

## A DISCRETE VIEW OF FAÀ DI BRUNO

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ABSTRACT. It is shown how solving a discrete convolution problem gives a unique insight into the famous Faà di Bruno formula for the  $n$ th derivative of a composite function.

**1. Introduction.** How are the following two problems related?

**Problem 1.1.** The chain rule for differentiating a composite function  $f(g(x))$  is familiar. Finding the  $n$ th derivative of  $f(g(x))$  is more problematic but can be achieved using the classic Faà di Bruno formula:

$$(1.1) \quad \frac{d^n}{dx^n} f(g(x)) \\ = \sum \frac{n!}{a_1! a_2! \cdots a_n!} f^{(k)}(g(x)) \left( \frac{g'(x)}{1!} \right)^{a_1} \left( \frac{g''(x)}{2!} \right)^{a_2} \cdots \left( \frac{g^{(n)}(x)}{n!} \right)^{a_n},$$

where the sum is taken over all nonnegative integer solutions  $a_1, a_2, \dots, a_n$  of the equation  $a_1 + 2a_2 + \cdots + na_n = n$ , and where  $k = a_1 + a_2 + \cdots + a_n$ . Johnson [5] takes the reader through the historical journey as this formula found its way into real analysis, combinatorial analysis, matrix methods and various other applications.

**Problem 1.2.** Given two sequences of numbers or polynomials  $\mathbf{k} = \{k_n\}_{n \geq 0}$  and  $\mathbf{r} = \{r_n\}_{n \geq 0}$ , the expression

$$u_n = \sum_{j=0}^n k_{n-j} r_j = \sum_{j=0}^n k_j r_{n-j}$$

is called the *convolution* of these two sequences and is denoted by  $\mathbf{u} = \mathbf{k} \star \mathbf{r}$ . An interesting task presents itself when one attempts to solve, for the sequence  $\mathbf{r}$ , the convolution equation  $\mathbf{u} = \mathbf{r} \star \mathbf{r}$ , where  $\mathbf{u}$

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is a given sequence. That is, we want to find the  $r_j$  in terms of the  $u_i$  in the system of equations

$$(1.2) \quad u_n = \sum_{j=0}^n r_{n-j} r_j, \quad n = 0, 1, 2, \dots$$

This convolution problem arises naturally in various settings [1, 4, 7]. The most immediate application is illustrated with the sequence of Legendre polynomials which, when convoluted with itself, results in the sequence of Chebyshev polynomials of the second kind.

Our goal is twofold: (a) to show how the discrete convolution solution provides a unique insight into the structure of the Faà formula, and (b) to see why Problem 1.1 can be used to solve Problem 1.2.

**2. The discrete convolution equation.** Consider two sequences  $\{u_n\}_{n \geq 0}$  and  $\{r_n\}_{n \geq 0}$  satisfying (1.2), where  $u_0 = 1$  and  $r_0 = 1$ . We want to find the solution sequence  $\{r_n\}_{n \geq 0}$  when  $\{u_n\}_{n \geq 0}$  is given. Equation (1.2) can be solved recursively:

$$(2.1) \quad \begin{aligned} u_1 = r_0 r_1 + r_1 r_0 &\implies r_1 = \frac{1}{2} u_1, \\ u_2 = 2 r_2 + r_1 r_1 &\implies r_2 = \frac{1}{2} (u_2 - r_1^2) = \frac{1}{2} u_2 - \frac{1}{2^3} u_1^2, \\ u_3 = 2 r_3 + 2 r_1 r_2 &\implies r_3 = \frac{1}{2} u_3 - \frac{1}{2^2} u_1 u_2 + \frac{1}{2^4} u_1^3, \\ u_4 = 2 r_4 + 2 r_1 r_3 + r_2^2 &\implies r_4 = \frac{1}{2} u_4 - \frac{1}{2^2} u_1 u_3 + \frac{3}{2^4} u_1^2 u_2 - \frac{1}{2^3} u_2^2 \\ &\quad - \frac{5}{2^7} u_1^4. \end{aligned}$$

Similarly, we find

$$(2.2) \quad \begin{aligned} r_5 &= \frac{1}{2} u_5 - \frac{1}{2^2} u_1 u_4 - \frac{1}{2^2} u_2 u_3 + \frac{3}{2^4} u_1^2 u_3 \\ &\quad + \frac{3}{2^4} u_1 u_2^2 - \frac{5}{2^5} u_1^3 u_2 + \frac{7}{2^8} u_1^5, \\ r_6 &= \frac{1}{2} u_6 - \frac{1}{2^2} u_1 u_5 - \frac{1}{2^2} u_2 u_4 - \frac{1}{2^3} u_3^2 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{3}{2^4} u_1^2 u_4 + \frac{3}{2^3} u_1 u_2 u_3 \\
 &+ \frac{1}{2^4} u_2^3 - \frac{5}{2^5} u_1^3 u_3 - \frac{15}{2^6} u_1^2 u_2^2 + \frac{35}{2^8} u_1^4 u_2 - \frac{21}{2^{10}} u_1^6,
 \end{aligned}$$

and so on.

These equations illustrate that each  $r_n$  is expressed as a sum of products

$$u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n},$$

where  $a_i \geq 0$  and

$$a_1 + 2a_2 + \cdots + na_n = n,$$

one summand for each partition of  $n$ . Recall that a partition of a positive integer  $n$  is a representation of  $n$  as a sum of positive integers, where their order does not matter and the number of summands is only bounded by  $n$ . However, it is common practice to write individual summands in nondecreasing order:

$$(2.3) \quad n = \underbrace{1 + 1 + \cdots + 1}_{a_1} + \underbrace{2 + \cdots + 2}_{a_2} + \cdots + \underbrace{n}_{a_n},$$

where  $a_k, k = 1, 2, \dots, n$ , is the number of times the integer  $k$  appears in the partition. To illustrate, the 11 summands for  $r_6$  in equation(2.2) reflect all 11 partitions of 6, which we list respectively:

$$\begin{aligned}
 6 &= 1 + 5 = 2 + 4 = 3 + 3 = 1 + 1 + 4 = 1 + 2 + 3 \\
 &= 2 + 2 + 2 = 1 + 1 + 1 + 3 = 1 + 1 + 2 + 2 \\
 &= 1 + 1 + 1 + 1 + 2 = 1 + 1 + 1 + 1 + 1 + 1.
 \end{aligned}$$

For example, the partition  $1 + 1 + 1 + 3$  of 6 corresponds to the summand  $-5 \cdot 2^{-5} u_1^3 u_3$ , where  $a_1 = 3, a_3 = 1$  and all other  $a$ 's are zeros.

In order to describe our sum  $r_n$  for general  $n$ , we follow Riordan [6] and adopt the following notation. For each partition (2.3) of the integer  $n$ , we associate vector  $\vec{\mathbf{p}}(n) = \langle a_1, a_2, \dots, a_n \rangle$ . For example, the partition  $1 + 1 + 1 + 2 + 2 + 5$  of the number 12 is associated with the 12-component vector  $\langle 3, 2, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 \rangle$ , which is shortened further to  $\langle 3, 2, 0, 0, 1 \rangle = 3\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_5$ , where  $\mathbf{e}_j$  is the unit coordinate vector whose components are all 0 except for the  $j$ th which is 1.

We express the intrinsic connection between two sequences  $\{u_n\}$  and

$\{r_n\}$  that are related through the convolution formula (1.2) as follows:

$$(2.4) \quad r_n = r_n(u_1, u_2, \dots, u_n) = \sum_{\vec{\mathbf{p}}(n) \in \mathcal{P}(n)} B_{\vec{\mathbf{p}}(n)} \mathbf{u}^{\vec{\mathbf{p}}(n)},$$

where summation is taken over the set  $\mathcal{P}(n)$  of all partitions  $\vec{\mathbf{p}}(n)$  of the integer  $n$ ,  $\mathbf{u}^{\vec{\mathbf{p}}(n)} = u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$ , and  $B_{\vec{\mathbf{p}}(n)}$  is its coefficient. In our illustration of the partition of 6 above,  $\mathbf{u}^{(3,0,1)} = u_1^3 u_3$ , and its coefficient in the sum representation (2.2) of  $r_6$  is  $B_{(3,0,1)} = B_{3\mathbf{e}_1 + \mathbf{e}_3} = -5 \cdot 2^{-5}$ .

To determine the coefficients  $B_{\vec{\mathbf{p}}(n)}$  in (2.4), we rewrite the convolution equation (1.2) as

$$u_n = 2r_n + \sum_{k=1}^{n-1} r_k r_{n-k},$$

which leads to the full-history recurrence:

$$(2.5) \quad r_n = \frac{1}{2} \left( u_n - \sum_{k=1}^{n-1} r_k r_{n-k} \right), \quad r_0 = 1, \quad r_1 = \frac{u_1}{2}.$$

This equation shows that  $r_n$  depends on  $u_n$  linearly, but the other components  $u_1, \dots, u_{n-1}$  appear in the formula for  $r_n$  through summation of their products. Since  $r_n = r_n(u_1, u_2, \dots, u_n)$  depends only on the first  $n$  values of the sequence  $\{u_n\}_{n \geq 1}$ ,  $u_0 = 1$ , equation (2.5) has a unique solution which can be found recursively. However, the amount of work and resources needed to do this grows as a rolling snowball. We will eventually avoid the full-history recurrence and express the coefficients  $B_{\vec{\mathbf{p}}(n)}$  in an explicit way.

Let us isolate  $u_n$  in equation (2.4) and write

$$(2.6) \quad r_n = \frac{1}{2} u_n + \sum_{\vec{\mathbf{p}}(n)} B_{\vec{\mathbf{p}}(n)} \mathbf{u}^{\vec{\mathbf{p}}(n)}, \quad n = 1, 2, \dots,$$

where  $\mathbf{u}^{\vec{\mathbf{p}}(n)} = u_1^{a_1} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}}$  and summation is taken over all partitions  $\vec{\mathbf{p}}(n)$  of the number  $n$  except  $n$  itself. We find from equations (2.5) and (2.6) that

$$r_n = \sum_{\vec{\mathbf{p}}(n) \in \mathcal{P}(n)} B_{\vec{\mathbf{p}}(n)} \mathbf{u}^{\vec{\mathbf{p}}(n)} = \frac{1}{2} \left( u_n - \sum_{k=1}^{n-1} r_k r_{n-k} \right)$$

$$= \frac{1}{2} u_n - \frac{1}{2} \sum_{k=1}^{n-1} \underbrace{\left( \sum_{\vec{p}(k)} B_{\vec{p}(k)} \mathbf{u}^{\vec{p}(k)} \right)}_{r_k} \underbrace{\left( \sum_{\vec{p}(n-k)} B_{\vec{p}(n-k)} \mathbf{u}^{\vec{p}(n-k)} \right)}_{r_{n-k}}.$$

Comparing like powers of  $u_i$ , we obtain

$$(2.7) \quad B_{\vec{p}(n)} = -\frac{1}{2} \sum_{k=1}^{n-1} \sum_{\vec{p}(k)} B_{\vec{p}(k)} B_{\vec{p}(n-k)},$$

where the inner sum is taken over all partitions  $\vec{p}(k)$  of the integer  $k$  ( $0 < k < n$ ). For example, we know that for  $r_6$ ,  $B_{\langle 3,0,1 \rangle} = B_{3\mathbf{e}_1 + \mathbf{e}_3} = -5 \cdot 2^{-5}$ . This coefficient is evaluated as

$$B_{\langle 3,0,1 \rangle} = -\frac{1}{2} \left( B_{\langle 2,0,1 \rangle} B_{\langle 1 \rangle} + B_{\langle 1,0,1 \rangle} B_{\langle 2 \rangle} + B_{\langle 0,0,1 \rangle} B_{\langle 3 \rangle} + B_{\langle 3 \rangle} B_{\langle 0,0,1 \rangle} + B_{\langle 2 \rangle} B_{\langle 1,0,1 \rangle} + B_{\langle 1 \rangle} B_{\langle 2,0,1 \rangle} \right) = -\frac{5}{2^5}$$

because, from (2.1),

$$B_{\langle 2,0,1 \rangle} = \frac{3}{2^4}, \quad B_{\langle 1 \rangle} = \frac{1}{2}, \quad B_{\langle 1,0,1 \rangle} = -\frac{1}{2^2}, \\ B_{\langle 2 \rangle} = -\frac{1}{2^3}, \quad B_{\langle 0,0,1 \rangle} = \frac{1}{2}, \quad B_{\langle 3 \rangle} = \frac{1}{2^4}.$$

Notice that vector entries in summands cannot exceed corresponding entries in the original vector, in this case,  $\langle 3, 0, 1 \rangle$ . An interesting pattern of coefficients is hidden in the full-history recurrence (2.7), which is empirically obtained (and readily verifiable).

Some particular coefficients can be determined immediately. If  $n$  has a partition  $\vec{p}(n) = r\mathbf{e}_j$  (partitioned into  $r$  integers equal to  $j$ ), then

$$(2.8) \quad B_{r\mathbf{e}_j} = c_r^{(0)}, \quad n = jr, \quad n = 1, 2, \dots,$$

where the sequence  $\{c_j^{(0)}\}_{j \geq 1}$  is generated by the function

$$\sqrt{1+z} - 1 = \sum_{j \geq 1} c_j^{(0)} z^j \\ = \frac{z}{2} - \frac{z^2}{2^3} + \frac{z^3}{2^4} - \frac{5z^4}{2^7} \\ + \frac{7z^5}{2^8} - \frac{21z^6}{2^{10}} + \frac{33z^7}{2^{11}} - \frac{429z^8}{2^{15}} + \dots$$

These binomial coefficients  $c_n^{(0)} = \binom{1/2}{n}$  satisfy the full-history recurrence

$$(2.9) \quad c_{n+1}^{(0)} = -\frac{1}{2} \sum_{i=1}^n c_i^{(0)} c_{n-i+1}^{(0)}, \quad n \geq 1,$$

which is a consequence of the Vandermonde convolution [2] of the sequence  $\{c_n^{(0)}\}$  with itself, analogous to (2.7). Applied to the  $j$ th entry of the vector  $\vec{\mathbf{p}}(n) = r\mathbf{e}_j$ , this recurrence verifies (2.8). For example, the coefficient  $B_{\langle 6 \rangle}$  of  $u_1^6$  is  $c_6^{(0)}$  and is computed (using either (2.7) or (2.9)) as

$$\begin{aligned} B_{\langle 6 \rangle} &= -\frac{1}{2} \left( B_{\langle 1 \rangle} B_{\langle 5 \rangle} + B_{\langle 2 \rangle} B_{\langle 4 \rangle} + B_{\langle 3 \rangle} B_{\langle 3 \rangle} + B_{\langle 4 \rangle} B_{\langle 2 \rangle} + B_{\langle 5 \rangle} B_{\langle 1 \rangle} \right) \\ &= -\frac{1}{2} \left( \frac{7}{2^8} + \frac{5}{2^{10}} + \frac{1}{2^8} + \frac{5}{2^{10}} + \frac{7}{2^9} \right) \\ &= -\frac{21}{2^{10}}. \end{aligned}$$

Actually, the coefficients  $B_{\vec{\mathbf{p}}(n)}$  are products of  $c_j^{(0)}$  and various coefficients  $c_j^{(k)}$ , where  $\{c_j^{(k)}\} = \{\binom{1/2-k}{j}\}$  is the sequence of binomial coefficients generated by the function  $(1+z)^{-(2k-1)/2}$ ,  $k = 1, 2, \dots$ . We proceed to derive explicit expressions for coefficients  $B_{\vec{\mathbf{p}}(n)}$ . Generalizing the formula  $B_{r\mathbf{e}_j} = c_r^{(0)}$ , the data that we generated suggest that for any distinct unit coordinate vectors  $\mathbf{e}_t$  and for any positive integers  $p, q, r, s$ , we have

$$\begin{aligned} B_{q\mathbf{e}_i+r\mathbf{e}_j} &= c_q^{(r)} c_r^{(0)}, \\ B_{p\mathbf{e}_i+q\mathbf{e}_j+r\mathbf{e}_k} &= c_p^{(q+r)} c_q^{(r)} c_r^{(0)} \\ B_{s\mathbf{e}_i+p\mathbf{e}_j+q\mathbf{e}_k+r\mathbf{e}_t} &= c_s^{(p+q+r)} c_p^{(q+r)} c_q^{(r)} c_r^{(0)}. \end{aligned}$$

In general, the coefficients are given by the formula

$$(2.10) \quad B_{\vec{\mathbf{p}}(n)} = B_{\sum_{j=1}^n a_j \mathbf{e}_{i_j}} = c_{a_n}^{(\sum_{j=1}^{n-1} a_j)} \dots c_{a_3}^{(a_1+a_2)} c_{a_2}^{(a_1)} c_{a_1}^{(0)}.$$

For example, if  $\mathbf{v} = 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + 3\mathbf{e}_t + \mathbf{e}_l$ , then

$$B_{\mathbf{v}} = c_2^{(6)} c_1^{(5)} c_1^{(4)} c_3^{(1)} c_1^{(0)} = \frac{143}{2^3} \left( \frac{-9}{2} \right) \left( \frac{-7}{2} \right) \left( \frac{-5}{2^4} \right) \frac{1}{2} = -\frac{45045}{2^{10}}.$$

This is a very revealing and appealing structural pattern of coefficients  $B_{\bar{p}(n)}$ . Notice that the coefficient of each unit coordinate vector serves as a subscript of a term  $c^{(\cdot)}$ , and the sum of a subscript and superscript in any term  $c^{(\cdot)}$  is equal to the superscript in the preceding  $c^{(\cdot)}$ . Since the unit coordinate vectors in equation (2.10) can be permuted, we see that the symbols  $a_i$  in the expression on the right-hand side of (2.10) can be permuted. We give an independent proof of this.

**Theorem 2.1.** *The symbols  $a_i$  in the formula*

$$(2.11) \quad c_{a_n}^{(\sum_{j=1}^{n-1} a_j)} \dots c_{a_3}^{(a_1+a_2)} c_{a_2}^{(a_1)} c_{a_1}^{(0)}$$

*can be permuted.*

*Proof.* Any permutation of symbols  $a_i$  in formula (2.11) can be achieved by repeatedly transposing pairs of adjacent symbols. It can be shown (see proof below) that

$$(2.12) \quad c_{a_j}^{(a_i+b)} c_{a_i}^{(b)} = c_{a_i}^{(a_j+b)} c_{a_j}^{(b)},$$

from which the theorem follows. □

*Proof of equation (2.12).* The equation  $c_d^{(a+b)} c_a^{(b)} = c_a^{(b+d)} c_d^{(b)}$  can be rewritten using binomial coefficients:

$$(2.13) \quad \binom{1/2 - a - b}{d} \binom{1/2 - b}{a} = \binom{1/2 - b - d}{a} \binom{1/2 - b}{d}.$$

We will show that each side of (2.13) is equal to

$$\binom{a+d}{a} \binom{1/2 - b}{a+d}.$$

Recall the notation

$$a^{\underline{b}} = a(a-1) \cdots (a-b+1)$$

for  $b$ th falling factorial and the definition of the binomial coefficient

$$\binom{a}{b} = \frac{a^{\underline{b}}}{b!}.$$

Looking at the left-hand side of (2.13), we compute

$$\begin{aligned} d! \binom{1/2 - a - b}{d} &= \left(\frac{1}{2} - a - b\right) \cdot \left(\frac{1}{2} - a - b - 1\right) \cdots \\ &\quad \left(\frac{1}{2} - a - b - d + 1\right), \\ a! \binom{1/2 - b}{a} &= \left(\frac{1}{2} - b\right) \cdot \left(\frac{1}{2} - b - 1\right) \cdots \left(\frac{1}{2} - a - b + 1\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \binom{1/2 - a - b}{d} \binom{1/2 - b}{a} &= \frac{1}{a! d!} \left(\frac{1}{2} - b\right) \cdot \left(\frac{1}{2} - b - 1\right) \cdots \\ &\quad \left(\frac{1}{2} - a - b + 1\right) \\ &\quad \times \left(\frac{1}{2} - a - b\right) \cdot \left(\frac{1}{2} - a - b - 1\right) \cdots \left(\frac{1}{2} - a - b - d + 1\right) \\ &= \frac{(a+d)!}{a! d!} \frac{1}{(a+d)!} \left(\frac{1}{2} - b\right)^{a+d} = \binom{a+d}{a} \binom{1/2 - b}{a+d}. \end{aligned}$$

The reasoning for the right-hand side of (2.13) is similar. This completes the proof.  $\square$

The identity we have established can be written as

$$c_d^{(a+b)} c_a^{(b)} = \binom{a+d}{a} c_{a+d}^{(b)}.$$

We apply this identity successively to formula (2.11) working from left to right. The first two applications result in

$$\begin{aligned} B_{\bar{\mathbf{p}}(n)} &= \binom{a_n + a_{n-1}}{a_n} c_{a_n + a_{n-1}}^{(\sum_{i=1}^{n-2} a_i)} c_{a_{n-2}}^{(\sum_{i=1}^{n-3} a_i)} \cdots c_{a_2}^{(a_1)} c_{a_1}^{(0)} \\ &= \binom{a_n + a_{n-1}}{a_n} \binom{a_n + a_{n-1} + a_{n-2}}{a_n + a_{n-1}} c_{a_n + a_{n-1} + a_{n-2}}^{(\sum_{i=1}^{n-4} a_i)} \cdots c_{a_2}^{(a_1)} c_{a_1}^{(0)} \\ &= \binom{a_n + a_{n-1} + a_{n-2}}{a_n, a_{n-1}, a_{n-2}} c_{a_n + a_{n-1} + a_{n-2}}^{(\sum_{i=1}^{n-4} a_i)} \cdots c_{a_2}^{(a_1)} c_{a_1}^{(0)}. \end{aligned}$$



Continuing this process, we obtain a product of  $n - 1$  (integer) binomial coefficients, which simplifies to a multinomial expression, multiplied by the last factor  $c_{\sum_{i=1}^n a_i}^{(0)} = \binom{1/2}{\sum_{i=1}^n a_i}$ . We summarize our work as follows.

**Theorem 2.2.** *The coefficients (2.10) in the solution (2.4) of the convolution problem are given by*

$$(2.14) \quad B_{\vec{p}(n)} = B_{\sum_{i=1}^n a_i \mathbf{e}_i} = \binom{\sum_{i=1}^n a_i}{a_1, a_2, \dots, a_n} \binom{1/2}{\sum_{i=1}^n a_i},$$

where  $\sum_{i=1}^n a_i \mathbf{e}_i$  is associated with a partition of the integer  $n$  as described previously.

*Proof.* Equation (2.8) gives the coefficient  $\binom{1/2}{a_j}$  of  $u_j^{a_j}$  when the partition of  $n = ja_j$  involves  $a_j$  copies of a single integer  $j$ . If the partition of  $n$  involves only two integers  $k$  and  $j$ , used  $a_k$  and  $a_j$  times respectively, then we can think of  $u_k^{a_k} u_j^{a_j}$  as a single entity (containing  $a_k + a_j$  factors), which, by (2.8), has coefficient  $\binom{1/2}{a_k + a_j}$ , and there are  $\binom{a_k + a_j}{a_k, a_j}$  such expressions. This results in the total coefficient  $\binom{a_k + a_j}{a_k, a_j} \binom{1/2}{a_k + a_j}$  for  $u_k^{a_k} u_j^{a_j}$ . Using this reasoning, we arrive at (2.14). Although the partition of  $n$  includes all of the integers  $1, \dots, n$ , the frequency values  $a_i$  are allowed to be zero. Indeed, if  $a_1 = n$  or if  $a_n = 1$ , then  $a_i = 0$  for all other  $i$ .  $\square$

**Example 2.3.** Consider a coefficient  $B_{\vec{p}}$  corresponding to the vector  $\vec{p} = 2\mathbf{e}_{i_1} + 5\mathbf{e}_{i_2} + 4\mathbf{e}_{i_3} + 2\mathbf{e}_{i_4}$ . Then

$$\begin{aligned} B_{\vec{p}} &= c_2^{(11)} c_5^{(6)} c_4^{(2)} c_2^{(0)} = \frac{483}{2^3} \left( \frac{-46189}{2^8} \right) \frac{315}{2^7} \left( \frac{-1}{2^3} \right) \\ &= \binom{13}{2, 5, 4, 2} \binom{1/2}{13} = 540540 \frac{52003}{8388608} = \frac{7027425405}{2^{21}}. \end{aligned}$$

**Example 2.4.** Let us compute  $r_7$  using the coefficient formula (2.14):

$$\begin{aligned} r_7 &= \binom{1/2}{1} u_7 + \binom{2}{1, 1} \binom{1/2}{2} (u_1 u_6 + u_2 u_5 + u_3 u_4) \\ &\quad + \binom{3}{1, 1, 1} \binom{1/2}{3} u_1 u_2 u_4 + \binom{3}{1, 2} \binom{1/2}{3} (u_1^2 u_5 + u_1 u_3^2 + u_2^2 u_3) \end{aligned}$$

$$\begin{aligned}
& + \binom{4}{1,3} \binom{1/2}{4} (u_1^3 u_4 + u_1 u_2^3) + \binom{4}{1,1,2} \binom{1/2}{4} u_1^2 u_2 u_3 \\
& + \binom{5}{2,3} \binom{1/2}{5} u_1^3 u_2^2 + \binom{5}{1,4} \binom{1/2}{5} u_1^4 u_3 \\
& + \binom{6}{1,5} \binom{1/2}{6} u_1^5 u_2 + \binom{1/2}{7} u_1^7.
\end{aligned}$$

Each summand corresponds to exactly 1 of the 15 partitions of  $n = 7$ . For example, the summand  $\binom{5}{2,3} \binom{1/2}{5} u_1^3 u_2^2$  corresponds to the partition  $1+1+1+2+2$ . Evaluating the coefficients, we obtain

$$\begin{aligned}
r_7 &= \frac{1}{2} u_7 - \frac{1}{2^2} (u_1 u_6 + u_2 u_5 + u_3 u_4) \\
& + \frac{3}{2^3} u_1 u_2 u_4 + \frac{3}{2^4} (u_1^2 u_5 + u_1 u_3^2 + u_2^2 u_3) \\
& - \frac{5}{2^5} (u_1^3 u_4 + u_1 u_2^3) - \frac{15}{2^5} u_1^2 u_2 u_3 \\
& + \frac{35}{2^7} u_1^3 u_2^2 + \frac{35}{2^8} u_1^4 u_3 - \frac{63}{2^9} u_1^5 u_2 + \frac{33}{2^{11}} u_1^7.
\end{aligned}$$

**3. The nexus.** We reconcile the convolution solution (2.4),

$$r_n = r_n(u_1, u_2, \dots, u_n) = \sum_{\bar{\mathbf{p}}(n) \in \mathcal{P}(n)} B_{\bar{\mathbf{p}}(n)} \mathbf{u}^{\bar{\mathbf{p}}(n)},$$

with the Faà di Bruno formula (1.1). The coefficient formula (2.14) in Theorem 2.2 may be written as

$$B_{\bar{\mathbf{p}}(n)} = B_{\sum_{i=1}^n a_i \mathbf{e}_i} = \binom{k}{a_1, a_2, \dots, a_n} \binom{1/2}{k},$$

where  $k = \sum_{i=1}^n a_i$ . The binomial factor  $\binom{1/2}{k}$  is the  $k$ th coefficient, equal to  $f^{(k)}(0)/k!$ , in the Maclaurin expansion of  $f(x) = \sqrt{1+x}$ . Letting  $U(z) = \sum_{n \geq 0} u_n z^n$ , we see that  $u_k = U^{(k)}(0)/k!$  for  $k \geq 1$ . Since

$$\mathbf{u}^{\bar{\mathbf{p}}(n)} = u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n},$$

equation (2.4) becomes

$$(3.1) \quad r_n = \sum_{\bar{\mathbf{p}}(n) \in \mathcal{P}(n)} \binom{k}{a_1, a_2, \dots, a_n} \frac{f^{(k)}(0)}{k!} \left( \frac{U'(0)}{1!} \right)^{a_1} \left( \frac{U''(0)}{2!} \right)^{a_2} \cdots \left( \frac{U^{(n)}(0)}{n!} \right)^{a_n},$$

where  $f(U(z) - 1) = \sqrt{U(z)}$ . This matches the Faà di Bruno formula (1.1) evaluated at  $x = 0$  and divided by  $n!$ . Thus, we see how the sum in the Faà formula varies over all partitions of  $n$  when viewed through the discrete convolution solution  $r_n$  in (3.1). See also the treatment by Flanders [3].

The aforementioned function  $U(z) = \sum_{n \geq 0} u_n z^n$  is a generating function for the sequence  $\mathbf{u} = \{u_n\}_{n \geq 0}$ . Letting  $R(z) = \sum_{n \geq 0} r_n z^n$  be the generating function for the sequence  $\mathbf{r} = \{r_n\}_{n \geq 0}$ , we have  $U(z) = R(z)^2$ , which corresponds to the convolution equation  $\mathbf{u} = \mathbf{r} \star \mathbf{r}$ . Hence,  $r_n$  is the coefficient of  $z^n$  in the Maclaurin expansion of  $U(z)^{1/2}$ . For example, the generating function for the sequence  $\{P_n(x)\}_{n \geq 0}$  of Legendre polynomials is

$$P(x, z) = (1 - 2xz + z^2)^{-1/2} = \sum_{n \geq 0} P_n(x) z^n,$$

and  $P(x, z)^2$  generates the Chebyshev polynomials of the second kind [2].

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