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ANGULAR VALUE DISTRIBUTION CONCERNING SHARED VALUES

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ABSTRACT. In this paper, we investigate the number of sharing values of a meromorphic function and its derivative in one angular domain instead of the whole complex plane and obtain the following results: Let f be a meromorphic function of lower order > 2 in the complex plane. Then there exists a direction H: $\arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ such that for any positive number ε, f and f' share at most two distinct finite values without counting multiplicities in the angular region $\{z : |\arg z - \theta_0| < \varepsilon\}$. This improve a result of Weichuan and Mori.

1. Introduction and main result. In this paper, by a *meromorphic function*, we mean that the function is meromorphic in the whole complex plane C. It is assumed that the reader is familiar with the basic result and notations of the Nevanlinna's value distribution theory (see [**1, 9**]), such as $T(r;f)$, $N(r,f)$ and $m(r,f)$. Meanwhile, the lower order μ and the order λ of a meromorphic function f are, in turn, defined as below:

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

Let D be a domain in the complex plane $\mathbf{C}$, and let

$$E_D(a,f) = \{z \in D : f(z) = a,\ \text{counting multiplicity}\},$$

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and

$$\overline{E}_D(a,f) = \{z : z \in D, f(z) = a\}(\text{ as a set in } \mathbf{C}).$$

We say that two meromorphic functions f and g share the value a IM (ignoring multiplicity) in D if $\overline{E}_D(a,f) = \overline{E}_D(a,g)$.

The problems about the uniqueness of meromorphic functions and their derivatives with shared values have been studied by several authors (see [**5, 10, 11**]). Mues, Steinmetz and Gundersen proved the following theorem.

Theorem A [**11**]. *Let $f(z)$ be a meromorphic function, a_1, a_2, a_3 distinct finite complex numbers. If a_1, a_2, a_3 are* IM *shared values of f and f' in $\mathbf{C}$, then $f \equiv f'$.*

From Theorem A, we can immediately obtain Theorem $A^{'}$.

Theorem A'**.** *Let f be a non-constant meromorphic function. If $f \not\equiv f'$, then f and f' share at most two finite distinct values* IM *in the complex plane* $\mathbf{C}$.

Theorem A' shows that the number of sharing values of $f(z)$ and $f'(z)$ are two at most in the complex plane $\mathbf{C}$ except $f(z) \equiv f'(z)$.

People have established a connection between normality criteria and shared values (see [**3, 6, 8**]). Naturally, we ask whether we can extend Theorem A' to some angular domains and establish a connection between angular value distribution (singular directions) and shared values of a meromorphic function. Lin and Mori [**7**] dealt with this subject under certain value-sharing condition in a sector instead of the plane $\mathbf{C}$ and proved the following theorem.

Theorem B**.** *Let $f(z)$ be a meromorphic function of infinite order and*

$$\limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} < +\infty.$$

Then there exists a direction $\arg z = \theta$ $(0 \le \theta < 2\pi)$ *such that, for every small positive number $\varepsilon < \pi/2$, $f(z)$ and $f'(z)$ share at most two distinct finite values in the angular domain* $\{z : |\arg z - \theta| < \varepsilon\}$.

The direction $\arg z = \theta$ in Theorem B is called *one* SV *direction* by Lin and Mori [**7**]. Theorem B only discussed the infinite order meromorphic functions of finite hyper order. In this paper, we shall prove that Theorem B is valid for any transcendental meromorphic functions of lower order $\mu > 2$.

Theorem 1.1. *Let f be a meromorphic function of lower order $\mu > 2$ in the complex plane* $\mathbf{C}$. *Then there exists a direction H:* $\arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ *such that, for every positive number ε, f and f' share two distinct finite values* IM *at most in* $\{z | \arg z - \theta_0| < \varepsilon\}$.

2. Some lemmas.

In order to prove Theorem 1.1, we will collect and prove some lemmas in this section.

Lemma 2.1. ([**4**]). *Let $\mathcal{F}$ be a family of meromorphic functions such that, for every function $f \in \mathcal{F}$, its zeros of multiplicity are at least k. If $\mathcal{F}$ is not a normal family at the origin* 0, *then for $0 \le \alpha < k$, there exist*:

(i) *a number* $r (0 < r < 1)$;
(ii) *a sequence of complex numbers* $z_n \to 0, |z_n| < r$;
(iii) *a sequence of functions* $f_n \in \mathcal{F}$;
(iv) *a sequence of positive numbers* $\rho_n \to 0$

such that

$$g_n(z) = {\rho_n}^{-\alpha} f_n(z_n + \rho_n z)$$

converges locally uniformly with respect to a spherical metric of a nonconstant meromorphic function $g(z)$ on $\mathbf{C}$, *and, moreover, g is of order at most two.*

For convenience, we will use the following notation

$$\begin{aligned} LD(r, f : c_1, c_2) = c_1 &\left[m\left(r, \frac{f'}{f}\right) + \sum_{i=1}^{3} m\left(r, \frac{f'}{f - a_i}\right)\right] \\ &+ c_2 \left[m\left(r, \frac{f''}{f'}\right) + \sum_{i=1}^{3} m\left(r, \frac{f''}{f' - ta_i}\right)\right]. \end{aligned}$$

Lemma 2.2. ([**6**]). *Let f, g be nonconstant meromorphic functions in the unit disc, which share distinct finite complex numbers a_1, a_2, a_3 and $a_4 = \infty$. If $a \ne a_j$ and $f(0) \ne a$, a_j, $(j = 1, 2, 3, 4)$, $f'(0) \ne 0, \infty$*

and $f(0) \neq g(0)$, *then*

$$T(r,f) \le T(r,g) + \log \frac{\prod_{i=1}^{3} |f(0)-a_i|}{|f'(0)||f(0)-g(0)|} + O(1)\left[m\left(r, \frac{f'}{f-a}\right) + \sum_{i=1}^{3} m\left(r, \frac{f'}{f-a_i}\right) + 1\right],$$

where $O(1)$ *is a complex number depending only on* a *and* a_i $(i = 1,2,3)$.

Lemma 2.3. *Let* f *be a meromorphic function in a domain* $D = \{z : |z| < R\}$, *let* a_1, a_2 *and* a_3 *be three distinct finite complex numbers, and let* t *be a positive real number. If*

$$\overline{E}_D(a_i, f) = \overline{E}_D(ta_i, f') \quad \text{for } i = 1,2,3;$$

and if $a \neq a_j$ *and* $f(0) \neq a_j, \infty$ $(j = 1,2,3,)$, $f'(0) \neq 0, at$ *and* $f''(0) \neq 0$, $f'(0) \neq tf(0)$, *then, for* $0 < r < R$, *we have*

$$T(r,f) \le LD(r,f:2,3) + \log \frac{\prod_{i=1}^{3} |f(0)-a_i|^2 |f'(0)-ta_i|^3}{|tf(0)-f'(0)|^5 |f'(0)|^2} + 3\log \frac{1}{|f''(0)|} + \left(\log^+ t + m\left(r, \frac{f''}{f'-ta}\right) + 1\right) O(1),$$

where $O(1)$ *is a complex number depending only on* a *and* a_i $(i = 1,2,3)$.

Proof. Firstly, we distinguish two cases to deduce the following inequality:

$$2T(r,f) \le T(r,f') + \overline{N}(r,f) + LD(r,f:1,0) + \log \frac{\prod_{i=1}^{3} |f(0)-a_i|}{|(tf-f')(0)||f'(0)|} + O(1) + \log^+ t. \tag{2.1}$$

Case 1. $a_1a_2a_3 \neq 0$. Since $\overline{E}_D(a_i, f) = \overline{E}_D(ta_i, f')$ $(i = 1,2,3)$ with $t \neq 0$, we get that $f - a_1$, $f - a_2$, $f - a_3$ has only simple zeros in D. By the assumption, we see that $f'(z) \not\equiv tf(z)$. Therefore, we have

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f-a_j}\right) \le N\left(r, \frac{1}{tf-f'}\right) \le T(r, tf-f') + \log \frac{1}{|tf(0)-f'(0)|}$$

$$
\begin{aligned}
&\le N(r,f') + m(r,f) + m\left(r,\frac{f'}{f}\right)\\
&\quad + \log^+ t + O(1) + \log\frac{1}{|tf(0)-f'(0)|}\\
&\le T(r,f) + \overline{N}(r,f) + m\left(r,\frac{f'}{f}\right)\\
&\quad + \log^+ t + O(1) + \log\frac{1}{|tf(0)-f'(0)|}.
\end{aligned}
$$

Note that

$$
\sum_{j=1}^{3} m\left(r,\frac{1}{f-a_j}\right) = m\left(r,\frac{1}{f'}\sum_{j=1}^{3}\frac{f'}{f-a_j}\right) + O(1), \tag{2.2}
$$

we have

$$
\begin{aligned}
\sum_{j=1}^{3} T\left(r,\frac{1}{f-a_j}\right) \le{}& T(r,f) + \overline{N}(r,f) + m\left(r,\frac{1}{f'}\right)\\
&+ LD(r,f:1,0) + \log^+ t\\
&+ \log\frac{1}{|(tf-f')(0)|} + O(1).
\end{aligned}
$$

By Nevanlinna's first fundamental theorem, we have

$$
\begin{aligned}
2T(r,f) \le{}& T(r,f') + \overline{N}(r,f)\\
&+ LD(r,f:1,0) + \log\frac{\prod_{i=1}^{3}|f(0)-a_i|}{|(tf-f')(0)||f'(0)|}\\
&+ O(1) + \log^+ t.
\end{aligned}
$$

Case 2. $a_1a_2a_3 = 0$. Without loss generality, we set $a_3 = 0$. By assumption, we have that $f - a_j (j = 1, 2,)$ has only simple zeros, and the zeros of f are of multiplicity ≥ 2. Thus,

$$
\begin{aligned}
\sum_{1}^{3} N\left(r,\frac{1}{f-a_i}\right) &= \sum_{1}^{2} N\left(r,\frac{1}{f-a_i}\right) + \overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'}\right)\\
&\le N\left(r,\frac{1}{tf-f'}\right) + N\left(r,\frac{1}{f'}\right)
\end{aligned}
$$

$$\le T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)+m\left(r,\frac{f'}{f}\right)$$
$$+\log^+ t+O(1)+\log\frac{1}{|tf(0)-f'(0)|}.$$

Combining this with (2.2), we also have

$$2T(r,f)\le T(r,f')+\overline{N}(r,f)+LD(r,f:1,0)$$
$$+\log\frac{\prod_{i=1}^{3}|f(0)-a_i|}{|(tf-f')(0)||f'(0)|}+O(1)+\log^+ t.$$

Thus, inequality (2.1) is proved.

On the other hand, note that $\overline{E}_D(a_i,f)=\overline{E}_D(ta_i,f')$, $i=1,2,3$, and $\overline{E}_D(\infty,f)=\overline{E}_D(\infty,f')$, $f(0)\ne a_j,\infty$ $(j=1,2,3)$. It follows that $f'(0)\ne ta_j,\infty$ $(j=1,2,3)$. By application of Lemma 2 to f' and tf, we have

$$T(r,f')\le T(r,f)+LD(r,f:0,1))$$
$$+\log\frac{\prod_{i=1}^{3}|f'(0)-ta_i|}{|tf(0)-f'(0)||f''(0)|}$$
$$+(\log^+ t+m(r,\frac{f''}{f'-ta})+1)O(1). \tag{2.3}$$

Now, substituting (2.3) into (2.1), we have

$$T(r,f)\le\overline{N}(r,f)+LD(r,f:1,1)$$
$$+\log\frac{\prod_{i=1}^{3}|f(0)-a_i||f'(0)-ta_i|}{|f''(0)||tf(0)-f'(0)|^2|f'(0)|}$$
$$+\left(\log^+ t+m\left(r,\frac{f''}{f'-ta}\right)+1\right)O(1).$$

Notice that

$$2\overline{N}(r,f)\le N(r,f)+\overline{N}(r,f)+m(r,f')=T(r,f').$$

Hence,

$$2T(r,f)\le T(r,f')+2LD(r,f:1,1)$$
$$+2\log\frac{\prod_{i=1}^{3}|f(0)-a_i||f'(0)-ta_i|}{|f''(0)||tf(0)-f'(0)|^2|f'(0)|}$$

$$+\left(\log^+ t + m\left(r, \frac{f''}{f' - ta}\right)\right)O(1).$$

Combining with (2.3), we have

$$\begin{aligned}T(r,f) \le{}& LD(r,f:2,3)\\&+\log\frac{\prod_{i=1}^3 |f(0)-a_i|^2|f'(0)-ta_i|^3}{|tf(0)-f'(0)|^5|f'(0)|^2}\\&+3\log\frac{1}{|f''(0)|}\\&+\left(\log^+ t + m\left(r,\frac{f''}{f'-ta}\right)+1\right)O(1).\end{aligned}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. ([**2**]). *Let $f(z)$ be a meromorphic function in* $\mathbf{C}$. *Let*

$$\beta_p(r) = \sup_{2\le t\le r}\left\{\frac{T_0(t,f)}{(\log t)^p}\right\}, \qquad \varepsilon(r) = \left(\frac{1}{\beta_p(r)}\right)^{1/q} \tag{2.4}$$

with $p \ge 2$ and $q \ge 3$. If $\lim_{r\to\infty}\beta_p(r) = \infty$, then there exists a sequence of a positive number $\{r_n\}_1^\infty$ and a sequence of points $\{z_n\}_1^\infty$ in $\mathbf{C}$ *such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty}|z_n| = +\infty$ and*

$$A(\varepsilon(|z_n|)|z_n|, z_n, f) \ge \frac{1}{64\pi^2}\beta_p(r_n)\}^{1-2/q}(\log r_n)^{p-2} \quad (n = 1,2,\ldots), \tag{2.5}$$

where

$$A(r,a,f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^r\left(\frac{|f'(a+\rho e^{i\theta})|}{1+|f(a+\rho e^{i\theta})|^2}\right)^2 d\rho\, d\theta, \quad |z_n| \le r_n \tag{2.6}$$

and

$$T_0(r,f) = \int_0^r \frac{A(t)}{t}\,dt, \tag{2.7}$$

$$A(t) = \frac{1}{\pi}\int_0^{2\pi}\int_0^t\left(\frac{|f'(\rho e^{i\theta})|}{1+|f(\rho e^{i\theta})|^2}\right)^2 d\rho\, d\theta.$$

We also need the following lemmas.

Lemma 2.5. ([**9**]). *Let $f(z)$ be a meromorphic function in* disc $D(0,R)$ *centered at* 0 *with radius R. If $f(0)\neq 0,\infty$, then we have for* $0<r<\rho<R$

$$m\left(r,\frac{f^{(k)}}{f}\right)<c_k\Big\{1+\log^+\log^+\left|\frac{1}{f(0)}\right|+\log^+\frac{1}{r}+\log^+\frac{1}{\rho-r}$$
$$+\log^+\rho+\log^+T(\rho,f)\Big\},$$

where k is a positive integer, and c_k is a constant depending only on k.

Lemma 2.6. ([**9**]). *Let $T(r)$ be a continuous, non-decreasing, non-negative function, and let $a(r)$ be a non-increasing, non-negative function on $[r_0,R]$ $(0<r_0<R<\infty)$. If there exist constants b,c such that*

$$T(r)<a(r)+b\log^+\frac{1}{\rho-r}+c\log^+T(\rho),$$

for $r_0<r<\rho<R$, then

$$T(r)<2a(r)+B\log^+\frac{2}{R-r}+C,$$

where B,C are two constants dependent only on b,c.

Lemma 2.7. ([**12**]). *Let $f(z)$ be a meromorphic function in a domain $D=\{z:|z|<R\}$. If $f(0)\neq\infty$, then we have for* $0<r<R$,

$$|T(t,f)-T_0(t,f)-\log^+|f(0)||\leq\frac{1}{2}\log 2. \tag{2.8}$$

where $\log^+|f(0)|$ *will be replaced by* $\log|c(0)|$ *when* $f(0)=\infty$, *and $c(0)$ is the coefficient of the Laurent series of $f(z)$ at* 0, *and $T_0(t,f)$ is defined as* (2.7).

3. Proof of theorem.

Proof. Now we are to prove Theorem 1.1. Let f be meromorphic in **C** with the lower order greater than 2. Then there exists a sequence of positive numbers $\{l_n\}_1^\infty$ such that

$$\lim_{n\to\infty}l_n=\infty\quad\text{and}\quad\lim_{n\to\infty}\frac{\log T(l_n,f)}{\log l_n}>2.$$

Thus, we have

$$\lim_{n\to\infty} \frac{\log T_0(l_n, f)}{\log l_n} > 2$$

by combining with (2.8). Hence, we get that $\lim_{r\to\infty} \beta_p(r) = \infty$ ($\beta_p(r)$ as defined in Lemma 2.4, $p \geq 3$). By Lemma 2.4, there are $z_n \in \mathbf{C}$ and

$$r_n \in (1, \infty)\ (|z_n| \leq r_n,\ \lim_{n\to\infty} |z_n| = +\infty)$$

such that (2.5) holds. We write

$$z_n = |z_n| e^{i\theta_n}, \quad \theta_n \in [0, 2\pi). \tag{3.1}$$

Thus, there is a convergent subsequence of $\{\theta_n\}$, and, without loss of generality, we may assume that

$$\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi). \tag{3.2}$$

Let $\varepsilon_n = |z_n|\varepsilon(|z_n|)$, then there exists a convergent subsequence of ε_n, and, without loss of generality, we still denote it by ε_n, such that

$$\lim_{n\to\infty} \varepsilon_n = s,$$

where s is a non-negative real number or $s = \infty$ and $\varepsilon(|z_n|)$ is as defined in Lemma 2.4.

For any $\varepsilon > 0$, if there are three distinct complex numbers a_1, a_2, a_3 such that

$$\overline{E}_{A(\theta_0,\varepsilon)}(a_j, f) = \overline{E}_{A(\theta_0,\varepsilon)}(a_j, f'), \quad j = 1, 2, 3,$$

where $A(\theta_0, \varepsilon) = \{z| \arg z - \theta_0| < \varepsilon\}$. Then we *claim* that one of the following two cases hold:

(1) If $s = 0$, then there exists a constant $M > 0$ such that

$$\frac{\varepsilon_n |f'(z_n + \varepsilon_n z)|}{1 + |f(z_n + \varepsilon_n z)|^2} \leq M, \quad n = 1, 2, 3 \ldots . \tag{3.3}$$

(2) If $s > 0$ or $s = \infty$, then there exists a constant $M_1 > 0$ such that

$$\frac{|f'(z_n + \varepsilon_n z)|}{1 + |f(z_n + \varepsilon_n z)|^2} \leq M_1, \quad n = 1, 2, 3 \ldots, \tag{3.4}$$

where $|z| \leq 1$ and $\varepsilon_n = |z_n|\varepsilon(|z_n|)$.

In the case that $s = 0$, from (3.3), we obtain

$$A(\varepsilon_n, z_n, f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon_n}\left(\frac{|f'(z_n+\varepsilon_n e^{i\theta}|}{1+|f(z_n+\varepsilon_n e^{i\theta})|^2}\right)^2 \rho d\rho\, d\theta \le 2M.$$

Combining with (2.5), we have

$$\frac{1}{64\pi^2}\{\beta_p(r_n)\}^{1-2/q}(\log r_n)^{p-2} \le 2M.$$

Note that $p \ge 2$, $q \ge 3$ and $\beta_p(r)$ are non-decreasing functions on the interval $(2, +\infty)$. This contradicts the assumption that $\lim_{r\to\infty}\beta_p(r) = \infty$.

In the case that $s > 0$ or $s = \infty$, from (3.4), we obtain

$$A(\varepsilon_n, z_n, f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon_n}\left(\frac{|f'(z_n+\varepsilon_n e^{i\theta}|}{1+|f(z_n+\varepsilon_n e^{i\theta})|^2}\right)^2 \rho d\rho\, d\theta \le 2M\varepsilon_n^2.$$

Combining with (2.5), we have

$$\frac{1}{64\pi^2}\{\beta_p(r_n)\}^{1-2/q}(\log r_n)^{p-2} \le 2M\varepsilon_n^2 = 2M|z_n|^2\varepsilon(|z_n|)^2,$$

where $|z_n| \le r_n, p \ge 2$ and $q \ge 3$.

Noting that $\beta_p(r)$ is a non-decreasing function on the interval $(2, +\infty)$, we have

$$\frac{1}{64\pi^2}\{\beta_p(|z_n|)\}^{1-2/q}(\log|z_n|)^{p-2} \le 2M|z_n|^2\varepsilon(|z_n|)^2.$$

Hence,

$$\begin{aligned}\{\beta_p(z_n)\}^{1-2/q}(\log z_n)^{p-2} &\le 128\pi^2 M|z_n|^2\varepsilon(|z_n|)^2\\ &= 128\pi^2 M|z_n|^2\{\beta_p(z_n)\}^{-2/q}.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\frac{\log\beta_p(z_n)}{\log|z_n|} \le 2.$$

Therefore, we can deduce that

$$\lim_{n\to\infty}\frac{\log T_0(|z_n|, f)}{\log|z_n|} \le 2.$$

By using Lemma 2.7, we get

$$\lim_{n\to\infty} \frac{\log T(|z_n|, f)}{\log |z_n|} \le 2.$$

This contradicts the assumption that the lower order of f is greater than 2. Thus, the proof of theorem is complete if we prove claims (1) and (2). □

Proof of Claim. Now we prove part (1) of the clam.

Suppose that the claim (3.3) fails. Then there exists a sequence of points ω_n, $\omega_n = z_n + \varepsilon_n z_n^*$ with $|z_n^*| \le 1$ such that

$$\lim_{n\to\infty} \frac{\varepsilon_n |f'(z_n + \varepsilon_n z_n^*)|}{1 + |f(z_n + \varepsilon_n z_n^*)|^2} = \infty.$$

Set

$$f_n(z) = f(z_n + \varepsilon_n z).$$

Then, by Marty's criteria, we have that a sequence of a function $\{f_n(z)\}$ is not normal at $|z| < 1$. We take $\alpha = 0$ in Lemma 2.1. According to Lemma 2.1, there exist

(i) a sequence of point $\{z'_n\} \subset \{|z| < 1\}$;
(ii) a subsequence of $\{f_n(z)\}_1^\infty$. Without loss of generality, we still denote it by $\{f_n(z)\}$;
(iii) positive numbers ρ_n with $\rho_n \to 0 (n \to \infty)$ such that

$$h_n(z) = \rho_n^{-\alpha} f_n(z'_n + \rho_n z) = f_n(z'_n + \rho_n z) \to g(z)$$

in a spherical metric uniformly on a compact subset of $\mathbf{C}$ as $n \to \infty$, where $g(z)$ is a non-constant meromorphic function.

Thus, for any positive integer k, we have

$$h_n^{(k)}(\xi) = {\rho_n}^k f_n^{(k)}(z'_n + \rho_n \xi) \longrightarrow g^{(k)}(\xi).$$

We claim $g''(\xi) \not\equiv 0$. Otherwise, $g(z) = cz + d$, $(c, d \in \mathbf{C}$ and $c \neq 0)$. We can choose ξ_0, with $g(\xi_0) = a_1$. By Hurwitz's theorem, there exists a sequence $\xi_n \to \xi_0$ such that

$$h_n(\xi_n) = f_n(z'_n + \rho_n \xi_n) = g(\xi_0) = a_1.$$

Notice that f and f' share a_1 IM in $\{z : |\arg z - \theta_0| < \varepsilon\}$, and $\varepsilon_n \to s = 0$ (when $n \to \infty$), and we have

$$\begin{aligned} c = g'(\xi_0) &= \lim_{n\to\infty} h_n'(\xi_n) \\ &= \lim_{n\to\infty} \rho_n \varepsilon_n f'(z_n + \varepsilon_n(z_n' + \rho_n \xi_n)) \\ &= \lim_{n\to\infty} \rho_n \varepsilon_n f(z_n + \varepsilon_n(z_n' + \rho_n \xi_n)) = \lim_{n\to\infty} \rho_n \varepsilon_n a_1 = 0. \end{aligned}$$

This gives a contradiction. Hence, we can choose $\xi_0 \in C$, such that

$$g(\xi_0) \neq 0, \quad a_1, a_2, a_3, \infty, \quad g'(\xi_0) \neq 0, \infty, \quad g''(\xi_0) \neq 0, \infty.$$

Let

$$p_n(z) = f_n(z_n' + \rho_n \xi_0 + z).$$

Then, for every sufficiently large n ($n \geq n_0$), we have on $|z| \leq 1$

$$p_n(z) = a_i \overset{\text{IM}}{\Longleftrightarrow} p_n'(z) = \varepsilon_n a_i \quad (i = 1, 2, 3).$$

Note that

$$\begin{aligned} p_n(0) &= f_n(z_n' + \rho_n \xi_0) = h_n(\xi_0) \longrightarrow g(\xi_0) \neq a_1, a_2, a_3, \infty, \\ p_n'(0) &= f_n'(z_n' + \rho_n \xi_0) = \frac{h_n'(\xi_0)}{\rho_n}, \qquad h_n'(\xi_0) \to g'(\xi_0), \\ p_n''(0) &= f_n''(z_n' + \rho_n \xi_0) = \frac{h_n''(\xi_0)}{\rho_n^2}, \qquad h_n''(\xi_0) \to g''(\xi_0), \end{aligned}$$

$$\varepsilon_n p_n(0) - p_n'(0) = \frac{\varepsilon_n \rho_n h_n(\xi_0) - h_n'(\xi_0)}{\rho_n}.$$

Thus, we have

$$\begin{aligned} \text{(3.5)} \quad & \log \frac{\prod_{i=1}^3 |p_n(0) - a_i|^2 |p_n'(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p_n'(0)|^5 |p_n'(0)|^2} + 3 \log \frac{1}{|p_n''(0)|} \\ &= \log \frac{\prod_{i=1}^3 |p_n(0) - a_i|^2 |p_n'(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p_n'(0)|^5 |p_n'(0)|^2 |p_n''(0)|^3} \\ &= 4 \log \rho_n + \log \frac{\prod_{i=1}^3 |h_n(\xi_0) - a_i|^2 |h_n'(\xi_0) - \rho_n \varepsilon_n a_i|^3}{|\rho_n \varepsilon_n h_n(\xi_0) - h_n'(\xi_0)|^5 |h_n'(\xi_0)|^2 |h_n''(\xi_0)|^3}. \end{aligned}$$

Since $\rho_n \to 0$ and $\varepsilon_n \to 0$, we deduce

$$\text{(3.6)} \quad \log \frac{\prod_{i=1}^3 |h_n(\xi_0) - a_i|^2 |h_n'(\xi_0) - \rho_n \varepsilon_n a_i|^3}{|\rho_n h_n(\xi_0) - h_n'(\xi_0)|^5 |h_n'(\xi_0)|^2 |h_n''(\xi_0)|^3}$$

$$\longrightarrow \log \frac{\prod_{i=1}^{3} |g(\xi_0) - a_i|^2}{|g'(\xi_0)|^{-2} |g''(\xi_0)|^3},$$

when $n \to \infty$.

By applying Lemma 2.3 to $p_n(z)$ with (3.5) and (3.6), we have

$$\begin{aligned} T(r, p_n) &\le LD(r, p_n; 2, 3) \\ &\quad + O(1)\left(\log^+ |z_n| + m\left(r, \frac{p_n''}{p_n' - \varepsilon_n a}\right) + 1\right) \end{aligned}$$

for $0 < r \le 3$ and sufficiently large n, where $a \ne a_j$ ($j = 1, 2, 3$) and $a \in C$.

By Lemmas 2.5 and 2.6, we have

$$T(r, p_n) \le O(1)(1 + \log^+ |z_n|).$$

Hence,

$$T_0(r, p_n) \le O(1)(1 + \log^+ |z_n|).$$

Thus, we get

$$T_0(3\varepsilon(|z_n|)|z_n|, z_n + \varepsilon(|z_n|)|z_n|(z_n' + \rho_n \xi_0), f) \le O(1)(1 + \log^+ |z_n|).$$

It follows that

$$A(2\varepsilon(|z_n|)|z_n|, z_n + \varepsilon(|z_n|)|z_n|(z_n' + \rho_n \xi_0), f) \le O(1)(1 + \log^+ |z_n|).$$

Note that $z_n' + \rho_n \xi_0 \to 0$. We get

$$\begin{aligned} &\{z : |z - z_n| < \varepsilon(|z_n|)|z_n|\} \\ &\qquad \subseteq \{z : |z - z_n - \varepsilon(|z_n|)|z_n|(z_n' + \rho_n \xi_0) < 2\varepsilon(|z_n|)|z_n|\}. \end{aligned}$$

Thus, we have

$$A(\varepsilon_n, z_n, f) \le O(1)(1 + \log^+ |z_n|).$$

Combining this with (2.5), we have

$$\beta_p(r_n)^{1-2/q} (\log r_n)^{p-2} \le O(1)(1 + \log^+ |z_n|).$$

Notice that $|z_n| \le r_n$, $p \ge 3$ and $\lim_{n\to\infty} \beta_p(r_n) = \infty$. We obtain a contradiction. Therefore, part (1) of the claim is proved.

Next we prove part (2) of the claim. With a similar argument, we can get (3.4). Suppose that the claim (3.4) fails. Then there exists a sequence of points $\omega_n, \omega_n = z_n + \varepsilon_n z_n^*$ with $|z_n^*| \leq 1$ such that

$$\lim_{n\to\infty} \frac{|f'(z_n + \varepsilon_n z_n^*)|}{1 + |f(z_n + \varepsilon_n z_n^*)|^2} = \infty.$$

Set

$$f_n(z) = f(\omega_n + z).$$

Then, by Marty's criteria, we have that a sequence of a function $\{f_n(z)\}$ is not normal at $z = 0$. We take $\alpha = 0$ in Lemma 2.1. According to Lemma 2.1, there exist

(i) a sequence of point $\{z_n'\} \subset \{|z| < 1\}$;
(ii) a subsequence of $\{f_n(z)\}_1^\infty$. Without loss of generality, we still denote it by $\{f_n(z)\}$;
(iii) positive numbers ρ_n with $\rho_n \to 0 (n \to \infty)$ such that

$$h_n(z) = f_n(z_n' + \rho_n z) \to g(z)$$

in a spherical metric uniformly on a compact subset of $\mathbf{C}$ as $n \to \infty$, where $g(z)$ is a non-constant meromorphic function.

Thus, for any positive integer k, we have

$$h_n^{(k)}(\xi) = {\rho_n}^k f_n^{(k)}(z_n' + \rho_n \xi) \to g^{(k)}(\xi).$$

We claim $g''(\xi) \not\equiv 0$. Otherwise, $g(z) = cz + d$,$(c, d \in \mathbf{C}$ and $c \neq 0)$. We can choose ξ_0 with $g(\xi_0) = a_1$. By Hurwitz's theorem, there exists a sequence $\xi_n \to \xi_0$ such that

$$h_n(\xi_n) = f_n(z_n' + \rho_n \xi_n) = g(\xi_0) = a_1.$$

Notice that f and f' share a_1 IM in $\{z : |\arg z - \theta_0| < \varepsilon\}$, and

$$\begin{aligned} &\lim_{n\to\infty} \arg(\omega_n + z_n' + \rho_n \xi_n) \\ &= \lim_{n\to\infty} \left(\arg z_n + \arg\left(1 + \frac{\varepsilon_n z_n^* + z_n' + \rho_n \xi_n}{z_n}\right)\right) \\ &= \theta_0. \end{aligned}$$

We have

$$c = g'(\xi_0) = \lim_{n\to\infty} h_n'(\xi_n)$$

$$\begin{aligned}
&= \lim_{n\to\infty} \rho_n f'(\omega_n + z'_n + \rho_n\xi_n) \\
&= \lim_{n\to\infty} \rho_n f(\omega_n + z'_n + \rho_n\xi_n) \\
&= \lim_{n\to\infty} \rho_n a_1 = 0.
\end{aligned}$$

This gives a contradiction. Hence, we can choose $\xi_0 \in C$ such that

$$g(\xi_0) \neq a_1, a_2, a_3, \infty, \qquad g'(\xi_0) \neq 0, \infty, \qquad g''(\xi_0) \neq 0, \infty.$$

Let

$$p_n(z) = f_n(z'_n + \rho_n\xi_0 + \varepsilon_n z).$$

Then, for every sufficiently large $n (n \geq n_0)$, we have on $|z| \leq 1$,

$$p_n(z) = a_i \overset{\text{IM}}{\Longleftrightarrow} p'_n(z) = \varepsilon_n a_i \quad (i = 1, 2, 3)$$

and

$$\begin{aligned}
&p_n(0) = f_n(z'_n + \rho_n\xi_0) = h_n(\xi_0) \to g(\xi_0) \neq a_1, a_2, a_3, \infty \\
&p'_n(0) = \varepsilon_n f'_n(z'_n + \rho_n\xi_0) = \varepsilon_n \frac{h'_n(\xi_0)}{\rho_n}, \\
&\qquad h'_n(\xi_0) \to g'(\xi_0), \\
&p''_n(0) = \varepsilon_n^2 f''_n(z'_n + \rho_n\xi_0) = \varepsilon_n^2 \frac{h''_n(\xi_0)}{\rho_n^2}, \\
&\qquad h''_n(\xi_0) \to g''(\xi_0), \\
&\varepsilon(|z_n|)|z_n|p_n(0) - p'_n(0) = \varepsilon(|z_n|)|z_n|\left(h_n(\xi_0) - \frac{h'_n(\xi_0)}{\rho_n}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\log \frac{\prod_{i=1}^3 |p_n(0) - a_i|^2 |p'_n(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p'_n(0)|^5 |p'_n(0)|^2} + 3\log\frac{1}{|p''_n(0)|} \\
&\quad = \log \frac{\prod_{i=1}^3 |p_n(0) - a_i|^2 |p'_n(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p'_n(0)|^5 |p'_n(0)|^2 |p''_n(0)|^3} \\
&\quad = 4\log\frac{1}{\varepsilon_n} + 4\log\rho_n \\
&\qquad + \log \frac{\prod_{i=1}^3 |h_n(\xi_0) - a_i|^2 |h'_n(\xi_0) - \rho_n a_i|^3}{|\rho_n h_n(\xi_0) - h'_n(\xi_0)|^5 |h'_n(\xi_0)|^2 |h''_n(\xi_0)|^3}
\end{aligned}$$

and

$$\text{(3.7)}\quad \log \frac{\prod_{i=1}^{3} |h_n(\xi_0) - a_i|^2 |h_n'(\xi_0) - \rho_n a_i|^3}{|\rho_n h_n(\xi_0) - h_n'(\xi_0)|^5 |h_n'(\xi_0)|^2 |h_n''(\xi_0)|^3)} \longrightarrow \log \frac{\prod_{i=1}^{3} |g(\xi_0) - a_i|^2}{|g'(\xi_0)|^{-2} |g''(\xi_0)|^3},$$

when $n \to \infty$.

By applying Lemma 2.3 to $p_n(z)$ with (3.7) and $\varepsilon_n \to s$, $s > 0$, we obtain for $0 < r \le 3$ and every sufficiently large n that

$$T(r, p_n) \le LD(r, p_n; 2, 3) + O(1)\left(\log^+ |z_n| + \log^+ \frac{1}{\varepsilon(|z_n|)} + m\left(r, \frac{p_n''}{p_n' - ta}\right) + 1\right),$$

where $a \ne a_j$ $(j = 1, 2, 3)$ and $a \in C$. By Lemmas 2.5 and 2.6, we have

$$T(r, p_n) \le O(1)\left(1 + \log^+ |z_n| + \log^+ \frac{1}{\varepsilon(|z_n|)}\right).$$

Hence,

$$T_0(r, p_n) \le O(1)\left(1 + \log^+ |z_n| + \log^+ \frac{1}{\varepsilon(|z_n|)}\right).$$

Thus, we get

$$T_0(3\varepsilon(|z_n|)|z_n|, z_n + \varepsilon(|z_n|)|z_n| z_n^* + z_n' + \rho_n \xi_0, f) \le O(1)\left(1 + \log^+ |z_n| + \log^+ \frac{1}{\varepsilon(|z_n|)}\right).$$

Note that $\varepsilon(|z_n|)|z_n| \to s$, $s \ne 0$ and $|z_n^*| \le 1$, $z_n' + \rho_n \xi_0 \to 0$ (when $n \to \infty$). Thus, we have

$$\{z : |z - z_n| < \varepsilon(|z_n|)|z_n|\} \subseteq \{z : |z - z_n - \varepsilon(|z_n|)|z_n| z_n^* - z_n' - \rho_n \xi_0| < 3\varepsilon(|z_n|)|z_n|\}.$$

Hence, we can get

$$A(\varepsilon_n, z_n, f) \le O(1)\left(1 + \log^+ |z_n| + \log^+ \frac{1}{\varepsilon(|z_n|)}\right).$$

Combining this with (2.6), we have

$$\beta_p(r_n)^{1-2/q}(\log r_n)^{p-2} \leq O(1)(1+\log^+|z_n|+\log^+\beta_p(r_n)).$$

Notice that $|z_n| \leq r_n$,$p \geq 3$ and $\lim_{n\to\infty}\beta_p(r_n) = \infty$. We obtain a contradiction. Therefore, part (2) of the claim is proved, and so is Theorem 1.1. □

As the end of this section, we conjecture that the conditions of Theorem 1.1, "f is a meromorphic function of lower order > 2 in the complex plane" can be replaced by "$f(z) \not\equiv f'(z)$."

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