

BILINEAR INTEGRATION AND APPLICATIONS TO OPERATOR AND SCATTERING THEORY

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ABSTRACT. We show how to integrate operator valued functions with respect to a spectral or orthogonally scattered measure. Such measures typically have a variation which has either the value zero or infinity on any set and cannot therefore be treated by the approaches of Bartle or Dobrakov. Bilinear integrals of this type arise from trace class operators between Banach function spaces and in the connection between stationary-state scattering theory and time-dependent scattering theory in Hilbert space.

1. Introduction. In a fundamental work of mathematical physics [1], Amrein, Georgescu and Jauch establish the connection between stationary-state scattering theory and time-dependent scattering theory by appealing to an interchange in the order of certain operator valued integrals. The study of bilinear integration originated with the work of Bartle [2], but in the foundational paper [1] on the connection between the two approaches to quantum scattering, the integrals were defined as a type of operator valued *Riemann integral* because they do not fit into Bartle's scheme as described in [6, Chapter II], or even more sophisticated later approaches to bilinear integration [7, 8]. The essential difficulty may be described as follows.

Suppose that $(\mathcal{H}, (\cdot | \cdot))$ is a Hilbert space and $P : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ is the spectral measure associated with a selfadjoint operator $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ with domain $\mathcal{D}(A)$. By this, we mean that P has values in the bounded selfadjoint projection operators and is σ -additive for the strong operator topology of $\mathcal{L}(\mathcal{H})$, $P(\mathbb{R}) = \text{Id}$, $P(A \cap B) = P(A)P(B)$

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for every $A, B \in \mathcal{B}(\mathbb{R})$ and the identity

$$(1.1) \quad A = \int_{\mathbb{R}} \lambda dP(\lambda)$$

is valid [19]. The unique spectral measure P satisfying (1.1) is concentrated on the spectrum $\sigma(A) \subseteq \mathbb{R}$ of the selfadjoint operator A , that is, $P(B) = 0$ for every Borel subset B of \mathbb{R} disjoint from $\sigma(A)$. For any vector $h \in \mathcal{H}$ and pairwise disjoint Borel subsets B_1, B_2, \dots of \mathbb{R} we have

$$\begin{aligned} \|P(\cup_{n=1}^{\infty} B_n)h\|_{\mathcal{H}}^2 &= (P(\cup_{n=1}^{\infty} B_n)h | P(\cup_{n=1}^{\infty} B_n)h) \\ &= \sum_{n,m=1}^{\infty} (P(B_n)h | P(B_m)h) \\ &= \sum_{n,m=1}^{\infty} (P(B_n \cap B_m)h | h) \\ &= \sum_{n=1}^{\infty} \|P(B_n)h\|_{\mathcal{H}}^2. \end{aligned}$$

Hence, the \mathcal{H} -valued measure $Ph : B \mapsto P(B)h$, $B \in \mathcal{B}(\mathbb{R})$, necessarily has *finite 2-variation*.

On the other hand, the $L^2([0, 1])$ -valued measure $m : B \mapsto \chi_B$, $B \in \mathcal{B}([0, 1])$, has the property that

$$\sum_{n=1}^{\infty} \|m((1/(n+1), 1/n])\|_2 = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} = \infty,$$

so that m has infinite *variation* in $L^2([0, 1])$. It is not hard to see that the variation $V(m) : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ of m defined by

$$V(m)(B) = \sup_{\mathcal{P} \in \Pi(B)} \sum_{E \in \mathcal{P}} \|m(E)\|_2$$

has values that are either zero or infinity. Here $\Pi(B)$ denotes the set of all finite partitions of B into Borel sets.

The $L^2([0, 1])$ -valued measure m is equal to $Q\mathbf{1}$ for the spectral measure $Q : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2([0, 1]))$ such that $Q(B)$ is the selfadjoint projection operator $h \mapsto \chi_B \cdot h$, $h \in L^2([0, 1])$, for each $B \in \mathcal{B}(\mathbb{R})$ and $\mathbf{1}$ is the constant function on the interval $[0, 1]$ equal to 1. The

spectral measure Q is the spectral measure of the position operator of a quantum particle on a line. Any spectral measure P in a Hilbert space \mathcal{H} is unitarily equivalent to a spectral measure like Q , so like the vector measure m , the variation $V(Ph)$ will have values that are either zero or infinity for $h \in \mathcal{H}$, except in trivial cases.

In scattering theory [1], there is an operator valued function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathcal{L}(\mathcal{H})$ and a selfadjoint spectral measure $P : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ with respect to which it is necessary to establish the equality of the integrals

$$\int_0^\infty \left(\int_{\mathbb{R}} f(t, \sigma) d(Ph)(\sigma) \right) dt$$

and

$$\int_{\mathbb{R}} \left(\int_0^\infty f(t, \sigma) dt \right) d(Ph)(\sigma)$$

for any vector $h \in \mathcal{H}$. Denote the Lebesgue measure on \mathbb{R} by λ . Using Fubini's strategy, our aim is to prove their equality with the integral

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f d(\lambda \otimes (Ph)).$$

The measure $\lambda \otimes (Ph) : C \times B \mapsto \lambda(C)(Ph)(B)$, $B \in \mathcal{B}(\mathbb{R})$, $C \in \mathcal{B}_f(\mathbb{R}_+)$ (Borel subsets of \mathbb{R}_+ with finite Lebesgue measure) is \mathcal{H} -valued and the values of the operator valued function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathcal{L}(\mathcal{H})$ act on the values of the \mathcal{H} -valued measure $\lambda \otimes (Ph)$ —here lies the pervasive difficulty, even in Hilbert space.

Now let us consider the abstract setting in which (Ω, \mathcal{S}) is a measurable space and X, Y Banach spaces. A σ -additive map $m : \mathcal{S} \rightarrow X$ is called an X -valued *vector measure*, that is, for all pairwise disjoint sets $B_n \in \mathcal{S}$, $n = 1, 2, \dots$, the sum $\sum_{n=1}^\infty m(B_n)$ converges in the norm of X and

$$m\left(\bigcup_{n=1}^\infty B_n\right) = \sum_{n=1}^\infty m(B_n).$$

Given an operator valued function $f : \Omega \rightarrow \mathcal{L}(X, Y)$, leaving measurability conditions aside for the moment, we wish to consider the integral $\int_\Omega f dm \in Y$ in generality sufficient to treat scattering theory.

For an element $y^* \in Y^*$ of the space Y^* dual to Y , the X^* -valued function $y^* \circ f : \Omega \rightarrow X^*$ is defined by

$$\langle x, (y^* \circ f)(\omega) \rangle = \langle f(\omega)x, y^* \rangle, \quad x \in X, \omega \in \Omega.$$

It is reasonable to expect that the identity

$$\left\langle \int_{\Omega} f \, dm, y^* \right\rangle = \int_{\Omega} \left\langle y^* \circ f, dm \right\rangle$$

ought to hold for each $y^* \in Y^*$, in which the right-hand side is the integral of the X^* -valued function $y^* \circ f$ acting on the range of the X -valued measure m . However, the total variation $V(m)(\Omega)$ of m satisfies the equation

$$V(m)(\Omega) = \sup_{\|s\|_{\infty} \leq 1} \int_{\Omega} \langle s, dm \rangle$$

by the Hahn-Banach theorem. The supremum is taken over all X^* -valued \mathcal{S} -simple functions $s = \sum_{k=1}^n x_k^* \chi_{B_k}$ with $\|x_k^*\| \leq 1$ and pairwise disjoint $B_k \in \mathcal{S}$ for $k = 1, \dots, n$ and $n = 1, 2, \dots$, where

$$\int_{\Omega} \langle s, dm \rangle = \sum_{k=1}^n x_k^*(m(B_k)).$$

As mentioned above, in the case that $m = Ph$ for a spectral measure P , the variation $V(m)$ may have only the values 0 and ∞ . Because there may be so few sets on which m has finite variation, neither of the approaches [2] nor [7] is applicable to m .

For spectral measures and their application to scattering theory, we may instead adopt the features outlined in [13] concerning bilinear integration in tensor products. In this approach, we may consider $\int_{\Omega} f \otimes dm \in \mathcal{L}(X, Y) \widehat{\otimes}_{\tau} X$ for some suitable tensor product topology τ for which the product map $J : \mathcal{L}(X, Y) \otimes X \rightarrow Y$ defined by

$$J\left(\sum_{j=1}^n T_j \otimes x_j\right) = \sum_{j=1}^n T_j x_j$$

has a continuous extension \widehat{J} from some completion $\mathcal{L}(X, Y) \widehat{\otimes}_{\tau} X$ into Y . Then we may define

$$(1.2) \quad \int_{\Omega} f \, dm := \widehat{J}\left(\int_{\Omega} f \otimes dm\right).$$

For an $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple function $f = \sum_{k=1}^n T_k \chi_{B_k}$, we have

$$\int_{\Omega} f \, dm := \widehat{J} \left(\int_{\Omega} f \otimes dm \right) = \sum_{k=1}^n T_k(m(B_k)) \in Y.$$

The choice of the tensor product topology τ needs to be adapted to the problem at hand. For example, the *projective tensor product norm* is given by

$$\|u\|_{\pi} = \inf \left\{ \sum_{j=1}^n \|T_j\| \cdot \|x_j\| : u = \sum_{j=1}^n T_j \otimes x_j \right\}.$$

Every element $u \in \mathcal{L}(X, Y) \widehat{\otimes}_{\pi} X$ has a representation

$$u = \sum_{j=1}^{\infty} T_j \otimes x_j, \quad T_j \in \mathcal{L}(X, Y), \quad x_j \in X, \quad j = 1, 2, \dots,$$

with

$$\sum_{j=1}^{\infty} \|T_j\| \cdot \|x_j\| < \infty$$

[22, III.7], so that $\widehat{J}u = \sum_{j=1}^{\infty} T_j x_j \in Y$.

Now we look at a number of concrete examples in operator and probability theory where the phenomenon under consideration arises.

Example 1.1. Let $Y = \mathbb{R}$ and $X = L^1([0, 1])$. Lebesgue measure on $[0, 1]$ is denoted by λ . Let $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a real valued measurable function and $m(B) = \chi_B$ for every $B \in \mathcal{B}([0, 1])$. We suppose that, for almost every $s \in [0, 1]$, the function $t \mapsto \varphi(s, t)$, $t \in [0, 1]$, is essentially bounded, that is, there exists $M_s > 0$ such that $|\varphi(s, t)| \leq M_s$ for almost every $t \in [0, 1]$. Moreover, if

$$K_{\varphi} = \operatorname{ess\,sup}_{t \in [0, 1]} \int_0^1 |\varphi(s, t)| \, ds < \infty,$$

then the linear map $T_{\varphi} : \psi \mapsto \int_0^1 \varphi(\cdot, t) \psi(t) \, dt$, $\psi \in L^1([0, 1])$, is a bounded linear operator on $L^1([0, 1])$, because by the Fubini-Tonelli theorem, we have

$$\|T_{\varphi} \psi\|_1 \leq \int_0^1 \int_0^1 |\varphi(s, t)| |\psi(t)| \, dt \, ds$$

$$\begin{aligned}
&= \int_0^1 \left(\int_0^1 |\varphi(s, t)| ds \right) |\psi(t)| dt \\
&\leq K_\varphi \|\psi\|_1
\end{aligned}$$

for every $\psi \in L^1([0, 1])$. According to [5, Theorem II.2.12], a weakly compact linear map acting on $L^1([0, 1])$ is *representable*. For example, the identity map on $L^1([0, 1])$ is not representable because it has the distribution $\delta(s - t)$ as its kernel. The representability of operators defined on L^p -spaces is discussed in detail in the papers [21].

Suppose that $K_\varphi < \infty$ and let $\Phi_\varphi : [0, 1] \rightarrow L^\infty([0, 1])$ be the function defined by $\Phi_\varphi(s) = [\varphi(s, \cdot)]$ for almost all $s \in [0, 1]$ (in [21], Φ_φ is called the *Halmos function* associated with φ). We assume that $\Phi_\varphi(s)$ is zero for those $s \in [0, 1]$ where $\varphi(s, \cdot)$ is not essentially bounded. Here $[\psi]$ denotes the collection all functions equal to the measurable function ψ almost everywhere on $[0, 1]$. Because $m : \mathcal{B}([0, 1]) \rightarrow L^1([0, 1])$ and $L^\infty([0, 1]) = L^1([0, 1])^*$, the bilinear integral $\int_0^1 \Phi_\varphi dm$ is a real number.

For any product $A = B \times C$ in $[0, 1]^2 = [0, 1] \times [0, 1]$ of Borel subsets B, C of $[0, 1]$, the equality

$$\int_0^1 \Phi_{\chi_A} dm = \langle \chi_B, \chi_C \rangle = \lambda(B \cap C)$$

holds and, for any simple function s on $[0, 1]^2$ based on product sets, the equality

$$\int_0^1 \Phi_s dm = \int_0^1 s(t, t) dt$$

is valid. Because $V(m) = \lambda$, if Φ_φ is strongly λ -measurable in the Banach space $L^\infty([0, 1])$, then there exist simple functions s_n , $n = 1, 2, \dots$, based on product sets such that $\Phi_{s_n} \rightarrow \Phi_\varphi$ λ almost everywhere in $L^\infty([0, 1])$ and with $\varphi(t, t) = \lim_{n \rightarrow \infty} s_n(t, t)$, almost everywhere

$$\int_0^1 \Phi_\varphi dm = \lim_{n \rightarrow \infty} \int_0^1 \Phi_{s_n} dm = \int_0^1 \varphi(t, t) dt.$$

The left hand side of the equation is defined in the manner of Bartle [2], [6, Section 8] or Dobrakov [7]. The result applies to the case where the integral kernel φ is continuous almost everywhere in $[0, 1]^2$.

Hence, a bounded linear operator $T_\varphi : L^1([0, 1]) \rightarrow L^1([0, 1])$ with integral kernel φ for which the bilinear integral $\int_0^1 \Phi_\varphi dm$ exists, may be viewed as a generalized “trace class” operator on $L^1([0, 1])$. If Lebesgue measure on $[0, 1]$ is replaced by counting measure on a finite index set $J \subset [0, 1]$, then

$$\int_0^1 \Phi_\varphi dm = \sum_{j \in J} \varphi(j, j)$$

is just the trace of the matrix $\{\varphi(j, k)\}_{j, k \in J}$.

In the simple case of the Volterra integral operator $T_\varphi : L^1([0, 1]) \rightarrow L^1([0, 1])$ with $(T_\varphi f)(s) = \int_0^s f(t) dt$, $f \in L^1([0, 1])$, $0 \leq s \leq t$, the integral kernel is given by

$$\varphi(s, t) = \begin{cases} 1, & \text{if } t \leq s, \\ 0, & \text{if } t > s, \end{cases}$$

so that $\Phi_\varphi(s) = \chi_{[0, s]}$. The function Φ_φ is not essentially separably valued in $L^\infty([0, 1])$, so it is not strongly λ -measurable in $L^\infty([0, 1])$ and $\int_0^1 \Phi_\varphi dm$ cannot be defined in the sense of [2, 7]. Using Definition 2.2 below, the value $\int_0^1 \Phi_\varphi dm = \frac{1}{2}$ is obtained in [11].

Linear operators lying in the image of the projective tensor product $L^1([0, 1]) \widehat{\otimes}_\pi L^\infty([0, 1])$ in $\mathcal{L}(L^1([0, 1]))$ are called *nuclear* operators on $L^1([0, 1])$ in the terminology of Grothendieck [22, III.7]. Such operators are represented as T_φ with φ given by

$$[\varphi] = \sum_{j=1}^\infty \psi_j \otimes \phi_j,$$

and

$$\sum_{j=1}^\infty \|\psi_j\|_1 \|\phi_j\|_\infty < \infty.$$

Then Φ_φ is m -integrable, and

$$\text{tr}(T_\varphi) = \sum_{j=1}^\infty \langle \psi_j, \phi_j \rangle = \int_0^1 \Phi_\varphi dm = \int_0^1 \varphi(t, t) dt.$$

Note that, if $\Phi_{\tilde{\varphi}}$ is strongly λ -measurable in $L^\infty([0, 1])$ and $\tilde{\varphi} = \varphi$ almost everywhere on $[0, 1]^2$, then $\Phi_{\tilde{\varphi}} = \Phi_\varphi$ almost everywhere by Fubini's theorem and $\int_0^1 \tilde{\varphi}(t, t) dt = \int_0^1 \varphi(t, t) dt$, although the diagonal $\{(t, t) : t \in [0, 1]\}$ is a set of measure zero in $[0, 1]^2$.

In the next example, we must deal with the situation in which the vector measure m has infinite variation.

Example 1.2. Let $Y = \mathbb{R}$, $1 < p < \infty$ and $X = L^p([0, 1])$. Let $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a measurable function and $m(B) = \chi_B$ for every $B \in \mathcal{B}([0, 1])$. We suppose that, for almost every $s \in [0, 1]$, the function $t \mapsto \varphi(s, t)$, $t \in [0, 1]$, is an element of $L^q([0, 1])$ for $1/p + 1/q = 1$, that is, there exists an $M_s > 0$ such that

$$\int_0^1 |\varphi(s, t)|^q dt \leq M_s^q$$

for almost every $s \in [0, 1]$. Suppose that

$$T_\varphi : \psi \mapsto \int_0^1 \varphi(\cdot, t)\psi(t) dt,$$

$\psi \in L^p([0, 1])$, is a bounded linear operator on $L^p([0, 1])$ and $\Phi_\varphi(s) = [\varphi(s, \cdot)]$ as above.

If the bilinear integral

$$\int_0^1 \Phi_\varphi dm$$

existed, we would still expect T_φ to be a type of “trace class” operator, see [17]. Recall that a bounded linear operator T on a separable Hilbert space \mathcal{H} is called *trace class* if, for any (some) orthonormal basis $\langle e_j \rangle_{j \in \mathbb{N}}$ of \mathcal{H} , we have

$$\sum_{j=1}^{\infty} |(Te_j, e_j)| < \infty.$$

If T is a selfadjoint trace class operator, then T is compact and the set

$$\sigma(T) = \{t_j : j = 1, 2, \dots\}$$

of real eigenvalues of T satisfies

$$\sum_{j \in \mathbb{N}} |t_j| < \infty,$$

and

$$\text{tr}(T) = \sum_{j \in \mathbb{N}} t_j$$

is the *trace* of T . For example, in the case $p = 2$, the equality

$$(1.3) \quad \text{tr}(T_\varphi) = \sum_{j=1}^{\infty} (T_\varphi \psi_j, \psi_j) = \int_0^1 \Phi_\varphi dm = \int_0^1 \varphi(t, t) dt$$

should hold for any orthonormal basis ψ_1, ψ_2, \dots of $L^2([0, 1])$ whenever

$$\int_0^1 \Phi_\varphi dm$$

exists.

Unfortunately, for every $1 < p < \infty$, the variation $V(m)$ of the L^p -valued measure m only has values 0 and ∞ , so neither of the bilinear integrals described in [2] nor [7] is applicable in the present context.

According to a result of Grothendieck, the image of the projective tensor product $L^2([0, 1]) \widehat{\otimes}_\pi L^2([0, 1])$ in $\mathcal{L}(L^2([0, 1]))$ coincides with the space of trace class operators [22, III.7]. The more familiar example of the image of the Hilbert space tensor product $L^2([0, 1]) \widehat{\otimes}_2 L^2([0, 1])$ in $\mathcal{L}(L^2([0, 1]))$ coincides with the space of *Hilbert-Schmidt operators* T_φ with

$$\int_0^1 \int_0^1 |\varphi(s, t)|^2 ds dt < \infty.$$

In [13, Proposition 4.2], it was shown that the bilinear integral

$$\int_0^1 \Phi_\varphi \otimes dm$$

converges in $L^2([0, 1]) \widehat{\otimes}_\pi L^2([0, 1])$ if and only if T_φ is a trace class operator on $L^2([0, 1])$. It follows that equation (1.3) is valid for $p = 2$, provided that we interpret the bilinear integral

$$\int_0^1 \Phi_\varphi dm$$

as

$$J\left(\int_0^1 \Phi_\varphi \otimes dm\right)$$

for the product map $J : L^2([0, 1]) \widehat{\otimes}_\pi L^2([0, 1]) \rightarrow \mathbb{C}$ defined by

$$J[u] = \sum_{j=1}^{\infty} (\psi_j | \phi_j),$$

for $[u] = \sum_{j=1}^{\infty} \psi_j \otimes \phi_j$ and $\sum_{j=1}^{\infty} \|\psi_j\|_2 \|\phi_j\|_2 < \infty$. For any function v equal to $\sum_{j=1}^{\infty} \psi_j \otimes \phi_j$ wherever the sum converges, it follows that

$$\int_0^1 |v(t, t)| dt \leq \sum_{j=1}^{\infty} \int_0^1 |\psi_j(t) \phi_j(t)| dt \leq \sum_{j=1}^{\infty} \|\psi_j\|_2 \|\phi_j\|_2$$

by the Cauchy-Schwarz inequality and the trace of T_φ is well defined.

The next example arises from the theory of stochastic processes but exhibits features similar to integration with respect to spectral measures.

Example 1.3. Let λ denote Lebesgue measure on $[0, 1]$. Suppose that $(\Omega, \mathcal{S}, \mathbb{P})$ is a probability measure space and $m : \mathcal{B}([0, 1]) \rightarrow L^2(\mathbb{P})$ is a Gaussian random measure with mean zero and variance λ . By this, we mean that m is a vector measure such that, for every finite collection B_1, \dots, B_n , $n = 1, 2, \dots$, of pairwise disjoint Borel subsets of $[0, 1]$, the random variables $m(B_1), \dots, m(B_n)$ are independent, and for every $B \in \mathcal{B}([0, 1])$, the element $m(B)$ of $L^2(\mathbb{P})$ is a real Gaussian random variable with mean zero and variance $\mathbb{E}(m(B)^2) = \lambda(B)$. Then, for all pairwise disjoint Borel subsets B_1, B_2, \dots of $[0, 1]$, we have

$$\sum_{j=1}^{\infty} \|m(B_j)\|_2^2 = \sum_{j=1}^{\infty} \lambda(B_j) = \lambda\left(\bigcup_{j=1}^{\infty} B_j\right),$$

so that m has finite 2-variation. Because $\|m(B)\|_2 = \sqrt{\lambda(B)}$ for every $B \in \mathcal{B}([0, 1])$, the variation $V(m)$ of m only has values zero and infinity.

If $X : [0, 1] \times \Omega \rightarrow E$ is an adapted process with values in a UMD Banach space E , then there exists $c > 0$ such that

$$\begin{aligned} c^{-1} \left\| \int_0^1 X \, dm \right\|_{L^2(\mathbb{P}, E)} &\leq \left\| \int_0^1 X \otimes dm \right\|_{L^2(\mathbb{P} \otimes \mathbb{P}, E)} \\ &\leq c \left\| \int_0^1 X \, dm \right\|_{L^2(\mathbb{P}, E)}. \end{aligned}$$

When the stochastic integral

$$\int_0^1 X \, dm$$

exists in $L^2(\mathbb{P}, E)$, then it may be represented as

$$\int_0^1 X \, dm = \widehat{J} \int_0^1 X \otimes dm,$$

where $J : L^0(\mathbb{P}, E) \otimes L^0(\mathbb{P}) \rightarrow L^0(\mathbb{P}, E)$ is the product map

$$\begin{aligned} (J(f \otimes g))(\omega) &= f(\omega)g(\omega), \quad f \in L^0(\mathbb{P}, E), \\ g &\in L^0(\mathbb{P}), \quad \omega \in \Omega, \end{aligned}$$

defined on the space $L^0(\mathbb{P}, E)$ of strongly \mathbb{P} -measurable E -valued functions with the topology of convergence in probability in E . By restricting J to the linear space

$$\mathcal{X} = \left\{ \int_0^1 X \otimes dm : X \text{ is simple and adapted} \right\},$$

and taking its continuous extension from the closure $\overline{\mathcal{X}}^{L^2(\mathbb{P} \otimes \mathbb{P}, E)}$ of \mathcal{X} in $L^2(\mathbb{P} \otimes \mathbb{P}, E)$ into $L^2(\mathbb{P}, E)$, we obtain the continuous linear map \widehat{J} above, see [10] for the details.

The examples show that the bilinear integral (1.2) arises naturally in operator theory on Hilbert space. We shall first consider the situation for general Banach spaces in Section 2. In Theorem 2.3, we obtain an analogue of Fubini's theorem for bilinear integrals by a simple application of the scalar case. Trace class operators on a Banach function space are considered in Example 1.2. The bilinear

trace features in the proof of the Cwikel-Lieb-Rosenblum inequality for dominated semigroups in quantum physics [12].

A toy example of quantum scattering theory from [10, Section 4] is considered in Example 2.5, where only approximation in the uniform operator norm is needed. The example shows the need for a deeper analysis in order to treat unbounded selfadjoint operators.

In Section 3, some results concerning different notions of measurability of operator valued functions are gathered. Measurability in the strong operator topology arises most in applications, but it is not a metrizable topology. The strong operator analogue of Pettis's measurability theorem [5, Theorem II.1.2] is given in Theorem 3.5. It does not seem to be in the literature.

Stationary-state and time-dependent approaches to quantum scattering are related by the application of a type of Fubini theorem for bilinear integrals in Hilbert space. The auxiliary Banach space E is constructed in Section 4, and Theorem 4.5 gives a general result on the interchange of integrals relevant to scattering theory, improving on the results of [9]. The proof of integrability in E requires certain technical facts concerning projective tensor products due to Grothendieck [22, Section III.6].

2. Bilinear integration. Let X, Y be Banach spaces. A locally convex space E is said to be *bilinear admissible* for X, Y if

- (a) E contains the vector space $\mathcal{L}(X, Y) \otimes X$ as a dense subspace,
- (b) the composition map $J : \mathcal{L}(X, Y) \times X \rightarrow Y$ defined by

$$J(T, x) = Tx, \quad T \in \mathcal{L}(X, Y), \quad x \in X,$$

has a continuous linear extension from $\mathcal{L}(X, Y) \otimes X$ to E , also denoted by J .

- (c) For $x \in X$, $x^* \in X^*$ and $y^* \in Y^*$, the linear functional defined by

$$x \otimes y^* \otimes x^* : T \otimes u \longmapsto \langle Tx, y^* \rangle \langle u, x^* \rangle, \\ T \in \mathcal{L}(X, Y), \quad u \in X,$$

is continuous on $\mathcal{L}(X, Y) \otimes X$ for the relative topology of E .

- (d) The family of all linear functionals $x \otimes y^* \otimes x^*$ for $x \in X$, $x^* \in X^*$ and $y^* \in Y^*$ separates points of E .

If $Y = X$, then we merely say E is *bilinear admissible* for X . If τ is a completely separating tensor product topology [13] on $\mathcal{L}(X, Y) \otimes X$, then we may take $E = \mathcal{L}(X, Y) \widehat{\otimes}_\tau X$, the completion of the linear space $\mathcal{L}(X, Y) \otimes X$ in the locally convex topology τ . Sometimes the *quasicompletion* [15, subsection 23.1] is taken.

Remark 2.1. If the Banach space X has the *approximation property* [22, Section III.9], then $X \otimes X^*$ separates points of the projective tensor product $X^* \widehat{\otimes}_\pi X$ [16, subsection 43.2 (12)], and this is precisely the property needed to define the *trace* of a nuclear operator on X [17]. In Example 1.2, the separation property is what we need to define the generalized trace

$$\int_0^1 \Phi_\varphi \, dm$$

when X is a Banach function space such as $L^p([0, 1])$, $1 \leq p < \infty$.

Definition 2.2. Suppose that the locally convex space E is bilinear admissible for the Banach spaces X and Y . Let (Ω, \mathcal{S}) be a measurable space. A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is said to be *m-integrable in E* for a vector measure $m : \mathcal{S} \rightarrow X$, if for each $x \in X$, $x^* \in X^*$, $y^* \in Y^*$, the scalar function $\langle fx, y^* \rangle$ is integrable with respect to the scalar measure $\langle m, x^* \rangle$ and, for each $S \in \mathcal{S}$, there exists an element $(f \otimes m)(S)$ of E such that

$$(2.1) \quad \langle (f \otimes m)(S), x \otimes y^* \otimes x^* \rangle = \int_S \langle fx, y^* \rangle d\langle m, x^* \rangle$$

for every $x \in X$, $x^* \in X^*$ and $y^* \in Y^*$.

If f is *m-integrable in E*, then $fm(S) \in Y$ is defined for each $S \in \mathcal{S}$ by

$$fm(S) = J((f \otimes m)(S)).$$

We also denote $fm(S)$ by

$$\int_S f \, dm$$

or

$$\int_S f(\omega) \, dm(\omega).$$

In the case where X is the set of scalars, fm is the indefinite (Pettis) integral of a Y -valued function with respect to a scalar measure m . We shall use the term *Bochner* integral to distinguish the stronger integration process when f is approximated in the norm of Y , see Section 3.

Because the linear space $X \otimes Y^* \otimes X^*$ separates points of E , the vector $(f \otimes m)(S) \in E$ is well defined for each $S \in \mathcal{S}$. The same definition is adopted if \mathcal{S} is generated by the δ -ring \mathcal{S}_0 and $m : \mathcal{S}_0 \rightarrow X$ is a vector measure on \mathcal{S}_0 . Definition 2.2 facilitates a simple version of Fubini's theorem in the operator context.

Theorem 2.3. *Suppose that the locally convex space E is bilinear admissible for the Banach spaces X and Y . Let (Ω, \mathcal{S}) be a measurable space and $(\Gamma, \mathcal{E}, \mu)$ a σ -finite measure space, and $m : \mathcal{S} \rightarrow X$ a vector measure.*

Suppose that $f : \Omega \times \Gamma \rightarrow \mathcal{L}(X, Y)$ is $m \otimes \mu$ -integrable in E . If

- (i) *for m -almost all $\omega \in \Omega$, the $\mathcal{L}(X, Y)$ -valued function $f(\omega, \cdot)$ is μ -integrable, and*
- (ii) *for μ -almost all $\gamma \in \Gamma$, the $\mathcal{L}(X, Y)$ -valued function $f(\cdot, \gamma)$ is m -integrable in E ,*

then the function

$$\omega \mapsto \int_{\Gamma} f(\omega, \gamma) d\mu(\gamma)$$

is m -integrable in E , the function

$$\gamma \mapsto \int_{\Omega} f(\omega, \gamma) dm(\omega)$$

is integrable in $\mathcal{L}(X, Y)$ with respect to μ , and the equalities

$$\begin{aligned} \int_{\Omega \times \Gamma} f d(m \otimes \mu) &= \int_{\Omega} \left(\int_{\Gamma} f(\omega, \gamma) d\mu(\gamma) \right) dm(\omega) \\ &= \int_{\Gamma} \left(\int_{\Omega} f(\omega, \gamma) dm(\omega) \right) d\mu(\gamma). \end{aligned}$$

hold.

Proof. Let

$$\Phi(\omega) = \int_{\Gamma} f(\omega, \gamma) d\mu(\gamma)$$

for all $\omega \in \Omega$ for which $f(\omega, \cdot)$ is μ -integrable and $\Phi(\omega) = 0$ otherwise. For each $x \in X$, $y^* \in Y^*$, $x^* \in X^*$ and $S \in \mathcal{S}$, we have

$$\langle \Phi(\omega)x, y^* \rangle = \int_{\Gamma} \langle f(\omega, \gamma)x, y^* \rangle d\mu(\gamma),$$

so that

$$\begin{aligned} \int_S \langle \Phi(\omega)x, y^* \rangle d\langle m, x^* \rangle &= \int_S \left(\int_{\Gamma} \langle f(\omega, \gamma)x, y^* \rangle d\mu(\gamma) \right) d\langle m, x^* \rangle \\ &= \langle (f \otimes (m \otimes \mu))(S \times \Gamma), x \otimes y^* \otimes x^* \rangle \end{aligned}$$

by the scalar version of Fubini's theorem. It follows that Φ is m -integrable in E and

$$\int_S \Phi(\omega) \otimes dm(\omega) = f \otimes (m \otimes \mu)(S \times \Gamma);$$

hence,

$$\int_{\Omega} \Phi(\omega) dm(\omega) = J(f \otimes (m \otimes \mu))(\Omega \times \Gamma) = \int_{\Omega \times \Gamma} f d(m \otimes \mu).$$

A similar appeal to the scalar version of Fubini's theorem applies to the other iterated integral. \square

Because we are dealing with *vector valued* integrals, the existence of integrals (i) and (ii) almost everywhere is not ensured by the integrability of f with respect to the product measure $m \otimes \mu$.

Example 2.4. Let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. The space of all μ -equivalence classes of \mathcal{S} -measurable scalar functions is denoted by $L^0(\mu)$. It is equipped with the topology of convergence in μ -measure and vector operations pointwise μ -almost everywhere. Any Banach space X that is a subspace of $L^0(\mu)$ with the properties that

- (i) X is an order ideal of $L^0(\mu)$, that is, if $g \in X$, $f \in L^0(\mu)$ and $|f| \leq |g|$ μ almost everywhere, then $f \in X$, and
- (ii) if $f, g \in X$ and $|f| \leq |g|$ μ almost everywhere, then $\|f\|_X \leq \|g\|_X$,

is called a *Banach function space* (based on $(\Omega, \mathcal{S}, \mu)$). We suppose that X contains constant functions and $m : \mathcal{S} \mapsto \mathcal{X}_S$, $S \in \mathcal{S}$, is σ -additive in X , for example, X is σ -order continuous, see [18, Corollary 3.6]. If X is reflexive and μ is non-atomic, then it follows from [18, Corollary 3.23] that the values of the variation $V(m)$ of m are either zero or infinity.

Suppose that $\varphi : \Omega \times \Omega \rightarrow \mathbb{C}$ is a jointly measurable function and $T_\varphi : X \rightarrow X$ is a bounded linear operator such that

$$T_\varphi f = \int_{\Omega} \varphi(\cdot, t) f(t) d\mu(t)$$

for a dense set of $f \in X$. Suppose also that $\Phi_\varphi(s) = [\varphi(s, \cdot)] \in X^*$ for μ -almost all $s \in \Omega$, that is, there exists $K_s > 0$ such that

$$\left| \int_{\Omega} \varphi(s, t) f(t) d\mu(t) \right| \leq K_s \|f\|_X, \quad f \in X.$$

A bounded operator T_φ for which there exists a bilinear (X, \mathbb{C}^*) -admissible space E such that Φ_φ is m -integrable in E , is a type of *generalized trace class operator* and

$$\int_{\Omega} \Phi_\varphi dm = J \int_{\Omega} \Phi_\varphi \otimes dm$$

is the trace of T_φ . For example, if X has the approximation property [22, Section III.9] and $E = X^* \widehat{\otimes}_\pi X$, then T_φ is a nuclear operator and

$$\int_{\Omega} \Phi_\varphi dm$$

is actually the trace of T_φ [17]. There are closed subspaces of ℓ^p , $1 \leq p < \infty$, $p \neq 2$, without the approximation property. Other examples of E -traces are given in [11].

As mentioned in Example 1.1, the value

$$\int_0^1 \Phi_\varphi dm = \frac{1}{2}$$

is obtained in [11] for the Volterra integral operator T_φ on $L^2([0, 1])$ with a careful choice of the auxiliary space E . The Volterra integral operator is Hilbert-Schmidt but not trace class on $L^2([0, 1])$. The

bilinear integral

$$\int_0^1 \Phi_\varphi dm$$

is a type of *singular integral* because the diagonal $\{(t, t) : t \in [0, 1]\}$ has measure zero in $[0, 1]^2$, so that the integral

$$\int_0^1 \varphi(t, t) dt$$

is not well defined for a general integral kernel φ associated with the operator T_φ . The auxiliary space E determines the averaging process of φ around the diagonal, and the density of the measure

$$A \mapsto \int_A \Phi_\varphi dm, \quad A \in \mathcal{B}([0, 1]),$$

with respect to Lebesgue measure represents the average $t \mapsto \tilde{\varphi}(t, t)$ of φ around the diagonal.

The second basic example from [10, Section 4] is relevant to the connection between stationary-state and time-dependent scattering theory [1] where H_0 represents the free Hamiltonian operator and V represents an interaction potential. Attention is first restricted to bounded operators.

Example 2.5. Let $(\mathcal{H}, (\cdot | \cdot))$ be a separable Hilbert space. Suppose H_0 and V are *bounded* selfadjoint operators. Then $H = H_0 + V$ is also a bounded selfadjoint operator, and the function $f_\epsilon : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathcal{L}(\mathcal{H})$ defined for $\epsilon > 0$ by

$$f_\epsilon(t, \sigma) = e^{itH} V e^{-i(\sigma - i\epsilon)t}$$

for $t \geq 0$ and $\sigma \in \mathbb{R}$ is uniformly bounded in $\mathcal{L}(\mathcal{H})$. Let P be the spectral measure associated with the selfadjoint operator H_0 and $h \in \mathcal{H}$. Lebesgue measure on \mathbb{R}_+ is denoted by λ . We would like to verify the identities

(2.2)

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f_\epsilon(t, \sigma) d(\lambda \otimes (Ph))(t, \sigma) = \int_0^\infty e^{-\epsilon t} e^{itH} V e^{-itH_0} h dt$$

$$(2.3) \quad = \int_{\mathbb{R}} \left(\int_0^{\infty} e^{-\epsilon t} e^{itH} V e^{-it\sigma} dt \right) d(Ph)(\sigma)$$

that help to establish the connection between stationary-state and time-dependent scattering theory in the case of unbounded selfadjoint operators H_0 and V [1]. The \mathcal{H} -valued measure $\lambda \otimes (Ph) : \mathcal{S} \rightarrow \mathcal{H}$ is given by

$$\lambda \otimes (Ph)(A \times B) = \lambda(A)(Ph)(B), \quad A \in \mathcal{B}_f(\mathbb{R}_+), \quad B \in \mathcal{B}(\mathbb{R}).$$

Here $\mathcal{B}_f(\mathbb{R}_+)$ is the collection of Borel subsets of \mathbb{R}_+ with finite Lebesgue measure and \mathcal{S} is the δ -ring generated by the collection $\{A \times B : A \in \mathcal{B}_f(\mathbb{R}_+), B \in \mathcal{B}(\mathbb{R})\}$ of product sets in $\mathbb{R}_+ \times \mathbb{R}$. We check that the projective tensor product

$$E = \mathcal{L}(\mathcal{H}) \widehat{\otimes}_{\pi} \mathcal{H}$$

of \mathcal{H} and $\mathcal{L}(\mathcal{H})$ with the uniform operator norm is bilinear admissible for \mathcal{H} and f_{ϵ} is $(\lambda \otimes (Ph))$ -integrable in E , with the appropriate modification for integration with respect to a vector measure defined on a δ -ring. Because a Hilbert space \mathcal{H} necessarily has the approximation property [22, Section III.9], $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ separates points of E [16, subsection 43.2 (12)].

Let $B(t) = \{\sigma : (t, \sigma) \in B\}$ be the section at $t \geq 0$ of the Borel subset B of $\mathbb{R}_+ \times \mathbb{R}$. We check that the function

$$\Phi_{\epsilon}^B : t \mapsto e^{-\epsilon t} (e^{itH} V) \otimes (e^{-itH_0} P(B(t))h), \quad t \geq 0,$$

is Bochner integrable in E . The function $t \mapsto P(B(t))h$, $t \geq 0$, is strongly measurable so by Lusin's theorem on each interval $[0, T]$, there exists a Borel set Σ of arbitrarily large measure on which it is continuous in \mathcal{H} . Then Φ_{ϵ}^B is continuous in E on Σ because H and H_0 are assumed to be *bounded* selfadjoint operators so that the unitary groups e^{itH} and e^{-itH_0} , $t \in \mathbb{R}$, are continuous for the *uniform operator topology*. Consequently, Φ_{ϵ}^B is strongly measurable in E . Because

$$\begin{aligned} \int_0^{\infty} \|\Phi_{\epsilon}^B\|_E dt &\leq \int_0^{\infty} e^{-\epsilon t} \|e^{itH} V\| \cdot \|e^{-itH_0} P(B(t))h\| dt \\ &\leq \frac{\|V\| \|h\|}{\epsilon}, \end{aligned}$$

it follows that Φ_ϵ is Bochner integrable in E . For every $u, v, w \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \int_0^\infty \Phi_\epsilon^B dt, u \otimes v \otimes w \right\rangle &= \int_0^\infty e^{-\epsilon t} (e^{itH} V u | v) (e^{-itH_0} P(B(t)) h | w) dt \\ &= \int_B \langle f_\epsilon(t, \sigma), u \otimes v \rangle d(\lambda \otimes ((Ph)|w))(t, \sigma). \end{aligned}$$

According to Definition 2.2, the function f_ϵ is $(\lambda \otimes (Ph))$ -integrable in E and, for every Borel subset B of $\mathbb{R}_+ \times \mathbb{R}$, we have

$$\int_B f_\epsilon d(\lambda \otimes (Ph)) = \int_0^\infty \Phi_\epsilon^B dt.$$

Furthermore, for each $t \geq 0$, the $\mathcal{L}(\mathcal{H})$ -valued function $f_\epsilon(t, \cdot)$ is (Ph) -integrable in E the vector measure Ph and, for each $B \in \mathcal{B}(\mathbb{R})$, we have

$$\int_B f_\epsilon(t, \sigma) \otimes d(Ph) = e^{-\epsilon t} e^{itH} V \otimes e^{-itH_0} P(B) h \in E,$$

so that

$$\begin{aligned} \int_{\mathbb{R}} f_\epsilon(t, \sigma) d(Ph)(\sigma) &= J \int_{\mathbb{R}} f_\epsilon(t, \sigma) \otimes d(Ph) \\ &= e^{-\epsilon t} e^{itH} V e^{-itH_0} h. \end{aligned}$$

For each $\sigma \in \mathbb{R}$, the function $f_\epsilon(\cdot, \sigma)$ is Bochner integrable for the uniform norm of $\mathcal{L}(\mathcal{H})$ and the $\mathcal{L}(\mathcal{H})$ -valued function

$$\sigma \mapsto \int_0^\infty f_\epsilon(t, \sigma) dt, \quad \sigma \in \mathbb{R},$$

is (Ph) -integrable in E . For every Borel subset C of \mathbb{R} , we have

$$\int_C \left(\int_0^\infty f_\epsilon(t, \sigma) dt \right) d(Ph) = J \int_0^\infty \Phi_\epsilon^{\mathbb{R}_+ \times C} dt.$$

The scalar version of Fubini's theorem and the assumption that $\mathcal{H} \otimes \mathcal{H}$ separates points of the bilinear admissible space E ensure the identity

$$\int_{\mathbb{R}} \left(\int_0^\infty f_\epsilon(t, \sigma) dt \right) d(Ph) = \int_0^\infty \left(\int_{\mathbb{R}} f_\epsilon(t, \sigma) d(Ph) \right) dt$$

relevant to scattering theory.

We can still make sense of Example 2.5 if H_0 and V are unbounded operators, but it is clear that the auxiliary space $E = \mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$ will no longer suffice, because the unitary groups e^{itH} and e^{-itH_0} , $t \in \mathbb{R}$, are only continuous for the *strong operator topology*—it is too much to expect that the function Φ_ϵ^B defined in the example above will be Bochner integrable in the Banach space $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$. We first need to consider the approximation of $\mathcal{L}(\mathcal{H})$ -valued functions in the strong operator topology.

3. Measurability for operator valued functions. If the linear map $H_0 : \mathcal{D}(H_0) \rightarrow \mathcal{H}$ is an unbounded selfadjoint operator with domain $\mathcal{D}(H_0)$ in a separable Hilbert space \mathcal{H} , then the unitary group e^{-itH_0} , $t \in \mathbb{R}$, is only continuous for the strong operator topology, so we have to approximate operator valued functions formed from $t \mapsto e^{-itH_0}$ in the strong operator topology rather than the uniform operator topology.

The space $\mathcal{L}_s(\mathcal{H})$ of linear operators on \mathcal{H} with the strong operator topology, or the topology of simple convergence, is a locally convex Hausdorff space whose topology is given by the fundamental family of seminorms $p_h : T \mapsto \|Th\|_{\mathcal{H}}$, $T \in \mathcal{L}(\mathcal{H})$, for $h \in \mathcal{H}$. Measurability in such locally convex spaces can be treated by the methods of Saab [20] and Chi [4]. We shall spell out the arguments explicitly in the context of $\mathcal{L}(X, Y)$ with X and Y Banach spaces and X is separable. Then bounded absolutely convex subsets of $\mathcal{L}_s(X, Y)$ are metrizable in the relative strong operator topology.

Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space. An \mathcal{S} -measurable function is called *simple* if it has finitely many values. A function $f : \Omega \rightarrow E$ with values in a Banach space E is said to be *strongly μ -measurable* if it is the limit μ -almost everywhere of E -valued \mathcal{S} -simple functions.

A strongly μ -measurable function $f : \Omega \rightarrow E$ is *Bochner μ -integrable* in E if and only if

$$\int_{\Omega} \|f(\omega)\|_E d\mu(\omega) < \infty.$$

Equivalently, there exist E -valued \mathcal{S} -simple functions s_j , $j = 1, 2, \dots$, converging to f μ almost everywhere such that

$$\int_{\Omega} \|s_j - s_k\|_E d\mu \rightarrow 0 \quad \text{as } j, k \rightarrow \infty.$$

Then

$$\int_B f \, d\mu = \lim_{n \rightarrow \infty} \int_B s_n \, d\mu$$

for each $B \in \mathcal{S}$. In particular, f is Pettis μ -integrable, and the integrals agree [5, Definition II.3.2]. The space $L^1(\mu, E)$ of equivalence classes $[f]$ of E -valued Bochner μ -integrable functions is a Banach space with the norm

$$\|[f]\|_1 = \int_{\Omega} \|f\| \, d\mu.$$

Definition 3.1. Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space and X, Y Banach spaces. The linear space of all bounded linear operators from X to Y with the topology of strong convergence is written as $\mathcal{L}_s(X, Y)$.

- (a) A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is said to be *scalarly μ -measurable* in $\mathcal{L}(X, Y)$ if and only if, for every $x \in X, y^* \in Y^*$, the scalar function $\langle fx, y^* \rangle : \omega \mapsto \langle fx, y^* \rangle, \omega \in \Omega$, is μ -measurable.
- (b) A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is said to be *strongly μ -measurable* in $\mathcal{L}_s(X, Y)$ if and only if there exist $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple functions $s_j, j = 1, 2, \dots$, converging to f μ almost everywhere in the strong operator topology of $\mathcal{L}(X, Y)$.
- (c) A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is said to be *uniformly μ -measurable* in $\mathcal{L}(X, Y)$ if and only if there exist $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple functions $s_j, j = 1, 2, \dots$, converging to f μ almost everywhere in the uniform operator topology on $\mathcal{L}(X, Y)$.

In Example 2.5, we dealt with operator valued functions which are λ -measurable in the uniform operator norm of $\mathcal{L}(X, Y)$ with respect to Lebesgue measure λ on \mathbb{R}_+ . In practical applications, such measurability is too much to expect. Additional information about spaces of operator valued functions may be found in [3].

Example 3.2. In general, the notions (a), (b) and (c) of measurability of an operator valued function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ need to be distinguished from the requirement that the X -valued function $fx : \omega \mapsto f(\omega)x, \omega \in \Omega$, is strongly measurable for each $x \in X$. For example, if δ_s denotes the evaluation map at $s \in [0, 1]$ on $X = \ell^2([0, 1])$,

and we let $Y = \mathbb{C}$ and $f(t) = \delta_t$ for $t \in [0, 1]$, then

$$\int_0^1 |f(t)x| dt = \int_0^1 |x(t)| dt = 0$$

for every $x \in \ell^2([0, 1])$, because $x(t) = 0$ for only countably many $t \in [0, 1]$, so $fx : [0, 1] \rightarrow \mathbb{C}$ is equal to zero almost everywhere and is certainly measurable.

If f were measurable in either of the senses (b) or (c), then, with respect to Lebesgue measure λ on $[0, 1]$, we would have $\lambda \circ f^{-1}(\{0\}) = 1$ by Theorem 3.7, which is certainly not the case.

The following lemma appeals to an old argument due to Dunford [5, Lemma II.3.1].

Lemma 3.3. *Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space and X, Y Banach spaces. Suppose that the operator valued function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ has the property that fx is strongly μ -measurable in Y and*

$$\int_{\Omega} \|f(\omega)x\|_Y d\mu(\omega) < \infty, \quad \text{for each } x \in X.$$

Then, for each μ -measurable subset B of Ω , the Y -valued map

$$x \longmapsto \int_B f(\omega)x d\mu(\omega), \quad x \in X,$$

is a bounded linear operator on X denoted by

$$\int_B f d\mu.$$

Proof. The linear map $L : X \rightarrow L^1(\mu, Y)$ defined by $Lx = fx$, $x \in X$, has the property that, if $x_n \rightarrow x$ and $Lx_n \rightarrow g$ in $L^1(\mu, Y)$ as $n \rightarrow \infty$, then $Lx_{n_k} \rightarrow g$ almost everywhere as $k \rightarrow \infty$ for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. Because $f(t)x_{n_k} \rightarrow f(t)x$ as $k \rightarrow \infty$ for each $t \in [0, 1]$, it follows that $g = fx = Lx$, so that L is a closed linear map on the Banach space X . By the closed graph theorem, L is a continuous linear map and

$$\left\| \int_B f(\omega)x \, d\mu(\omega) \right\| \leq \int_{\Omega} \|f(\omega)x\|_Y \, d\mu(\omega) = \|Lx\| \leq \|L\|_{\mathcal{L}(X, L^1(\mu, Y))} \|x\|$$

for every $x \in X$. □

It is worthwhile noting when an operator valued function is approximable by operator valued simple functions.

Proposition 3.4. *Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space and X, Y Banach spaces. Suppose that the operator valued function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ has the property that fx is strongly μ -measurable in Y , the σ -algebra generated by $\{fx : x \in X\}$ is countably generated and*

$$\int_{\Omega} \|f(\omega)x\|_Y \, d\mu(\omega) < \infty, \quad \text{for each } x \in X.$$

Then there exist $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple functions s_n , $n = 1, 2, \dots$, such that for each $x \in X$, we have $s_n x \rightarrow fx$ μ almost everywhere and

$$\int_{\Omega} \|f(\omega)x - s_n(\omega)x\|_Y \, d\mu(\omega) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Let $\Omega_n \in \mathcal{S}$ be sets for which $0 < \mu(\Omega_n) < \infty$ and $\Omega \setminus \cup_{n=1}^{\infty} \Omega_n$ is μ -null. Because $\sigma(\{fx : x \in X\})$ is countably generated, for each $n = 1, 2, \dots$, there exist pairwise disjoint subsets $A_{n,j} \in \mathcal{S}$, $j = 1, 2, \dots$ of Ω_n , such that

$$\sigma(\{fx : x \in X\}) \cap \Omega_n = \sigma(\{A_{n,j} : j = 1, 2, \dots\}).$$

Let $\mathcal{F}_{n,k}$ be the algebra of all unions of set $A_{n,j}$, $j = 1, \dots, k$. Then $\mathcal{F}_{n,k}$, $k = 1, 2, \dots$, is an increasing family of finitely generated algebras of subsets of Ω_n such that

$$\bigvee_{k=1}^{\infty} \mathcal{F}_{n,k} = \sigma(\{fx : x \in X\}) \cap \Omega_n.$$

For each $x \in X$ and $n, k = 1, 2, \dots$, the conditional expectation $\mathbb{E}(fx | \mathcal{F}_{n,k})$ with respect to the probability measure $\mu(\Omega_n)^{-1} \mu$ on $\mathcal{S} \cap \Omega_n$

is the μ -equivalence class of the \mathcal{S} -simple function given by

$$\mathbb{E}(fx|\mathcal{F}_{n,k}) = \sum_{F \in \mathcal{F}_{n,k}} \frac{\int_F fx \, d\mu}{\mu(\Omega_n \cap F)} \chi_F.$$

By the Martingale convergence theorem for vector valued functions [5, Theorem V.2.9], for each $n = 1, 2, \dots$, the simple functions $\mathbb{E}(fx|\mathcal{F}_{n,k})$ converge fx in Y μ almost everywhere on Ω_n as $k \rightarrow \infty$ and

$$\int_{\Omega_n} \|fx - \mathbb{E}(fx|\mathcal{F}_{n,k})\|_Y \, d\mu \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

According to Lemma 3.3, we may define the $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple function $\mathbb{E}(f|\mathcal{F}_{n,k})$ by the formula

$$\mathbb{E}(f|\mathcal{F}_{n,k}) = \sum_{F \in \mathcal{F}_{n,k}} \frac{\int_F f \, d\mu}{\mu(\Omega_n \cap F)} \chi_F.$$

Then $s_n = \sum_{k=1}^n \mathbb{E}(f|\mathcal{F}_{n,k})$, $n = 1, 2, \dots$, are the required $\mathcal{L}(X, Y)$ -valued simple functions. \square

Next we have a version of Pettis's measurability theorem [5, Theorem II.1.2] for operator valued functions.

Theorem 3.5. *Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space and X, Y Banach spaces with X separable. Let \mathcal{D} be a countable subset of X whose linear span is dense in X . A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is strongly μ -measurable in $\mathcal{L}_s(X, Y)$ if and only if*

- (i) fx is μ -essentially separably valued in Y for each $x \in \mathcal{D}$, that is, $(fx)(\Omega \setminus N_x)$ is a norm separable subset of Y for some null set N_x , $x \in \mathcal{D}$, and
- (ii) f is scalarly μ -measurable in $\mathcal{L}(X, Y)$.

Proof. If $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is strongly μ -measurable in $\mathcal{L}_s(X, Y)$, then properties (i) and (ii) follow by applying the proof of [5, Theorem II.1.2] to the Y -valued function fx for each $x \in X$.

Now suppose that conditions (i) and (ii) hold. By Pettis's measurability theorem in Banach spaces [5, Theorem II.1.2], fx is strongly μ -measurable in Y for each $x \in X$. Because X is separable, there exists a countable dense subset \mathcal{U} of the closed unit ball of X . Then

$\|f(\omega)\|_{\mathcal{L}(X,Y)} = \sup_{x \in U} \|f(\omega)x\|$ for each $\omega \in \Omega$. It follows that $\|f\|_{\mathcal{L}(X,Y)}$ is a finite \mathcal{S} -measurable function and, up to a μ -null set, we can divide Ω into a countable family of sets Γ on which $\|f\|_{\mathcal{L}(X,Y)} \leq c$ for some $c > 0$ and $0 < \mu(\Gamma) < \infty$.

By Proposition 3.4, there exist $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple functions $s_n, n = 1, 2, \dots$ such that for each $x \in X$, we have $s_n x \rightarrow f x$ μ almost everywhere on Γ as $n \rightarrow \infty$ and

$$\int_{\Gamma} \|f(\omega)x - s_n(\omega)x\|_Y d\mu(\omega) \rightarrow 0.$$

The set

$$\Gamma_0 = \cup_{x \in \mathcal{D}} \{\omega \in \Gamma : s_n(\omega)x \rightarrow f(\omega)x \text{ as } n \rightarrow \infty\}$$

is a set of full measure and, because f is uniformly bounded on Γ in the uniform norm of $\mathcal{L}(X, Y)$, it follows that $s_n(\omega) \rightarrow f(\omega)$ in $\mathcal{L}_s(X, Y)$ as $n \rightarrow \infty$ for every $\omega \in \Gamma_0$. We construct $\mathcal{L}(X, Y)$ -valued \mathcal{S} -simple functions converging to f μ almost everywhere by piecing together these simple functions. Hence, $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is strongly μ -measurable in $\mathcal{L}_s(X, Y)$. □

Corollary 3.6. *Let $(\Omega, \mathcal{S}, \mu)$ be a σ -finite measure space and X, Y Banach spaces with X separable. A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ is strongly μ -measurable in $\mathcal{L}_s(X, Y)$ if and only if $f x$ is strongly μ -measurable in Y for each $x \in X$.*

Example 3.2 shows that the assumption that X is separable cannot be omitted, even if $Y = \mathbb{R}$.

Let T be a Hausdorff topological space and $(\Omega, \mathcal{S}, \mu)$ a σ -finite measure space. A function $f : \Omega \rightarrow T$ is called *Borel μ -measurable* if, for every open subset U of T , the set $f^{-1}(U)$ belongs to the μ -completion of \mathcal{S} , that is, there exists a μ -null set N such that $f^{-1}(U) = S \Delta N$ for $S \in \mathcal{S}$.

For Banach spaces X, Y , the space of bounded linear operators $u : X \rightarrow Y$ endowed with a locally convex topology τ is written as $\mathcal{L}_{\tau}(X, Y)$. The strong operator topology s is given by the family of seminorms $u \mapsto \|ux\|_Y, u \in \mathcal{L}(X, Y)$ for all $x \in X$.

The topology κ of precompact convergence on $\mathcal{L}(X, Y)$ has a neighborhood base of zero consisting of sets $\{u \in \mathcal{L}(X, Y) : u(K) \subset U\}$ for all relatively compact subsets K of X and all open neighborhoods U of zero in Y .

The weak topology σ is given by the family of seminorms $u \mapsto |\langle ux, y^* \rangle|$, $u \in \mathcal{L}(X, Y)$ for all $x \in X$, $y^* \in Y^*$.

Theorem 3.7. *Let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space and X, Y Banach spaces with X separable. The following conditions are equivalent for a function $f : \Omega \rightarrow \mathcal{L}(X, Y)$.*

- (i) *f is strongly μ -measurable in $\mathcal{L}_s(X, Y)$.*
- (ii) *f is Borel μ -measurable in $\mathcal{L}_s(X, Y)$ and $\mu \circ f^{-1}$ is a radon measure on $\mathcal{L}_s(X, Y)$.*
- (iii) *f is Borel μ -measurable in $\mathcal{L}_\kappa(X, Y)$ and $\mu \circ f^{-1}$ is a radon measure on $\mathcal{L}_\kappa(X, Y)$.*
- (iv) *f is Borel μ -measurable in $\mathcal{L}_\sigma(X, Y)$ and $\mu \circ f^{-1}$ is a radon measure on $\mathcal{L}_\sigma(X, Y)$.*

Proof. Suppose first that (i) holds. Then f is Borel μ -measurable for each of the spaces $\mathcal{L}_s(X, Y)$, $\mathcal{L}_\kappa(X, Y)$, $\mathcal{L}_\sigma(X, Y)$. To see this, observe that f is the limit μ almost everywhere \mathcal{S} -simple functions s_n , $n = 1, 2, \dots$, exist. Because finite subsets are closed sets in each of the topologies, every simple function s_n , $n = 1, 2, \dots$, is Borel measurable in each of the topologies, and f is the pointwise limit μ -almost everywhere in Ω .

We may replace Y by the closed linear span Z of $\bigcup\{f(\Omega \setminus N_x) : x \in \mathcal{D}\}$ for a countable dense subset \mathcal{D} of X and null sets $N_x \subset \Omega$ for which the subset $f(\Omega \setminus N_x)$ is separable in Y for $x \in \mathcal{D}$. Then Z is a separable closed subspace Y , and f has values in $\mathcal{L}(X, Z)$ off the μ -null set $\bigcup_{x \in \mathcal{D}} N_x$, so we suppose that Y is itself separable.

On each set $\Gamma_c = \{\|f\|_{\mathcal{L}(X, Y)} \leq c\}$, $c > 0$, the topologies s and κ coincide with pointwise convergence in Y on the countable set \mathcal{D} by the Banach-Steinhaus Theorem. Then the measure $(\chi_{\Gamma_c} \cdot \mu) \circ f^{-1}$ is a finite Borel measure on a complete separable metric space (a Polish space) $\{\|u\|_{\mathcal{L}(X, Y)} \leq c\}$ for each of the topologies s and κ , so conditions (ii), (iii) and (iv) hold [23], because the topology σ is weaker (coarser) than either s or κ .

Now suppose that (iv) holds. Appealing to Theorem 3.5, to prove (i), it suffices to establish that fx has μ -essentially separable range for each $x \in X$. The evaluation map $\delta_x u = ux$, $u \in \mathcal{L}(X, Y)$, is continuous for the weak topology σ into $(Y, \sigma(Y, Y^*))$ for each $x \in X$, so

$$(\mu \circ f^{-1}) \circ \delta_x^{-1} = \mu \circ (fx)^{-1}$$

is a Radon measure for the weak topology $\sigma(Y, Y^*)$ of Y . The weak and norm topologies of the Banach space Y are Radon-equivalent [23], so $\mu \circ (fx)^{-1}$ is a Radon measure on Y . Its support is contained in the countable union of compact sets, which is norm separable in Y , so fx has μ -essentially separable range and condition (i) follows from Theorem 3.5. \square

Again, Example 3.2 shows that the assumption that X is separable cannot be omitted, even if $Y = \mathbb{R}$.

Remark 3.8.

- (i) On each set $\Gamma_c = \{\|f\|_{\mathcal{L}(X, Y)} \leq c\}$, convergence for the strong operator topology in $\{u : \|u\|_{\mathcal{L}(X, Y)} \leq c\}$ is determined by the metric

$$(u, v) \longmapsto \sum_{j=1}^{\infty} 2^{-j} \|ux_j - vx_j\|_Y, \quad u, v \in \mathcal{L}(X, Y),$$

for some enumeration $\{x_j : j = 1, 2, \dots\}$ of a countable dense subset \mathcal{U} of the closed unit ball of X . If f is strongly μ -measurable in $\mathcal{L}_s(X, Y)$, then we can use this metric to obtain a version of Lusin’s theorem (if μ is a finite Radon measure) and Egorov’s convergence theorem in $\mathcal{L}_s(X, Y)$ or $\mathcal{L}_\kappa(X, Y)$, if μ is a finite measure. In a more general context, this idea appeared in [20].

- (ii) Suppose that \mathcal{U} is a countable dense subset of the close unit ball of the Banach space X . For each $x \in \mathcal{U}$,

$$\epsilon \mu(\{\|fx\| > \epsilon\}) \leq \int_{\Omega} \|fx\|_Y d\mu < \infty,$$

and

$$\mu(\{\|f\| > 0\}) \subseteq \bigcup_{x \in \mathcal{U}} \mu(\{\omega \in \Omega : \|f(\omega)x\| > 0\}).$$

It follows that for *any* measure space $(\Omega, \mathcal{S}, \mu)$, the set $\{\|f\| > 0\}$ is σ -finite if

$$\int_{\Omega} \|fx\|_Y d\mu < \infty$$

for every $x \in X$, provided that X is separable, so in this case, the preceding arguments are valid for an *arbitrary* measure space.

4. Bilinear integrals in scattering theory. In this section we return to the situation considered in Example 2.5 but with the more physically realistic case of unbounded selfadjoint operators.

Let $(\mathcal{H}, (\cdot | \cdot))$ be a separable Hilbert space. Suppose $H_0 : \mathcal{D}(H_0) \rightarrow \mathcal{H}$ and $V : \mathcal{D}(V) \rightarrow \mathcal{H}$ are selfadjoint operators with respective dense domains $\mathcal{D}(H_0) \subset \mathcal{D}(V)$. We suppose that $H = H_0 + V$ is also a selfadjoint operator on $\mathcal{D}(H_0)$. For example, this is the situation for the free Hamiltonian $H_0 = -\hbar^2/(2m)\Delta$ of a particle moving in \mathbb{R}^3 subject to a Coulomb potential $V(x) = c/|x|$, $x \in \mathbb{R}^3 \setminus \{0\}$, $c \neq 0$.

The function $f_\epsilon : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathcal{L}(\mathcal{D}(H_0), \mathcal{H})$ given by

$$f_\epsilon(t, \sigma) = e^{itH} V e^{-i(\sigma - i\epsilon)t}$$

for $t \geq 0$ and $\sigma \in \mathbb{R}$ is defined on $\mathcal{D}(H_0)$. Let P_0 be the spectral measure associated with the selfadjoint operator H_0 and $h \in \mathcal{D}(H_0)$. As above, the Lebesgue measure on \mathbb{R}_+ is denoted by λ . The spectral measure P_0 commutes with the selfadjoint operator H_0 in the sense that, for each Borel subset B of \mathbb{R} , the inclusion $P_0(B)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ is valid and $H_0 P_0(B)h = P_0(B)H_0 h$ for every $h \in \mathcal{D}(H_0)$. We give $\mathcal{D}(H_0)$ the graph norm

$$h \mapsto (\|h\|_{\mathcal{H}}^2 + \|H_0 h\|_{\mathcal{H}}^2)^{1/2}, \quad h \in \mathcal{D}(H_0),$$

under which it is itself a Hilbert space.

Now we seek a suitable space E , bilinear admissible for $\mathcal{D}(\mathcal{H}_0)$, \mathcal{H} , for which f_ϵ is $(\lambda \otimes (Ph))$ -integrable in E with respect to the $\mathcal{D}(\mathcal{H}_0)$ -valued measure $\lambda \otimes (Ph)$.

Let X, Y be Banach spaces. For $y^* \in Y^*$, we have

$$\left| \left\langle \sum_{j=1}^n T_j x_j, y^* \right\rangle \right| = \left| \left\langle \sum_{j=1}^n x_j, T_j^* y^* \right\rangle \right| \leq \sum_{j=1}^n \|x_j\|_X \cdot \|T_j^* y^*\|_{X^*},$$

for all $T_j \in \mathcal{L}(X, Y)$ and $x_j \in X$, $j = 1, \dots, n$ and all $n = 1, 2, \dots$. Hence, if we let

$$\|u\|_\tau = \sup_{\|y^*\| \leq 1} \inf \left\{ \sum_{j=1}^n \|x_j\|_X \cdot \|T_j^* y^*\|_{X^*} : u = \sum_{j=1}^n T_j \otimes x_j \right\}$$

over all representations $u = \sum_{j=1}^n T_j \otimes x_j$, $n = 1, 2, \dots$, of $u \in \mathcal{L}(X, Y) \otimes X$, then the inequality $\|Ju\|_Y \leq \|u\|_\tau$ holds for the product map $Ju = \sum_{j=1}^n T_j x_j$ by the Hahn-Banach theorem. The completion of the linear space $\mathcal{L}(X, Y) \widehat{\otimes}_\tau X$ with respect to the norm $\|\cdot\|_\tau$ is written as $\mathcal{L}(X, Y) \widehat{\otimes}_\tau X$.

Lemma 4.1. *For each $u = \sum_{j=1}^n T_j \otimes x_j \in \mathcal{L}(X, Y) \otimes X$, let $\mathcal{T}_u = \sum_{j=1}^n x_j \otimes T_j^* \in \mathcal{L}(Y^*, X \widehat{\otimes}_\pi X^*)$ denote the linear map $y^* \mapsto \sum_{j=1}^n x_j \otimes (T_j^* y^*)$, $y^* \in Y^*$.*

Then the linear mapping $k : u \mapsto \mathcal{T}_u$, $u \in \mathcal{L}(X, Y) \otimes X$, is the restriction to $\mathcal{L}(X, Y) \otimes X$ of an isometry of $\mathcal{L}(X, Y) \widehat{\otimes}_\tau X$ onto a closed linear subspace of $\mathcal{L}(Y^, X \widehat{\otimes}_\pi X^*)$ in the uniform norm.*

Proof. For $u \in \mathcal{L}(X, Y) \otimes X$, we have $\|u\|_\tau = \|\mathcal{T}_u\|_{\mathcal{L}(Y^*, X \widehat{\otimes}_\pi X^*)}$. \square

Lemma 4.2. *Let X, Y be Banach spaces with Y^* norm separable. If B is a bounded, absolutely convex subset of $\mathcal{L}(X, Y) \otimes X$ with τ -closure \overline{B} , then $k\overline{B}$ is a closed subset of $\mathcal{L}_s(Y^*, X \widehat{\otimes}_\pi X^*)$.*

Proof. Every element v of the closure of $k\overline{B}$ in $\mathcal{L}_s(Y^*, X \widehat{\otimes}_\pi X^*)$ can be represented as $v = \sum_{j=1}^\infty \lambda_j x_j \otimes T_j^*$ in $\mathcal{L}_s(Y^*, X \widehat{\otimes}_\pi X^*)$, where $\sum_{j=1}^\infty |\lambda_j| < \infty$, the finite sum $\sum_{j=1}^n \lambda_j x_j \otimes T_j^*$ belongs to B for each $n = 1, 2, \dots$, and $T_j^* \rightarrow 0$ in $\mathcal{L}_s(Y^*, X^*)$, $x_j \rightarrow 0$ in X as $j \rightarrow \infty$.

To see this, we note that Y^* is separable, so the relative topology of $\mathcal{L}_s(Y^*, X \widehat{\otimes}_\pi X^*)$ on $k\overline{B}$ coincides with the relative topology of $X \widehat{\otimes}_\pi F$, where F is the metrizable locally convex space $\mathcal{L}(Y^*, X^*)$ endowed with the topology of pointwise convergence on a countable dense subset of the unit ball of Y^* [16, (1) page 138]. Then, the given representation for v follows from a result of Grothendieck [22, Theorem III.6.4] applied to $X \widehat{\otimes}_\pi F$.

Because

$$\begin{aligned} \left\| \sum_{j=n}^{\infty} \lambda_j T_j \otimes x_j \right\|_{\tau} &\leq \sup_{\|y^*\| \leq 1} \sum_{j=n}^{\infty} |\lambda_j| \|x_j\|_X \|T_j^* y^*\|_{X^*} \\ &\leq \sup_j \{ \|x_j\|_X \|T_j\|_{\mathcal{L}(X, Y)} \} \sum_{j=n}^{\infty} |\lambda_j| \longrightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, it follows that $v \in k\overline{B}$, hence $k\overline{B}$ is closed in $\mathcal{L}_s(Y^*, X \widehat{\otimes}_{\pi} X^*)$. □

A Banach space X has the *approximation property* if $X^* \otimes X$ is dense in $\mathcal{L}_{\kappa}(X)$. Each element $\sum_{j=1}^n x_j^* \otimes x_j$ of $X^* \otimes X$ defines the finite rank operator

$$x \mapsto \sum_{j=1}^n \langle x, x_j^* \rangle x_j, \quad x \in X.$$

Lemma 4.3. *The Banach space $E = \mathcal{L}(X, Y) \widehat{\otimes}_{\tau} X$ is bilinear admissible for the Banach spaces X and Y , provided that X has the approximation property.*

Proof. Properties (a)–(c) of bilinear admissability clearly hold, so it remains to establish property (d). The family of all linear maps $x \otimes y^* \otimes x^* : T \otimes x \mapsto \langle Tx, y^* \rangle \langle x, x^* \rangle$ for $x \in X$, $x^* \in X^*$ and $y^* \in Y^*$ separates points of E .

Let $\mathfrak{B}_e((X \otimes Y^*)_s, X_s^*)$ denote the linear space of all separately continuous bilinear forms on $(X \otimes Y^*) \times X^*$ for the topologies of simple convergence $\sigma(X \otimes Y^*, \mathcal{L}(X, Y))$ and $\sigma(X^*, X)$. It is equipped with the topology e of bi-equicontinuous convergence [16, page 167]. Because X has the approximation property, the canonical linear map

$$\psi : \mathcal{L}_s(Y^*, X^*) \widehat{\otimes}_{\pi} X \rightarrow \mathfrak{B}_e((X \otimes Y^*)_s, X_s^*)$$

is one-to-one [16, subsection 43.2 (12)].

The topology τ has a fundamental system of neighborhoods of zero closed for the weaker topology of $\mathcal{L}_s(Y^*, X^*) \widehat{\otimes}_{\pi} X$, so E embeds in $\mathfrak{B}_e((X \otimes Y^*)_s, X_s^*)$ too [16, subsection 18.4 (4)]. Because $(X \otimes Y^*) \times X^*$ separates the space $\mathfrak{B}_e((X \otimes Y^*)_s, X_s^*)$ of bilinear forms, it follows that $X \otimes Y^* \otimes X^*$ separates points of E . □

Remark 4.4. The approximation property for X is needed to make sense of the integral

$$\int_{\Omega} f \, dm$$

of an $\mathcal{L}(X, Y)$ -valued function f with respect to an X -valued measure m . Similarly, the approximation property is needed to define the trace of a nuclear operator on X in the case $Y = \mathbb{C}$. All Banach spaces of practical interest, including Hilbert spaces, possess the approximation property, see [17].

The following result improves [9, Theorem 4.4] in the sense that we obtain a stronger form of integrability in the conclusion (c) below.

Theorem 4.5. *Let $(\mathcal{H}, (\cdot | \cdot))$ be a separable Hilbert space and $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ a selfadjoint operator with spectral measure P_A . Let $E = \mathcal{L}(\mathcal{D}(A), \mathcal{H}) \widehat{\otimes}_{\tau} \mathcal{D}(A)$ where the Hilbert space $\mathcal{D}(A)$ is endowed with the graph norm.*

Let $(\Gamma, \mathcal{E}, \mu)$ be a σ -finite measure space. Suppose that the measurable function $u : \mathbb{R} \times \Gamma \rightarrow \mathbb{C}$ has the property that, for every $h \in \mathcal{D}(A)$, the function $u(\cdot, \gamma)$ is $P_A h$ -integrable in $\mathcal{D}(A)$.

Let $f : \Gamma \rightarrow \mathcal{L}(\mathcal{D}(A), \mathcal{H})$ be a strongly μ -measurable in $\mathcal{L}_s(\mathcal{D}(A)\mathcal{H})$ function for which there exist positive μ -measurable functions α, β, v , on Γ , with the following properties:

- (i) $|u(\sigma, \gamma)| \leq v(\gamma)$ for every $\sigma \in \mathbb{R}$ and $\gamma \in \Gamma$,
- (ii) $\|f(\gamma)h\| \leq \alpha(\gamma)\|Ah\| + \beta(\gamma)\|h\|$ for every $h \in \mathcal{D}(A)$, $\gamma \in \Gamma$, and
- (iii) $\int_{\Gamma} v(\gamma)(\alpha(\gamma) + \beta(\gamma)) \, d\mu(\gamma) < \infty$.

Then, for each $h \in \mathcal{D}(A)$,

(a) *the function*

$$\gamma \mapsto f(\gamma) \int_S u(\sigma, \gamma) \, d(P_A h)(\sigma), \quad \gamma \in \Gamma,$$

is Bochner μ -integrable in \mathcal{H} for each $S \in \mathcal{B}(\mathbb{R})$,

(b) *the function*

$$\gamma \mapsto u(\sigma, \gamma)f(\gamma)g, \quad \gamma \in \Gamma,$$

is Bochner μ -integrable in \mathcal{H} for each $\sigma \in \mathbb{R}$ and $g \in \mathcal{D}(A)$, and

(c) the $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$ -valued function

$$\sigma \longmapsto \int_T u(\sigma, \gamma) f(\gamma) d\mu(\gamma), \quad \sigma \in \mathbb{R},$$

is strongly measurable in $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$ and $(P_A h)$ -integrable in E with respect to the $\mathcal{D}(A)$ -valued measure $P_A h$, for each set $T \in \mathcal{E}$.

Moreover, the equality

$$\begin{aligned} (4.1) \quad \int_S \left(\int_T u(\sigma, \gamma) f(\gamma) d\mu(\gamma) \right) d(P_A h)(\sigma) \\ = \int_T f(\gamma) \left(\int_S u(\sigma, \gamma) d(P_A h)(\sigma) \right) d\mu(\gamma) \end{aligned}$$

holds for every $S \in \mathcal{B}(\mathbb{R})$ and $T \in \mathcal{E}$.

Proof. Property (a) follows from the strong μ -measurability of f in $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$ and the norm estimates

$$\begin{aligned} & \int_{\Gamma} \left\| f(\gamma) \int_S u(\sigma, \gamma) d(P_A h)(\sigma) \right\| d\mu(\gamma) \\ & \leq \int_{\Gamma} \alpha(\gamma) \left\| A \int_S u(\sigma, \gamma) d(P_A h)(\sigma) \right\| d\mu(\gamma) \\ & \quad + \int_{\Gamma} \beta(\gamma) \left\| \int_S u(\sigma, \gamma) d(P_A h)(\sigma) \right\| d\mu(\gamma) \\ & \leq \|P_A h\|_{sv(\mathcal{D}(A))} \int_{\Gamma} v(\gamma) (\alpha(\gamma) + \beta(\gamma)) d\mu(\gamma) \\ & < \infty. \end{aligned}$$

Here $\|P_A h\|_{sv(\mathcal{D}(A))}$ is the semivariation norm [5, page 2] of the $\mathcal{D}(A)$ -valued measure $P_A h$. The estimates (i)–(iii) also give property (b).

Now let B be an element of the σ -algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{E}$. We check that the $\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$ -valued function

$$(4.2) \quad \Phi_B : \gamma \longmapsto f(\gamma) \otimes \int_{B(\gamma)} u(\sigma, \gamma) d(P_A h)(\sigma), \quad \gamma \in \Gamma,$$

is integrable to the Banach space $E = \mathcal{L}(\mathcal{D}(A), \mathcal{H}) \widehat{\otimes}_\tau \mathcal{D}(A)$ and

$$(4.3) \quad \left\| \int_\Gamma \Phi_B(\gamma) d\mu(\gamma) \right\|_\tau \leq \|P_A h\|_{sv(\mathcal{D}(A))} \int_\Gamma v(\gamma)(\alpha(\gamma) + \beta(\gamma)) d\mu(\gamma).$$

Firstly, the inequality

$$\begin{aligned} \int_\Gamma v(\gamma) \|f(\gamma)^*\|_{\mathcal{L}(\mathcal{H}, \mathcal{D}(A))} d\mu(\gamma) &= \int_\Gamma v(\gamma) \|f(\gamma)\|_{\mathcal{L}(\mathcal{D}(A), \mathcal{H})} d\mu(\gamma) \\ &= \sup_{\|\psi\|_\infty \leq 1} \int_\Gamma v(\gamma) \|f(\gamma)\psi(\gamma)\|_{\mathcal{H}} d\mu(\gamma) \\ &\leq \int_\Gamma v(\gamma)(\alpha(\gamma) + \beta(\gamma)) d\mu(\gamma) \end{aligned}$$

follows from assumption (ii), where the supremum is taken over $\mathcal{D}(A)$ -valued \mathcal{E} -simple functions ψ . It follows that

$$\int_\Gamma \|\Phi_B(\gamma)\|_\tau d\mu(\gamma) \leq \|P_A h\|_{sv(\mathcal{D}(A))} \int_\Gamma v(\gamma)(\alpha(\gamma) + \beta(\gamma)) d\mu(\gamma).$$

We are not assuming that f is strongly μ -measurable for the *uniform norm* of $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$, so some caution is needed.

Now $k \circ \Phi_B : \Gamma \rightarrow \mathcal{L}(\mathcal{D}(A), \mathcal{H} \otimes \mathcal{H})$ defines a linear map $L_B : \mathcal{D}(A) \rightarrow L^1(\mu, \mathcal{H} \otimes \mathcal{H})$ by

$$L_B g = [k \circ \Phi_B g], \quad g \in \mathcal{D}(A).$$

The bounded linear operator L_B is the limit in $\mathcal{L}(\mathcal{D}(A), L^1(\mu, \mathcal{H} \widehat{\otimes}_\pi \mathcal{H}))$ of $(P_n \otimes I)L_B$, $n = 1, 2, \dots$, for conditioning operators $P_n : L^1(\mu, \mathcal{H}) \rightarrow L^1(\mu, \mathcal{H})$, $n = 1, 2, \dots$, on finitely generated σ -algebras, as in the proof of Proposition 3.4. Then $(P_n \otimes I)L_B = [k \circ \Phi_{B,n}]$ for

$$\Phi_{B,n} : \gamma \longmapsto (P_n f)(\gamma) \otimes \int_{B(\gamma)} u(\sigma, \gamma) d(P_A h)(\sigma), \quad \gamma \in \Gamma,$$

and $P_n f$ is an $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$ -valued \mathcal{E} -simple function for each $n = 1, 2, \dots$, so $\Phi_{B,n}$ has values in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$. Moreover,

$$\|(P_n \otimes I)L_B\|_{\mathcal{L}(\mathcal{D}(A), L^1(\mu, \mathcal{H} \widehat{\otimes}_\pi \mathcal{H}))} \leq \int_\Gamma \|\Phi_B\|_\tau d\mu,$$

$$\int_B (P_n \otimes I)L_B d\mu \in \mathcal{L}(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$$

for each $n = 1, 2, \dots$, and

$$(4.4) \quad \lim_{n \rightarrow \infty} \int_C (P_n \otimes I) L_B d\mu = \int_C L_B d\mu$$

in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H} \widehat{\otimes}_\pi \mathcal{H})$ for each $C \in \mathcal{E}$.

We want to show that for each $C \in \mathcal{E}$, the linear map

$$g \mapsto \int_C L_B g d\mu, \quad g \in \mathcal{D}(A),$$

belongs to the *uniform closure* kE of $k(\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A))$ in the space $\mathcal{L}(\mathcal{D}(A), \mathcal{H} \widehat{\otimes}_\pi \mathcal{H})$ of linear operators. An appeal to Lemma 4.2 and equation (4.4) shows that this is true on any set $\Gamma_n = \{\|\Phi_B\|_\tau \leq n\}$ and, because the vector measure $L_B \mu : \mathcal{E} \rightarrow \mathcal{L}_s(\mathcal{D}(A), \mathcal{H} \widehat{\otimes}_\pi \mathcal{H})$ has finite variation

$$V(L_B \mu)(\Gamma) = \int_\Gamma \|\Phi_B(\gamma)\|_\tau d\mu(\gamma)$$

in the uniform operator norm of $\mathcal{L}(\mathcal{D}(A), \mathcal{H} \widehat{\otimes}_\pi \mathcal{H})$, we have $L_B \mu : \mathcal{E} \rightarrow kE$. The equality $L_B \mu = k \circ (\Phi_B \cdot \mu)$ ensures that Φ_B is μ -integrable in E .

Because the Banach space E is bilinear admissible for $\mathcal{D}(A), \mathcal{H}$, the integral

$$\int_B (u(\sigma, \gamma) f(\gamma)) \otimes d((P_A h) \otimes \mu)(\sigma, \gamma) \in E$$

is uniquely defined by the scalar equation

$$\begin{aligned} & \left(\int_B (u(\sigma, \gamma) f(\gamma)) \otimes d((P_A h) \otimes \mu)(\sigma, \gamma) \mid u \otimes v \otimes w \right) \\ &= \int_B (u(\sigma, \gamma) (f(\gamma) u | v)) d((P_A h | w) \otimes \mu)(\sigma, \gamma) = \left(\int_\Gamma \Phi_B d\mu \mid u \otimes v \otimes w \right) \end{aligned}$$

for each $B \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{E}$, $u, w \in \mathcal{D}(A)$ and $v \in \mathcal{H}$. Moreover, statement (c) holds, and

$$\begin{aligned} & \int_S \left(\int_T u(\sigma, \gamma) f(\gamma) d\mu(\gamma) \right) \otimes d(P_A h)(\sigma) \\ &= \int_{S \times T} (u(\sigma, \gamma) f(\gamma)) \otimes d((P_A h) \otimes \mu)(\sigma, \gamma), \end{aligned}$$

for all $S \in \mathcal{B}(\mathbb{R})$ and $T \in \mathcal{E}$. The equality (4.1) of the iterated integrals follows from the scalar version of Fubini's theorem and the bilinear admissability of E . \square

Corollary 4.6. *Let $(\mathcal{H}, (\cdot | \cdot))$ be a separable Hilbert space and $H_0 : \mathcal{D}(H_0) \rightarrow \mathcal{H}$ be a selfadjoint operator with spectral measure P_0 . Let $E = \mathcal{L}(\mathcal{D}(H_0), \mathcal{H}) \widehat{\otimes}_\tau \mathcal{D}(H_0)$ where the Hilbert space $\mathcal{D}(H_0)$ is endowed with the graph norm.*

Suppose that $V : \mathcal{D}(V) \rightarrow \mathcal{H}$ is a selfadjoint operator with dense domain $\mathcal{D}(H_0) \subset \mathcal{D}(V)$. We suppose that $H = H_0 + V$ is also a selfadjoint operator on $\mathcal{D}(H_0)$, $\epsilon > 0$ and the function $f_\epsilon : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathcal{L}(\mathcal{D}(H_0), \mathcal{H})$ is given by

$$f_\epsilon(t, \sigma) = e^{itH} V e^{-i(\sigma - i\epsilon)t}, \quad t \in \mathbb{R}_+, \sigma \in \mathbb{R}.$$

Then for each $h \in \mathcal{D}(H_0)$,

- (a) *the function $t \mapsto e^{-\epsilon t} e^{itH} V e^{-iH_0 t} g$, $t \in \mathbb{R}_+$, is Bochner integrable in \mathcal{H} for each $g \in \mathcal{D}(H_0)$,*
- (b) *the function $t \mapsto f_\epsilon(t, \sigma)g$, $t \in \mathbb{R}_+$, is Bochner μ -integrable in \mathcal{H} for each $\sigma \in \mathbb{R}$ and $g \in \mathcal{D}(H_0)$, and*
- (c) *the $\mathcal{L}(\mathcal{D}(H_0), \mathcal{H})$ -valued function*

$$\sigma \mapsto \int_T f_\epsilon(t, \sigma) dt, \quad \sigma \in \mathbb{R},$$

is strongly measurable in $\mathcal{L}(\mathcal{D}(H_0), \mathcal{H})$ and $(P_0 h)$ -integrable in E with respect to the $\mathcal{D}(H_0)$ -valued measure $P_0 h$, for each set $T \in \mathcal{B}(\mathbb{R}_+)$.

Moreover, the equality

$$(4.5) \quad \int_S \left(\int_T f_\epsilon(t, \sigma) dt \right) d(P_0 h)(\sigma) = \int_T e^{-\epsilon t} e^{itH} V e^{-iH_0 t} P_0(S) h dt$$

holds for every $S \in \mathcal{B}(\mathbb{R})$ and $T \in \mathcal{B}(\mathbb{R}_+)$.

Proof. Statements (a) and (b) follow directly from the facts that e^{-itH} and e^{-itH_0} are continuous unitary semigroups on \mathcal{H} and $\mathcal{D}(H_0)$. The selfadjoint operator V is closed and has domain containing $\mathcal{D}(H_0)$. By the closed graph theorem it is bounded in the graph norm of the Hilbert space $\mathcal{D}(H_0)$. Conditions (i)–(iii) of Theorem 4.5 are satisfied

for the function $u(t, \sigma) = e^{-\epsilon t} e^{-i\sigma t}$ and

$$f : t \mapsto e^{itH} V, \quad t \in \mathbb{R}_+, \sigma \in \mathbb{R},$$

where we take $v(t) = e^{-\epsilon t}$, $t \in \mathbb{R}_+$, and α, β are constants. \square

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