathbf{g})$ is determined by α_i.

3.2. Ruan cohomology rings of X^s and X^{sf}. Now, for the case $\mathcal{X} = X^s$, the decomposition of moduli space $\overline{\mathcal{M}}_{0,n}(X^s, d)$ is

$$\overline{\mathcal{M}}_{0,n}(X^s, d) = \coprod_{(\mathbf{g})\in\mathcal{T}_{X^s}^n} \overline{\mathcal{M}}_{0,(\mathbf{g})}(X^s, d),$$

where d means $d[\Gamma^s]$ with $d \in \mathbb{Z}$. If $(\mathbf{g}) = (X^s, \ldots, X^s, T_1, \ldots, T_k)$ with $\mathsf{x} = (T_1, \ldots, T_k)$ consisting of k twisted sectors in X^s, then we denote $\overline{\mathcal{M}}_{0,(\mathbf{g})}(X^s, d)$ by $\overline{\mathcal{M}}_{0,l,k}(X^s, d, \mathsf{x})$.

Now we can define the three-point function on the Chen-Ruan cohomology group of X^s. For $\alpha_1, \alpha_2, \alpha_3 \in H^*_{CR}(X^s)$, the three-point function is

$$F^s(\alpha_1, \alpha_2, \alpha_3) \triangleq \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} + \sum_{d \geq 1} q^d \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,(\mathbf{g}),d}^{X^s}$$

$$= \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} + \sum_{d \geq 1} q^d \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d}^{X^s},$$

where q is a formal variable and $(\mathbf{g})$ is determined by $\alpha_1, \alpha_2, \alpha_3$. Here, the first term

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,0}^{X^s}$$

is the three-point function defining the Chen-Ruan product $\star_{CR}$. By introducing twisting factors (cf., [1]), one can turn a twisted form α, i.e., a form on a twisted sector, into a formal form $\widetilde{\alpha}$ on the global orbifold. Then, we have

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} = \int_{X^s}^{\mathrm{orb}} \widetilde{\alpha}_1 \wedge \widetilde{\alpha}_2 \wedge \widetilde{\alpha}_3.$$

The three-point function F^s and the orbifold Poincaré pairing $\langle \cdot, \cdot \rangle_{CR}^{X^s}$ on $H_{CR}^*(X^s)$ induce a ring structure on $H_{CR}^*(X^s)$.

Definition 3.1. Define the product on $H_{CR}^*(X^s)$ by

$$\langle \alpha_1 \star_R \alpha_2, \alpha_3 \rangle_{CR}^{X^s} = F^s(\alpha_1, \alpha_2, \alpha_3).$$

Then $\star_R$ is an associative product on $H_{CR}^*(X^s)$. We call it the *Ruan product* on X^s. This cohomology ring is denoted by $RH_{CR}^*(X^s)$.

Similarly, for X^{sf}, we can define the three-point function $F^{sf}(\beta_1, \beta_2, \beta_3)$ for $\beta_1, \beta_2, \beta_3 \in H_{CR}^*(X^{sf})$ and the corresponding Ruan product as above. We denote the resulting ring by $RH_{CR}^*(X^{sf})$.

4. Computing three-point functions.

4.1. Three-point function on X^s. The Chen-Ruan cohomology of X^s has a basis

- $\mathbb{1}^s, H^s, x^s, H^s x^s, (x^s)^2, (x^s)^3 = 2H^s(x^s)^2$ of $H^*(X^s)$, and
- $\mathbb{1}_{0,k}^s, x_{0,k}^s$ of $H^*([L_0^s]_k)$, $1 \leq k \leq r-1$, and
- $\mathbb{1}_{\infty,k}^s, x_{\infty,k}^s$ of $H^*([L_\infty^s]_k)$, $1 \leq k \leq r-1$, and

- $\mathbb{1}^s_{p^s,k}$ of $H^*([p^s]_k)$, $1 \le k \le r - 1$, and
- $\mathbb{1}^s_{q^s,k}$ of $H^*([q^s]_k)$, $1 \le k \le r - 1$.

Here x^s is the first Chern class of the hyperplane bundle over X^s, and H^s is a generator of $H^2(\Gamma^s)$. In the following, we will call those classes in $H^*_{CR}(X^s) \backslash H^*(X^s)$ *twisted classes* and the others *non-twisted classes*. We normalize H^s by letting

$$\int_{\Gamma^s}^{\mathrm{orb}} H^s = 1.$$

Note that we have

$$\int_{\Gamma^s}^{\mathrm{orb}} x^s = 0.$$

Since the three-point function is linear, we always take α_i as one of the basis elements. By comparing the virtual dimension and the sum of the degrees of the insertions and the divisor axiom we have:

Proposition 4.1. *The three-point function $F^s(\alpha_1, \alpha_2, \alpha_3)$ is determined by $\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{X^s}_{CR}$ and the n-point degree $d \ge 1$ orbifold Gromov-Witten invariants with the following insertions*:

(4.1)
$$
\begin{cases}
(\emptyset) \\
(H^s x^s) \\
(\mathbb{1}^s, \mathbb{1}^s, (x^s)^3) \\
(\alpha_1) & \alpha_1 = \mathbb{1}^s_{0,k} \ or \ \mathbb{1}^s_{\infty,k}, \\
(\alpha_1, \mathbb{1}^s, H^s x^s) & \alpha_1 = \mathbb{1}^s_{0,k} \ or \ \mathbb{1}^s_{\infty,k}, \\
(\alpha_1, \mathbb{1}^s, (x^s)^2) & \alpha_1 = \mathbb{1}^s_{0,k} \ or \ \mathbb{1}^s_{\infty,k}, \\
(\alpha_1, \mathbb{1}^s) & \alpha_1 = x^s_{0,k} \ or \ x^s_{\infty,k}, \\
(\alpha_1, \alpha_2) & \alpha_1, \alpha_2 \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k}\}_{k=1}^{r-1}, \\
(\alpha_1, \alpha_2, \mathbb{1}^s) & \alpha_1, \alpha_2 \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k}, x^s_{0,k}, x^s_{\infty,k}\}_{k=1}^{r-1}, \deg \alpha_1 < \deg \alpha_2, \\
(\alpha_1, \alpha_2, \alpha_3) & \alpha_i \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k}\}_{k=1}^{r-1}, \ i = 1, 2, 3, \\
(\alpha_1, \alpha_2, \alpha_3) & at \ least \ one \ of \ \alpha_i \ belongs \ to \in \{\mathbb{1}^s_{p^s,k}, \mathbb{1}^s_{p^s,k}\}_{k=1}^{r-1}.
\end{cases}
$$

4.1.1. Chen-Ruan three-point functions. By using the de Rham model [**1, 12**], we get:

Proposition 4.2. *For X^s, the nontrivial Chen-Ruan products of the twisted classes are given by*

$$\mathbb{1}^s_{0,i} \star_{CR} \mathbb{1}^s_{0,j} = \delta_{i+j,r} \Theta_{r,w},$$

$$\mathbb{1}^s_{0,i} \star_{CR} x^s_{0,j} = x^s \delta_{i+j,r} \Theta_{r,w},$$

$$\mathbb{1}^s_{\infty,i} \star_{CR} \mathbb{1}^s_{\infty,j} = \delta_{i+j,r} \Theta_{r,z},$$

$$\mathbb{1}^s_{\infty,i} \star_{CR} x^s_{\infty,j} = x^s \delta_{i+j,r} \Theta_{r,z},$$

$$\mathbb{1}^s_{p^s,i} \star_{CR} \mathbb{1}^s_{p^s,j} = \delta_{i+j,r} \Theta_{p^s},$$

$$\mathbb{1}^s_{q^s,i} \star_{CR} \mathbb{1}^s_{q^s,j} = \delta_{i+j,r} \Theta_{q^s}.$$

Here $\Theta_{r,w}$ and $\Theta_{r,z}$ are the Thom forms of the normal bundles of L^s_0 and L^s_∞ in X^s, and Θ_{p^s} and Θ_{q^s} are the Thom forms of the normal bundles of p^s and q^s in X^s, respectively. Moreover,

$$\beta \star_{CR} \alpha = 0,$$

if β is a twisted class, $\alpha \in H^*(X^s)$ and $\alpha \neq \mathbb{1}^s$.

The Chen-Ruan product on $H^*(X^s)$ is just the original cup product.

This proposition implies that:

Proposition 4.3. *Suppose at least one of the α_i is a twisted class in the three-point function $\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{X^s}_{CR}$. Then only the following functions are nontrivial*:

$$\langle \mathbb{1}^s_{0,i}, \mathbb{1}^s_{0,j}, x^s \rangle^{X^s}_{CR} = \langle \mathbb{1}^s_{0,i}, x^s_{0,j}, \mathbb{1}^s \rangle^{X^s}_{CR} = \delta_{i+j,r} \frac{1}{r};$$

$$\langle \mathbb{1}^s_{\infty,i}, \mathbb{1}^s_{\infty,j}, x^s \rangle^{X^s}_{CR} = \langle \mathbb{1}^s_{\infty,i}, x^s_{\infty,j}, \mathbb{1}^s \rangle^{X^s}_{CR} = \delta_{i+j,r} \frac{1}{r};$$

$$\langle \mathbb{1}^s_{p^s,i}, \mathbb{1}^s_{p^s,j}, \mathbb{1}^s \rangle^{X^s}_{CR} = \langle \mathbb{1}^s_{q^s,i}, \mathbb{1}^s_{q^s,j}, \mathbb{1}^s \rangle^{X^s}_{CR} = \delta_{i+j,r} \frac{1}{r}.$$

4.1.2. Invariants without insertions. Now we consider the orbifold Gromov-Witten invariants with no insertions.

The top stratum $\mathcal{M}_{0,0,0}(X^s, d[\Gamma^s])$ of the moduli space $\overline{\mathcal{M}}_{0,0,0}(X^s, d[\Gamma^s])$, consists of only "smooth" maps (maps from $\mathbb{P}^1$ to X^s). As well, $\overline{\mathcal{M}}_{0,0,0}(X^s, d[\Gamma^s])$ has a subspace $\overline{\mathcal{M}}^{\text{smooth}}_{0,0,0}(X^s, d[\Gamma^s])$, consisting of all smooth holomorphic curves (whose domains contain no orbifold singularities). This space gives us a compactification of $\mathcal{M}_{0,0,0}(X^s, d[\Gamma^s])$,

and its complement in $\overline{\mathcal{M}}_{0,0,0}(X^s, d[\Gamma^s])$ consists of all other lower strata which have codimension at least 2.

Recall that $X^s = \widetilde{X}^s/\mu_r$ and $\Gamma^s = \widetilde{\Gamma}^s/\mu_r$. We now compare the moduli space $\overline{\mathcal{M}}_{0,0,0}^{\text{smooth}}(X^s, d[\Gamma^s])$ with $\overline{\mathcal{M}}_{0,0}(\widetilde{X}^s, d[\widetilde{\Gamma}^s])$. Note that μ_r acts naturally on the latter space by acting on the image of the stable map. Following the argument in [**3**, subsection 6.3], we have

Lemma 4.4. *For* $d \geq 1$,

$$\overline{\mathcal{M}}_{0,0,0}^{\text{smooth}}(X^s, d[\Gamma^s]) = \begin{cases} \emptyset & \text{if } r \nmid d, \\ \overline{\mathcal{M}}_{0,0}(\widetilde{X}^s, m[\widetilde{\Gamma}^s])/\mu_r & \text{if } d = mr. \end{cases}$$

From Lemma 4.4, we get:

Proposition 4.5. *For* $d \geq 1$, *if* $r \nmid d$, $\langle \ \rangle_{0,0,d[\Gamma^s]}^{X^s}$ *vanishes. Otherwise, if* $d = mr$,

$$\langle \ \rangle_{0,0,d[\Gamma^s]}^{X^s} = \frac{1}{m^3}.$$

Proof. We have

$$\langle \ \rangle_{0,0,d[\Gamma^s]}^{X^s} = \int_{[\overline{\mathcal{M}}_{0,0,0}(X^s, d[\Gamma^s])]^{\text{vir}}} 1 = \int_{[\overline{\mathcal{M}}_{0,0,0}^{\text{smooth}}(X^s, d[\Gamma^s])]^{\text{vir}}} 1.$$

From Lemma 4.4, we know

$$\overline{\mathcal{M}}_{0,0,0}^{\text{smooth}}(X^s, d[\Gamma^s]) = \overline{\mathcal{M}}_{0,0}(\widetilde{X}^s, d[\widetilde{\Gamma}^s])/\mu_r.$$

Then, following the computation in [**3**, subsection 6.4], we get the results.   □

Now suppose $\alpha_i \in H^2(X^s)$, $i = 1, 2, 3$. Then, by the divisor axiom and Proposition 4.5, we have

$$\sum_{m \geqslant 1} q^{mr[\Gamma^s]} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,0,mr[\Gamma^s]}^{X^s} = \alpha_1(r[\Gamma^s])\alpha_2(r[\Gamma^s])\alpha_3(r[\Gamma^s]) \frac{q^{r[\Gamma^s]}}{1 - q^{r[\Gamma^s]}}.$$

4.2. 3-point function on X^{sf}. The Chen-Ruan cohomology of X^{sf} has a basis

- $\mathbb{1}^{sf}, H^{sf}, x^{sf}, H^{sf}x^{sf}, (x^{sf})^2, (x^{sf})^3 = 2H^{sf}(x^{sf})^2$ of $H^*(X^{sf})$, and
- $\mathbb{1}^{sf}_{0,k}, x^{sf}_{0,k}$ of $H^*([L^{sf}_0]_k)$, $1 \le k \le r-1$, and
- $\mathbb{1}^{sf}_{\infty,k}, x^{sf}_{\infty,k}$ of $H^*([L^{sf}_\infty]_k)$, $1 \le k \le r-1$, and
- $\mathbb{1}^{sf}_{p^{sf},k}$ of $H^*([p^{sf}]_k)$, $1 \le k \le r-1$ and
- $\mathbb{1}^{sf}_{q^{sf},k}$ of $H^*([q^{sf}]_k)$, $1 \le k \le r-1$.

Here x^{sf} is the first Chern class of the hyperplane bundle over X^{sf} and H^{sf} is a generator of $H^2(\Gamma^{sf})$. We will also call those classes in $H^*_{CR}(X^{sf}) \backslash H^*(X^{sf})$ twisted classes and the others non-twisted classes. We normalize H^{sf} such that

$$\int_{\Gamma^{sf}}^{\mathrm{orb}} H^{sf} = 1.$$

We also have $\int_{\Gamma^{sf}}^{\mathrm{orb}} x^{sf} = 0$.

Following the argument for X^s we have:

Proposition 4.6. *The three-point function* $F^{sf}(\beta_1, \beta_2, \beta_3)$ *is determined by* $\langle \beta_1, \beta_2, \beta_3 \rangle^{X^{sf}}_{CR}$ *and n-point degree* $d \ge 1$ *orbifold Gromov-Witten invariants with the following insertions*:
(4.2)
$$\begin{cases}
(\emptyset) \\
(\mathbb{1}^{sf}, H^{sf}x^{sf}) \\
(\mathbb{1}^{sf}, \mathbb{1}^{sf}, (x^{sf})^3) \\
(\beta_1) & \beta_1 = \mathbb{1}^{sf}_{0,k} \text{ or } \mathbb{1}^{sf}_{\infty,k}, \\
(\beta_1, \mathbb{1}^{sf}, H^{sf}x^{sf}) & \beta_1 = \mathbb{1}^{sf}_{0,k} \text{ or } \mathbb{1}^{sf}_{\infty,k}, \\
(\beta_1, \mathbb{1}^{sf}, (x^{sf})^2) & \beta_1 = \mathbb{1}^{sf}_{0,k} \text{ or } \mathbb{1}^{sf}_{\infty,k}, \\
(\beta_1, \mathbb{1}^{sf}) & \beta_1 = x^{sf}_{0,k} \text{ or } x^{sf}_{\infty,k}, \\
(\beta_1, \beta_2) & \beta_1, \beta_2 \in \{\mathbb{1}^{sf}_{0,k}, \mathbb{1}^{sf}_{\infty,k}\}^{r-1}_{k=1}, \\
(\beta_1, \beta_2, \mathbb{1}^{sf}) & \beta_1, \beta_2 \in \{\mathbb{1}^{sf}_{0,k}, \mathbb{1}^{sf}_{\infty,k}, x^{sf}_{0,k}, x^{sf}_{\infty,k}\}^{r-1}_{k=1}, \deg\beta_1 < \deg\beta_2, \\
(\beta_1, \beta_2, \beta_3) & \beta_i \in \{\mathbb{1}^{sf}_{0,k}, \mathbb{1}^{sf}_{\infty,k}\}^{r-1}_{k=1}, i = 1, 2, 3, \\
(\beta_1, \beta_2, \beta_3) & \text{at least one of } \beta_i \text{ belongs to } \{\mathbb{1}^{sf}_{p^{sf},k}, \mathbb{1}^{sf}_{q^{sf},k}\}^{r-1}_{k=1}.
\end{cases}$$

And, for $\langle \beta_1, \beta_2, \beta_3 \rangle^{X^{sf}}_{CR}$, *when at least one of* β_i *is twisted, only the*

following functions are nontrivial:

$$\langle \mathbb{1}_{0,i}^{sf}, \mathbb{1}_{0,j}^{sf}, x^{sf} \rangle_{CR}^{X^{sf}} = \langle \mathbb{1}_{0,i}^{sf}, x_{0,j}^{sf}, \mathbb{1}^{sf} \rangle_{CR}^{X^{sf}} = \delta_{i+j,r} \frac{1}{r},$$

$$\langle \mathbb{1}_{\infty,i}^{sf}, \mathbb{1}_{\infty,j}^{sf}, x^{sf} \rangle_{CR}^{X^{sf}} = \langle \mathbb{1}_{\infty,i}^{sf}, x_{\infty,j}^{sf}, \mathbb{1}^{sf} \rangle_{CR}^{X^{sf}} = \delta_{i+j,r} \frac{1}{r},$$

$$\langle \mathbb{1}_{p^{sf},i}^{sf}, \mathbb{1}_{p^{sf},j}^{sf}, \mathbb{1}^{sf} \rangle_{CR}^{X^{sf}} = \langle \mathbb{1}_{q^{sf},i}^{sf}, \mathbb{1}_{q^{sf},j}^{sf}, \mathbb{1}^{sf} \rangle_{CR}^{X^{sf}} = \delta_{i+j,r} \frac{1}{r}.$$

We also have

$$\langle \ \rangle_{0,0,d[\Gamma^{sf}]}^{X^{sf}} = \begin{cases} 0 & \text{if } r \nmid d, \\ 1/m^3 & \text{if } d = mr. \end{cases}$$

We will compute the remaining invariants with insertions in equations (4.1) and (4.2) in the next section by using the virtual localization technique.

5. Localization and orbifold Gromov-Witten invariants with insertions from (4.1) and (4.2).

5.1. Torus action. We introduce a T^2-action on $\widetilde{V}$:

$$(t_1, t_2) \cdot (x, y, z, w) = (t_1 t_2 x, t_1^{-1} t_2 y, t_2 z, t_2 w), \quad (t_1, t_2) \in T^2.$$

This T^2-action extends naturally to the two small resolutions $\widetilde{V}^s$ and $\widetilde{V}^{sf}$ of $\widetilde{V}$, and hence to $\widetilde{X}^s = \mathbb{P}(\widetilde{V}^s \oplus \mathcal{O})$ and $\widetilde{X}^{sf} = \mathbb{P}(\widetilde{V}^{sf} \oplus \mathcal{O})$ with trivial action on the trivial bundle. The resulting T^2-action commutes with the μ_r-action on $\widetilde{X}^s$ and $\widetilde{X}^{sf}$. Hence, we get a T^2-action on X^s and X^{sf}, respectively.

The fixed points of the T^2-action on X^s are the three special points in F_0^s, the fiber of $X^s = \mathbb{P}(V^s \oplus \mathcal{O})$ over 0^s, and the three special points in F_∞^s, the fiber of $X^s = \mathbb{P}(V^s \oplus \mathcal{O})$ over ∞^s. In addition, the fixed lines connecting these fixed points are projective lines in F_0^s and F_∞^s, and Γ^s. The fixed points of the T^2-action on X^{sf} are the three special points in F_0^{sf}, the fiber of $X^{sf} = \mathbb{P}(V^{sf} \oplus \mathcal{O})$ over 0^{sf}, and the three special points in F_∞^{sf}, the fiber of $X^{sf} = \mathbb{P}(V^{sf} \oplus \mathcal{O})$ over ∞^{sf}. In addition, the fixed lines connecting these fixed points are the projection lines in F_0^{sf} and F_∞^{sf}, and Γ^{sf}. Note that the degree 2 homology classes represented by those projective lines do not lie in the subgroup generated by $[\Gamma^s]$ (respectively, $[\Gamma^{sf}]$).

We denote by $[0^s]_k \subset [L_0^s]_k$ the twisted sector corresponding to $0^s \subset L_0^s$. The same notation applies to $[\infty^s]_k$, $[0^{sf}]_k$ and $[\infty^{sf}]_k$.

In the following, we focus on X^s. All computations on X^{sf} are parallel.

5.2. Fixed loci components and their virtual normal bundles.
For a map f from an irreducible genus 0 (orbi-)curve $\mathcal{C}$ to X^s, we have:

Lemma 5.1. *Suppose that $d \geq 1$, and $f : \mathcal{C} \to X^s$ is a degree $d[\Gamma^s]$ J-holomorphic map with two marked points which is invariant with respect to the T^2-action, then either*:

(i) $\mathcal{C} = \mathbb{P}^1$, *and the degree satisfies* $d \equiv 0 \pmod r$, *or*
(ii) $\mathcal{C} \cong \Gamma^s$. *The two marked points are just the two orbifold points of $\mathcal{C}$ and mapped to $[0^s]_k, [\infty^s]_{r-k}$, respectively. The degree satisfies $d \equiv -k \pmod r$. This map is realized by*

$$
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{\quad d \quad} & \tilde{\Gamma}^s \\
\downarrow{\scriptstyle /\mu_r} & & \downarrow{\scriptstyle /\mu_r} \\
\mathcal{C} & \xrightarrow{\quad d \quad} & \Gamma^s.
\end{array}
$$

The corresponding group homomorphism $\mu_r \to \mu_r$ is given by: $\xi \mapsto \xi^{-k}$.

For every connected component of the fixed loci in $\overline{\mathcal{M}}_{0,l,k}(X^s, d, \times)$, we can assign a labeled graph T to it. From Lemma 5.1, a vertex $v \in V_T$ corresponds to a connected component $\mathcal{C}_v$ of $f^{-1}\{0^s, \infty^s\}$. Note that each $\mathcal{C}_v$ can be either a point of $\mathcal{C}$ or a non-empty union of irreducible components of $\mathcal{C}$. An edge $e \in E_T$ corresponds to a irreducible component $\mathcal{C}_e$ of genus 0 mapped to Γ^s. We endow T with additional specifications: the vertex v will be labeled by $p_v = 0^s$ or ∞^s; the set $S_v \subset \{x_1, \ldots, x_l, y_1, \ldots, y_k\}$ of marked points lying on $\mathcal{C}_v$; and the nodes N_v on $\mathcal{C}_v$ which connect $\mathcal{C}_v$ and $\mathcal{C}_e$ for edges incident to v. The edges will be labeled by the degrees $d_e \in \mathbb{N}$. The valency of v is defined as $\mathrm{val}(v) = \#N_v$, and we denote $\#S_v$ by $h(v)$.

Then the connected components of $\overline{\mathcal{M}}_{0,l,k}(X^s, d, \times)^{T^2}$ are naturally labeled by equivalence classes of labeled connected graphs T with specifications obeying the following conditions:

- if $e \in E_T$ connects vertices $v, u \in V_T$ then $p_v \neq p_u$;
- $\sum_{e \in E_T} d_e = d$;
- $\{x_1, \cdots, x_l, y_1, \ldots, y_k\} = \coprod_{v \in V_T} S_v$.

Each component $\overline{\mathcal{M}}_T$ is isomorphic to the quotient space of the product of moduli spaces of stable curves over the set of vertices of T with orbifold structure recorded by S_v, N_v and the balance condition, modulo the action of the natural automorphism group specified by

$$1 \longrightarrow \prod_{E_T} \mathbb{Z}_e \longrightarrow \mathbf{A}_T \longrightarrow \mathrm{Aut}(T) \longrightarrow 1,$$

where $\mathrm{Aut}\,(T)$ is the automorphic group of T (as a labeled graph), and $\mathbb{Z}_e = \mathbb{Z}_{d_e}$.

A flag $F = (v, e)$ consists of a vertex v and an edge e with v being a vertex of e.

Let $\mathfrak{G}$ be the collection of graphs.

From [**9**], we get an exact sequence:

$$0 \longrightarrow H^0(\mathcal{C}, T\mathcal{C}) \longrightarrow H^0(\mathcal{C}, f^*TX^s) \longrightarrow \mathcal{T}^0$$
$$\longrightarrow H^1(\mathcal{C}, T\mathcal{C}) \longrightarrow H^1(\mathcal{C}, f^*TX^s) \longrightarrow \mathcal{T}^1 \longrightarrow 0,$$

where $\mathcal{T}^0 - \mathcal{T}^1$ is the virtual tangent bundle of $\overline{\mathcal{M}}_{0,l,k}(X^s, d[\Gamma^s], \times)$ in K-theory and is T^2-equivariant.

Write $E = f^*TX^s$. In terms of cohomological data, the equivariant Euler class of the virtual normal bundle $N^{\mathrm{vir}}_{\overline{\mathcal{M}}_T}$ is given by

$$\frac{1}{e_{T^2}(N^{\mathrm{vir}}_{\overline{\mathcal{M}}_T})} = \frac{e_{T^2}(H^0(\mathcal{C}, T\mathcal{C})^m) \cdot e_{T^2}(H^1(\mathcal{C}, E)^m)}{e_{T^2}(H^1(\mathcal{C}, T\mathcal{C})^m) \cdot e_{T^2}(H^0(\mathcal{C}, E)^m)}.$$

Here, the superscript m denotes the moving part. The explicit form is (cf., [**9**])

$$(5.1) \quad \frac{1}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})} = \prod_{\substack{\mathrm{val}(v)=1 \\ h(v)=0}} \omega_F \cdot \prod_{\substack{F=(v,e) \\ \mathrm{val}(v)+h(v) \geqslant 3}} \frac{1}{\omega_F - \psi_F}$$

$$\cdot \prod_{\substack{\mathrm{val}(v)=2 \\ h(v)=0}} \frac{1}{\omega_{F_{v,1}} + \omega_{F_{v,2}}} \cdot \prod_{V_T} e_{T^2}(E)^{\mathrm{val}(v)}$$

$$\cdot \frac{\prod_{V_T} e_{T^2}(H^1(\mathcal{C}_v, E)^m) \cdot \prod_{E_T} e_{T^2}(H^1(\mathcal{C}_e, E)^m)}{\prod_{V_T} e_{T^2}(H^0(\mathcal{C}_v, E)^m) \cdot \prod_{E_T} e_{T^2}(H^0(\mathcal{C}_e, E)^m)}.$$

Since there may be an orbifold structure on the domain curve $\mathcal{C}$, in the formula above, we should take the invariant subspace of some certain finite group.

5.3. Vanishing results. Denote by λ, u the equivariant factors corresponding to the two factors in T^2, respectively. Now for classes $\alpha_j \in H^*_{CR}(X^s)$, $1 \leq j \leq n = l + k$, let

$$\Omega = \prod_j ev_j^*(\alpha_{j,T^2}),$$

where α_{j,T^2} is the equivariant lifting of α_j. By virtual localization, we can write $\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d}^{X^s}$ as

$$I(\Omega) = I^\Omega(\lambda, u) \triangleq \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d}^{X^s}$$

$$= \sum_{T \in \mathfrak{G}} \frac{1}{|\mathbf{A}_T|} \int_{[\overline{\mathcal{M}}_T]^{\mathrm{vir}}} \frac{i_T^*(\Omega)}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})}$$

$$= \sum_{T \in \mathfrak{G}} I_T(\lambda, u),$$

where $i_T : \overline{\mathcal{M}}_T \to \overline{\mathcal{M}}_{0,0,k}(X^s, d, \mathsf{x})$ is the natural inclusion of the fixed loci component. Since the left hand side of the above equality is independent of u, we have

$$I(\Omega) = \lim_{u \to 0} I^\Omega(\lambda, u) = \sum_{T \in \mathfrak{G}} \lim_{u \to 0} I_T(\lambda, u).$$

Lemma 5.2. *Suppose $n \geq 1$ and $d \geq 1$. If T contains more than one edge,*

$$\lim_{u \to 0} I_T(\lambda, u) = 0.$$

Proof. Note that t_2, the second factor of T^2, acts on Γ^s trivially. Hence, the induced t_2-action on the domain curve $\mathcal{C}$ is trivial. Hence, the factor

$$\frac{e_{T^2}(H^0(\mathcal{C}, T\mathcal{C})^m)}{e_{T^2}(H^1(\mathcal{C}, T\mathcal{C})^m)},$$

of $1/e_{T^2}(N^{\mathrm{vir}}_{\overline{\mathcal{M}}_T})$ contains no factor u. Now we look at the factor $e_{T^2}(H^1(\mathcal{C}, E)^m)/e_{T^2}(H^0(\mathcal{C}, E)^m)$, where $E = f^* T X^s$. On $\mathcal{C}$, we have (cf., [**9**])

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \bigoplus_{v \in V_T} \mathcal{O}_{\mathcal{C}_v} \oplus \bigoplus_{e \in E_T} \mathcal{O}_{\mathcal{C}_e} \longrightarrow \bigoplus_{F \in F_T} \mathcal{O}_{\mathcal{C}_F} \longrightarrow 0.$$

Then we have

$$0 \longrightarrow H^0(\mathcal{C}, E) \longrightarrow \bigoplus_{v \in V_T} H^0(\mathcal{C}_v, E)$$

$$\oplus \bigoplus_{e \in E_T} H^0(\mathcal{C}_e, E)$$

$$\longrightarrow \bigoplus_{F \in F_T} E_{p(F)} \longrightarrow H^1(\mathcal{C}, E)$$

$$\longrightarrow \bigoplus_{v \in V_T} H^1(\mathcal{C}_v, E) \oplus \bigoplus_{e \in E_T} H^1(\mathcal{C}_e, E) \longrightarrow 0.$$

Note that, when $\mathcal{C}$ has an orbifold structure, these spaces are the invariant subspaces of a certain finite group. From this exact sequence, we get

$$\frac{e_{T^2}(H^1(\mathcal{C}, E)^m)}{e_{T^2}(H^0(\mathcal{C}, E)^m)}$$

$$= \frac{e_{T^2}(\bigoplus_{F \in F_T} E_{p(F)}) \cdot e_{T^2}(\bigoplus_{v \in V_T} H^1(\mathcal{C}_v, E)^m) \cdot e_{T^2}(\bigoplus_{e \in E_T} H^1(\mathcal{C}_e, E)^m)}{e_{T^2}(\bigoplus_{v \in V_T} H^0(\mathcal{C}_v, E)^m) \cdot e_{T^2}(\bigoplus_{e \in E_T} H^0(\mathcal{C}_e, E)^m)}.$$

Observe that, for every edge e, $H^0(\mathcal{C}_e, E)^m = H^0(\mathcal{C}_e, f^* T\Gamma^s)^m$. Hence, the term $e_{T^2}(\bigoplus_{e \in E_T} H^0(\mathcal{C}_e, E)^m)$ contains no power of u as a factor.

By the assumption, T contains at least 2 edges, V_T must contains a vertex v such that $\mathrm{val}(v) \geq 2$. For this $\mathcal{C}_v$, if $p_v = 0^s$, then since the μ_r-action on $\mathbb{C}_w$ is trivial, the t_2-action on $\mathbb{C}_w$ gives us a factor

$u^{\mathrm{val}(v)-1}$ coming from

$$\frac{e_{T^2}(\bigoplus_{F=(v,e)} E_{p(F)})}{e_{T^2}(H^0(\mathcal{C}_v, E)^m)}.$$

If $p_v = \infty^s$, then the μ_r-action on $E_{p_v} = \mathbb{C}_z$ is trivial, and the t_2-action on $E_{p_v} = \mathbb{C}_z$ gives us a factor $u^{\mathrm{val}(v)-1}$.

Hence, $e_{T^2}(H^0(\mathcal{C}, E)^m)/e_{T^2}(H^1(\mathcal{C}, E)^m)$ contains a positive power of u as a factor. Therefore,

$$\lim_{u \to 0} I_T(\lambda, u) = 0.$$

This completes the proof.   □

5.4. Orbifold Gromov-Witten invariants of X^s with degree $d \geq 1$.
In this section, we compute orbifold Gromov-Witten invariants with insertion in equation (4.1). First note that the T^2-equivariant Chen-Ruan cohomology of X^s is defined as the T^2-equivariant cohomology of the inertia orbifold ΛX^s. We observe that:

Lemma 5.3. *Suppose that $x^s_{T^2}$, x^s_{0,k,T^2}, x^s_{∞,k,T^2}, $\mathbb{1}^s_{p^s,k,T^2}$ and $\mathbb{1}^s_{q^s,k,T^2}$ are the equivariant lifting of x^s, $x^s_{0,k}$, $x^s_{\infty,k}$, $\mathbb{1}^s_{p^s,k}$ and $\mathbb{1}^s_{q^s,k}$, respectively. Then we have*

$$\begin{cases} i^*_{[0^s]_j,T^2}(x^s_{T^2}) = i^*_{[\infty^s]_j,T^2}(x^s_{T^2}) = 0 \\ i^*_{[0^s]_j,T^2}(x^s_{0,k,T^2}) = i^*_{[\infty^s]_j,T^2}(x^s_{\infty,k,T^2}) = 0 & 1 \leq k \leq r-1, \\ i^*_{[0^s]_j,T^2}(\mathbb{1}^s_{p^s,k,T^2}) = i^*_{[\infty^s]_j,T^2}(\mathbb{1}^s_{q^s,k,T^2}) = 0 & 1 \leq k \leq r-1, \end{cases}$$

for any $0 \leq j \leq r-1$, where $[0^s]_0 = 0^s$ and $[\infty^s]_0 = \infty^s$.

For a fixed component $\overline{\mathcal{M}}_T$ of $\overline{\mathcal{M}}_{0,l,k}(X^s, d, \mathsf{x})$, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,l,k}(X^s, d, \mathsf{x}) & \xrightarrow{ev_j} & \Lambda X^s \\ {\scriptstyle i_T} \uparrow & & \uparrow {\scriptstyle i} \\ \overline{\mathcal{M}}_T & \xrightarrow{ev_j} & p_j, \end{array}$$

where p_j is the image of the jth marked point, hence a fixed point in ΛX^s with respect to the T^2-action. So, by Lemma 5.1, we have $p_j = [0^s]_k$ or $[\infty^s]_k$, $0 \leq k \leq r - 1$.

Now, for the invariant $\langle \alpha_1, \ldots, \alpha_n \rangle^{X^s}_{0,n,d}$ with at least one of α_j being one of

$$\{x^s\} \cup \{x^s_{0,k}, x^s_{\infty,k}, \mathbb{1}^s_{p^s,k}, \mathbb{1}^s_{q^s,k} \mid 1 \leq k \leq r - 1\},$$

by Lemma 5.3, we have

$$i^*_T(\Omega) = i^*_T\left(\prod_j ev^*_j \alpha_{j,T^2}\right) = \prod_j ev^*_j(i^*_{p_j} \alpha_{j,T^2}) = 0.$$

This implies that

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{X^s}_{0,n,d} = \sum_{T \in \mathfrak{G}} I_T(\lambda, u) = 0,$$

if at least one of α_j belongs to

$$\{x^s\} \cup \{x^s_{0,k}, x^s_{\infty,k}, \mathbb{1}^s_{p^s,k}, \mathbb{1}^s_{q^s,k} \mid 1 \leq k \leq r - 1\}.$$

Therefore, to compute invariants with insertions in (4.1), we only have to deal with the following insertions:

$$\begin{cases} (\alpha_1) & \alpha_1 = \mathbb{1}^s_{0,k} \text{ or } \mathbb{1}^s_{\infty,k}, \\ (\alpha_1, \alpha_2) & \alpha_1, \alpha_2 \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k} \mid 1 \leq k \leq r - 1\}, \\ (\alpha_1, \alpha_2, \alpha_3) & \alpha_i \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k} \mid 1 \leq k \leq r - 1\}, \quad i = 1, 2, 3. \end{cases}$$

Lemmas 5.1 and 5.2 imply that we only have to consider one-edge graphs, and the corresponding stable map on $\mathcal{C}_e$ is a degree $d \geq 1$ covering of Γ^s. So the domain curve must have at least two orbifold points, hence at least two marked points. Therefore,

$$\langle \alpha_1 \rangle^{X^s}_{0,1,d} = 0,$$

for twisted class $\alpha_1 \in H^*_{CR}(X^s)$ and $d \geq 1$.

We next consider 2-point and 3-point Gromov-Witten invariants on twisted classes.

5.4.1. $\langle \alpha_1, \alpha_2 \rangle^{X^s}_{0,2,d}$ with $\alpha_1, \alpha_2 \in \{\mathbb{1}^s_{0,k}, \mathbb{1}^s_{\infty,k} \mid 1 \leq k \leq r - 1\}$ and $d \geq 1$. For this case, by Lemmas 5.1 and 5.2, to get nonzero invariants, we only have to consider the following two cases:

- $\alpha_1 = \mathbb{1}_{0,i}^s, \alpha_2 = \mathbb{1}_{\infty,r-i}^s$, with $d \equiv -i \pmod{r}$, or
- $\alpha_1 = \mathbb{1}_{\infty,i}^s, \alpha_2 = \mathbb{1}_{0,r-i}^s$, with $d \equiv i \pmod{r}$.

Since Gromov-Witten invariants are antisymmetric on insertions and $\deg \alpha_j = 0$, we have

$$\langle \mathbb{1}_{0,i}^s, \mathbb{1}_{\infty,r-i}^s \rangle_{0,2,d}^{X^s} = \langle \mathbb{1}_{\infty,r-i}^s, \mathbb{1}_{0,i}^s \rangle_{0,2,d}^{X^s}.$$

Therefore, we only have to compute the first case.

For the first case, the moduli space $\overline{\mathcal{M}}_{0,0,2}(X^s, d, \mathsf{x})$ contains one component of the fixed locus, which is indexed by the one-edge graph $T = (V_T, E_T)$ with $V_T = \{v_1, v_2\}$, $E = \{e\}$. In addition, the domain curve $\mathcal{C}$ is just Γ^s with two marked points being the two orbifold points $p(v_1) = 0^s$ and $p(v_2) = \infty^s$. We also have $d_e = d$. The stable map $f : \mathcal{C} \to \Gamma^s$ is realized by:

$$
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{\tilde{f}} & \tilde{\Gamma}^s \\
{\scriptstyle /\mu_r} \downarrow & & \downarrow {\scriptstyle /\mu_r} \\
\mathcal{C} & \xrightarrow{f} & \Gamma^s
\end{array}
$$

The corresponding homomorphism $\mu_r \to \mu_r$ is $\xi \mapsto \xi^{-i}$. From equation (5.1), we know that the T^2-equivariant Euler class of the virtual normal bundle of this fixed component is

$$\frac{1}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})} = \frac{e_{T^2}(H^1(\mathcal{C}, f^*TX^s)^m)}{e_{T^2}(H^0(\mathcal{C}, f^*TX^s)^m)}.$$

Note that

$$H^i(\mathcal{C}, f^*TX^s) = \left(H^i(\mathbb{P}^1, \tilde{f}^*T\tilde{X}^s) \right)^{\mu_r}$$

for $i = 0, 1$, and $\tilde{f}$ is a degree d covering. The normal bundle of Γ^s in X^s is $L_1^s \oplus L_2^s = V^s \to \Gamma^s$. Hence,

$$H^0(\mathcal{C}, f^*TX^s) = H^0(\mathcal{C}, f^*T\Gamma^s),$$

and

$$H^1(\mathcal{C}, f^*TX^s) = H^1(\mathcal{C}, f^*L_1^s \oplus f^*L_2^s).$$

Suppose $d = ar + r - i, a \geq 0$. Then we have

$$\frac{1}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})} = \frac{e_{T^2}(H^1(\mathcal{C}, f^*L_1^s \oplus f^*L_2^s)^m)}{e_{T^2}(H^0(\mathcal{C}, f^*T\Gamma^s)^m)}$$

$$= \frac{\prod_{s=1}^{a}((sr/d)\lambda + u) \cdot \prod_{s=1}^{a}((-sr/d)\lambda + u)}{[(-1)^a(a!)^2 r^{2a}\lambda^{2a}]/d^{2a}}.$$

Remark 5.4. Note that

$$H^1(\mathcal{C}, f^*L_1^s \oplus f^*L_2^s) = H^1(\mathbb{P}^1, \widetilde{f}^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))^{\mu_r},$$

and

$$\widetilde{f}^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) = \mathcal{O}(-d) \oplus \mathcal{O}(-d)$$

over $\mathbb{P}^1$. To compute $H^1(\mathbb{P}^1, \widetilde{f}^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))$, we use the *Serre duality*:

$$H^1(\mathbb{P}^1, \mathcal{O}(-d) \oplus \mathcal{O}(-d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(d-1) \oplus \mathcal{O}(d-1))^{\vee},$$

where $\vee$ stands for dual. Then we take the μ_r-invariant subspace with respect to the induced μ_r-action to get $H^1(\mathcal{C}, f^*L_1^s \oplus f^*L_2^s)$.

For $\Omega = ev_1^*(\mathbb{1}_{0,i,T^2}^s) \wedge ev_2^*(\mathbb{1}_{\infty,r-i,T^2}^s)$, we have

$$i_T^*(\Omega_{T^2}) = 1.$$

Note that $|\mathbf{A}_T| = d$. Summarizing, we get

$$\langle \mathbb{1}_{0,i}^s, \mathbb{1}_{\infty,r-i}^s \rangle_{0,2,d}^{X^s} = \lim_{u \to 0} \frac{1}{d} \cdot \frac{\prod_{s=1}^{a}((sr/d)\lambda + u) \cdot \prod_{s=1}^{a}((-sr)/d\lambda + u)}{[(-1)^a(a!)^2 r^{2a}\lambda^{2a}]/d^{2a}}$$

$$= \frac{1}{d} \cdot \frac{\prod_{s=1}^{a}(sr/d)\lambda \cdot \prod_{s=1}^{a}(-sr)/d\lambda}{[(-1)^a(a!)^2 r^{2a}\lambda^{2a}]/d^{2a}} = \frac{1}{d}.$$

5.4.2. $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d}^{X^s}$ with $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbb{1}_{0,k}^s, \mathbb{1}_{\infty,k}^s \mid 1 \leq k \leq r-1\}$ and $d \geq 1$. By Lemmas 5.1 and 5.2, we only have to consider the following two cases:

(1) $\alpha_1 = \mathbb{1}_{0,i}^s, \alpha_2 = \mathbb{1}_{0,j}^s, \alpha_3 = \mathbb{1}_{\infty,k}^s, d \equiv k \pmod{r}$, and
(2) $\alpha_1 = \mathbb{1}_{0,i}^s, \alpha_2 = \mathbb{1}_{\infty,j}^s, \alpha_3 = \mathbb{1}_{\infty,k}^s, d \equiv r - i \pmod{r}$.

with $i + j + k = r$ or $2r$.

Case (1). For this case, there is only one fixed component indexed by a one-edge graph T with

$$V_T = \{v_1, v_2\} \quad \text{and} \quad E_T = \{e\}.$$

The domain curve is $\mathcal{C} = \mathcal{C}_{v_1} \cup \mathcal{C}_e$. The first two marked points sit on $\mathcal{C}_{v_1}$. We have

$$|\mathbf{A}_T| = d \quad \text{with } d = ar + k.$$

Denote by $F = (v_1, e)$ the flag with vertex v_1. Then, from (5.1) and the analysis in subsection 5.4.1 we have

$$\frac{1}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})} = \frac{1}{\omega_F}$$

$$\cdot \frac{e_{T^2}(H^1(\mathcal{C}_{v_1}, f^*(L_1^s \oplus L_2^s \oplus T\Gamma^s))^m) \cdot e_{T^2}(H^1(\mathcal{C}_e, f^*(L_1^s \oplus L_2^s))^m)}{e_{T^2}(H^0(\mathcal{C}_e, f^*T\Gamma^s)^m)}.$$

For $H^1(\mathcal{C}_{v_1}, f^*(L_1^s \oplus L_2^s \oplus T\Gamma^s))$, we have (cf., [**1**, subsection 3.4])

$$H^1(C_{v_1}, f^*(L_1^s \oplus L_2^s \oplus T\Gamma^s)) = \begin{cases} T_{0^s}\Gamma^s & i + j + k = r, \\ L_1^s|_{0^s} & i + j + k = 2r. \end{cases}$$

For this case, we also have $i_T^*(\Omega_{T^2}) = 1$. Hence,

$$\langle \mathbb{1}_{0,i}^s, \mathbb{1}_{0,j}^s, \mathbb{1}_{\infty,k}^s \rangle_{0,3,d}^{X^s}$$

$$= \lim_{u \to 0} \frac{1}{d} \cdot \frac{i_T^*(\Omega_{T^2})}{e_{T^2}(N_{\overline{\mathcal{M}}_T}^{\mathrm{vir}})}$$

$$= \lim_{u \to 0} \frac{1}{d} \cdot \frac{e_{T^2}(H^1(\mathcal{C}_{v_1}, f^*(L_1^s \oplus L_2^s \oplus T_{0^s}\Gamma^s)))}{\omega_F}$$

$$\cdot \frac{e_{T^2}(H^1(\mathcal{C}_e, f^*(L_1^s \oplus L_2^s))^m)}{e_{T^2}(H^0(\mathcal{C}_e, f^*T\Gamma^s)^m)}$$

$$= \begin{cases} \lim_{u \to 0} \frac{1}{d} \cdot \frac{e_{T^2}(T_{0^s}\Gamma^s) \cdot e_{T^2}(H^1(\mathcal{C}_e, f^*(L_1^s \oplus L_2^s)))}{\omega_F \cdot e_{T^2}(H^0(\mathcal{C}_e, f^*T\Gamma^s)^m)} & i + j + k = r, \\ \lim_{u \to 0} \frac{1}{d} \cdot \frac{e_{T^2}(L_1^s|_{0^s}) \cdot e_{T^2}(H^1(\mathcal{C}_e, f^*(L_1^s \oplus L_2^s)))}{\omega_F \cdot e_{T^2}(H^0(\mathcal{C}_e, f^*T\Gamma^s)^m)} & i + j + k = 2r. \end{cases}$$

$$= \begin{cases} \lim_{u \to 0} \frac{1}{d} \cdot \frac{-\lambda \cdot \prod_{s=1}^a (\frac{sr}{d}\lambda + u) \cdot \prod_{s=1}^a ((-sr/d)\lambda + u)}{(-\lambda/d) \cdot [(-1)^a (a!)^2 r^{2a} \lambda^{2a}]/d^2a} & i + j + k = r, \\ \lim_{u \to 0} \frac{1}{d} \cdot \frac{(\lambda + u) \cdot \prod_{s=1}^a ((sr/d)\lambda + u) \cdot \prod_{s=1}^a ((-sr/d)\lambda + u)}{(-\lambda/d) \cdot [(-1)^a (a!)^2 r^{2a} \lambda^{2a}]/d^2a} & i + j + k = 2r. \end{cases}$$

$$= \begin{cases} 1 & i + j + k = r, \\ -1 & i + j + k = 2r. \end{cases}$$

Case (2). Similar to Case (1), we have

$$\langle \mathbb{1}_{0,i}^s, \mathbb{1}_{\infty,j}^s, \mathbb{1}_{\infty,k}^s \rangle_{0,3,d}^{X^s} = \begin{cases} -1 & i+j+k = r, \\ 1 & i+j+k = 2r. \end{cases}$$

5.5. Orbifold Gromov-Witten invariants of X^{sf} with degree $d \geq 1$. We can also apply the virtual localization technique to compute $\langle \beta_1, \ldots, \beta_n \rangle_{0,n,d}^{X^{sf}}$, with $(\beta_1, \ldots, \beta_n)$ being one of equation (4.2) and $d \geq 1$. We have

$$\langle \beta_1, \ldots, \beta_n \rangle_{0,n,d}^{X^{sf}}$$

$$= \begin{cases} 1/d & n = 2, \text{ and } \beta_1 = \mathbb{1}_{0,i}^{sf}, \ \beta_2 = \mathbb{1}_{\infty,r-i}^{sf}, \\ 1/d & n = 2, \text{ and } \beta_1 = \mathbb{1}_{\infty,i}^{sf}, \ \beta_2 = \mathbb{1}_{0,r-i}^{sf}, \\ 1 & n = 3, \text{ and } \beta_1 = \mathbb{1}_{0,i}^{sf}, \ \beta_2 = \mathbb{1}_{0,j}^{sf}, \ \beta_3 = \mathbb{1}_{\infty,r-i-j}^{sf}, \\ -1 & n = 3, \text{ and } \beta_1 = \mathbb{1}_{0,i}^{sf}, \ \beta_2 = \mathbb{1}_{0,j}^{sf}, \ \beta_3 = \mathbb{1}_{\infty,2r-i-j}^{sf}, \\ -1 & n = 3, \text{ and } \beta_1 = \mathbb{1}_{0,i}^{sf}, \ \beta_2 = \mathbb{1}_{\infty,j}^{sf}, \ \beta_3 = \mathbb{1}_{\infty,r-i-j}^{sf}, \\ 1 & n = 3, \text{ and } \beta_1 = \mathbb{1}_{0,i}^{sf}, \ \beta_2 = \mathbb{1}_{\infty,j}^{sf}, \ \beta_3 = \mathbb{1}_{\infty,2r-i-j}^{sf}, \\ 0 & \text{otherwise}, \end{cases}$$

for $d \geq 1$.

6. Isomorphism between Ruan cohomology rings of X^s and X^{sf}. In this section, we give an additive homomorphism $\phi : H_{CR}^*(X^s) \to H_{CR}^*(X^{sf})$ between the Chen-Ruan cohomology groups, which preserves the orbifold Poincare pairing. Then we show that, under the identification

$$[\Gamma^s] \longleftrightarrow -[\Gamma^{sf}],$$

i.e.,

$$q \longleftrightarrow q^{-1},$$

we can identify the three-point functions F^s and F^{sf}.

6.1. Isomorphism between Chen-Ruan cohomology groups of X^s and X^{sf}. In this section we define an additive homomorphism between the Chen-Ruan cohomology of X^s and the Chen-Ruan cohomology of X^{sf}.

We define a map:

$$\phi : H^*_{CR}(V^s_r) \longrightarrow H^*_{CR}(V^{sf}_r).$$

On the twisted classes, we define

$$\phi(\mathbb{1}^s_{0,k}) = \mathbb{1}^{sf}_{0,r-k}, \qquad \phi(\mathbb{1}^s_{\infty,k}) = \mathbb{1}^{sf}_{\infty,r-k},$$

$$\phi(x^s_{0,k}) = x^{sf}_{0,r-k}, \qquad \phi(x^s_{\infty,k}) = x^{sf}_{\infty,r-k},$$

and

$$\phi(\mathbb{1}^s_{p^s,k}) = \mathbb{1}^{sf}_{p^{sf},r-k}, \qquad \phi(\mathbb{1}^s_{q^s,k}) = \mathbb{1}^{sf}_{q^{sf},r-k}.$$

On $H^*(X^s)$, for degree 0 and 6 forms, ϕ is defined in an obvious way. For $\alpha \in H^2(X^s)$, $\phi(\alpha)$ is defined to be the unique extension of

$$\alpha|_{X^s-\Gamma^s} = \alpha|_{X^{sf}-\Gamma^{sf}}$$

over X^{sf}. For $\beta \in H^4(X^s)$, define $\phi(\beta) \in H^4(X^{sf})$ to be the extension as above such that

$$\int^{\text{orb}}_{X^s} \alpha \wedge \beta = \int^{\text{orb}}_{X^{sf}} \phi(\alpha) \wedge \phi(\beta),$$

for any $\alpha \in H^2(X^s)$.

6.2. Isomorphism of Ruan cohomology rings. We first note that

Lemma 6.1. *Suppose that $\alpha_i \in H^2(X^s)$ and $\beta_i = \phi(\alpha_i)$, $1 \le i \le 3$. Then*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{X^s}_{CR} - \langle \beta_1, \beta_2, \beta_3 \rangle^{X^{sf}}_{CR} = r^3 \alpha_1([\Gamma^s]) \alpha_2([\Gamma^s]) \alpha_3([\Gamma^s])$$
$$= -r^3 \beta_1([\Gamma^{sf}]) \beta_2([\Gamma^{sf}]) \beta_3([\Gamma^{sf}]).$$

Proof. If one of α_i, say α_1, is $n_i x^s$, then the left hand side is zero, and the right hand side is also zero since $x^s([\Gamma^s]) = 0$. If $\alpha_i = n_i H^s$, $i = 1, 2, 3$, then the proof is the same as the proof of Lemma 6.14 in [**3**]. We omit it here. ☐

Now we state our main theorem.

Theorem 6.2. *Let $\alpha_i \in H^*_{CR}(X^s)$, $1 \le i \le 3$ and $\beta_i = \phi(\alpha_i)$. Then*

$$F^s(\alpha_1, \alpha_2, \alpha_3) = F^{sf}(\beta_1, \beta_2, \beta_3),$$

under the identification $[\Gamma_r^s] \leftrightarrow -[\Gamma_r^{sf}]$, *i.e.,* $q \leftrightarrow q^{-1}$. *Hence, we have an isomorphism of Ruan cohomology*

$$RH_{CR}^*(X^s) \cong RH_{CR}^*(X^{sf}).$$

Proof. We first assume that all $\alpha_i \in H^*(X^s)$. If one of α_i, say α_1, has degree ≥ 4, the quantum correction term in $F^s(\alpha_1, \alpha_2, \alpha_3)$ vanishes. Therefore, we only need to verify

$$\langle \phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3) \rangle_{CR}^{X^{sf}} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s}.$$

We can choose α_1 to be supported away from Γ^s. Then, we have

$$\begin{aligned}
\langle \phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3) \rangle_{CR}^{X^{sf}} &= \int_{X^{sf}} \phi(\alpha_1) \wedge \phi(\alpha_2) \wedge \phi(\alpha_3) \\
&= \int_{X^s} \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \\
&= \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s}.
\end{aligned}$$

Now we assume that $\deg \alpha_i = 2$. For this case, the difference,

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3),$$

consists of two parts.

$$(1) \quad \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} - \langle \beta_1, \beta_2, \beta_3 \rangle_{CR}^{X^{sf}} = r^3 \alpha_1([\Gamma^s]) \alpha_2([\Gamma^s]) \alpha_3([\Gamma^s]),$$

$$\begin{aligned}
(2) \quad &\alpha_1(r[\Gamma^s]) \alpha_2(r[\Gamma^s]) \alpha_3(r[\Gamma^s]) \frac{q^{r[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} \\
&- \beta_1(r[\Gamma^{sf}]) \beta_2(r[\Gamma^{sf}]) \beta_3(r[\Gamma^{sf}]) \frac{q^{r[\Gamma^{sf}]}}{1 - q^{r[\Gamma^{sf}]}} \\
&= r^3 \alpha_1([\Gamma^s]) \alpha_2([\Gamma^s]) \alpha_3([\Gamma^s]) \frac{q^{r[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} \\
&+ r^3 \alpha_1([\Gamma^s]) \alpha_2([\Gamma^s]) \alpha_3([\Gamma^s]) \frac{q^{-r[\Gamma^s]}}{1 - q^{-r[\Gamma^s]}} \\
&= -r^3 \alpha_1([\Gamma^s]) \alpha_2([\Gamma^s]) \alpha_3([\Gamma^s]).
\end{aligned}$$

Here we use $[\Gamma^s] \leftrightarrow -[\Gamma^{sf}]$. Part (1) cancels part (2). Therefore,

$$F^s(\alpha_1, \alpha_2, \alpha_3) = F^{sf}(\beta_1, \beta_2, \beta_3).$$

We next assume that at least one of α_i is twisted. Then, by Proposition 4.1, Proposition 4.3 and Proposition 4.6, and the computation in Section 5, we only have to consider the following cases.

Case (1). $(\alpha_1, \alpha_2, \alpha_3) = (\mathbb{1}_{0,i}^s, \mathbb{1}_{0,r-i}^s, x^s)$, $(\mathbb{1}_{\infty,i}^s, \mathbb{1}_{\infty,r-i}^s, x^s)$, $(\mathbb{1}_{0,i}^s, x_{0,r-i}^s, \mathbb{1}^s)$, $(\mathbb{1}_{\infty,i}^s, x_{\infty,r-i}^s, \mathbb{1}^s)$, $(\mathbb{1}_{p^s,i}^s, \mathbb{1}_{p^s,r-i}^s, \mathbb{1}^s)$ or $(\mathbb{1}_{q^s,i}^s, \mathbb{1}_{q^s,r-i}^s, \mathbb{1}^s)$. For all these cases, the difference of three-point functions is

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3)$$
$$= \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR}^{X^s} - \langle \phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3) \rangle_{CR}^{X^{sf}} = \frac{1}{r} - \frac{1}{r} = 0.$$

Case (2). $(\alpha_1, \alpha_2, \alpha_3) = (\mathbb{1}_{0,i}^s, \mathbb{1}_{\infty,r-i}^s, H^s)$. For this case, note that $H^s([\Gamma^s]) = 1$, and by Lemma 6.1, we have $\phi(H^s)([\Gamma^{sf}]) = -1$. Hence, the difference of three-point functions is

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3) = \sum_{d>0} \langle \mathbb{1}_{0,i}^s, \mathbb{1}_{\infty,r-i}^s, H^s \rangle_{0,3,d}^{X^s} q^{d[\Gamma^s]}$$

$$- \sum_{d>0} \langle \mathbb{1}_{0,r-i}^{sf}, \mathbb{1}_{\infty,i}^{sf}, \phi(H^s) \rangle_{0,3,d}^{X^{sf}} q^{d[\Gamma^{sf}]}$$

$$= \sum_{d \equiv r-i \,(\mathrm{mod}\, r)} \frac{d}{d} q^{d[\Gamma^s]}$$

$$- \sum_{d \equiv i \,(\mathrm{mod}\, r)} \frac{-d}{d} q^{d[\Gamma^{sf}]}$$

$$= \frac{q^{(r-i)[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} + \frac{q^{i[\Gamma^{sf}]}}{1 - q^{r[\Gamma^{sf}]}}$$

$$= \frac{q^{(r-i)[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} + \frac{q^{-i[\Gamma^s]}}{1 - q^{-r[\Gamma^s]}} = 0.$$

Case (3). $(\alpha_1, \alpha_2, \alpha_3) = (\mathbb{1}_{\infty,i}^s, \mathbb{1}_{0,r-i}^s, H^s)$. For this case, as in Case (2), the difference is

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3) = \sum_{d \equiv i \,(\mathrm{mod}\, r)} \frac{d}{d} q^{d[\Gamma^s]}$$

$$- \sum_{d \equiv r-i \,(\mathrm{mod}\, r)} \frac{-d}{d} q^{d[\Gamma^{sf}]}$$

$$= \frac{q^{i[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} + \frac{q^{r-i[\Gamma^{sf}]}}{1 - q^{r[\Gamma^{sf}]}} = \frac{q^{i[\Gamma^s]}}{1 - q^{r[\Gamma^s]}} + \frac{q^{i-r[\Gamma^s]}}{1 - q^{-r[\Gamma^s]}} = 0.$$

Case (4). $(\alpha_1, \alpha_2, \alpha_3) = (\mathbb{1}^s_{0,i}, \mathbb{1}^s_{0,j}, \mathbb{1}^s_{\infty,k})$, $i + j + k \equiv 0 \pmod{r}$. For this case, the difference is

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3)$$

$$= \begin{cases} \sum_{d \equiv k \pmod{r}} q^{d[\Gamma^s]} \\ \quad - \sum_{d \equiv r-k \pmod{r}} -q^{d[\Gamma^{sf}]} & i+j+k = r, \\ \sum_{d \equiv k \pmod{r}} -q^{d[\Gamma^s]} \\ \quad - \sum_{d \equiv r-k \pmod{r}} q^{d[\Gamma^{sf}]} & i+j+k = 2r, \end{cases}$$

$$= \begin{cases} \frac{q^{k[\Gamma^s]}}{1-q^{r[\Gamma^s]}} + \frac{q^{r-k[\Gamma^{sf}]}}{1-q^{r[\Gamma^{sf}]}} & i+j+k = r, \\ -\frac{q^{k[\Gamma^s]}}{1-q^{r[\Gamma^s]}} - \frac{q^{r-k[\Gamma^{sf}]}}{1-q^{r[\Gamma^{sf}]}} & i+j+k = 2r, \end{cases}$$

$$= 0.$$

Case (5). $(\alpha_1, \alpha_2, \alpha_3) = (\mathbb{1}^s_{0,i}, \mathbb{1}^s_{\infty,j}, \mathbb{1}^s_{\infty,k})$, $i + j + k \equiv 0 \pmod{r}$. For this case, the difference is

$$F^s(\alpha_1, \alpha_2, \alpha_3) - F^{sf}(\beta_1, \beta_2, \beta_3)$$

$$= \begin{cases} \sum_{d \equiv r-i \pmod{r}} -q^{d[\Gamma^s]} \\ \quad - \sum_{d \equiv i \pmod{r}} q^{d[\Gamma^{sf}]} & i+j+k = r, \\ \sum_{d \equiv r-i \pmod{r}} q^{d[\Gamma^s]} \\ \quad - \sum_{d \equiv i \pmod{r}} -q^{d[\Gamma^{sf}]} & i+j+k = 2r, \end{cases}$$

$$= \begin{cases} -\frac{q^{(r-i)[\Gamma^s]}}{1-q^{r[\Gamma^s]}} - \frac{q^{i[\Gamma^{sf}]}}{1-q^{r[\Gamma^{sf}]}} & i+j+k = r, \\ \frac{q^{(r-i)[\Gamma^s]}}{1-q^{r[\Gamma^s]}} + \frac{q^{i[\Gamma^{sf}]}}{1-q^{r[\Gamma^{sf}]}} & i+j+k = 2r, \end{cases}$$

$$= 0.$$

Summarizing all of these above cases, we get

$$F^s(\alpha_1, \alpha_2, \alpha_3) = F^{sf}(\beta_1, \beta_2, \beta_3).$$

It is easy to see that ϕ preserves the Poincaré pairing on the Chen-Ruan cohomology groups of X^s and X^{sf}. This completes the proof of the theorem. $\square$

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