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COPY-PASTE TREES AND THEIR GROWTH RATES

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ABSTRACT. In this paper, we describe a copy-and-paste method for constructing a class of infinite self-similar trees. A copy-paste tree is constructed by repeatedly attaching copies of a finite tree (called a *generator*) to certain designated attachment vertices. We show that each generator has an associated nonnegative matrix which can be used to determine a formula for the growth function of the copy-paste tree. In our main theorem, we use results from Perron-Frobenius theory to show that every copy-paste tree has exponential growth, with growth rate equal to the spectral radius of its associated matrix.

1. Introduction and motivation. This work was motivated by the study of the growth rate of self-similar graphs that arise from a repetitive construction process called a vertex replacement rule, where certain vertices of a graph are systematically deleted and replaced by copies of certain finite graphs, see [**8**]. The copy-paste construction method described in this current paper is a special case of a vertex replacement rule applied to trees. Vertex replacement rules themselves were motivated by studying the horospheres of the geodesic flow of a two-dimensional singular branching space having non-positive curvature, see [**1**].

Many other recursive construction schemes of graphs have been studied, see [**3, 5, 10**], and the references therein. These constructions, when applied to trees, often yield trees having finitely many cones (i.e., there are finitely many automorphism classes of cones, where a cone is obtained by taking a vertex and all its descendants). See [**4**]. The copy-paste method described in this paper also produces trees that have finitely many cones, but not all such trees can be obtained from the copy-paste procedure. The reader should be aware that, unlike most

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constructions, we allow for edges of any integer length (not all edges have length one). Lastly, it should be mentioned that the results of this paper can be adjusted to compute the growth rates of trees that do not have finitely many cones, as is demonstrated in the last example of the last section.

The topic of this paper is a class of self-similar infinite rooted trees, which we call *copy-paste trees*, that are constructed by an iterative process of attaching copies of a given tree (called a generator) to designated vertices (called *attachment vertices*). In our main result (Theorem 2.6), we show that every copy-paste tree has exponential growth with growth rate equal to the spectral radius of a matrix whose entries are determined by the number of the generator's attachment vertices and their respective distances to the root.

2. Definitions and examples. A *tree* is a connected graph with no circuits. The trees that we consider in this paper are equipped with more structure than a combinatorial object. In particular, if one considers a tree as a simplicial complex, we consider its realization. We impose a metric on each tree by assuming each edge is isometric to an open interval, making any tree a metric space by using the associated intrinsic metric. In other words, the distance $d(x, y)$ between any two points x and y (not necessarily vertices) is realized by the minimal length of a (simple) path connecting x to y. In particular, if x and y are vertices, then $d(x, y)$ is the sum of the lengths of the unique edges which are part of a simple path from x to y. Furthermore, we assume that every edge has an integer length and allow for different edges to have different integer lengths.

Let T be a rooted tree (i.e., a tree with one distinguished vertex r called the root). Denote the set of vertices as $V(T)$ and the set of edges as $E(T)$. Given points x and y of T (not necessarily vertices), if x lies on a simple path from y to the root, then x is an *ancestor* of y, and y is a *descendant* of x. We define the *height* of T by

$$h(T) = \sup_{v \in V(T)} d(v, r).$$

A rooted tree is called *infinite* if its height is unbounded. For integers $i \geq 0$, define the *vertex count* function of T by

$$c_i(T) = |\{v \in V(T) \mid d(v,r) = i\}|,$$

where $|\cdot|$ denotes cardinality.

Definition 2.1. Let T be an infinite rooted tree. The *growth function* of T is given by

$$f(T,n) = \sum_{i=0}^{n} c_i(T).$$

Definition 2.2. An infinite rooted tree T has *exponential growth* if there exists $\lambda > 1$ and positive constants α_1 and α_2 such that

$$\alpha_1 \lambda^n \leq f(T,n) \leq \alpha_2 \lambda^n,$$

for all $n \in \mathbb{N}$. The constant λ is called the *growth rate* of T. Observe that T has exponential growth with growth rate λ if, and only if,

$$\lim_{n\to\infty} \frac{\ln f(T,n)}{n} = \ln \lambda.$$

The topic of this paper is a class of self-similar infinite rooted trees called *copy-paste trees* that are constructed according to the rules described below in Definition 2.3.

Define a *generator* to be a finite rooted tree H, having root r, with a subset $V_{\text{att}}(H)$ of $V(H) \setminus \{r\}$, called the set of *attachment vertices* of H, such that $|V_{\text{att}}(H)| > 1$.

Definition 2.3. An infinite tree T_H is a *copy-paste tree* if it is obtained by the following inductive procedure:

Step 1. Start with a single generator H. (The root r of this first copy of H will be the root of T_H.)

Step 2. To each attachment vertex of H glue another copy of H by identifying the root of the copy to the attachment vertex.

Step 3. To each attachment vertex of a copy of H that appeared in Step 2, attach another copy of H by identifying the root of this latest copy to this attachment vertex.

Step 4. Proceed inductively as follows: for each attachment vertex on a copy of H that appeared at the previous stage, attach another copy of H by identifying the root of this latest copy to this attachment vertex.

The limit of this procedure yields an infinite rooted tree T_H. We define the set of *attachment vertices* of T_H, denoted $V_{\text{att}}(T_H)$, to be the set of all vertices of T_H that were attachment vertices at any stage of the construction of T_H.

Example 2.4. Let H_1 and H_2 be the generators shown in Figures 1 and 2, respectively. Suppose $V_{\text{att}}(H_1) = \{v_1, v_2\}$, $V_{\text{att}}(H_2) = \{w_1, w_2, w_3, w_4\}$ and all edges have length 1. Figure 1 shows the first three stages in the construction of T_{H_1}. Note that $T_{H_1} = T_{H_2}$, the infinite full binary tree, but $V_{\text{att}}(T_{H_1}) \neq V_{\text{att}}(T_{H_2})$.

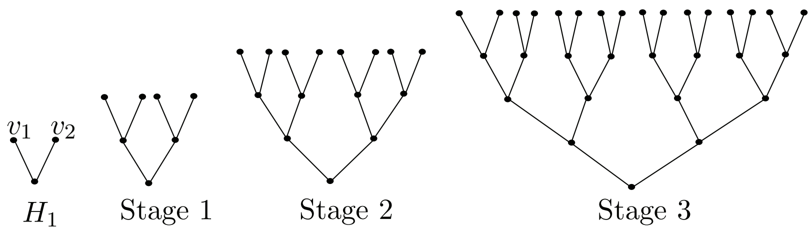

FIGURE 1. First stages in the construction of T_{H_1}.

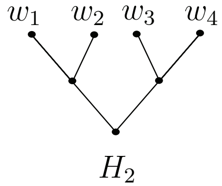

FIGURE 2. H_2 is another generator of the infinite full binary tree.

Example 2.5. Let H be the generator shown in Figure 3. Suppose $V_{\text{att}}(H) = \{v_1, v_2\}$. Suppose also that edge (r, w) has length 2, and all other edges of H have length 1. Figure 3 also shows T_H.

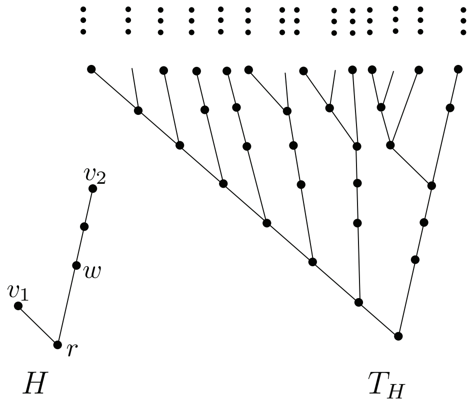

FIGURE 3. A generator H for the copy-paste tree T_H.

Our main result about copy-paste trees is the following theorem:

Theorem 2.6. *Let T_H be a copy-paste tree with generator H and let*

$$A_H = \left[\begin{array}{c|ccccc} 0_{N-1,1} & I_{N-1,N-1} & & & \\ \hline b_N & b_{N-1} & \cdots & b_2 & b_1 \end{array}\right],$$

where $b_i(H) = |\{v \in V_{\text{att}}(H) : d(v,r) = i\}|$ and $I_{N-1,N-1}$ is the size $N-1$ identity matrix. Then T_H has exponential growth with growth rate $\rho(A_H)$, where $\rho(A_H)$ is the spectral radius of A_H (i.e., $\rho(A_H)$ is the maximum magnitude of the eigenvalues of A_H).

Note that Theorem 2.6 requires generator H to have at least two attachment vertices, i.e., $|V_{\text{att}}(H)| > 1$. Applying the copy-and-paste method to a rooted tree with only one attachment vertex results in linear growth.

A key step in the proof of Theorem 2.6 is determining a formula for a copy-paste tree's growth function using powers of the nonnegative matrix A_H. We conclude this section with some notation, definitions, and results on nonnegative matrices.

We denote the (i,j) entry of A by $a_{i,j}$, and the (i,j) entry of A^n by $(A^n)_{i,j}$. A real matrix A is *nonnegative* (*positive*) if $a_{i,j} \geq 0$ ($a_{i,j} > 0$) for every (i,j). We write $A \geq 0$ if A is nonnegative, and $A > 0$ if A

is positive. A nonnegative square matrix A is called *irreducible* if for any (i, j) there exists k such that $(A^k)_{i,j} > 0$. A nonnegative square matrix A is called *primitive* if $A^k > 0$ for some k.

The following theorems can be found in **[2, 6, 9]**.

Theorem 2.7 (Perron-Frobenius)**.** *Let A be an irreducible matrix.*

(a) *A has a positive real eigenvalue $r = \rho(A)$ such that $r \geq |\lambda_i|$ for any eigenvalue λ_i of A. (The eigenvalue r is called the* Perron eigenvalue *of A.)*
(b) *Furthermore, r has algebraic and geometric multiplicity 1 and has a positive eigenvector* $\mathbf{x}$.
(c) *Any nonnegative eigenvector of A is a multiple of* $\mathbf{x}$.
(d)
$$\min_i \sum_j a_{i,j} \leq r \leq \max_i \sum_j a_{i,j}.$$
(e) *If A is primitive, then $r > |\lambda_i|$ for any eigenvalue λ_i of A, $\lambda_i \neq r$.*

Theorem 2.8. *Let A be a primitive matrix with Perron eigenvalue r. Then there exists a matrix L with positive entries so that*

$$\lim_{k \to \infty} \left(\frac{A}{r} \right)^k = L.$$

The associated directed graph of an $n \times n$ nonnegative matrix A, denoted $\mathcal{G}(A)$, consists of n vertices $v_1, \ldots, v_n$, where there exists a directed edge from v_i to v_j, denoted (v_i, v_j), if, and only if, $a_{i,j} \neq 0$. A graph is *strongly connected* if, for any two vertices v_i and v_j, there is a directed path from v_i to v_j. The following theorem gives necessary and sufficient conditions for a matrix to be irreducible or primitive.

Theorem 2.9. *Let A be a nonnegative matrix.*

(i) *Then A is irreducible if, and only if, $\mathcal{G}(A)$ is strongly connected.*
(ii) *Suppose also that A is irreducible. Then A is primitive if, and only if, $\mathcal{G}(A)$ has two cycles of relatively prime lengths.*

3. Computing the growth function of a copy-paste tree. Throughout this section, let H be a generator of height N. Recall from Theorem 2.6 that, for $i = 1, \ldots, N$,

$$b_i(H) = |\{v \in V_{\text{att}}(H) : d(v, r) = i\}|.$$

The following definitions will be used to construct a partition of the vertices of T_H that leads to a recursive formula for $c_i(T_H)$. If only one generator H and its corresponding copy-paste tree T_H are under consideration, we will often write T for T_H, b_i for $b_i(H)$, and c_i for $c_i(T_H)$.

Definition 3.1. Define the closed ball of radius n about any point (not necessarily a vertex) $x \in T$, denoted $\mathcal{B}_n(T, x)$, by

$$\mathcal{B}_n(T, x) = \{z \in T : d(z, x) \leq n\}.$$

Definition 3.2. For $i \geq 1$, define the **i**-*shell*, denoted S_i, by

$$S_i = \mathcal{B}_i(T, r) \setminus \mathcal{B}_{i-1}(T, r).$$

For example, Figure 4 shows the shells $S_1, \ldots, S_6$ for the copy-paste tree of Example 2.5 (depicted in Figure 3). Note that in this example not all points of distance i to the root are vertices in S_i. Vertices are denoted in Figure 4 with filled in dots. Also note that $|V(S_n)| = c_n(T)$.

An immediate consequence of Definitions 2.3 and 3.2 is the following:

Observation 3.3. Let T be a copy-paste tree and let $b \in V_{\text{att}}(T)$. Consider $D(T, b)$, which consists of b together with all of its descendants. Then for all $i \geq 1$, $(\mathcal{B}_i(T, b) \setminus \mathcal{B}_{i-1}(T, b)) \cap D(T, b)$ is isometric to S_i.

The next lemma is technical but important.

Lemma 3.4. *Let T be a copy-paste tree with a generator H of height N. For all $n > N$, there exists a partition $\mathcal{P}_n$ of S_n such that each element of $\mathcal{P}_n$ is isometric to one of the shells $S_1, \ldots, S_N$.*

For $1 \leq i \leq N$ and $1 \leq n \leq N$, let $t_i(S_n) = \delta_{i,n}$, where δ is the Kronecker delta. For $1 \leq i \leq N$ and $n > N$, let $t_i(S_n)$ be the number

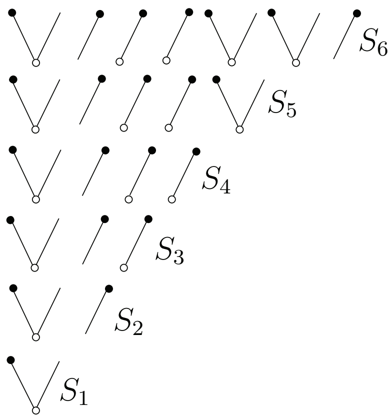

FIGURE 4. First shells of the copy-paste tree of Example 2.5.

of isometric copies of S_i in the partition $\mathcal{P}_n$ of S_n. Then the following formulas hold:

(a) $c_n(T) = t_1(S_n)c_1(T) + t_2(S_n)c_2(T) + \cdots + t_N(S_n)c_N(T)$.
(b) $t_i(S_{N+1}) = b_{N+1-i}$, *for* $i = 1, \ldots, N$.
(c) *For* $n > N$,

$$t_j(S_{n+1}) = \begin{cases} t_N(S_n)t_1(S_{N+1}) & j = 1, \\ t_{j-1}(S_n) + t_N(S_n)t_j(S_{N+1}) & 2 \leq j \leq N. \end{cases}$$

Proof. Let $n > N$. For each $x \in S_n$ (not necessarily a vertex), let $a(x) \in T$ be the unique ancestor of x that is exactly N closer to the root than x. Since $h(H) = N$, any path from x to $a(x)$ must contain at least one attachment vertex. If $a(x)$ is itself an attachment vertex, let $k(x) = a(x)$. If $a(x)$ is not an attachment vertex, then $a(x)$ lies in a unique copy of H that was used in the construction of T, which we will denote $H(a(x))$. In this case, let $k(x)$ be the unique attachment vertex of $H(a(x))$ that is an ancestor of x. Consult Example 3.5 for an illustration.

Define an equivalence relation on S_n by xRy if $k(x) = k(y)$. (In Figure 5, yRx.) Let $\mathcal{P}_n$ denote the partition of S_n formed by the equivalence classes of R, and let $\mathcal{P}_n(x)$ denote the partition element of

$\mathcal{P}_n$ containing x. Observe that

$$\mathcal{P}_n(x) = (\mathcal{B}_i(T, k(x)) \setminus \mathcal{B}_{i-1}(T, k(x))) \cap D(T, k(x))$$

for some $1 \leq i \leq N$ which, by Observation 3.3, is isometric to S_i for some $1 \leq i \leq N$. The formula in part (a) immediately follows.

We next use the partition to find formulas for counting the number of isometric copies of $S_1, \ldots, S_N$ in the S_{N+1} shell. Let $x \in S_{N+1}$. Clearly, $d(a(x), r) \leq 1$ so $a(x)$ is in the initial copy of H (the one with r). By part (a), $\mathcal{P}_n(x)$ is isometric to S_i for some $1 \leq i \leq N$. Therefore $k(x)$ is an attachment vertex of the original copy of H that is $N+1-i$ away from r. Finally, there is a one-to-one correspondence between the number of partition elements in S_{N+1} and the attachment vertices of H by the bijection $\mathcal{P}_n(x) \leftrightarrow k(x)$. Thus the formula in part (b) holds.

Now we establish the formula in part (c). Let $n > N$. Both S_n and S_{n+1} are partitioned into elements isometric to $S_1, \ldots, S_N$ by part (a). For every partition element of S_n there are two cases.

Case 1. If a partition element P of S_n is isometric to S_i for $1 \leq i \leq N-1$, then such an element will uniquely correspond to a partition element Q of S_{n+1} that is isometric to S_{i+1} for $1 \leq i \leq N-1$. This unique partition element of S_{n+1} corresponds to descendants of P.

Case 2. If a partition element P of S_n is isometric to S_N, then its descendants in S_{n+1} will consist of an isometric copy of S_{N+1} which in turn can be partitioned into $t_i(S_{N+1})$ elements for each i with $1 \leq i \leq N$.

Note that those partition elements of S_{n+1} that are isometric to S_1 can only arise in the second case. So $t_1(S_{n+1}) = t_N(S_n)t_1(S_{N+1})$ since there are $t_N(S_n)$ distinct isometric copies of S_N in S_n. Those partition elements of S_{n+1} that are isometric to S_j for $2 \leq j \leq N$ can arise in either case. Therefore, the total number of partition elements in S_{n+1} isometric to S_j for $2 \leq j \leq N$ is

$$t_j(S_{n+1}) = t_{j-1}(S_n) + t_N(S_n)t_j(S_{N+1}). \qquad \square$$

Example 3.5. Figure 5 shows the copy-paste tree of Example 2.5 (depicted in Figure 3). Recall that $V_{\text{att}}(H) = \{v_1, v_2\}$ and $N = 4$. For w, x and y as indicated in Figure 5, $a(w)$, $a(x)$ and $a(y)$ are in the first copy of H used in the construction of T_H. Since $a(w) = k(w) = v_1$, w is in an isometric copy of S_4. Note that $k(w) \neq k(y)$. Hence w and y are not related and are not in the same partition element of S_5. Since $k(x) = k(y) = v_2$, x and y are related and are in the same isometric copy of S_1. The partition elements of S_5, S_6 and S_7 are circled in Figure 5.

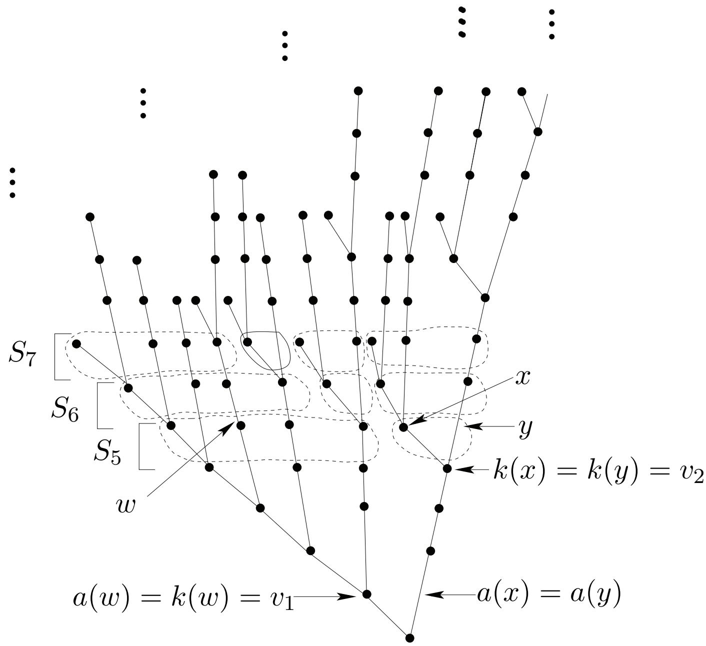

FIGURE 5. Partition elements of S_5, S_6 and S_7.

Definition 3.6. Define the *copy-paste matrix* A_H associated to a generator H of height N to be the $N \times N$ nonnegative matrix

$$A_H = \begin{bmatrix} t_1(S_2) & t_2(S_2) & t_3(S_2) & \cdots & t_N(S_2) \\ t_1(S_3) & t_2(S_3) & t_3(S_3) & \cdots & t_N(S_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_1(S_{N+1}) & t_2(S_{N+1}) & t_3(S_{N+1}) & \cdots & t_N(S_{N+1}) \end{bmatrix}. \tag{3.1}$$

Recall that for $1 \leq i \leq N$ and $1 \leq j \leq N$, $t_i(S_j) = \delta_{i,j}$, where δ is the Kronecker delta. Using this fact together with Lemma 3.4 (b), the copy-paste matrix in (3.1) can be simplified to the block matrix

$$A_H = \left[\begin{array}{c|cccc} 0_{N-1,1} & I_{N-1,N-1} & & & \\ \hline b_N & b_{N-1} & \cdots & b_2 & b_1 \end{array}\right], \tag{3.2}$$

where $I_{N-1,N-1}$ is the size $N-1$ identity matrix.

Lemma 3.7 lists several properties of copy-paste matrices.

Lemma 3.7. *Let A be a copy-paste matrix of the form* (3.2) *with spectral radius $\rho(A)$.*

(a) *If λ is an eigenvalue of A, then $\begin{bmatrix} 1 & \lambda & \cdots \lambda^{N-1} \end{bmatrix}^t$ is an eigenvector of A corresponding to λ (where t denotes transpose).*
(b) *For all $N \geq 2$, the determinant of $A - I$ is given by*

$$|A - I| = (-1)^N \left(1 - \sum_{i=1}^{N} b_i\right).$$

(c) *1 is not an eigenvalue of A.*
(d) *If $b_N \neq 0$, then A is irreducible.*
(e) *If $b_N \neq 0$, then $\rho(A) > 1$.*
(f) *If $b_N = b_{N-1} = \cdots = b_{N-j} = 0$, $b_{N-(j+1)} \neq 0$ and*

$$A' = \left[\begin{array}{c|cccc} 0_{N-(j+2),1} & I_{N-(j+2),N-(j+2)} & & & \\ \hline b_{N-(j+1)} & b_{N-(j+2)} & \cdots & b_2 & b_1 \end{array}\right], \tag{3.3}$$

then

$$|A - \lambda I| = (-\lambda)^{j+1} |A' - \lambda I| \quad \textit{and} \quad \rho(A) = \rho(A'),$$

where I is the appropriate sized identity matrix.

Proof.

(a) Let $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^t$ be an eigenvector of A associated to λ. Since $A\mathbf{v} = \lambda\mathbf{v}$, we have that

$$v_i = \lambda v_{i-1} \quad \text{for } i = 2, \ldots, N \tag{3.4}$$

and

$$b_N v_1 + b_{N-1} v_2 + \cdots + b_1 v_N = \lambda v_N. \tag{3.5}$$

Since $\mathbf{v}$ is an eigenvector of A, then $v_1 \neq 0$. For if $v_1 = 0$, then $v_2 = \cdots = v_N = 0$ by equation (3.4). Without loss of generality, $v_1 = 1$. From equation (3.4) it follows that

$$v_i = \lambda^{i-1}, \quad \text{for } i = 2, \ldots, N. \tag{3.6}$$

Substituting equation (3.6) into equation (3.5) yields

$$b_N + b_{N-1}\lambda + b_{N-2}\lambda^2 + \cdots + b_1 \lambda^{N-1} = \lambda^N. \tag{3.7}$$

Using equation (3.7), it is now straightforward to verify

$$A \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{N-1} \end{bmatrix}^t = \lambda \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{N-1} \end{bmatrix}^t.$$

(b) We will argue by induction on N. When $N = 2$, we have that

$$|A - I| = \begin{vmatrix} -1 & 1 \\ b_2 & b_1 - 1 \end{vmatrix} = 1 - b_1 - b_2.$$

So the base case holds. Next assume the determinant formula is valid for any $(k-1) \times (k-1)$ matrix of the form given by equation (3.2), where $k \geq 3$. Let

$$A = \left[\begin{array}{c|cccc} 0_{N-1,1} & I_{N-1,N-1} & & & \\ \hline b_k & b_{k-1} & \cdots & b_2 & b_1 \end{array} \right].$$

Using a cofactor expansion along the first column of $A-I$ we have that

$$|A-I| = -1\begin{vmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b_{k-1} & b_{k-2} & b_{k-3} & \cdots & b_1 - 1 \end{vmatrix} + (-1)^{k+1} b_k \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{vmatrix}. \tag{3.8}$$

By applying the induction hypothesis and properties of triangular matrices, we see

$$|A-I| = (-1)(-1)^{k-1}\left(1 - \sum_{i=1}^{k-1} b_i\right) + (-1)^{k+1} b_k = (-1)^k\left(1 - \sum_{i=1}^{k} b_i\right),$$

which is the desired result.

(c) Since

$$\sum_{i=1}^{N} b_i \geq 2,$$

it follows by part (b) that $|A-I|$ is not zero and hence 1 is not an eigenvalue of A.

(d) By Theorem 2.9, it suffices to show that $\mathcal{G}(A)$, the associated directed graph of A, is strongly connected. Let $x_1, \ldots, x_N$ denote the vertices of $\mathcal{G}(A)$. Observe that, for $i = 1, \ldots, N-1$, there is a directed edge in $\mathcal{G}(A)$ from x_i to x_{i+1}. Since $a_{N,1} = b_N \neq 0$, there is also a directed edge in $\mathcal{G}(A)$ from x_N to x_1. Therefore $\mathcal{G}(A)$ has a cycle of length N, namely, $x_N, x_1, x_2, \ldots, x_{N-1}, x_N$, that includes every vertex of $\mathcal{G}(A)$. Hence $\mathcal{G}(A)$ is strongly connected.

(e) Since A is irreducible, it follows from the Perron-Frobenius theorem (Theorem 2.7) that $\rho(A) \geq 1$. The result now follows by part (c). (f) It is straightforward to check that $|A-\lambda I| = (-\lambda)^{j+1}|A' -$

$\lambda I|$ by computing the determinant of $(A - \lambda I)$ by a sequence of first column cofactor expansions. □

Theorem 3.8. *Let T be a copy-paste tree with generator H of height N, and let A be the copy-paste matrix associated to H.*

(a) *Then $(A^n)_{i,j} = t_j(S_{n+i})$ for all $n \geq 1$.*
(b) *Let $\mathbf{u} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}(n) = \begin{bmatrix} c_{n+1} & c_{n+2} & \cdots & c_{n+N} \end{bmatrix}^t$. Then the growth function of T satisfies*

$$f(T, kN) = 1 + \mathbf{u}\left[I + A^N + \cdots + A^{(k-1)N}\right]\mathbf{c}(0).$$

(c) *If A is primitive, then T has exponential growth with growth rate $\rho(A)$.*

Proof.

(a) We argue by induction on n. When $n = 1$, the result clearly holds. Next we assume the result holds for $n = k$, where $k \geq 1$. Note that multiplication in the order below results in a matrix whose first $N - 1$ rows are just the rows of A^k shifted up by one row.

$$A^{k+1} = AA^k = A\begin{bmatrix} t_1(S_{k+1}) & t_2(S_{k+1}) & \cdots & t_N(S_{k+1}) \\ t_1(S_{k+2}) & t_2(S_{k+2}) & \cdots & t_N(S_{k+2}) \\ t_1(S_{k+3}) & t_2(S_{k+3}) & \cdots & t_N(S_{k+3}) \\ t_1(S_{k+4}) & t_2(S_{k+4}) & \cdots & t_N(S_{k+4}) \\ \vdots & \vdots & \cdots & \vdots \\ t_1(S_{k+N}) & t_2(S_{k+N}) & \cdots & t_N(S_{k+N}) \end{bmatrix}.$$

Thus the entries in the first $N - 1$ rows of A^{k+1} are the desired result. To show the last row of A^{k+1} is the desired result we write A^{k+1} as the product $A^{k+1} = A^k A$ and use Lemma 3.4 (c).

(b) The following calculation uses Lemma 3.4 (a) to verify that $\mathbf{c}(n) = A^n\mathbf{c}(0)$:

$$A^n\begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} t_1(S_{n+1}) & \cdots & t_N(S_{n+1}) \\ \vdots & \vdots & \vdots \\ t_1(S_{n+N}) & \cdots & t_N(S_{n+N}) \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}$$

$$= \begin{bmatrix} c_1 t_1(S_{n+1}) + \cdots + c_N t_N(S_{n+1}) \\ \vdots \\ c_1 t_1(S_{n+N}) + \cdots + c_N t_N(S_{n+N}) \end{bmatrix}$$

$$= \begin{bmatrix} c_{n+1} \\ \vdots \\ c_{N+n} \end{bmatrix}.$$

Observe that

$$\mathbf{u}A^n\mathbf{c}(0) = \mathbf{u}\mathbf{c}(n) = \sum_{i=1}^{N} c_{i+n}.$$

Therefore,

$$f(T, kN) = \sum_{i=0}^{kN} c_i = 1 + \mathbf{u}\left[I + A^N + A^{2N} + \cdots + A^{(k-1)N}\right]\mathbf{c}(0). \tag{3.9}$$

(c) Suppose A is primitive. Let $r = \rho(A)$ and let L be the positive matrix (as described in Theorem 2.8) given by

$$L = \lim_{k\to\infty}\left(\frac{A}{r}\right)^k.$$

It follows from Lemma 3.7 (c) and Theorem 2.7 (d) that $r > 1$.

For every $n \geq 0$, there exists an integer $k_n \geq 0$ such that

$$k_n N \leq n \leq (k_n + 1)N. \tag{3.10}$$

Clearly,

$$0 < f(T, k_n N) \leq f(T, n) \leq f(T, (k_n + 1)N). \tag{3.11}$$

So, for $r > 1$, we have

$$\frac{f(T, k_n N)}{r^{k_n N} r^N} \leq \frac{f(T, n)}{r^n} \leq \frac{f(T, (k_n + 1)N)}{r^{k_n N}}. \tag{3.12}$$

Since 1 is not an eigenvalue of A (Lemma 3.7), the matrix $A^N - I$ is invertible. Therefore, equation (3.9) can be simplified as follows:

$$f(T, kN) = 1 + \mathbf{u}\left[I + A^N + \cdots + A^{(k-1)N}\right]\mathbf{c}(0)$$

$$= 1 + \mathbf{u}\left[(A^N - I)^{-1}(A^{kN} - I)\right]\mathbf{c}(0) \tag{3.13}$$

$$= 1 + \mathbf{u}\left[(A^N - I)^{-1}A^{kN}\right]\mathbf{c}(0) - \mathbf{u}\left[(A^N - I)^{-1}\right]\mathbf{c}(0).$$

From inequality (3.12), Definition 2.2 and from convergent sequence results, it follows that T has exponential growth with growth rate r if there exist real numbers ℓ_1 and ℓ_2, $0 < \ell_1 \leq \ell_2$, such that

$$\lim_{k\to\infty} \frac{f(T,kN)}{r^{kN}r^N} = \ell_1 \quad \text{and} \quad \lim_{k\to\infty} \frac{f(T,(k+1)N)}{r^{kN}} = \ell_2.$$

We will show that ℓ_1 exists. By equation (3.13),

(3.14)
$$\frac{f(T,kN)}{r^{kN}r^N} = \frac{1}{r^{kN}r^N} + \frac{\mathbf{u}\left[(A^N - I)^{-1}A^{kN}\right]\mathbf{c}(0)}{r^{kN}r^N} - \frac{\mathbf{u}\left[(A^N - I)^{-1}\right]\mathbf{c}(0)}{r^{kN}r^N}.$$

Since $r > 1$, the first and last terms of the right-hand side of equation (3.14) vanish as $k \to \infty$. Since

$$\lim_{k\to\infty} \frac{A^{kN}}{r^{kN}} = L,$$

we have

$$\ell_1 = \lim_{k\to\infty} \frac{f(T,kN)}{r^{kN}r^N} = \frac{\mathbf{u}\left[(A^N - I)^{-1}\right]L\mathbf{c}(0)}{r^N}.$$

Next, we show that $\ell_1 > 0$. From equation (3.9), we have that

$$\frac{f(T,kN)}{r^{kN}r^N} = \frac{1 + \mathbf{u}\left[I + A^N + \cdots + A^{(k-1)N}\right]\mathbf{c}(0)}{r^{kN}r^N}. \tag{3.15}$$

Since A is primitive, $r > 1$, and the entries of $\mathbf{u}$ and $\mathbf{c}(0)$ are positive, it immediately follows from equation (3.15) that

$$\frac{f(T,kN)}{r^{kN}r^N} \geq \frac{\mathbf{u}A^{(k-1)N}\mathbf{c}(0)}{r^{kN}r^N}. \tag{3.16}$$

Since

$$\lim_{k\to\infty} \frac{\mathbf{u}A^{(k-1)N}\mathbf{c}(0)}{r^{kN}r^N} = \lim_{k\to\infty} \frac{\mathbf{u}A^{(k-1)N}\mathbf{c}(0)}{r^{(k-1)N}r^{2N}} = \frac{\mathbf{u}L\mathbf{c}(0)}{r^{2N}} > 0,$$

it follows that $\ell_1 > 0$.

A similar argument establishes that ℓ_2 also exists and, by equation (3.12), $\ell_2 \geq \ell_1$. □

4. Copy-paste trees with exponential growth. The purpose of this section is to show, using a bootstrapping argument, that every copy-paste tree has exponential growth. In particular, we consider the case where the associated copy-paste matrix is reducible. We then construct related copy-paste trees having the same growth rate whose associated copy-paste matrices are primitive.

4.1. Reducing to the case of irreducible copy-paste trees. Let H be a generator of height N for a copy-paste tree T_H, and let M be the maximum of the distances of all the attachment vertices in H to the root. If $N > M$, then $b_N(H) = 0$, and hence the associated copy-paste matrix A_H has a column of all zeros making it reducible.

We build two related copy-paste trees with generators H' and $\widetilde{H}$ of height M with associated $M \times M$ matrices

$$\left[\begin{array}{c|ccccc} 0_{M-1,1} & I_{M-1,M-1} & & & & \\ \hline b_M & b_{M-1} & & \cdots & b_2 & b_1 \end{array}\right]. \tag{4.1}$$

This matrix satisfies the hypotheses of parts (d) and (e) of Lemma 3.7. We will construct H' and $\widetilde{H}$ in such a way so that

$$f(T_{H'}, n) \leq f(T_H, n) \leq f(T_{\tilde{H}}, n).$$

We will use this to show that T_H has the same growth rate as $T_{H'}$ and $T_{\tilde{H}}$.

To build H', start with a root r' and, for each attachment vertex $v \in V_{\text{att}}(H)$, create an attachment vertex $v' \in V_{\text{att}}(H')$ with one edge adjacent to both r' and v' having length $d(r, v)$ (i.e., obtain H' from H by keeping only the root and all attachment vertices and removing all interior vertices). Note that H' will have exactly $|V_{\text{att}}(H)| + 1$ vertices, with $|V_{\text{att}}(H)|$ leaves (i.e., vertices adjacent to exactly one edge) and $|V_{\text{att}}(H)|$ edges all adjacent to r'.

To construct $\widetilde{H}$, again start with a root $\widetilde{r}$ and for each attachment vertex $v \in V_{\text{att}}(H)$, create an attachment vertex $\widetilde{v} \in V_{\text{att}}(\widetilde{H})$ which is a leaf. However, unlike H', create $d(r, v)$ edges, each of length 1, and $d(r, v) - 1$ vertices so that these new vertices and edges form a path from $\widetilde{r}$ to $\widetilde{v}$ having length $d(r, v)$. Additionally, for every vertex $w \in V(H)$, create a non-attachment leaf $\widetilde{w}$ that is exactly M away from the root together with M edges of length one and $M - 1$ vertices so that these new vertices and edges form a path from $\widetilde{r}$ to $\widetilde{w}$. Recall that M is the maximum of the distances of all the attachment vertices in H to the root. Note that all vertices except for the leaves and the root are adjacent to exactly two edges. The following example illustrates the construction of H' and $\widetilde{H}$. Note that $A_{H'} = A_{\widetilde{H}}$ are given by (4.1).

Example 4.1. Let H be the generator shown below in Figure 6. Suppose edges (x, y) and (z, w) have length 2, and all other edges have length 1. Suppose also that $V_{\text{att}}(H) = \{v_1, v_2, v_3\}$. Note that H has height $N = d(s, r) = 6$. Since the distances of the attachment vertices to the root are $d(v_1, r) = d(v_2, r) = 4$ and $d(v_3, r) = 2$, we have that $M = 4$.

The tree H' has exactly four vertices: a root r' and three attachment vertices v'_1, v'_2 and v'_3. Edges (r', v'_1) and (r', v'_2) have length 4, and edge (r', v'_3) has length 2. The tree $\widetilde{H}$ has three attachment vertices: $\widetilde{v}_1, \widetilde{v}_2$ and $\widetilde{v}_3$, and 15 non-attachment leaves. All edges of $\widetilde{H}$ have length 1. (See Figure 6.)

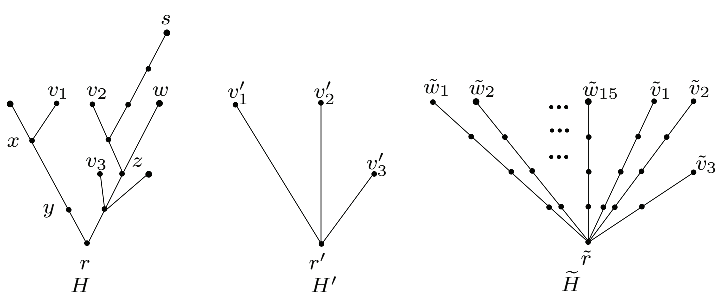

FIGURE 6. Trees H' and $\widetilde{H}$ corresponding to H.

Since $b_M \neq 0$ we have that $A_{H'}$ is irreducible. Furthermore, it is clear that $c_i(H') \leq c_i(H) \leq c_i(\widetilde{H})$, from which it follows that

$$f(T_{H'}, n) \leq f(T_H, n) \leq f(T_{\widetilde{H}}, n). \tag{4.2}$$

If $A_{H'}$ is primitive, then, by Theorem 3.8, the copy-paste trees $T_{H'}$ and $T_{\tilde{H}}$ have exponential growth with growth rate $\rho(A_{H'})$. Hence, by equation (4.2), it follows that T_H has the same growth rate.

4.2. Reducing to the case of a primitive copy-paste tree. Although the matrix $A_{H'}$ is irreducible, it may not be primitive. Consider for instance the copy-paste matrix

$$A_{H'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

of Example 4.1. One can readily check that the directed graph of $A_{H'}$ contains cycles of lengths 2 and 4, and hence, by Theorem 2.9, $A_{H'}$ is not primitive.

In this section, we find sufficient conditions for $A_{H'}$ to be primitive and then consider the case where $A_{H'}$ is not primitive. In the case where $A_{H'}$ is not primitive, we construct related copy-paste trees with matrices that are primitive which will be used to compute the growth rate of $T_{H'}$ (which in turn has the same growth rate as T_H).

Definition 4.2. Let H be a generator and let $m = \gcd(d(v_1, r), \dots, d(v_n, r))$, where each $v_i \in V_{\text{att}}(H)$. We say that H is *unscaleable* if $m = 1$, and that H is *scaleable* if $m > 1$.

Recall that H_1 and H_2 of Example 2.4 are both generators for the infinite full binary tree. The generator H_1 is unscaleable, whereas H_2 is scaleable with $m = 2$.

Lemma 4.3. *Let T_H be a copy-paste tree with unscaleable generator H of height $N > 1$. Then $A_{H'}$ is primitive.*

Proof. Let $\mathcal{G}(A_{H'})$ be the associated directed graph of $A_{H'}$, and let $x_1, \dots, x_M$ denote the vertices of $\mathcal{G}(A_{H'})$. Recall that in the proof of Lemma 3.7 (d), it was shown that $b_M(H) \neq 0$ implies that $\mathcal{G}(A_{H'})$ has a cycle of length M that contains every vertex, namely,

$$x_M, x_1, x_2, \dots, x_{M-1}, x_M.$$

Observe that, if $b_i(H) \neq 0$ for some $1 \leq i \leq M$, then (x_M, x_{M-i+1}) is a directed edge in $\mathcal{G}(A_{H'})$. Using the cycle described in the previous paragraph, it follows that

$$x_M, x_{M-i+1}, x_{M-i+2}, \ldots, x_{(M-i)+(i-1)}, x_M$$

is a cycle of length i in $\mathcal{G}(A_{H'})$.

Since H is unscaleable, there exist attachment vertices v and w of H such that $\gcd(d(v,r), d(w,r)) = 1$. Let $j = d(v,r)$ and $k = d(w,r)$. Then $b_j(H) \neq 0$, $b_k(H) \neq 0$ and $\gcd(j,k) = 1$. Thus $\mathcal{G}(A_{H'})$ has a cycle of length j and a cycle of length k. Since j and k are relatively prime, Theorem 2.9 implies $A_{H'}$ is primitive. □

If H is scaleable, we will construct two additional trees H'_{sc} and $\widetilde{H}_{\text{sc}}$. Construct H'_{sc} from H' by scaling each edge of H' by a factor of $1/m$, where $m = \gcd(d(v_1,r), \ldots, d(v_n,r))$ for each $v_i \in V_{\text{att}}(H)$. (Note that the distance between any two vertices in H'_{sc} is still an integer.)

Construct $\widetilde{H}_{\text{sc}}$ as follows. For each attachment (respectively, non-attachment) leaf of $\widetilde{H}$, construct an attachment (respectively, non-attachment) leaf of $\widetilde{H}_{\text{sc}}$ connecting it to the root $\widetilde{s}$ via a path of M/m edges, each of length 1, so that there are $M/m - 1$ interior vertices associated with this path. The following example illustrates these constructions.

Example 4.4. Let H be the generator of Example 4.1 (depicted in Figure 6). As before, edges (x,y) and (z,w) have length 2, and all other edges have length 1. Since the distances of the attachment vertices of H to the root are $d(v_1,r) = d(v_2,r) = 4$ and $d(v_3,r) = 2$, H is scaleable with $m = 2$.

The tree H'_{sc} has exactly four vertices: a root r' and three attachment vertices v'_1, v'_2 and v'_3. Note that $d(r',v'_1) = d(r',v'_2) = 2$ and $d(r',v'_3) = 1$. The tree $\widetilde{H}_{\text{sc}}$ has three attachment vertices: $\widetilde{v}_1, \widetilde{v}_2$ and $\widetilde{v}_3$, and 15 non-attachment leaves $\widetilde{w}_1, \ldots, \widetilde{w}_{15}$. All edges of $\widetilde{H}_{\text{sc}}$ have length 1, $d(\widetilde{r},\widetilde{v}_1) = d(\widetilde{r},\widetilde{v}_2) = 2$, $d(\widetilde{r},\widetilde{v}_3) = 1$ and $d(\widetilde{r},\widetilde{w}_i) = 2$ for all $i = 1, \ldots, 15$. Figure 7 shows the corresponding scaled trees H'_{sc} and $\widetilde{H}_{\text{sc}}$.

Observation 4.5. For a scaleable H, the properties below follow immediately from the construction of H'_{sc}. Recall that c_i is the number of vertices in a tree that are distance i to the root.

(i) H'_{sc} and $\widetilde{H}_{\text{sc}}$ have height M/m and $A_{H'_{\text{sc}}} = A_{\tilde{H}_{\text{sc}}}$.
(ii) $b_{im}(H') = b_i(H'_{\text{sc}})$ and $c_{im}(H') = c_i(H'_{\text{sc}})$. Hence, $c_i(T_{H'_{\text{sc}}}) = c_{im}(T_{H'})$ for $i \geq 0$.
(iii) $b_{im}(\widetilde{H}) = b_i(\widetilde{H}_{\text{sc}})$ and $c_{im}(\widetilde{H}) = c_i(\widetilde{H}_{\text{sc}})$. Hence, $c_i(T_{\tilde{H}_{\text{sc}}}) = c_{im}(T_{\tilde{H}})$ for $i \geq 0$.

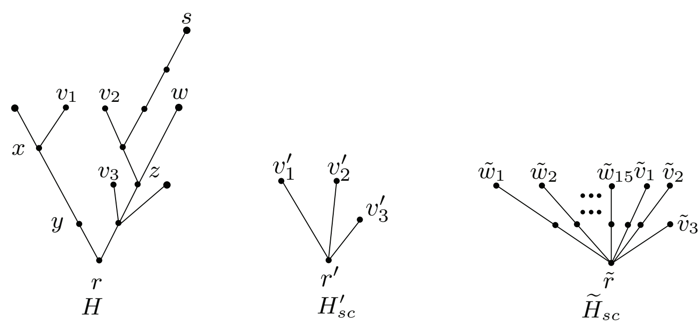

FIGURE 7. Scaled trees corresponding to H.

Lemma 4.6 describes the effects of scaling on the spectral radius of the copy-paste matrix and on the growth rate of the copy-paste tree.

Lemma 4.6. *Let H be a generator with associated generator H' of height $M > 1$. Suppose H' is scaleable with $m = \gcd(d(v_1, r), \ldots, d(v_n, r)) > 1$, and let H'_{sc} be the scaled generator described above.*

(i) *H'_{sc} and $\widetilde{H}_{\text{sc}}$ are unscaleable which implies that $A_{H'_{\text{sc}}} = A_{\tilde{H}_{\text{sc}}}$ is primitive. Moreover, $T_{H'_{\text{sc}}}$ and $T_{\tilde{H}_{\text{sc}}}$ have growth rate $\rho(A_{H'_{\text{sc}}})$.*
(ii) *If s is an eigenvalue of $A_{H'}$, then s^m is an eigenvalue of $A_{H'_{\text{sc}}}$. Conversely, if s is an eigenvalue of $A_{H'_{\text{sc}}}$, then $\sqrt[m]{s}$ is an eigenvalue of $A_{H'}$. Moreover, $\rho(A_{H'}) = \sqrt[m]{\rho(A_{H'_{\text{sc}}})}$.*
(iii) *The copy-paste trees $T_{H'}$ and $T_{\tilde{H}}$ have exponential growth rate $\rho(A_{H'})$.*
(iv) *T_H has exponential growth with growth rate $\rho(A_H) = \rho(A_{H'})$.*

Proof.

(i) One can show that, if $H'_{\rm sc}$ (respectively, $\widetilde{H}_{\rm sc}$) is scaleable, then m is not the greatest common divisor of $\{d(v_1, r), \ldots, d(v_n, r)\}$. Therefore, by Lemma 4.3 we have that $A_{H'_{\rm sc}}$ is primitive and, by Theorem 3.8, $T_{H'_{\rm sc}}$ and $T_{\tilde{H}_{\rm sc}}$ have exponential growth rate $\rho(A_{H'_{\rm sc}})$.

(ii) Let s be an eigenvalue of $A_{H'}$. By Lemma 3.7 (i), $\mathbf{v} = \begin{bmatrix}1 & s & s^2 & \cdots & s^{M-1}\end{bmatrix}^T$ is an eigenvector of $A_{H'}$ associated with s. Note that the last component in the vector equation $A_{H'}\mathbf{v} = s\mathbf{v}$ is

$$b_M + sb_{M-1} + s^2 b_{M-2} + \cdots + s^{M-1} b_1 = s^M, \tag{4.3}$$

where b_i denotes $b_i(H)$.

Since $m \mid M$, there exists $k \in \mathbb{Z}^+$ such that $M = km$. Since $m > 1$, the only possible nonzero entries in the last row of $A_{H'}$ have the form $b_{mi}(H)$ for some $i \in \mathbb{Z}^+$. Thus equation (4.3) can be written as

$$s^{mk} = b_{mk} + s^m b_{m(k-1)} + \cdots + s^{m(k-1)} b_m. \tag{4.4}$$

Since $b_i(H'_{\rm sc}) = b_{mi}(H') = b_{mi}(H)$ and $H'_{\rm sc}$ has height k, the copy-paste matrix of $H'_{\rm sc}$ is given by

$$A_{H'_{\rm sc}} = \left[\begin{array}{c|cccc} 0_{k-1,1} & I_{k-1,k-1} & & & \\ \hline b_{mk} & b_{m(k-1)} & \cdots & b_{2m} & b_m \end{array}\right]. \tag{4.5}$$

Using equation (4.4), a simple calculation verifies that s^m is an eigenvalue of $A_{H'_{\rm sc}}$.

Similarly, if s is an eigenvalue of $A_{H'_{\rm sc}}$, then

$$\mathbf{w} = \begin{bmatrix}1 & s & s^2 & \cdots & s^{k-1}\end{bmatrix}^T$$

is an eigenvector associated to s where $km = M$. Since the only possible nonzero entries in the last row of $A_{H'}$ have the form $b_{mi}(H)$, that is, $b_{M-j}(H) = b_{mk-j}(H) = 0$ for $j \neq im$, we have

$$\begin{aligned} b_{mk} &+ b_{M-1}s^{1/m} + b_{M-2}s^{2/m} + \cdots + b_{M-(m-1)}s^{1-1/m} \\ &+ b_{m(k-1)}s + b_{M-(m+1)}s^{1+1/m} + \cdots + b_{M-(2m-1)}s^{2-1/m} \\ &\qquad + \cdots + b_m s^{k-1} + \cdots + b_1 s^{k-1/m} = s^k, \end{aligned} \tag{4.6}$$

because the coefficients of all non-integer powers of s in the above expression are zero.

Another straightforward calculation verifies that

$$\begin{bmatrix} 1 & s^{1/m} & s^{2/m} & \cdots & s & \cdots & s^2 & \cdots & s^{k-1/m} \end{bmatrix}^T$$

is an eigenvector of $A_{H'}$ with corresponding eigenvalue $s^{1/m}$, where equation (4.6) gives the last entry in the matrix computation.

(iii) We will only prove the result for $T_{H'_{\text{sc}}}$. (The case of $T_{\tilde{H}_{\text{sc}}}$ is similar.) Because $T_{H'_{\text{sc}}}$ has exponential growth with growth rate $\lambda = \rho(A_{H'_{\text{sc}}})$, there exist positive constants α_1 and α_2 such that

$$\alpha_1 \lambda^n \leq f(T_{H'_{\text{sc}}}, n) \leq \alpha_2 \lambda^n.$$

Thus, using part (ii) of Observation 4.5,

$$f(T_{H'_{\text{sc}}}, n) = \sum_{i=0}^{n} c_i(T_{H'_{\text{sc}}}) = \sum_{i=0}^{n} c_{im}(T_{H'}) \leq \sum_{i=0}^{nm} c_i(T_{H'}) = f(T_{H'}, nm).$$

By Observation 4.5 (ii), it also follows that

$$\begin{aligned} f(T_{H'}, nm) &= c_0(T_{H'}) \\ &\quad + (c_1(T_{H'}) + \cdots + c_m(T_{H'})) \\ &\quad + (c_{m+1}(T_{H'}) + \cdots + c_{2m}(T_{H'})) \\ &\quad \vdots \\ &\quad + (c_{(n-1)m+1}(T_{H'}) + \cdots + c_{nm}(T_{H'})) \\ &\leq m(c_0(T_{H'}) + c_m(T_{H'}) + c_{2m}(T_{H'}) + \cdots + c_{nm}(T_{H'})) \\ &= m(c_0(T_{H'_{\text{sc}}}) + c_1(T_{H'_{\text{sc}}}) + \cdots + c_n(T_{H'_{\text{sc}}})) \\ &= m f(T_{H'_{\text{sc}}}, n). \end{aligned} \tag{4.7}$$

Therefore,

$$f(T_{H'_{\text{sc}}}, n) \leq f(T_{H'}, nm) \leq m f(T_{H'_{\text{sc}}}, n).$$

Hence,

$$\alpha_1 \lambda^n \leq f(T_{H'}, nm) \leq (m\alpha_2)\lambda^n,$$

and so

$$\alpha_1 (\lambda^{1/m})^{nm} \leq f(T_{H'}, nm) \leq (m\alpha_2)(\lambda^{1/m})^{nm}.$$

It now follows that $T_{H'}$ has exponential growth with growth rate $\sqrt[m]{\lambda}$ which equals $\rho(A_{H'})$ by (ii). Again, from a similar argument, it follows that $T_{\tilde{H}}$ has the same growth rate $\rho(A_{H'})$.

(iv) Since $T_{H'}$ and $T_{\tilde{H}}$ have the same growth rate, namely, $\rho(A_H) = \rho(A_{H'})$, it follows from equation (4.2) that T_H has growth rate $\rho(A_H) = \rho(A_{H'})$. □

To summarize, we have proved our main result:

Theorem 2.6. *Let T_H be a copy-paste tree with generator H and copy-paste matrix*

$$A_H = \left[\begin{array}{c|cccc} 0_{N-1,1} & I_{N-1,N-1} & & & \\ \hline b_N & b_{N-1} & \cdots & b_2 & b_1 \end{array}\right].$$

Then T_H has exponential growth with growth rate $\rho(A_H)$.

5. Examples. Next, we calculate the growth rates of the copy-paste trees considered throughout this paper.

Example 5.1. Let T be the infinite full binary tree (Example 2.4) with generators H_1 and H_2 shown in Figure 2. Note that $A_{H_1} = [2]$ and

$$A_{H_2} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}.$$

As expected, T has exponential growth with growth rate $\rho(A_{H_1}) = \rho(A_{H_2}) = 2$.

Example 5.2. Let H be the generator of the copy-paste tree T_H of Example 2.5, see Figure 3. Note that

$$A_H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 2.6, T_H has exponential growth with growth rate $\rho(A_H) \approx 1.3803$.

Example 5.3. Let H be the generator of the copy-paste tree T_H of Example 4.1 (depicted in Figure 6). Note that $N = 6$, $M = 4$ and

$$A_{H'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

(Since $N > M$, it is convenient to compute the growth rate by using the smaller-sized matrix $A_{H'}$.) Then T_H has exponential growth with growth rate $\rho(A_{H'}) = \sqrt{2}$.

Our next example illustrates a class of trees called *shuffled copy-paste trees*, which are obtained by permuting components of the shells of a given copy-paste tree. Such trees obviously have identical growth functions to the original copy-paste tree.

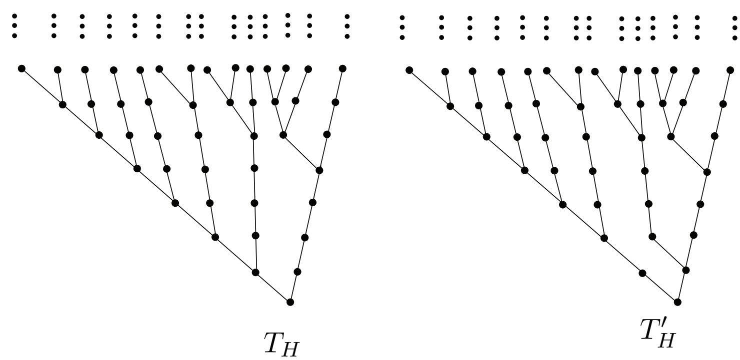

FIGURE 8. The permuted tree T'_H is not self-similar; it has the same growth rate as T_H.

Example 5.4. Consider the generator described in Example 2.5; however, to simplify the construction, we include a vertex in H at the midpoint of (r, w). To build T'_H, realize that the components of the shells of T_H can be permuted to obtain a new tree. In particular, at S_2, permute the edge on the right with the wedge shape on the left to obtain a new tree T'_H that is not self-similar.

Clearly, one could also create a shuffled copy-paste tree from T_H in Example 5.4, that does not have finitely many cones, by producing a tree having non-branching paths of any integer length having the same growth rate as T_H.

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