RIEMANN HYPOTHESIS FOR THE GOSS t-ADIC ZETA FUNCTION

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ABSTRACT. In this short note, we give a proof of the Riemann hypothesis for the Goss v-adic zeta function $\zeta_v(s)$, when v is a prime of $\mathbb{F}_q[t]$ of degree one.

The Riemann hypothesis says that the non-trivial complex zeros of the Riemann zeta function all lie on a line $\operatorname{Re}(s) = 1/2$ in the complex plane. An analog [2, 5, 7] of this statement was proved for the Goss zeta function for $\mathbb{F}_q[t]$, for q a prime by Wan [7] (see also [1]) and for general q by Sheats [4]. The proofs use the calculation of the Newton polygons associated to the power series these zeta functions represent, and their slopes are calculated or estimated by the degrees of power sums which make up the terms. In [1, 4], the exact degrees derivable (as noticed by Thakur) from the incomplete work of Carlitz were justified and the Riemann hypothesis was derived.

In this paper, we look at the v-adic Goss zeta function and prove an analog of the Riemann hypothesis, in the case where v is a prime of degree one in $\mathbb{F}_q[t]$, using the valuation formulas for the corresponding power sums given essentially (see below) in [6] and following the method of [1], [5, subsection 5.8]. We note that, for $q = 2$, this was already shown by Wan [7] using the earlier calculation at the infinite place. See below for details. We also note that the situation for higher degree v is different [6, Section 10] and is not fully understood.

We now give the relevant definitions and describe the results precisely.

In the function field-number field analogy, we have $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, $K_\infty = \mathbb{F}_q((1/t))$ and $\mathbb{C}_\infty$, the completion of an algebraic closure of K_∞ as analogs of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We consider the v-adic situation where v is an irreducible polynomial of A and K_v

is the completion of K at v, as an analog of the p-adic situation, where p is a prime in $\mathbb{Z}$ and $\mathbb{Q}_p$ is the field of p-adic numbers.

For v a prime of $\mathbb{F}_q[t]$, the *v-adic zeta function of Goss* is defined on the space $S_v := \mathbb{C}_v^* \times \varprojlim_j \mathbb{Z}/(q^{\deg v} - 1)p^j\,\mathbb{Z}$, where $\mathbb{C}_v$ is the completion of an algebraic closure of K_v and the $\varprojlim_j \mathbb{Z}/(q^{\deg v} - 1)p^j\mathbb{Z}$ is isomorphic to the product of $\mathbb{Z}_p$ with the cyclic group $\mathbb{Z}/(q^{\deg v} - 1)\mathbb{Z}$. See [**2**, subsection 8.3], [**5**, subsection 5.5] for motivation and details.

For $s = (x, y) \in S_v$, the Goss v-adic zeta function is then defined as

$$\zeta_v(s) = \sum_{d=0}^{\infty} x^d \sum_{\substack{a \in \mathbb{F}_q[t],\ \text{monic} \\ \deg(a)=d \\ (v,a)=1}} a^y.$$

Note that a^y here, by Fermat's little theorem, is a p-adic power of a one-unit at v, and thus makes sense. For $y \in \mathbb{Z}/(q^{\deg v} - 1)\mathbb{Z} \times \mathbb{Z}_p$ we write

$$S_{d,v}(y) := \sum_{\substack{a \in \mathbb{F}_q[t],\ \text{monic} \\ \deg(a)=d \\ (v,a)=1}} a^y,$$

and let

$$v_d(y) := \mathrm{val}_v(S_{d,v}(y)).$$

We focus on v of degree one, so that without loss of generality we assume that $v = t$.

In this case, the Riemann hypothesis is the statement that, for a fixed y, all zeros in x of $\zeta_v(s)$ are in $\mathbb{F}_q((t)) = K_t$ (inside a much bigger field $\mathbb{C}_t$), and they are simple zeros, i.e., with multiplicity 1.

We note that, for $q = 2$ and $v = t$, Wan [**7**] proved this and, for general q, Goss [**3**, Proposition 9] got the same result for y in $(q-1)S_v$. This last result can also be derived from the results obtained by Thakur in [**6**, see Corollary 8]. We drop these restrictions.

Let q be a power of a prime p. Let n be any positive integer. If we can write n as a sum of non-negative integers $n_0 + \cdots + n_d$, so that when we consider $n_0, \ldots, n_d$ in their base-p expansions, there is no carry over of coefficients when we perform the sum in base-p, we

express this by writing $n = n_0 \oplus \cdots \oplus n_d$. Of all such decompositions of n, choose those such that $n_d \leq \cdots \leq n_1$. Having done so, we can order such decompositions of n by saying $n_0 \oplus \cdots \oplus n_d \leq m_0 \oplus \cdots \oplus m_d$ if for some $i, n_d = m_d, \ldots, n_i = m_i$, but $n_{i-1} < m_{i-1}$, so we will say that the minimal such decomposition is "given by the greedy algorithm."

The next theorem is a slight generalization of Theorem 7 (iii) in [**6**], which is the special case $z \equiv m \mod (q-1)$.

Theorem 1. *Let m be a positive integer and $y = (z, m) \in \mathbb{Z}/$ $(q^{\deg v} - 1)\mathbb{Z} \times \mathbb{Z}_p$. Let v be a prime of $\mathbb{F}_q[t]$ of degree one. Then either $v_d(y) = \min(m_1 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, $(q-1)$ divides $m_i > 0$ for $0 < i < d$, and $q - 1$ divides $z - m_d$; or $v_d(k)$ is infinite, if there is no such decomposition. When the decomposition exists, the minimum is uniquely given by the greedy algorithm. (If the least non-negative residue $\mod q - 1$ of z is r, m_d is the least possible sum of p-powers chosen from the p-expansion of m which is r modulo $q - 1$.)*

Proof. For $v = t$, we have

$$S_{d,v}(y) = \sum_{\substack{f_i \in \mathbb{F}_q \\ f_0 \neq 0}} f_0^z \left(1 + \frac{f_1}{f_0}t + \cdots + \frac{f_{d-1}}{f_0}t^{d-1} + \frac{t^d}{f_0}\right)^m$$

$$= \sum_{\substack{f_i \in \mathbb{F}_q \\ f_0 \neq 0}} f_0^z \sum_{\substack{m = m_0 + \cdots + m_d \\ m_i \geq 0, i = 0, \ldots, d}} \binom{m}{m_0, \ldots, m_d} \left(\frac{f_1}{f_0}\right)^{m_1} \cdots$$

$$\left(\frac{f_{d-1}}{f_0}\right)^{m_{d-1}} \left(\frac{1}{f_0}\right)^{m_d} t^{m_1 + \cdots + dm_d}$$

$$= \sum_{\substack{f_i \in \mathbb{F}_q \\ f_0 \neq 0}} \sum_{\substack{m = m_0 + \cdots + m_d \\ m_i \geq 0, i = 0, \ldots, d}} \binom{m}{m_0, \ldots, m_d} f_0^{z - m_d} f_1^{m_1} \cdots$$

$$f_{d-1}^{m_{d-1}} t^{m_1 + \cdots + dm_d}.$$

Now, by the theorem of Lucas $\binom{m}{m_0, \ldots, m_d}$ is non-zero if and only if the sum $m = m_0 + \cdots + m_d$ is such that there is no carry over of digits for base p. Changing the order of the summation and using the relation

$$\sum_{f \in \mathbb{F}_q} f^h = \begin{cases} 0 & \text{if } (q-1) \nmid h \text{ or } h = 0 \\ -1 & (q-1) \mid h, \end{cases}$$

we see that

$$S_{d,v}(y) = \sum_{\substack{f_i \in \mathbb{F}_q \\ f_0 \neq 0}} \sum_{\substack{m = m_0 \oplus \cdots \oplus m_d \\ (q-1)|m_i>0, i=1,\ldots,d-1 \\ (q-1)|z-m_d \\ m_0, m_d \geq 0}} \binom{m}{m_0, \ldots, m_d} f_0^{z-m_d} f_1^{m_1}$$

$$\cdots f_{d-1}^{m_{d-1}} t^{m_1 + \cdots + dm_d}.$$

By Sheats [**4**], $\min(m_1 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, $(q-1)$ divides $m_i > 0$ for $0 < i < d$, and $q - 1$ divides $z - m_d$ is unique. From this, it follows that $v_d(y) = \min(m_1 + \cdots + dm_d)$ is this unique minimum.   □

Remark 2. We can go further in Theorem 1. We should not only choose m_d that way, but also choose m_{d-1} to be the least possible sum of p-powers chosen from the expansion of $m - m_d$ which is $0 \bmod q - 1$, and so on.

In the case that $q = p$ a prime, we have the following special case of Theorem 1 (see [**6,** Theorem 9]).

Theorem 3. *Let $q = p$ be a prime, v a prime of $\mathbb{F}_q[t]$ of degree one, $y = (z, m) \in \mathbb{Z}/(q^{\deg v} - 1)\mathbb{Z} \times \mathbb{Z}_p$ and $m > 0$ an integer. Also, let r be the least non-negative residue of $z \bmod (q - 1)$. Write*

$$m = \sum_{i=1}^{\ell} p^{e_i},$$

with e_i monotonically increasing and with not more than $p - 1$ of the consecutive values being the same, i.e., consider the base p-digit expansion sequentially one digit at a time. Then $v_d(y)$ is infinite if

$\ell < (p-1)(d-1) + r$, *and otherwise*

$$v_d(y) = d\sum_{i=1}^{r} p^{e_i} + \sum_{j=1}^{d-1} j \sum_{i=1}^{p-1} p^{e(d-1-j)(p-1)+r+i}.$$

Proof. We have $v_d(y) = \min(m_1 + \cdots + dm_d)$, where $m = m_0 + \cdots + m_d$, when this sum has no carry over base p and $q-1$ divides $m_u > 0$ for $0 < u < d$ and $q-1$ divides $z - m_d$. Note that $p^u \equiv 1 \mod (q-1)$ when q is prime. Hence, $p-1$ powers together give divisibility by $q-1$. Therefore, for the minimum we choose r digits for m_d and $m_{d-1}, \ldots, m_1$ are obtained by picking $p-1$ digits from the base p expansion of $m - m_d$ starting from the lowest digits (and dumping the rest of the expansion, if any, into m_0). □

With respect to the location of the zeros of the Goss v-adic zeta function and its multiplicity, when the degree of v is one, Wan [7] showed that for $q = 2$, the zeros are simple with corresponding x in $\mathbb{F}_q((t))$. For general q, Goss [3, Proposition 9] got the same result for y in $(q-1)S_v$. This last result can also be derived from the results obtained by Thakur in [6, Corollary 8]. Here we prove the full Riemann hypothesis for the v-adic zeta function for v of degree one, that is, all the zeros of $\zeta_v(x, y)$ are simple with x in $\mathbb{F}_q((t))$ inside $\mathbb{C}_t$.

For $k > 0$, let $\ell(k)$ be the sum of the digits of the base q expansion of k. Now, it is easy to show that, if $y = (z, m) \in \mathbb{Z}/(q^{\deg v} - 1)\mathbb{Z} \times \mathbb{Z}_p$, $m > 0$ an integer, and

$$d > \frac{\ell(m)}{q-1} + 1,$$

then $S_{d,v}(y) = 0$ (see [5, Theorem 5.1.2]). So, $\zeta_v(s)$ is a polynomial on x. The Newton polygon of this polynomial determines the v-adic valuation of its zeros. Namely, if the Newton polygon has a side of slope $-m$ whose horizontal projection is of length l, then the polynomial has precisely l roots with valuation m. Furthermore, if

$$\zeta_m(x) = \prod_{\substack{\zeta_v(\alpha)=0 \\ \mathrm{val}_v(\alpha)=m}} (x - \alpha),$$

then $\zeta_m(x) \in \mathbb{F}_q((t))[x]$. Now, as in [1], we have

Lemma 4. *Let $q = p^n$, $n \geq 1$. For $v = t$ prime of $\mathbb{F}_q[t]$ of degree one, and for $y = (z, m) \in \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}_p$ with $m \geq 0$ an integer, the zeros of*

$$\zeta_v(s) = \sum_{d=0}^{\infty} S_{d,v}(y)x^d$$

are simple with $x \in \mathbb{F}_q((t))$.

Proof. Let us assume that the Goss t-adic zeta function is a polynomial of degree d,

$$\zeta_v(s) = 1 + S_{1,v}(y)x + \cdots + S_{d,v}(y)x^d,$$

and recall that $v_d(y) := \mathrm{val}_v(S_{d,v}(y))$. Then,

$$v_d(y) = m_1(d) + 2m_2(d) + \cdots + dm_d(d),$$

with obvious notation.

If r is the least non-negative residue of z modulo $q-1$, by Theorem 1 and Remark 2, $m_d(d)$ is the least possible sum of p-powers chosen from the p-expansion of m which is r modulo $q-1$. Next, $m_{d-1}(d)$ is obtained as the least possible sum of p-powers chosen from the expansion of $m - m_d(d)$ which is $0 \bmod q - 1$, and so on.

For $q = p$, a prime number, the greedy solution is achieved in a simpler way, by the previous Theorem 3:

$$m_d(d) = \sum_{s=1}^{r} p^{e_s}, \, m_{d-1}(d) = \sum_{s=1}^{p-1} p^{e_{r+s}},$$

and so on.

Therefore, for q a prime or power of a prime, the valuations $v_d(y)$ of the coefficients of Goss v-adic zeta functions are:

$$v_d(y) = m_1(d) + 2m_2(d) + \cdots + dm_d(d)$$
$$v_{d-1}(y) = m_1(d-1) + 2m_2(d-1) + \cdots + (d-1)m_{d-1}(d-1)$$

$$\vdots = \vdots$$
$$v_1(y) = m_1(1)$$
$$v_0(y) = 0.$$

Now, notice that by construction we have

$$m_{d-j}(d) = m_{d-j-1}(d-1) \quad \text{for } j = 0, \ldots, d-2.$$

Using this, straight manipulation allows us to calculate the slope $\lambda(d)$ of the Newton polygon from $(d-1, v_{d-1}(y))$ to $(d, v_d(y))$,

$$\lambda(d) = v_d(y) - v_{d-1}(y)$$
$$= m_1(d) + m_1(d-1) + m_2(d-1) + \cdots + m_{d-1}(d-1).$$

Finally, $\lambda(d)$ is a strictly increasing function of d since

$$\lambda(d) - \lambda(d-1) = m_1(d) > 0$$

for $d > 1$. For $d = 1$, $\lambda(d) - \lambda(d-1) = m_1(1)$, if we put $\lambda(0) = 0$, and in this case $m_1(1)$ could be zero if z is even, i.e., when z is congruent to $0 \mod q-1$, but that just states that the first slope is zero. This implies that each slope of the Newton polygon has horizontal projection one and so the zeros of $\zeta_v(s)$ are simple with x in $\mathbb{F}_q((t))$. $\square$

Theorem 5 (Riemann hypothesis). *Let q be any prime power. For the prime $v = t$ of $\mathbb{F}_q[t]$, and for $y = (z, M) \in \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}_p$, the zeros of $\zeta_v(x, y)$ are simple and have the x in $K_v = \mathbb{F}_q((t))$ inside $\mathbb{C}_v$.*

Proof. By Lemma 4, we can assume that $M \in \mathbb{Z}_p$ is not a non-negative integer. Thus, we can choose sufficiently many appropriate p-adic digits to form a positive integer $m > 0$, which has a decomposition as in Theorem 1 with positive m_i's for all $i = 1, \ldots, d-1$, so that the valuation stabilizes (exactly as in [**1**], [**5**, subsection 5.8]), as it is independent of m_0. We can form pairs $y_m = (z, m)$, $m > 0$ that converge to (z, M). Hence, we can calculate the valuation as

$$v_d(y) = \mathrm{val}_v \left(\sum_{\substack{a \in \mathbb{F}_q[t], \text{ monic} \\ \deg(a)=d \\ (v,a)=1}} a^y \right)$$

$$= \mathrm{val}_v \left(\sum_{\substack{a \in \mathbb{F}_q[t], \text{ monic} \\ \deg(a)=d \\ (v,a)=1}} a^{y_m} \right)$$

$$= v_d(y_m).$$

Then the slope inequalities of Lemma 4 imply the required result in general. □

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