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MODULES WHOSE CERTAIN SUBMODULES ARE ESSENTIALLY EMBEDDED IN DIRECT SUMMANDS

YELIZ KARA AND ADNAN TERCAN

ABSTRACT. It is well known that, if the ring has acc on essential right ideals, then for every quasi-continuous module over the ring, the finite exchange property implies the full exchange property. In this paper, we obtain the former implication for the generalizations of quasi-continuous modules over a ring with acc on right annhilators of elements of the module. Moreover, we focus on direct sums and direct summands of weak C_{12} modules i.e., modules with the property that every semisimple submodule can be essentially embedded in a direct summand. To this end, we prove that since weak C_{12} is closed under direct sums. Amongst other results, we provide several counterexamples including the tangent bundle of a real sphere of odd dimension over its coordinate ring for the open problem of whether weak C_{12} implies the C_{12} condition.

1. Introduction. All rings are associative with unity and modules are unital right modules. We use R to denote such a ring and M to denote a right R-module. Recall that a module is called *extending* or CS or said to satisfy the C_1 condition if every submodule is essential in a direct summand; equivalently, every complement submodule is a direct summand. This condition has proved to be an important common generalization of the injective, semisimple and uniform module notions. There have been a number of generalizations of the extending property, including the following:

(1) M is a weak CS [**10**] if every semisimple submodule of M is essential in a direct summand of M;
(2) M is a C_{11}-module [**12**] if each submodule of M has a complement that is a direct summand of M;

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(3) M is a C_{12}-module [**12**] if each submodule can be essentially embedded in a direct summand of M, and
(4) M is *PI-extending* [**3**] if every projection invariant submodule (i.e., every submodule such that the image under all idempotent endomorphisms contained in itself) is essential in a direct summand of M.

In a similar way to weak CS-modules [**10**], weak C_{11} and weak C_{12} modules were introduced in [**6, 15**]. Recall that a module M is a *weak* C_{11} (C_{12})-*module* if each semisimple submodule of M has a complement that is a direct summand (if each semisimple submodule of M can be essentially embedded in a direct summand) of M.

In this paper, we prove that if the ring R has acc (ascending chain condition) on right annihilators $r(m) = \{r \in R \mid mr = 0\}$, where $m \in M$ and M_R satisfy C_{11} and C_3 (or C_{12} and C_2) then the finite exchange property implies the full exchange property. We also obtain the PI-extending version of the result on continuous modules [**9,** Proposition 3.5] when the endomorphism ring of the module is Abelian.

Further, we focus our attention on weak C_{12}-modules as a proper generalization of extending modules. It is well known that a direct summand of an extending module is extending, but a direct sum of extending modules is not an extending module, e.g., let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$, where p is any prime integer (see [**12,** page 1814]). In contrast to extending modules, we show that any direct sum of weak C_{12}-modules is a weak C_{12}-module. Since we are unable to settle at this time whether a direct summand of a weak C_{12}-module needs to be a weak C_{12}-module we obtain a positive answer for this question under some conditions. Recall that, whether weak C_{12} implies C_{12} was left as a problem in [**6,** page 496]. However, we provide several counterexamples which exhibit the failure of the problem. To this end, we observe that tangent bundles of all real spheres of odd dimensions over their coordinate rings have weak C_{12} but not the C_{12} property. We have then, for any module, the following implications:

$$\begin{array}{ccccc} CS & \Longrightarrow & C_{11} & \Longrightarrow & C_{12} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{weak } CS & \Longrightarrow & \text{weak } C_{11} & \Longrightarrow & \text{weak } C_{12} \end{array}$$

No other implications can be added to this table in general. To see why this is the case, we refer to [**12,** page 1814], [**12,** Proposition 3.6], [**10,** Example 1.1], Example 3.7, [**17,** Counter Example 3] and [**13,** Proposition 2.6].

Let R be any ring and M a right R-module. If $X \subseteq M$, then $X \leq M$ and $\operatorname{Soc} X$ denote X as a submodule of M and the socle of X, respectively. A module M is called *locally Noetherian* if every finitely generated submodule is Noetherian. Further, a ring is called *Abelian* if every idempotent is central. For any unexplained terminology, definitions and notation, see [**1, 4, 5, 7, 9**].

2. Generalizations of extending modules with C_2 or C_3. Let R be a ring and M a right R-module. Recall the following conditions for M:

C_3: for all direct summands K and L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.

C_2: for each direct summand N of M and each monomorphism $\varphi : N \to M$, the submodule $\varphi(N)$ is also a direct summand of M.

A CS-module with (C_3) C_2 is called a (*quasi*) *continuous* module. For good references on these notions, see [**5, 9**]. In this section, we mainly work with the general form of (quasi) continuous modules. We begin by proving a basic fact about indecomposable modules with Goldie dimension 1.

Lemma 2.1. *Let R be a right Noetherian ring and M a nonzero indecomposable right R-module. Then the following conditions are equivalent*:

(i) *M is uniform,*
(ii) *M has C_{11},*
(iii) *M has C_{12},*
(iv) *M is PI-extending.*

Proof.

(i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). By [**12,** Proposition, 3.2]

(iii) $\Rightarrow$ (iv). By hypothesis, every submodule of M is projection invariant in M. Let X be a nonzero submodule of M. Then there exists a nonzero element $x \in X$ and $R/r(x) \cong xR$. Thus, there exists a uniform submodule U of M such that $U \leq xR \leq X \leq M$. By hypothesis, there exists a monomorphism $\varphi : U \to M$ such that $\varphi(U)$ is essential in M. Since $\varphi(U) \leq_e M$ and φ is a monomorphism, we have U is essential in M, and hence X is essential in M.

(iv) $\Rightarrow$ (i). Let N be a nonzero submodule of M. Since M is indecomposable, N is a projection invariant submodule of M. By (iv), there exists a direct summand K of M such that N is essential in K. Hence, $K = M$. Thus, N is essential in M so M is uniform. □

Corollary 2.2. *Let R be a right locally Noetherian ring and M a nonzero indecomposable right R-module. Then the following statements are equivalent.*

(i) *M is uniform,*
(ii) *M has C_{11},*
(iii) *M has C_{12},*
(iv) *M is PI-extending.*

Proof. The proof is immediate by Lemma 2.1. □

Lemma 2.3. *Let R be a ring and M an R-module such that R satisfies* acc *on right ideals of the form $r(m)$ where $m \in M$. If M_R satisfies*

(a) *C_{11} and C_3,*

or

(b) *C_{12} and C_2,*

then M has an indecomposable decomposition.

Proof. The proof follows from **[12,** Lemma 4.6] and **[9,** Theorem 2.17]. □

Proposition 2.4. *The following statements are equivalent for a nonsingular C_{11}-module M which satisfies C_3.*

(i) *M has an indecomposable decomposition.*

(ii) *Every finitely generated submodule of M has finite uniform dimension.*
(iii) *Every cyclic submodule of M has finite uniform dimension.*
(iv) *R satisfies* acc *on right ideals of the form $r(m)$ where $m \in M$.*

Proof. (i) $\Rightarrow$ (ii). There exist an index set I and indecomposable submodules M_i $(i \in I)$ of M such that $M = \oplus_{i \in I} M_i$. Since M is C_{11}-module with C_3, M_i is also C_{11} by [**12,** Theorem 4.3]. It follows that M_i $(i \in I)$ is uniform. If L is a finitely generated submodule of M, then $L \subseteq \oplus_{i \in J} M_i$ for some finite subset J of I, and hence, L has finite uniform dimension.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (iv). Let $m \in M$. Suppose that $r(m)$ is essential in a right ideal A of R. Let $a \in A$. There exists an essential right ideal E of R such that $aE \subseteq r(m)$. It follows that $maE = 0$, and hence, $ma = 0$ so $a \in r(m)$. Thus, $r(m) = A$. Hence, $r(m)$ is a complement in R-module R for each $m \in M$. Moreover, $R/r(m) \cong mR$, gives that the R-module $R/r(m)$ has finite uniform dimension. Now (iv) follows by [**5,** Theorem 5.10].

(iv) $\Rightarrow$ (i). Follows by [**12,** Theorem 4.7] and [**9,** Theorem 2.17]. □

Combining Proposition 2.4 together with [**18,** Corollary 6] gives the following result on the exchange property of modules without further proof.

Theorem 2.5. *Let M be a nonsingular module such that mR has finite uniform dimension for each $m \in M$. If M satisfies C_{11} and C_3, then the finite exchange property implies the full exchange property.*

Corollary 2.6. *Let R be a ring with finite uniform dimension and M a nonsingular right R-module which satisfies C_{11} and C_3. Then the finite exchange property implies the full exchange property.*

Proof. Let M and R be as stated. Let $m \in M$. Then $r(m)$ is a complement in the right R-module R. Hence, the R-module $R/r(m)$ has finite uniform dimension. It is clear that $mR \cong R/r(m)$. Now apply Theorem 2.5 to get the result. □

Recall that, over a ring with acc on essential right ideals, the finite exchange property implies the full exchange property for every quasi-continuous module [**16**]. We have the following result in this trend.

Theorem 2.7. *Let R be a ring with* acc *on right ideals of the form $r(m)$ where $m \in M$ and M is a right R-module. If M_R satisfies*

(a) C_{11} *and* C_3,

or

(b) C_{12} *and* C_2,

then the finite exchange property implies the full exchange property.

Proof. By Lemma 2.3, M has an indecomposable decomposition. Then the result follows from [**18,** Corollary 6]. □

The next example shows that the assumptions of the above theorem do not imply the quasi-continuity of the module.

Example 2.8. Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}_p)\oplus\mathbb{Q}$ for any prime p. Then M satisfies C_{11} and C_2 as well as C_{12} and C_3 by [**12,** Proposition 3.2] and [**9,** Proposition 2.2]. Since $M_{\mathbb{Z}}$ is not a CS module, it is not a quasi-continuous module (see [**12,** Example 4.2]).

Corollary 2.9. *Let M be a locally Noetherian module with C_{11} and C_3. Then the finite exchange property implies the full exchange property.*

Proof. Let $m \in M$. Since $R/r(m) \cong mR$, $R/r(m)$ is a Noetherian right R-module. It follows that R satisfies acc on right ideals of the form $r(x)$, $x \in M$. Then Theorem 2.7 yields the result. □

One might ask whether a PI-extending module with the full exchange property and C_3 condition (or C_2) implies that the endomorphism ring of the module is Abelian or not? However, we provide examples which eliminate these possibilities.

Example 2.10. Let $R = \mathbb{Z}_{(p)}$ be the localization of integers $\mathbb{Z}$ at a prime p. Put $M_R = \mathbb{Z}_{(p)} \oplus \mathbb{Z}(p^{\infty})$. Then M_R is a PI-extending module with C_3 which does not satisfy C_2. Moreover, M has the full exchange property whose endomorphism ring has noncentral idempotents.

Proof. Observe that M_R is quasi-continuous by [**16,** Example 2.13]. Hence, M_R is PI-extending with C_3 by [**3,** Proposition 3.7]. Now, let us consider $f \in \operatorname{End}(M_R)$ such that $f(x,y) = (px, y)$ where $(x,y) \in M$. It is easy to see that f is a monomorphism and $f(M) = p\mathbb{Z}_{(p)} \oplus \mathbb{Z}(p^\infty)$ which is an essential submodule of M. By [**9,** Lemma 3.14], M_R does not satisfy the C_2 condition. For the last part, see [**16,** Example 2.13]. □

Example 2.11. Let $M_2(R)$ be the matrix ring as in [**2,** Example 2.7]. Let $T = M_2(R)$. Since T_T is a C_{11}-module then T_T is a PI-extending module by [**3,** Proposition 3.7]. Note that T_T has the full exchange property by [**9,** Theorem 3.24]. Moreover, it can be seen easily that the endomorphism ring of T is not Abelian.

The next few results concern the endomorphism ring of PI-extending modules. To this end, we refer to [**14, 15**] for the corresponding results in terms of C_{11}-modules and *weak* C_{11}-modules, respectively. We will use S and $J(S)$ to denote the endomorphism ring of a module M and the Jacobson radical of S, respectively. Further, Δ will stand for the ideal $\{\alpha \in S \mid \ker\alpha \text{ is essential in } M\}$.

Theorem 2.12. *Let M_R be a PI-extending module with the C_2 condition, and let S be an Abelian ring. Then S/Δ is a (von Neumann) regular ring and $\Delta = J(S)$.*

Proof. Let $\alpha \in S$, $K = \ker\alpha$. Let $f^2 = f \in S$, and let $y \in f(K)$. Then there exists an element k of K such that $y = f(k)$. So $\alpha(y) = \alpha(f(k))$. Therefore, $\alpha(y) = \alpha(f(k)) = f(\alpha(k)) = 0$. Hence, $y \in K$. It follows that K is a projection invariant submodule of M. By hypothesis, there exists a direct summand L of M such that L is a complement of K in M. Since $\alpha|_L$ is a monomorphism then $\alpha(L)$ is a direct summand of M, by C_2. Hence, there exists $\beta \in S$ such that $\beta\alpha = 1|_L$. Then

$$(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = 0,$$

and so $K \oplus L$ is a submodule of $\ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L$ is essential in M then $\alpha - \alpha\beta\alpha \in \Delta$. Therefore, S/Δ is a regular ring. This also proves that J is contained in Δ.

Let $a \in \Delta$. Since $\ker a \cap \ker(1-a) = 0$ and $\ker a$ is essential in M then $\ker(1-a) = 0$. Hence, $(1-a)M$ is a direct summand of M by C_2. However, $(1-a)M$ is essential submodule of M since $\ker a$ is a submodule of $(1-a)M$. Thus, $(1-a)M = M$, therefore $1-a$ is a unit in S. Hence, $a \in J$. It follows that $\Delta = J$. □

Corollary 2.13. *Let M be a nonsingular right R-module. If M is a PI-extending module with C_2 condition and S is Abelian, then S is a regular ring.*

Proof. Let $g \in \Delta$ and $N = \ker g$. Then, for any $x \in M$, we build up the following set

$$L = \{r \in R \mid xr \in N\}.$$

Then clearly L is a right ideal of R and also L is essential in R. Now, $g(x)L = 0$. Since M is nonsingular then $g(x) = 0$, and since x is arbitrary, $g = 0$. Therefore, $\Delta = 0$. Hence, the result follows from Theorem 2.8. □

Note that C_2 cannot be replaced by C_3 in Theorem 2.12 as the following example illustrates. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}$. Obviously, $M_{\mathbb{Z}}$ is a PI-extending module with C_3 and $S = \operatorname{End}(M_{\mathbb{Z}}) \cong \mathbb{Z}$ is Abelian. But $\Delta = \operatorname{End}(M_{\mathbb{Z}}) \cong \mathbb{Z}$ and $0 = J(S) \neq \Delta$.

We conclude this section with the following example which demonstrates both Theorem 2.7 and Theorem 2.12.

Example 2.14. Let M be the module as in Example 2.8, i.e., let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}_p) \oplus \mathbb{Q}$. Then $M_{\mathbb{Z}}$ is a PI-extending module with C_2 and the endomorphism ring of M is Abelian.

Proof. From Example 2.8 and [**3,** Proposition 3.7], $M_{\mathbb{Z}}$ is a PI-extending module. Moreover, [**12,** Example 4.2] shows that $M_{\mathbb{Z}}$ satisfies the C_2 property. Now,

$$S = \operatorname{End}(M) \cong \begin{bmatrix} \operatorname{End}(\mathbb{Z}/\mathbb{Z}p) & \operatorname{Hom}(\mathbb{Q}, \mathbb{Z}/\mathbb{Z}p) \\ \operatorname{Hom}(\mathbb{Z}/\mathbb{Z}p, \mathbb{Q}) & \operatorname{End}(\mathbb{Q}) \end{bmatrix} = \begin{bmatrix} \mathbb{Z}/\mathbb{Z}p & 0 \\ 0 & \mathbb{Q} \end{bmatrix}$$

which is Abelian. □

3. Weak C_{12} modules. In this section we focus our attention on weak C_{12} modules, i.e., modules with the property that each semisimple submodule can be essentially embedded in a direct summand. Recall that a direct sum of extending modules is not an extending module in general (see, for example, Example 2.14). However, we have the following closure property which shows that any direct sum of weak C_{12} modules are also a weak C_{12}-module.

Theorem 3.1. *Any direct sum of modules with the weak C_{12} property satisfies weak C_{12}.*

Proof. Let M_λ $(\lambda \in \Lambda)$ be a nonempty collection of modules, each satisfying weak C_{12}. Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$, and let N be any semisimple submodule of M. Let Λ' be a nonempty subset of Λ containing λ such that $M' = \oplus_{\lambda\in\Lambda'} M_\lambda$. Assume that M' satisfies the weak C_{12} property. Then for any semisimple submodule N' of M' there exists a direct summand K' of M' and a monomorphism $\alpha_1 : N' \to K'$ such that $\alpha_1(N')$ is essential in K'. Suppose $\Lambda \neq \Lambda'$. Then there exists $\mu \in \Lambda$, $\mu \notin \Lambda'$. Let $\Lambda'' = \Lambda' \cup \{\mu\}$ and $M'' = \oplus_{\lambda\in\Lambda''} M_\lambda = M' \oplus M_\mu$. Since M_μ is a weak C_{12} module, for any semisimple submodule N_μ of M_μ there exist a direct summand K_μ of M_μ and a monomorphism $\alpha_2 : N_\mu \to K_\mu$ such that $\alpha_2(N_\mu)$ is essential in K_μ. It is clear that $K' \cap K_\mu = 0$. Let $K'' = K' \oplus K_\mu$. Note that K'' is a direct summand of M''. Consider the semisimple submodule $N' \oplus N_\mu$ of M''. Define $\beta : N' \oplus N_\mu \to K' \oplus K_\mu$ by

$$\beta(n) = \beta(m_1 + m_2) = \alpha_1(m_1) + \alpha_2(m_2),$$

where $n \in N' \oplus N_\mu$, $m_1 \in N'$ and $m_2 \in N_\mu$. It is easy to check that β is a monomorphism. Furthermore,

$$\beta(N' \oplus N_\mu) = \alpha_1(N') \oplus \alpha_2(N_\mu)$$

is an essential submodule of $K' \oplus K_\mu$ by [**1,** Proposition 5.20]. When using the transfinite induction argument, there exist a direct summand K of M and a monomorphism $\gamma : N \to K$ such that $\gamma(N)$ is essential in K. Thus, M satisfies weak C_{12}. □

The proof of the above theorem brings us to the following observation. Although the statement of [**13,** Theorem 1.2] is true, its proof is not complete at all. Incidentally, the idea in the proof of Theorem 3.1

works exactly by replacing semisimple submodules with submodules to obtain [**13,** Theorem 1.2].

Corollary 3.2. *Any direct sum of modules which satisfy the weak C_{11} (respectively, one of the conditions, extending, uniform or C_{11}) satisfy weak C_{12}.*

Proof. Immediate by Theorem 3.1. □

After applying Theorem 3.1, we have the next easy fact on modules over Dedekind domains.

Corollary 3.3. *Let R be a Dedekind domain and M an R-module with finite uniform dimension. Then M is a weak C_{12} module.*

Proof. It follows from [**10,** Corollary 1.17] and Corollary 3.2. □

Recall that C_{11} and also C_{12} properties are not inherited by direct summands (for details see, [**11, 13, 15**]). We do not know whether direct summands of a weak C_{12}-module need to be weak C_{12} or not, so far. Now we deal with some special cases for the former question.

Corollary 3.4. *Let M be a right R-module and $M = U \oplus V$ where U and V are uniform submodules. Then every direct summand of M is a weak C_{12}-module.*

Proof. Let $0 \neq K$ be a direct summand of M. If $K = M$, then K has a weak C_{12} from Corollary 3.2. If $K \neq M$, then K is uniform. Hence, K has a weak C_{12}. □

Theorem 3.5. *Let M be a $\mathbb{Z}$-module such that M is a direct sum of uniform modules. Then every direct summand of M is a weak C_{12}-module.*

Proof. Let N be a direct summand of M. Then N is also a direct sum of uniform modules by [**12,** Theorem 5.5]. Now, Corollary 3.2 yields that N satisfies weak C_{12}. □

The next result provides a condition which ensures that a direct summand of a module is a weak C_{12}-module.

Theorem 3.6. *Let $M = M_1 \oplus M_2$. Then M_1 satisfies weak C_{12} if and only if, for every semisimple submodule N of M_1, there exist a direct summand K of M and φ monomorphism on N such that $M_2 \subseteq K$, $\varphi(N) \cap K = 0$ and $\varphi(N) \oplus K$ is an essential submodule of M.*

Proof. Suppose M_1 satisfies weak C_{12}. Let N be any semisimple submodule of M_1. There exist a direct summand L of M_1 and a monomorphism $\varphi : N \to L$ such that $\varphi(N)$ is an essential submodule of L. So $M_1 = L \oplus L'$ for some submodule L' of M_1. Clearly, $L' \oplus M_2$ is a direct summand of M, $(L' \oplus M_2) \cap \varphi(N) = 0$ and $(L' \oplus M_2) \oplus \varphi(N)$ is an essential submodule of M.

Conversely, suppose M_1 has the stated property. Let H be a semisimple submodule of M_1. By hypothesis, there exist a direct summand K of M and a monomorphism on H such that $M_2 \subseteq K$, $\varphi(H) \cap K = 0$ and $\varphi(H) \oplus K$ is an essential submodule of M. Now

$$K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2,$$

so that $K \cap M_1$ is a direct summand of M, and hence also of M_1. Let $M_1 = (K \cap M_1) \oplus X$ for some submodule X of M_1 and let $\pi : M \to X$ be the canonical projection with kernel K. Define $f : H \to X$ by $f(h) = \pi(\varphi(h))$ where $h \in H$. It is easy to check that f is a monomorphism. Let $0 \neq x \in X$. Then there exists an $r \in R$ such that $0 \neq xr \in \varphi(H) \oplus K$. It follows that $xr = \pi(xr) = f(h) + \pi(k)$ for some $h \in H$, $k \in K$. So

$$0 \neq xr = f(h) \in f(H).$$

Hence, $f(H)$ is an essential submodule of X. Thus, M_1 satisfies the weak C_{12} condition. □

A question posed in [**6,** page 496] which asks whether the weak C_{12} condition implies the C_{12} condition or not? Our final concern is to answer this question negatively by providing several counter examples. First, note that the next two examples are based on the Abelian group, i.e., the $\mathbb{Z}$-module and the torsion-free module over a principal ideal domain.

Example 3.7. The Specker group satisfies the weak C_{12} but does not satisfy C_{12}.

Proof. Let M be the Specker group $\prod_{i=1}^{\infty}\mathbb{Z}$. First note that $M_{\mathbb{Z}}$ does not satisfy C_{12} by [**12,** Lemma 3.4]. Now $M_{\mathbb{Z}}$ is nonsingular by [**7,** Proposition 1.22]. Hence, [**7,** Corollary 1.26] gives that $M_{\mathbb{Z}}$ has zero socle. So $M_{\mathbb{Z}}$ has the weak C_{12} condition. □

Example 3.8. Let R be a principal ideal domain. If R is not a complete discrete valuation ring, then there exists an indecomposable torsion-free R- module M of rank 2 by [**8,** Theorem 19]. For M, $\operatorname{Soc} M = 0$. Hence, clearly M satisfies the weak C_{12} condition. However, M_R has uniform dimension 2. By Lemma 2.1, M_R does not have the C_{12} property.

Although Examples 3.7 and 3.8 show the existence of counter examples for the aforementioned question, surprisingly we can provide more algebraic topology type examples in the following result.

Theorem 3.9. *Let $\mathbb{R}$ be the real field and n any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ indeterminates $x_1, \ldots, x_n$ over $\mathbb{R}$. Let R be the ring S/Ss, where $s = x_1^2 + \cdots + x_n^2 - 1$. Then the free R-module $M = \oplus_{i=1}^{n} R$ contains a submodule K_R which satisfies weak C_{12} but does not satisfy C_{12}.*

Proof. It is clear that R is a commutative Noetherian domain. Let $\varphi : M \to R$ be the homomorphism defined by

$$\varphi(a_1 + Ss, \ldots, a_n + Ss) = a_1x_1 + \cdots + a_nx_n + Ss$$

for all a_i in S, $1 \leq i \leq n$. Clearly, φ is an epimorphism and hence its kernel K is a direct summand of M, i.e., $M = K \oplus K'$ for some submodule $K' \cong R$. Obviously, K is not uniform. Note that K is the R-module of regular sections of the tangent bundle of the $(n-1)$-sphere S^{n-1}. Since the Euler characteristic $\chi(S^{n-1}) \neq 0$ it follows that the $(n-1)$-sphere cannot have a nonvanishing regular section of its tangent bundle (see, [**4,** Corollary VI.13.3]). Now K_R has zero socle and hence it satisfies the weak C_{12} condition. However, K_R has uniform dimension $n-1$ which yields that K_R does not satisfy the C_{12} property by Lemma 2.1. □

Remark 3.10.

(i) If n is 1 or 2 in Theorem 3.9, then K_R is isomorphic to 0 or R, respectively. In these cases, K_R has C_{12} and so too does the weak C_{12}.

(ii) If n is any even integer with $n \geq 4$, then the proof of Theorem 3.9 does not work. For example, S^3, S^5 and S^7 all have decomposable tangent bundles by the result of Adams (see [**4**]) and in these cases, K_R is isomorphic to a (finite) direct sum of uniform modules. Hence, K_R has C_{12} by [**13,** Theorem 1.2] and also satisfies the weak C_{12}.

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Department of Mathematics, Hacettepe University, Beytepe Campus, Ankara 06532, Turkey
Email address: **yelizkara@hacettepe.edu.tr**

Department of Mathematics, Hacettepe University, Beytepe Campus, Ankara 06532, Turkey
Email address: **tercan@hacettepe.edu.tr**