ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 1, 2016

SOME PROPERTIES OF THE SOLUTIONS OF THIRD ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The method of Riccati equations is used to study some properties of third order linear ordinary differential equations. Some criteria of asymptotic behavior and non stability of solution of this equation are obtained. Two oscillatory criteria are proved.

1. Introduction. Let $p(t), q(t)$ and $r(t)$ be real valued continuous functions on $[t_0, +\infty)$. Consider the equation

$$\phi'''(t) + p(t)\phi''(t) + q(t)\phi'(t) + r(t)\phi(t) = 0, \quad t \geq t_0. \tag{1.1}$$

Such problems, as the study of the question of asymptotic behavior and stability, of the question of oscillation or non oscillation of solutions of equation (1.1) by the properties of its coefficients, occupy an important place among the problems of the qualitative theory of differential equations, and many works are devoted to them (see **[1, 5, 6, 7, 8, 10, 11, 12]** and the references therein).

In the case of the constant coefficients of equation (1.1) the answers on the above mentioned questions are evident. In particular, equation (1.1) is non oscillatory or has two linearly independent oscillatory solutions, depending on whether all the roots of the characteristic equation

$$x^3 + px^2 + qx + r = 0,$$

are real or this equation has two complex conjugate roots. The spread of this simple yet important statement on non-autonomous equation (1.1) is a difficult problem. Important results in this direction were obtained in **[1, 5, 6, 11]**.

2010 AMS *Mathematics subject classification.* Primary 34C10, 34C99.

Keywords and phrases. Riccati equations, criterion of Raus-Hurwitz, asymptotic behavior, non stability, oscillation.

Received by the editors on June 10, 2013, and in revised form on April 16, 2014.

DOI:10.1216/RMJ-2016-46-1-147

In the paper [**5,** pages 441, 442], the next remarkable statement is proved.

Theorem 1.1. [**10**]. *If* $p(t) \equiv 0$, $q(t) \le 0$, $r(t) > 0$, $t \ge t_0$ *and*

$$I_L \equiv \int_{t_0}^{+\infty} \left[r(\tau) - \frac{2}{3\sqrt{3}}(-q(\tau))^{3/2}\right] d\tau = +\infty, \tag{1.2}$$

then equation (1.1) *has oscillatory solutions.*

The relation (1.3) is sharp in the sense that it is also necessary for oscillation of equation (1.1) when q and r are constants. In this paper, an oscillatory criteria for (1.1) is proved, from which follows (see Example 3.8, below), that equation (1.1) will have an oscillatory solution when $p(t) \equiv 0$, $q(t) \le 0$, $r(t) > 0$, $t \ge t_0$, even if $I_L = -\infty$.

The criterion Routh-Hurvitz asserts (see [**9,** page 290]), that if p, q and r are constant, then equation (1.1) is asymptotically stable if and only if

$$p > 0, \qquad q > 0, \qquad r > 0, \qquad pq > r. \tag{1.3}$$

Therefore, the failure of one or more of these inequalities can lead to instability of the autonomous equation (1.1). There is a question: Whether and when a non-autonomous equation (1.1) is unstable, if (1.3) is broken? The answer to this question in part is given in this paper (see Theorems 3.1–3.6 below).

2. Auxiliary propositions. In equation (1.1), make the change

$$\phi(t) = \exp\left\{\int_{t_0}^{t} y(\tau)\, d\tau\right\}\left[c + \int_{t_0}^{t} \psi(\tau)\, d\tau\right], \quad t \ge t_0, \tag{2.1}$$

where $c =$ const, and $y(t)$ and $\psi(t)$ are unknown twice continuously differentiable functions on the $[t_0, +\infty)$. We come to the equation (2.2)

$$\psi''(t) + p_0(t)\psi'(t) + q_0(t)\psi(t) + r_0(t)\left[c + \int_{t_0}^{t} \psi(\tau) d\tau\right] = 0, \quad t \ge t_0,$$

where

$$\begin{cases} p_0(t) &= 3y(t)+p(t); \\ q_0(t) &= 3y'(t)+3y^2(t)+2p(t)y(t)+q(t); \\ r_0(t) &= y''(t)+(3y(t)+p(t))y'(t)+y^3(t)+p(t)y^2(t) \\ & \quad +q(t)y(t)+r(t), \quad t\ge t_0. \end{cases} \tag{2.3}$$

Consider the Riccati equation

$$y''(t)+(3y(t)+p(t))y'(t) + y^3(t)+p(t)y^2(t)+q(t)y(t)+r(t)=0, \quad t\ge t_0. \tag{2.4}$$

Let $y_0(t)$ be a real valued solution of this equation on the $[t_0,+\infty)$. Put in (2.2) $y(t)=y_0(t)$, $t\ge t_0$. Taking into account (2.3), we obtain the equation

$$\psi''(t)+p_1(t)\psi'(t)+q_1(t)\psi(t)=0, \quad t\ge t_0, \tag{2.5}$$

where $p_1(t)=3y_0(t)+p(t)$, $q_1(t)=3y_0'(t)+3y_0^2(t)+2p(t)y_0(t)=q(t)$, $t \;\ge t_0$. In the future, where necessary, we will assume the functions $p(t)$ and $q(t)$ are required of times continuously differentiable. The addition of equation (2.4) $y(t)=y_0(t)$, $t\ge t_0$ and integrating from t_0 to t gives

$$y_0'(t)+\frac{3}{2}y_0^2(t)+p(t)y_0(t) + \int_{t_0}^{t}\Big[y_0^3(\tau)++p(\tau)y_0^2(\tau)+(q(\tau)-p'(\tau))y_0(\tau)+r(\tau)\Big]\,d\tau=c_0, \quad t\ge t_0, \tag{2.6}$$

where $c_0=y_0'(t_0)+(3/2)y_0^2(t_0)+p(t_0)y_0(t_0)$. Denote:

$$p_{\min}(t)=\min_{\tau\in[t_0,t]}\{p(\tau)\},$$
$$\widetilde{p}_{\max}(t)=\max_{\tau\in[t_0,t]}\{p'(\tau)\},$$
$$q_{\min}(t)=\min_{\tau\in[t_0,t]}\{q(\tau)\}.$$

Using Helder's inequality, it is not difficult to show that

$$\left(\int_{t_0}^t y_0(\tau)\,d\tau\right)^3 \le (t-t_0)^2\int_{t_0}^t |y_0^3(\tau)|\,d\tau, \quad t\ge t_0;$$
$$p_{\min}(t)\left(\int_{t_0}^t y_0(\tau)\,d\tau\right)^2 \le (t-t_0)\int_{t_0}^t p(\tau)y_0^2(\tau)\,d\tau, \quad t\ge t_0.$$

From this and the evident inequality

$$(q_{\min}(t)-\widetilde{p}_{\max}(t))\int_{t_0}^t |y_0(\tau)|\,d\tau \le \int_{t_0}^t (q(\tau)-p'(\tau))|y_0(\tau)|\,d\tau, \quad t\ge t_0,$$

it follows that

(2.7)

$$\begin{aligned} Q(Y_0(t),t,c_0) \le (t-t_0)^2\Bigg[\int_{t_0}^t \Big\{&|y_0^3(\tau)| + p(\tau)y_0^2(\tau) \\ &+ (q(\tau)-p'(\tau))|y_0(\tau)| + r(\tau)\Big\}\,d\tau - c_0\Bigg], \quad t\ge t_0, \end{aligned}$$

where $Q(Y,t,c) \equiv Y^3+(t-t_0)p_{\min}(t)Y^2+(t-t_0)(q_{\min}(t)-\widetilde{p}_{\max}(t))|Y|+(t-t_0)^2[\int_{t_0}^t r(\tau)\,d\tau - c]$, $-\infty < Y$, $c < +\infty$, $Y_0(t) \equiv \int_{t_0}^t y_0(\tau)\,d\tau$, $t\ge t_0$. For every $t\ge t_0$ and $c\in(-\infty,+\infty)$, we denote by $\underline{Y}(t,c)$ and $\overline{Y}(t,c)$ the lower bound and maximum of solutions of the system

$$\text{(2.8)} \qquad \begin{cases} Q(Y,t,c)\le 0; \\ Y>0, \end{cases}$$

respectively.

Lemma 2.1. *Let the conditions* $y_0(t) > 0$ $y_0'(t) + (3/2)y_0^2(t) + p(t)y_0(t) \ge 0$, $t\ge t_0$ *hold. Then, for every* $t\ge t_0$ *and* $c\ge c_0$, *the system* (2.8) *is solvable, and*

$$\text{(2.9)} \qquad \underline{Y}(t,c) \le Y_0(t) \le \overline{Y}(t,c), \quad t\ge t_0.$$

Proof. From the conditions of the lemma, from (2.6) and (2.7) it follows that, for every $t\ge t_0$ and $c\ge c_0$, the integral $Y_0(t)$ is a solution of system (2.8). Therefore, (2.9) is valid. The proof of the lemma is complete. □

Let $a(t)$, $b(t)$, $c(t)$, $a_1(t)$, $b_1(t)$, $c_1(t)$ be real valued continuous functions on $[t_0, +\infty)$. Consider the Riccati equations

$$z'(t) + a(t)z^2(t) + b(t)z(t) + c(t) = 0, \quad t \ge t_0, \tag{2.10}$$

$$z'(t) + a_1(t)z^2(t) + b_1(t)z(t) + c_1(t) = 0, \quad t \ge t_0 \tag{2.11}$$

and the differential inequalities

$$\eta'(t) + a(t)\eta^2(t) + b(t)\eta(t) + c(t) \ge 0, \quad t \ge t_0, \tag{2.12}$$

$$\eta'(t) + a_1(t)\eta^2(t) + b_1(t)\eta(t) + c_1(t) \ge 0, \quad t \ge t_0. \tag{2.13}$$

It is not difficult to show that, for $a(t) \ge 0$, $a_1(t) \ge 0$, $t \ge t_0$, these inequalities have a solution on $[t_0, +\infty)$ satisfying the arbitrary large initial value condition: $\eta(t_0) = \eta_{(0)}$.

Theorem*. Let $z_1(t)$ a solution of equation (2.11) on $[t_0, +\infty)$, $\eta_0(t)$ and $\eta_1(t)$ solution inequalities (2.12) and (2.13) and correspondingly on $[t_0, +\infty)$ with $\eta_0(t_0) \ge z_1(t_0)$ and $\eta_1(t_0) \ge z_1(t_0)$. Moreover, let the following conditions hold:

$$a(t) \ge 0,$$

$$z_{(0)} - z_1(t_0) + \int_{t_0}^{t} \exp\left\{ \int_{t_0}^{\tau} a_1(\xi)[\eta_0(\xi) + \eta_1(\xi) + b_1(\xi)]\, d\xi \right\}$$
$$\left[(a_1(\tau) - a(\tau))z_1^2(\tau) + (b_1(\tau) - b(\tau))z_1(\tau) + c_1(\tau) - c(\tau)\right]\, d\tau \ge 0,$$
$$t \ge t_0,$$

for some $z_{(0)} \in [z_1(t_0), \eta_1(t_0)]$. Then equation (2.10) has solution $z_0(t)$ on $[t_0, +\infty)$ with $z_0(t_0) \ge z_{(0)}$, and $z_0(t) \ge z_1(t)$, $t \ge t_0$.

For the proof of this theorem, see **[3,** pages 1228–1230].

It is evident that, for $a_1(t) \ge 0$, $t \ge t_0$ and $c_1(t) \equiv 0$ the function

$$z_1(t) \equiv \frac{\lambda_0 \exp\left\{ - \int_{t_0}^{t} b_1(\tau)\, d\tau \right\}}{1 + \lambda_0 \int_{t_0}^{t} a_1(\tau) \exp\left\{ - \int_{t_0}^{\tau} b_1(s)\, ds \right\}\, d\tau},$$
$$t \ge t_0, \quad (\lambda_0 = \mathrm{const} \ge 0)$$

is a solution of equation (2.11) on $[t_0, +\infty)$. Taking into account this fact and putting $a_1(t) = a(t)$, $b_1(t) = b(t)$, $t \ge t_0$, and $c_1(t) \equiv 0$, from Theorem$*$ we get

Corollary*. Let $a(t) \ge 0$, $c(t) \le 0$, $t \ge t_0$. Then, for every $z_{(0)} \ge 0$, equation (2.10) has solutions $z_0(t)$ on $[t_0, +\infty)$ with $z_0(t_0) = z_{(0)}$, and

$$z_0(t) \ge \frac{z_0(t_0)\exp\{-\int_{t_0}^t b_1(\tau)\,d\tau\}}{1 + z_0(t_0)\int_{t_0}^t a_1(\tau)\exp\{-\int_{t_0}^\tau b_1(s)\,ds\}\,d\tau}, \quad t \ge t_0.$$

Let $\mathcal{L}(t)$ and $\mathcal{S}(t)$ be arbitrary continuous differentiable functions on $[t_0, +\infty)$. Rewrite equation (2.4) in the form

$$\begin{aligned}(y'(t) + y^2(t) - \mathcal{L}(t)y(t) + \mathcal{S}(t))' + (y(t) + p(t) + \mathcal{L}(t))\\ \times(y'(t) + y^2(t) - \mathcal{L}(t)y(t) + \mathcal{S}(t))\\ +(\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) - \mathcal{S}(t))\\ y(t) + r(t) - \mathcal{S}'(t) - (p(t) + \mathcal{L}(t))\mathcal{S}(t) = 0, \quad t \ge t_0.\end{aligned} \tag{2.14}$$

Denote $u(t) \equiv y'(t) + y^2(t) - \mathcal{L}(t)y(t) + \mathcal{S}(t)$, $t \ge t_0$. Then equation (2.10) can be replaced by the following system

$$\left\{\begin{array}{l} u'(t) + (y(t) + p(t) + \mathcal{L}(t))u(t) + (\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t)+\\ +q(t) - \mathcal{S}(t))y(t) + r(t) - \mathcal{S}'(t) - (p(t) + \mathcal{L}(t))\mathcal{S}(t) = 0;\\ y'(t) + y^2(t) - \mathcal{L}(t)y(t) + \mathcal{S}(t) = u(t),\end{array}\right. \tag{2.15}$$

$t \ge t_0$. Solving the first equation of this system with respect to $u(t)$ and putting it into the second, we get

$$\begin{aligned} &y'(t) + y^2(t) - \mathcal{L}(t)y(t)\\ &= -\mathcal{S}(t) + \frac{1}{E(t)}\Bigg[c_1 - \int_{t_0}^t E(\tau)\Big\{\big[\mathcal{L}'(\tau) + \mathcal{L}^2(\tau) + p(\tau)\mathcal{L}(\tau) + q(\tau) - \mathcal{S}(\tau)\big]\\ &\qquad \big[y(\tau) + p(\tau) + \mathcal{L}(\tau)\big] - \big[\mathcal{L}'(\tau) + \mathcal{L}^2(\tau) + p(\tau)\mathcal{L}(\tau) + q(\tau)\big]\\ &\qquad \big[p(\tau) + \mathcal{L}(\tau)\big] - \mathcal{S}'(\tau) + r(\tau)\Big\}\,d\tau\Bigg], \quad t \ge t_0,\end{aligned} \tag{2.16}$$

where

$$E(t) \equiv \exp\left\{\int_{t_0}^{t} [y(\xi) + p(\xi) + \mathcal{L}(\xi)]\, d\xi\right\}, \quad t \geq t_0,$$
$$c_1 = y'(t_0) + y^2(t_0) - \mathcal{L}(t_0)y(t_0) + \mathcal{S}(t_0).$$

Lemma 2.2. *Let the conditions*

(1) $\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) \leq \mathcal{S}(t) \leq 0$, $t \geq t_0$;
(2) $\big[\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) - \mathcal{S}(t)\big]\big[p(t) + \mathcal{L}(t)\big] \leq 0$, $t \geq t_0$;
(3) $r(t) \leq \big[\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t)\big]\big[p(t) + \mathcal{L}(t)\big] + \mathcal{S}'(t)$, $t \geq t_0$,

hold. Then, for every $y_{(0)} > 0$ and $c_{(0)} > 0$, equation (2.4) *has a positive solution $y_0(t)$ on $[t_0, +\infty)$, satisfying the conditions*

$$y_0(t_0) = y_{(0)}; \tag{2.17}$$

$$y_0'(t_0) + y_0^2(t_0) - \mathcal{L}(t_0)y_0(t_0) + \mathcal{S}(t_0) = c_{(0)}, \tag{2.18}$$

and the following inequality holds

$$\int_{t_0}^{t} y_0(\tau)\, d\tau \geq \ln\left(1 + y_0(t_0)\int_{t_0}^{t} \exp\left\{\int_{t_0}^{\tau} \mathcal{L}(\xi)\, d\xi\right\}\right), \quad t \geq t_0. \tag{2.19}$$

Proof. Let $y_0(t)$ be a solution of equation (2.4) on the $[t_0, \nu)$, satisfying the initial value conditions (2.17), (2.18), where $[t_0, \nu)$ is the maximum interval of existence for $y_0(t)$. Show that $y_0(t) > 0$, $t \in [t_0, \nu)$. Suppose that it is not true. Then $y_0(t_1) \leq 0$ for some $t_1 \in [t_0, \nu)$. By virtue of continuity of $y_0(t)$, from this and (2.17) it follows that, for some $\overline{t} \in [t_0, \nu)$ $y_0(\overline{t}) = 0$ and $y_0(t) > 0$, $t \in [t_0, \overline{t})$. From conditions (1)–(3) and (2.18), it follows that, for $y(t) \equiv y_0(t)$ and $t = \overline{t}$, the right hand side of (2.16) is positive. Therefore, $y'(\overline{t}) > 0$. But, on the other hand, $y'(\overline{t}) = \lim_{\Delta t \to 0}(y_0(\overline{t} + \Delta t))/\Delta t \leq 0$. The contradiction thus obtained shows that $y_0(t) > 0$, $t \in [t_0, \nu)$.

We next show that $\nu = +\infty$. Suppose $\nu < +\infty$. As for the fact that $y_0(t)$ is positive, then

$$\phi_0(t) \equiv \exp\left\{\int_{t_0}^{t} y_0(\tau)\, d\tau\right\} \geq 1, \quad t \in [t_0, \nu). \tag{2.20}$$

By virtue of (2.1), $\phi_0(t)$ coincides with the solution $\widetilde{\phi}_0(t)$ of equation (1.1) on the $[t_0, \nu)$. Then, from (2.20), it follows that $\widetilde{\phi}_0(t) \neq 0$, $t \in [t_0, \widetilde{\nu})$ for some $\widetilde{\nu} > \nu$. By virtue of (2.1), from this it follows that $\widetilde{y}_0(t) \equiv \widetilde{\phi}'_0(t)/\widetilde{\phi}_0(t)$ is a solution of equation (2.4) on $[t_0, \widetilde{\nu})$ and coincides with $y_0(t)$ on $[t_0, \nu)$. Therefore, $[t_0, \nu)$ is not the maximal interval of existence for $y_0(t)$. The contradiction obtained shows that $\nu = +\infty$.

We next prove (2.19). Note that $y_0(t)$ is a solution of the Riccati equation

$$y'(t) + y^2(t) - \mathcal{L}(t)y(t) = u_0(t), \quad t \geq t_0,$$

where $u_0(t) \geq 0$ ($t \geq t_0$ is the right hand side of (2.16) for $y(t) = y_0(t)$, $t \geq t_0$. Therefore, by virtue of Corollary*, the inequality (2.19) holds. The proof of the lemma is complete. $\square$

We now indicate some special cases in which conditions (1)–(3) of Lemma 2.2 hold.

I. $\mathcal{L}(t) \equiv 0$, $q(t) \leq \mathcal{S}(t) = \lambda_0 + \int_{t_0}^t [r(\tau) - p(\tau)q(\tau)]\, d\tau \leq 0$, $[q(t) - \lambda_0 - \int_{t_0}^t (r(\tau) - p(\tau)q(\tau))\, d\tau]p(t) \geq 0$, $t \geq t_0$;
II. $\mathcal{L}(t) \equiv 0$, $p(t)q(t) \geq 0$, $q(t) \leq \mathcal{S}(t) = \lambda_0 + \int_{t_0}^t r(\tau)\, d\tau \leq 0$, $t \geq t_0$;
III. $\mathcal{L}(t) = -p(t)$, $q(t) - p'(t) \leq \mathcal{S}(t) = \lambda_0 + \int_{t_0}^t r(\tau)\, d\tau \leq 0$, $t \geq t_0$;
IV. $\mathcal{L}(t) = -p(t)$, $S(t) \equiv 0$, $q(t) \leq p'(t)$, $r(t) \leq 0$, $t \geq t_0$;
V. $\mathcal{L}(t) = -p(t)$, $S(t) = q(t)$, $p'(t) \geq 0$, $q(t) \leq 0$, $r(t) \leq q'(t)$ $t \geq t_0$;
VI. $\mathcal{L}(t) = -p(t)$, $S(t) = q(t) - p'(t)$, $q(t) \leq p'(t)$, $r(t) \leq -p''(t) + q'(t)$, $t \geq t_0$.

In system (2.15) we make the change $u(t) = v(t)y(t)$, $v(t) \neq 0$, $t \geq t_0$. Arrive at the system

$$\tag{2.21} \begin{cases} y'(t) + y^2(t) + \frac{1}{v(t)}\Big[\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) \\ -\mathcal{S}(t) + v'(t) + (p(t) + \mathcal{L}(t))v(t)\Big]y(t) \\ +\frac{1}{v(t)}\Big[r(t) - \mathcal{S}'(t) - (p(t) + \mathcal{L}(t))\mathcal{S}(t)\Big] = 0; \\ y'(t) + y^2(t) - [\mathcal{L}(t) + v(t)]y(t) + \mathcal{S}(t) = 0, \quad t \geq t_0. \end{cases}$$

We require $\mathcal{L}(t)$, $\mathcal{S}(t)$ and $v(t)$ to be such that the coefficients of the first and second equations of system (2.21) are the same. We arrive at the system

$$\begin{cases} [\mathcal{L}(t)+v(t)]' + [\mathcal{L}(t)+v(t)]^2 + p(t) \\ \qquad [\mathcal{L}(t)+v(t)] + \mathcal{S}(t) = 0; \\ \mathcal{S}'(t) + [p(t)+\mathcal{L}(t)+v(t)]\mathcal{S}(t) = r(t), \quad t \geq t_0. \end{cases} \tag{2.22}$$

Consequently, if $\mathcal{L}(t)+v(t)$ and $\mathcal{S}(t)$ form a solution of the system (2.22), then every solution of the Riccati equation

$$y'(t) + y^2(t) - [\mathcal{L}(t)+v(t)]y(t) + \mathcal{S}(t) = 0, \quad t \geq t_0, \tag{2.23}$$

is a solution of equation (2.4).

Given $[t_0, +\infty)$ functions $x(t)$, $y(t)$, $z(t)$ such, that for every thrice continuous differentiable functions $\phi(t)$, we find

$$\begin{aligned} &[\phi''(t) - y(t)\phi'(t) + x(t)\phi(t)]' \\ &\qquad + z(t)[\phi''(t) - y(t)\phi'(t) + x(t)\phi(t)] \\ &= \phi'''(t) + p(t)\phi''(t) + q(t)\phi'(t) + r(t)\phi(t) = 0, \quad t \geq t_0. \end{aligned} \tag{2.24}$$

Expanding the brackets in this relation and resulting terms of like derivatives of $\phi(t)$, we get $(y(t) - z(t) + p(t))\phi''(t) + (y'(t) - x(t)z(t) + q(t))\phi'(t) + (-x'(t) - -x(t)z(t) + r(t))\phi(t) = 0$, $t \geq t_0$. Consequently,

$$\begin{cases} z(t) = y(t) + p(t); \\ x(t) = y'(t) + (y(t)+p(t))y(t) + q(t); \\ x'(t) + (y(t)+p(t))x(t) = r(t), \quad t \geq t_0. \end{cases} \tag{2.25}$$

Eliminating $x(t)$ and $z(t)$ from this system, we obtain the following Riccati equation

$$\begin{aligned} &y''(t) + (3y(t) + 2p(t))y'(t) + y^3(t) + 2p(t)y^2(t) \\ &\qquad + (p'(t) + p^2(t) + q(t))y(t) \\ &\qquad + q'(t) + p(t)q(t) - r(t) = 0, \quad t \geq t_0. \end{aligned} \tag{2.26}$$

Let $y_1(t)$ be a solution of this equation on $[t_0, +\infty)$, and let

$$p_2(t) \equiv -y_1(t), \qquad q_2(t) \equiv y_1'(t) + y_1^2(t) + p(t)y_1(t) + q(t), \tag{2.27}$$

$t \geq t_0$. Consider the equation

$$\phi''(t) + p_2(t)\phi'(t) + q_2(t)\phi(t) = 0, \;\; t \geq t_0. \tag{2.28}$$

Lemma 2.3. *Every solution of equation* (2.28) *is a solution of equation* (1.1).

Proof. Let $x_1(t), y_1(t)$ and $z_1(t)$ form a solution of the system (2.26). Then, by virtue of (2.24) and (2.25), every solution of the equation

$$\phi''(t) - y_1(t)\phi'(t) + x_1(t)\phi(t) = 0, \quad t \geq t_0,$$

is a solution of equation (1.1). Therefore by virtue of (2.28) to complete the proof of the lemma we need only to show that $q_2(t) = x_1(t),\ t \geq t_0$. By virtue of the second equation of the system (2.26) $x_1(t) = y_1'(t) + y_1^2(t) + p(t)y_1(t) + q(t) = q_2(t),\ t \geq t_0$. The proof of the lemma is complete. □

Apply Lemma 2.2 to equation (2.26). This brings us to the assertion

Lemma 2.4. *Let the following conditions hold*:

1°) $(\mathcal{L}(t) + p(t))' + (\mathcal{L}(t) + p(t))^2 + q(t) \leq \mathcal{S}(t) \leq 0,\ t \geq t_0$:
2°) $[(\mathcal{L}(t)+p(t))' + (\mathcal{L}(t)+p(t))^2 + q(t) - \mathcal{S}(t)] \times (2p(t)+\mathcal{L}(t)) \leq 0,$ $t \geq t_0$:
3°) $q'(t) + p(t)q(t) - r(t) \leq [(\mathcal{L}(t) + p(t))' + (\mathcal{L}(t) + p(t))^2 + q(t)](2p(t) + \mathcal{L}(t)) + \mathcal{S}'(t),\ t \geq t_0$.

Then, for every $y_{(1)} > 0$ *and* $c_{(1)} > 0$, *equation* (2.26) *has positive solution* $y_1(t)$ *on the* $[t_0, +\infty)$, *satisfying the conditions*

$$y_1(t_0) = y_{(1)};$$
$$y_1'(t_0) + y_1^2(t_0) + \mathcal{L}(t_0)y_1(t_0) + \mathcal{S}(t_0) = c_{(1)}.$$

Moreover, the inequality

$$\int_{t_0}^{t} y_1(\tau)\, d\tau \geq \ln\left(1 + y_1(t_0)\int_{t_0}^{t} \exp\left\{\int_{t_0}^{\tau} \mathcal{L}(\xi)d\xi\right\}\right), \quad t \geq t_0,$$

holds.

We indicate some particular cases in which the conditions 1°)–3°) of Lemma 2.4 are satisfied:

I.° $S(t) \equiv 0,\ (\mathcal{L}(t) + p(t))' + (\mathcal{L}(t) + p(t))^2 + q(t) = 0,\ q'(t) + p(t)q(t) - r(t) \leq 0,\ t \geq t_0$;

II.$^\circ$ $\mathcal{L}(t)\equiv 0$, $S(t)=p'(t)+p^2(t)+q(t)\le 0$, $q'(t)+p(t)q(t)-r(t)\le 2(p'(t)+p^2(t)+q(t))p(t)+(p'(t)+p^2(t)+q(t))'$, $t\ge t_0$;

III.$^\circ$ $\mathcal{L}(t)=-p(t)$, $\mathcal{S}(t)=q(t)\le 0$, $r(t)\ge 0$, $t\ge t_0$;

IV.$^\circ$ $\mathcal{L}(t)=-p(t)$, $\mathcal{S}(t)\equiv 0$, $q(t)\le 0$, $p(t)q(t)\le 0$, $q'(t)\le r(t)$, $t\ge t_0$;

V.$^\circ$ $\mathcal{L}(t)=-2p(t)$, $\mathcal{S}(t)\equiv 0$, $q'(t)+p(t)q(t)\le r(t)$, $-p'(t)+p^2(t)+q(t)\le 0$, $t\ge t_0$;

VI.$^\circ$ $\mathcal{L}(t)=-2p(t)$, $\mathcal{S}(t)=-p'(t)+p^2(t)+q(t)\le 0$, $p(t)q(t)-r(t)\le [-p'(t)+p^2(t)]'$, $\quad t\ge t_0$.

Let $a(t)$, $b(t)$ and $c(t)$ be continuous functions on $[t_0,+\infty)$.

Lemma 2.5. *Let the following conditions hold*:

1) $\int_{t_0}^{+\infty} c(\tau)\,d\tau=+\infty$;
2) $\int_{t_0}^{t}|a(\tau)|\,d\tau\le K\int_{t_0}^{t}c(\tau)\,d\tau$, *$t\ge t_0$, $K=$* const;
3) $b(t)\to 0$ *when $t\to\infty$.*

Then

$$\int_{t_0}^{t} a(\tau)b(\tau)\,d\tau = o\bigg(\int_{t_0}^{t} c(\tau)\,d\tau\bigg),\quad t\to+\infty. \tag{2.29}$$

Proof. From 1), it follows that $\int_{t_0}^{t}c(\tau)\,d\tau>0$, $t\ge t_1$, for some $t_1\ge t_0$. We show that, for every $\varepsilon>0$, there exists $t_\varepsilon>t_1$ such that

$$J(t)\equiv\frac{\int_{t_0}^{t}|a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t}c(\tau)\,d\tau}<\varepsilon,\quad t\ge t_\varepsilon. \tag{2.30}$$

By virtue of 3), choose $N>t_1$ so large that $|b(t)|<\varepsilon/(2K)$ for $t>N$. Then, by virtue of 2),

$$\begin{aligned} J(t)&=\frac{\int_{t_0}^{N}|a(\tau)b(\tau)|\,d\tau+\int_{N}^{t}|a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t}c(\tau)\,d\tau}\\ &\le\frac{\int_{t_0}^{N}|a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t}c(\tau)\,d\tau}+\frac{\varepsilon}{2K}\frac{\int_{N}^{t}|a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t}c(\tau)\,d\tau}\\ &\le\frac{\int_{t_0}^{N}|a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t}c(\tau)\,d\tau}+\frac{\varepsilon}{2K}. \end{aligned} \tag{2.31}$$

By virtue of 1), we choose $t_\varepsilon > N$ so large that

$$\frac{\int_{t_0}^{N} |a(\tau)b(\tau)|\,d\tau}{\int_{t_0}^{t} c(\tau)\,d\tau} < \frac{\varepsilon}{2} \quad \text{for } t \geq t_\varepsilon.$$

From this and from (2.31), equation (2.30) follows, which proves (2.29). The proof of the lemma is complete. $\square$

3. Some properties of the solutions of equation (1.1). Let $t_0 < t_1 < \cdots < t_n < \cdots$ be an infinitely large sequence.

Theorem 3.1. *Let the following conditions hold*:

A$_1$) $\int_{t_k}^{t} \exp\big\{ \int_{t_k}^{\tau} [p(s) - \int_{t_k}^{s} \exp\{-\int_{\xi}^{s} p(u)\,du\} q(\xi)\,d\xi]\,ds\big\} q(\tau)\,d\tau \leq 0$, $t \in [t_k, t_{k+1})$, $k = 0, 1, \ldots$;
B$_1$) $r(t) \leq 0$, $t \geq t_0$.

Then every solution $\phi_0(t)$ *of equation* (1.1) *with* $\phi_0(t_0) = 1$, $\phi_0'(t_0) > 0$ *satisfies the inequalities*

$$\phi_0(t) \geq 1 + \phi_0'(t_0)(t - t_0), \quad \phi'(t) > 0,\ t \geq t_0. \tag{3.1}$$

Proof. From A$_1$), it follows that the equation

$$\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) = 0, \quad t \geq t_0,$$

has nonnegative solution $\mathcal{L}_0(t)$ on the $[t_0, +\infty)$ (see [**4,** page 26, Theorem 4.1]). Then from B$_1$), it follows that, for $\mathcal{L}(t) \equiv \mathcal{L}_0(t)$, $S(t) \equiv 0$, Lemma 2.2 1)–3) hold. Therefore, by virtue of Lemma 2.2, equation (2.4) has positive solution $y_0(t)$, satisfying the condition $y_0(t_0) = \phi_0'(t_0) > 0$, and (because $\mathcal{L}_0(t) \geq 0$, $t \geq t_0$)

$$\int_{t_0}^{t} y_0(\tau)\,d\tau \geq \ln(1 + \phi_0'(t_0)(t - t_0)), \quad t \geq t_0.$$

By virtue of (2.1), it follows that the function

$$\phi_0(t) \equiv \exp\left\{ \int_{t_0}^{t} y_0(\tau)\,d\tau \right\}, \quad t \geq t_0,$$

is a solution of equation (1.1), satisfying the conditions in (3.1). The proof of the theorem is complete. $\square$

Theorem 3.2. *Let the following conditions hold*:

A$_2$) $q(t) \le 0$; $r(t) \le q'(t) + p(t)q(t)$, $t \ge t_0$.

Then every solution $\phi_0(t)$ *of equation* (1.1) *with* $\phi_0(t_0) = 1$ *and* $\phi_0'(t_0) > 0$ *satisfies the inequalities* (3.1). *Also, if*

B$_1$) $p(t) \ge 0$, $t \ge t_0$,

then

$$\exp\{\underline{Y}(t,c)\} \le \phi_0(t) \le \exp\{\overline{Y}(t,c)\}, \quad t \ge t_0, \tag{3.2}$$

where $c = \phi_0''(t_0) + (1/2)(\phi_0'(t_0))^2 + p(t_0)\phi_0'(t_0)$.

Proof. The conditions A$_2$) of Theorem 3.2 show that, for $\mathcal{L}(t) \equiv 0$, $\mathcal{S}(t) = q(t)$, $t \ge t_0$, conditions 1)–3) of Lemma 2.2 are satisfied. Therefore, equation (2.4) has positive solution $y_0(t)$ on $[t_0, +\infty)$ with $y_0(t_0) = \phi_0'(t_0) > 0$, and

$$\int_{t_0}^{t} y_0(\tau)\, d\tau \ge \ln\big(1 + \phi_0'(t_0)(t - t_0)\big), \quad t \ge t_0.$$

By virtue of (2.1), it follows that $\phi_0(t) \equiv \exp\{\int_{t_0}^t y_0(\tau)\, d\tau\}(t \ge t_0)$ is a solution of equation (1.1), satisfying the relations (3.1). Further, because $y_0(t)$ is a solution of equation (2.16), then from the conditions A$_2$) it follows that $y_0'(t) + y_0^2(t) \ge 0$, $t \ge t_0$. From this and from A$_2$), we get

$$y_0'(t) + \frac{3}{2} y_0^2(t) + p(t) y_0(t) \ge 0, \quad t \ge t_0.$$

By virtue of Lemma 2.1 equation (3.2) follows. The proof of the theorem is complete. $\square$

Remark 3.3. In view of I and II, Theorem 3.2 remains valid, if in it we change condition A$_2$) by one of the following groups of conditions.

a$_1^1$) $q(t) \le \lambda_0 + \int_{t_0}^t [r(\tau) - p(\tau)q(\tau)]\, d\tau \le 0$, $(q(t) - \lambda_0 - \int_{t_0}^t [r(\tau) - p(\tau)q(\tau)]\, d\tau)p(t) \ge 0$, $\lambda_0 = \text{const}$, $t \ge t_0$.

a$_1^2$) $q(t) \le \lambda_0 + \int_{t_0}^t r(\tau)\, d\tau \le 0$, $p(t)q(t) \ge 0$, $\lambda_0 = \text{const}$, $t \ge t_0$.

Taking into account III–VI, by analogy of Theorem 3.2, the following can be proven.

Theorem 3.4. *Let one of the following groups of conditions be satisfied*:

A$_3$) $q(t) - p'(t) \le \lambda_0 + \int_{t_0}^t r(\tau)\,d\tau \le 0,\ t \ge t_0,\ \lambda_0 = \mathrm{const}$;
B$_3$) $q(t) \le p'(t),\ r(t) \le 0,\ t \ge t_0$;
C$_3$) $p'(t) \ge 0,\ q(t) \le 0,\ r(t) \le q'(t),\ t \ge t_0$;
Γ_3) $q(t) \le p'(t),\ r(t) \le q'(t) - p''(t),\ t \ge t_0$.

Then every solution $\phi_0(t)$ *of equation* (1.1) *with* $\phi_0(t_0) = 1,\ \phi'(t_0) > 0$ *satisfies the inequalities*:

$$\phi_0(t) > 1 + \phi_0'(t_0)\int_{t_0}^t \exp\left\{-\int_{t_0}^\tau p(s)\,ds\right\}d\tau,$$
$$\phi'(t) > 0, \quad t \ge t_0.$$

Theorem 3.5. *Let condition* B$_1$) *of Theorem* 3.1 *and the condition*

A$_4$) $q(t) \le 0,\ t \ge t_0$,

be satisfied. Then, for every $\alpha > 0$, *equation* (1.1) *has a solution* $\phi_0(t)$ *such that*

(3.3)
$$\phi_0(t_0) = 1,$$
$$\phi_0(t) \ge 1 + \phi_0'(t_0)(t - t_0) + \alpha\int_{t_0}^t d\tau\int_{t_0}^\tau \exp\left\{\int_{t_0}^\xi p(s)\,ds\right\}d\xi, \quad \phi_0'(t) > 0,\ t \ge t_0,$$

and, if condition B$_1$) *is satisfied, then* (3.2) *is valid.*

Proof. By virtue of Corollary* and from condition A$_2$) it follows that the equation

$$\mathcal{L}'(t) + \mathcal{L}^2(t) + p(t)\mathcal{L}(t) + q(t) = 0, \quad t \ge t_0,$$

has a positive solution $\mathcal{L}_0(t)$ on $[t_0, +\infty)$, satisfying the inequality

$$\mathcal{L}_0(t) \ge \frac{\alpha\exp\left\{\int_{t_0}^t p(\xi)\,d\xi\right\}}{1 + \alpha\int_{t_0}^t \exp\left\{\int_{t_0}^\tau p(\xi)\,d\xi\right\}d\tau}, \quad t \ge t_0,\ \alpha > 0. \tag{3.4}$$

It is not difficult to see that, for $\mathcal{L}(t) = \mathcal{L}_0(t),\ t \ge t_0,\ \mathcal{S}(t) \equiv 0$ and if the condition B$_2$) is satisfied, then conditions 1)–3) of Lemma 2.2

are satisfied. Therefore, equation (2.4) has a positive solution $y_0(t)$ on $[t_0, +\infty)$ with $y_0(t_0) = \phi_0'(t_0) > 0$, and the inequality

(3.5)
$$\int_{t_0}^{t} y_0(\tau)\, d\tau \geq \ln\left(1 + \phi_0'(t_0) \int_{t_0}^{t} \exp\left\{ \int_{t_0}^{\tau} \mathcal{L}_0(\xi)\, d\xi \right\} d\tau\right), \quad t \geq t_0,$$

is satisfied. By virtue of (2.1),

$$\phi_0(t) \equiv \exp\left\{ \int_{t_0}^{t} y_0(\tau)\, d\tau \right\}, \quad t \geq t_0,$$

is a solution of equation (1.1). Then, from (3.4) and (3.5), equation (3.3) follows.

To prove the last part of the theorem we merely repeat the arguments relating to the proof of (3.2) of Theorem 3.1. The proof of the theorem is complete. □

Theorem 3.6. *Let condition* A_4) *of Theorem* 3.5 *be satisfied, and let*

A_3) $\int_{t_0}^{+\infty} \exp\{-\int_{t_0}^{\tau} p(\xi)\, d\xi\}\, d\tau = +\infty$; $r(t) \geq 0$, $t \geq t_0$.

Then equation (1.1) *is nonstable.*

Proof. Consider the equation

(3.6)
$$y'(t) + y^2(t) + p(t)y(t) = -q(t) + \int_{t_0}^{t} \exp\left\{ -\int_{\tau}^{t} (p(\xi) + y(\xi)) \right\} r(\tau)\, d\tau, \quad t \geq t_0.$$

Let $y_2(t)$ solution of this equation with $y_2(t_0) > 0$. From A_2) and B_3) it follows that the right hand side of (3.6) for $y(t) \equiv y_2(t)$ is nonnegative in the domain of existence of the $y_2(t)$. Then, using the method of proof of Lemma 2.2, we can easily show that $y_2(t)$ is continuable on $[t_0, +\infty)$. Note that $y_2(t)$ is a solution of the Riccati equation

$$y'(t) + y^2(t) + p(t)y(t) = u_2(t), \quad t \geq t_0,$$

where $u_2(t) \geq 0$, $t \geq t_0$, the right hand part of (3.6) for $y(t) = y_2(t)$,

$t \geq t_0$. By virtue of Corollary*, from this the inequality

$$y_2(t) \geq \frac{y_2(t_0)\exp\left\{-\int_{t_0}^t p(\xi)\,d\xi\right\}}{1 + y_2(t_0)\int_{t_0}^t \exp\left\{-\int_{t_0}^\tau p(\xi)\,d\xi\right\}\,d\tau}, \quad t \geq t_0$$

follows. Consequently,

$$\int_{t_0}^t y_2(\tau)\,d\tau \geq \ln\left(1 + y_2(t_0)\int_{t_0}^t \exp\left\{-\int_{t_0}^\tau p(\xi)\,d\xi\right\}d\tau\right), \quad t \geq t_0. \tag{3.7}$$

Note that $\mathcal{L}_0(t) + v_0(t) \equiv y_2(t)$ and

$$\mathcal{S}_0(t) \equiv \int_{t_0}^t \exp\left\{-\int_{t_0}^\tau (p(\xi) + y_2(\xi))\,d\xi\right\} r(\tau)\,d\tau, \quad t \geq t_0,$$

form a solution of the system (2.22). Consider the equation

$$\phi''(t) - (\mathcal{L}_0(t) + v_0(t))\phi'(t) + \mathcal{S}_0(t)\phi(t) = 0, \quad t \geq t_0. \tag{3.8}$$

Let $\phi_j(t)$, $j = 1, 2$, be linearly independent real-valued solutions of this equation. Then $\phi_\pm(t) \equiv \phi_1(t) \pm i\phi_2(t)$ linearly independent complex solutions of the same equation do not vanish on $[t_0, +\infty)$. Therefore, $y_\pm(t) \equiv \phi'_\pm(t)/\phi_\pm(t)$, $t \geq t_0$, are solutions of equation (2.23) for $\mathcal{L}(t) + v(t) = \mathcal{L}_0(t) + v_0(t)$, $\mathcal{S}(t) = \mathcal{S}_0(t)$, $t \geq t_0$. Then $y_\pm(t)$ is a solutions of equation (2.4) on $[t_0, +\infty)$. By virtue of (2.1) it follows that

$$\phi_\pm(t) = \phi_\pm(t_0)\exp\left\{\int_{t_0}^t y_\pm(\tau)\,d\tau\right\}$$

is a solution of equation (1.1). Therefore, to complete the proof of the theorem, it is enough to show that equation (3.8) is nonstable. By virtue of Liuvill's formula the Wronskian $W(t)$ of the solutions $\phi_\pm(t)$ is equal to:

$$W(t) = W(t_0)\exp\left\{\int_{t_0}^t y_2(\tau)\,d\tau\right\}, \quad t \geq t_0 \ (W(t_0) \neq 0).$$

From this, A$_3$) and (3.7) the unboundedness of $W(t)$ follows. Consequently, equation (3.9) is unstable. The proof of the theorem is complete. □

Theorem 3.7. *Let condition* B$_5$) *of Theorem* 3.6 *be satisfied, and let*:

A$_6$) $p(t) \geq 0$, $t \geq t_0$;
B$_6$) $q(t) \leq 0$, $\int_{t_0}^{+\infty} |q(\tau)|\, d\tau < +\infty$;

$$\int_{t_0}^{+\infty} [r(\tau) - p(\tau)q(\tau) - q'(\tau)]\, d\tau = +\infty,$$
$$\int_{t_0}^{t} \big|p'(\tau) - p^2(\tau) - q(\tau)\big|\, d\tau$$
$$= O\bigg(\int_{t_0}^{t} [r(\tau) - p(\tau)q(\tau) - q'(\tau)]\, d\tau\bigg), \quad t \to +\infty.$$

Then equation (1.1) *has two linearly independent oscillatory solutions which are solutions of a linear ordinary differential equation with one coefficient of greatest derivative.*

Proof. Put $\mathcal{L}(t) = -p(t)$, $\mathcal{S}(t) = q(t)$, $t \geq t_0$. Then, if the conditions B$_5$), A$_6$) and B$_6$) are satisfied, the conditions 1)–3) of Lemma 2.4 will be satisfied. Therefore, equation (2.26) has positive solution $y_1(t)$ on $[t_0, +\infty)$. Consequently, by virtue of Lemma 2.3 to prove the theorem, it is enough to show that equation (2.28) is oscillatory.

In (2.28), we make a change:

$$\phi(t) = \exp\bigg\{-\int_{t_0}^{t} \frac{p_2(\tau)}{2}\, d\tau\bigg\}\psi(t), \quad t \geq t_0.$$

This brings us to

$$\psi''(t) + Q(t)\psi(t) = 0, \quad t \geq t_0, \tag{3.9}$$

where

$$Q(t) = q_2(t) - \frac{p_2'(t)}{2} - \frac{p_2^2(t)}{4}, \quad t \geq t_0.$$

It is evident that equation (2.28) is oscillatory if and only if equation (3.13) is the same. A sufficient condition of oscillation of (3.13) is (see [**4,** page 958])

$$\int_{t_0}^{+\infty} Q(\tau)\, d\tau = +\infty. \tag{3.10}$$

By virtue of (2.27),

$$Q(t) = \frac{3}{2}y_1'(t) + \frac{3}{4}y_1^2(t) + p(t)y_1(t) + q(t), \quad t \geq t_0. \tag{3.11}$$

By virtue of A_6), two cases are possible:

a) $\int_{t_0}^{+\infty} p(\tau)y_1(\tau)\,d\tau = +\infty$;
b) $\int_{t_0}^{+\infty} p(\tau)y_1(\tau)\,d\tau < +\infty$.

In case B_6) a), (3.11) follows (3.10). Let case b) take place. By virtue of (2.14)–(2.16), equation (2.26) is equivalent to the following equation

$$y'(t) + y^2(t) + p(t)y(t) + q(t) = \frac{1}{E_1(t)}\left\{c_1 + \int_{t_0}^{t} E_1(\tau)r(\tau)\,d\tau\right\}, \tag{3.12}$$

$t \geq t_0$, where

$$E_1(t) \equiv \exp\left\{\int_{t_0}^{t}\big[y(\xi) + p(\xi)\big]\,d\xi\right\}, \quad t \geq t_0,$$

$c_1 \equiv y'(t_0) + y^2(t_0) + +p(t_0)y(t_0) + q(t_0) > 0$. From this, it follows that

$$y_1'(t) + y_1^2(t) - \mathcal{L}(t)y_1(t) = u_1(t), \quad t \geq t_0,$$

where $u_1(t)(\geq 0)$ right hand part of (3.12) for $y(t) = y_1(t)$, $t \geq t_0$. Then

$$\widetilde{Q}(t) \equiv y_1'(t) + y_1^2(t) + p(t)y_1(t) + q(t) \geq 0, \quad t \geq t_0. \tag{3.13}$$

We show that

$$I \equiv \int_{t_0}^{+\infty} \widetilde{Q}(\tau)\,d\tau = +\infty. \tag{3.14}$$

Suppose that this is not true. Then, by virtue of (3.13), the inequality $I < +\infty$ holds. From this, A_6) and B_6), it follows that

$$\lim_{t\to+\infty} y_1(t) = 0, \tag{3.15}$$

$$\int_{t_0}^{+\infty} y_1^2(\tau)\,d\tau < +\infty. \tag{3.16}$$

Substituting $y(t) = y_1(t)$, $t \ge t_0$, in (2.26) and integrating from t_0 to t, we will have:

$$
\begin{aligned}
(3.17)\quad & y_1'(t) + \frac{3}{2}y_1^2(t) + 2p(t)y_1(t) \\
&\quad + \int_{t_0}^{t} y_1^3(\tau)\,d\tau + 2\int_{t_0}^{t} p(\tau)y_1^2(\tau)\,d\tau \\
&\quad + \int_{t_0}^{t} \big[p^2(\tau) - p'(\tau) + q(\tau)\big]y_1(\tau)\,d\tau \\
&\quad + \int_{t_0}^{t} [q'(\tau) + p(\tau)q(\tau) - r(\tau)]\,d\tau = c_2, \quad t \ge t_0,
\end{aligned}
$$

where $c_2 = y_1'(t_0) + \frac{3}{2}y_1^2(t_0) + 2p(t_0)y_1(t_0)$. By virtue of Lemma 2.5 from B_6) and (3.15) it follows that

$$
\int_{t_0}^{t} \big[p^2(\tau) - p'(\tau) + q(\tau)\big]y_1(\tau)\,d\tau = o\bigg(\int_{t_0}^{t} [r(\tau) - q'(\tau) - p(\tau)q(\tau)]\,d\tau\bigg),
$$
$$
t \to +\infty.
$$

From this, B_6), (3.15), (3.16) and b),

$$
y_1'(t) + \frac{3}{2}y_1^2(t) + 2p(t)y_1(t) \longrightarrow +\infty, \quad t \to +\infty
$$

follows. Then, by virtue of b) and (3.16), we will have $y_1(t) \to +\infty$ when $t \to +\infty$, which contradicts (3.15). The contradiction thus obtained proves (3.14). From (3.14), B_6) and b), (3.10) follows. The proof of the theorem is complete. □

Example 3.8. Consider equation

$$
(3.18)\qquad \phi'''(t) + (1 + \sqrt{3} + \sin t)\phi''(t) + \lambda\phi(t) = 0,
$$
$$
t \ge t_0, \quad \lambda = \mathrm{const} > 0.
$$

It is not difficult to see that, for this equation, all conditions of Theorem 3.7 are satisfied. Therefore, it has two linearly independent oscillatory solutions, the zeroes of which separate each other.

In (3.18), we make a change:

$$\phi(t) = \exp\Big\{ -\frac{1}{2}\int_{t_0}^{t} p(\tau)\, d\tau\Big\}\psi(t), \quad t \ge t_0.$$

This brings us to the equation

$$\psi'''(t) + q_1(t)\psi'(t) + r_1(t)\psi(t) = 0, \quad t \ge t_0, \tag{3.19}$$

where $q_1(t) \equiv -(1/3)(1+\sqrt{3}+\sin t)^2 - \cos t \le 0$, $r_1(t) \equiv \lambda + (2/27)(1+\sqrt{3}+\sin t)^3 - \sin t/3 > 0$, $t \ge t_0$. For

$$I_1 \equiv \int_0^{2\pi} \Big\{(1+\sqrt{3}+\sin\tau)^3 - [(1+\sqrt{3}+\sin\tau)^2 + 3\cos\tau]^{3/2}\Big\}\, d\tau$$
$$\approx -4.065,$$

and for equation (3.19) condition (1.3) is satisfied only for $\lambda > -I_1/27\pi \approx 0.047$ (for $0 < \lambda < -I_1/27\pi$, we have $I_{\mathrm{L}} = -\infty$). Consequently, Lazer's theorem is not applicable to (3.18), where $0 < \lambda \le -I_1/27\pi$ (note that Theorem 8 of [5] [a generalization of Lazer's theorem]) also cannot be applicable to (3.18); see [**11,** page 392]. Meanwhile, Theorem 3.6 is applicable to (3.18) for all $\lambda > 0$.

Remark 3.9. The oscillatory theorems for the cases $q_1(t) \le 0$ and $r_1(t) \ge 0$, where the equality (1.3) (for $q(t) \equiv q_1(t)$ and $r(t) \equiv r_1(t)$) may not hold, are obtained in the work [**1**]. They relate to the case when

$$I_2 \equiv \int_{t_0}^{+\infty} r_1(\tau)\, d\tau < +\infty$$

(in Example 3.8, $I_2 = +\infty$). It is not difficult to see that, if $p(t) \ge 0$, $q(t) \equiv 0$ and $r(t) \ge 0$, $r(t) \not\equiv 0$ are periodic functions, then the conditions B_6) of Theorem 3.7 are satisfied, while for I_L (for $q(t) \equiv q_1(t)$ and $r(t) \equiv r_1(t)$) the following cases are possible.

1) $I_L = +\infty$,
2) I_L diverges,
3) $I_L = -\infty$.

The following theorem is a supplement to [**5,** page 134, Theorem 4].

Theorem 3.10. *Let the following conditions hold*:

A_7)

$$\int_{t_k}^{t} \exp\left\{-\int_{t_k}^{\tau} ds \int_{t_k}^{s} q(\xi)d\xi\right\} q(\tau)\, d\tau \le 0,$$
$$t \in [t_k, t_{k+1}), \; k = 0, 1, 2, \ldots,$$

where t_k*,* $k = 0, 1, \ldots$*, is the same as in Theorem* 3.1*,*

B_7) $q'(t) + p(t)q(t) - r(t) \le 0$, $t \ge t_0$.

Then, if all nontrivial solutions of equation (1.1) *oscillate, except one multiplied by arbitrary constant, then equation* (1.1) *has two linearly independent oscillatory solutions, which are solutions of a second order linear ordinary differential equation with one coefficient of greatest derivative.*

Proof. By virtue of [**4,** Theorem 4.1], from A_7), it follows that the equation

$$(\mathcal{L}(t) + p(t))' + (\mathcal{L}(t) + p(t))^2 + q(t) = 0, \quad t \ge t_0,$$

has solution $\mathcal{L}_0(t)$ on $[t_0, +\infty)$. Then, from B_7) it follows that for $\mathcal{L}(t) \equiv \mathcal{L}_0(t)$ and $\mathcal{S}(t) \equiv 0$ the conditions 1°)–3°) of Lemma 2.4 are satisfied (see I°)). Therefore, equation (2.26) has solution $y_1(t)$ on $[t_0, +\infty)$.

By Lemma 2.3, it follows that all solutions of equation (2.28), the coefficients of which are determined by $y_1(t)$ according to formula (2.27), are solutions of equation (1.1).

Hence, to complete the proof of the theorem, it remains to show that equation (2.28) oscillates. Suppose equation (2.28) does not oscillate. Then it has two linearly independent non oscillatory solutions which are simultaneously solutions of equation (1.1). But, from the conditions of the theorem, it follows that equation (1.1) cannot have two linearly independent oscillatory solutions. The contradiction so obtained proves oscillation of equation (2.28). The proof of the theorem is complete. □

Remark 3.11. In view of II°–VI°, Theorem 3.10 remains valid if we replace conditions A_7) and B_7) by one of the following groups of conditions.

a$_7^1$) $q'(t) + p(t)q(t) - r(t) \le 2(p'(t)p^2(t) + q(t))p(t) + (p'(t)p^2(t) + q(t))'$, $p'(t)p^2(t) + q(t) \le 0$, $t \ge t_0$.
a$_7^2$) $q(t) \le 0$, $r(t) \ge 0$, $t \ge t_0$.
a$_7^3$) $q(t) \le 0$, $p(t)q(t) \le 0$, $q'(t) \le r(t)$, $t \ge t_0$.
a$_7^4$) $q'(t) + p(t)q(t) \le r(t)$, $-p'(t) + p^2(t) + q(t) \le 0$, $t \ge t_0$.
a$_7^5$) $p(t)q(t) - r(t) \le [-p'(t) + p^2(t)]'$, $-p'(t) + p^2(t) + q(t) \le 0$, $t \ge t_0$.

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