ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 5, 2015

ON ETEMADI'S SUBSEQUENCES AND THE STRONG LAW OF LARGE NUMBERS FOR RANDOM FIELDS

PRZEMYSŁAW MATUŁA AND MICHAŁ SEWERYN

ABSTRACT. We study different concepts of the convergence of sequences with multidimensional indices. Emphasis is placed on the use of Etemadi's subsequence method in the study of almost sure convergence of random fields. An improvement of the SLLN for pairwise independent random variables is also obtained.

1. Introduction and notation. Since Etemadi demonstrated his technique of proof of the strong law of large numbers (SLLN for short), see [**1**], it has been widely used in the context of almost sure convergence of dependent random variables. In the present paper, we aim to refine Etemadi's method and discuss consequences of its use in relation to different modes of convergence of random fields.

We begin with some notation concerning random fields. Let $\mathbb{N}^d$, $d \geq 1$, be a d-dimensional lattice and denote by $\underline{m} = (m_1, \ldots, m_d)$, $\underline{n} = (n_1, \ldots, n_d)$ the elements of this lattice. The set $\mathbb{N}^d$ is partially ordered by the relation $\underline{m} \leq \underline{n}$ if and only if for every $i = 1, \ldots, d$, we have $m_i \leq n_i$. We shall also write $\underline{m} < \underline{n}$ if $\underline{m} \leq \underline{n}$ and $\underline{m} \neq \underline{n}$; moreover, $\underline{m} \nleq \underline{n}$ if and only if for at least one i_0 we have $m_{i_0} > n_{i_0}$. Let us put

$$|\underline{n}| = \prod_{i=1}^{d} n_i$$

2010 AMS *Mathematics subject classification.* Primary 60F15, 60G50, 60G60.
Keywords and phrases. Strong law of large numbers, independent random variables, random fields, method of subsequences.
The research of the second author has been supported in part by US NIH grant R01CA-152158 (MS) and by the Mathematical Biosciences Institute and the National Science Foundation under grant DMS 0931642 (MS).
Received by the editors on October 1, 2013, and in revised form on December 2, 2013.
DOI:10.1216/RMJ-2015-45-5-1555

and

$$\|\underline{n}\| = \max_{1\le i\le d} |n_i|.$$

We study the convergence of sequences indexed by lattice points. To this aim, let us recall that $\underline{n} \to \infty$ may have different meanings; in other words, the term "$\underline{n}$ tends to infinity" may be understood as $|\underline{n}| \to \infty$ (equivalently $\|\underline{n}\| \to \infty$) or $\min_{1\le i\le d}(n_i) \to \infty$ (we shall write $\underline{n} \to_{\max} \infty$ and $\underline{n} \to_{\min} \infty$, respectively).

For the definition of convergence along a subsequence defined by the infinite subset $I \subset \mathbb{N}^d$, set

$$I_{\min}(\underline{n}) := I \cap \left\{\underline{k} \in \mathbb{N}^d : \underline{k} \ge \underline{n}\right\},$$
$$I_{\max}(\underline{n}) := I \cap \left\{\underline{k} \in \mathbb{N}^d : \underline{k} \not\le \underline{n}\right\},$$

and, in what follows, we shall require not only I to be infinite, but $I_{\min}$ as well ($I_{\max}$ to be automatically infinite).

Definition 1.1. Let $(a_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ be a field of real numbers indexed by positive lattice points and $I \subset \mathbb{N}^d$ an infinite set. We write $a_{\underline{n}} \to_{I,\max} a$ if and only if, for every $\varepsilon > 0$, there exists a $\underline{n}_0 \in I$ such that, for each $\underline{n} \in I_{\max}(\underline{n}_0)$, we have $|a_{\underline{n}} - a| < \varepsilon$. Moreover, we write $a_{\underline{n}} \to_{I,\min} a$ if and only if, for every $\varepsilon > 0$, there exists a $\underline{n}_0 \in I$ such that, for each $\underline{n} \in I_{\min}(\underline{n}_0)$, we have $|a_{\underline{n}} - a| < \varepsilon$. In what follows, we shall also write $a_{\underline{n}} \to_{\max} a$ ($a_{\underline{n}} \to_{\min} a$) for the choice of $I = \mathbb{N}^d$ (alternatively, $a_{\underline{n}} \to a$, $\underline{n} \to_{\max} \infty$ or $a_{\underline{n}} \to a$, $\underline{n} \to_{\min} \infty$).

A particularly interesting choice of subsets, I, are sectors defined as

$$I = S^d_\theta := \left\{(i_1, \dots, i_d) \in \mathbb{N}^d : \theta < i_l/i_k < 1/\theta,\ k, l = 1, \dots, d\right\},$$

where $\theta \in (0, 1)$. For convergence of random fields with indices belonging to such sectors (i.e., sectorial convergence) see **[3]** or **[5]**.

Another important choice of sets is:

$$\mathcal{E}_\alpha := \left\{\underline{n} \in \mathbb{N}^d : n_1 = [\alpha^{k_1}], \dots, n_d = [\alpha^{k_d}],\ \text{for some } \underline{k} = (k_1, \dots, k_d)\right\}, \tag{1.1}$$

where $\alpha > 1$. We shall call elements of $\mathcal{E}_\alpha$ *Etemadi's numbers*, which is related to the method of subsequences often used in the proofs of the laws of large numbers (eg., see [**1**]).

The elementary relations between different modes of convergence for sequences with multidimensional indices are established in the following proposition.

Proposition 1.2. *Let $(a_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ be a field of real numbers indexed by positive lattice points and $I \subset \mathbb{N}^d$ an infinite set. We have*

(i) $a_{\underline{n}} \to_{I,\max} a \Rightarrow a_{\underline{n}} \to_{I,\min} a$,
(ii) $a_{\underline{n}} \to_{\max} a \Leftrightarrow a_{\underline{n}} \to_{I,\max} a$, *for each infinite set* $I \subset \mathbb{N}^d$,
(iii) $a_{\underline{n}} \to_{\min} a \Leftrightarrow a_{\underline{n}} \to_{I,\min} a$, *for each infinite set* $I \subset \mathbb{N}^d$,
(iv) $a_{\underline{n}} \to_{S^d_\theta,\max} a \Leftrightarrow a_{\underline{n}} \to_{S^d_\theta,\min} a,\ \theta \in (0,1)$.

In this paper, we are going to prove that, for normalized sums of non-negative numbers (or random fields) from the convergence $a_{\underline{n}} \to_{\mathcal{E}_\alpha,\max} a$ for every $\alpha \in \mathcal{A}$, it follows that $a_{\underline{n}} \to_{\max} a$, where

(1.2) $\quad \mathcal{A}$ is a countable subset of real numbers such that $\inf \mathcal{A} = 1$.

In particular, we show that, if Etemadi's method of subsequences is used in the proof of the strong law of numbers for random fields, then we can obtain not only the almost sure convergence in the sense of $\min_{1\le i\le d}(n_i) \to \infty$, but $\max_{1\le i\le d}(n_i) \to \infty$ as well.

The paper is organized as follows. In the next section, we prove a convergence result for sequences of real numbers with multidimensional indices, whereas the last section is devoted to application of Etemadi's subsequence method to the problem of almost sure convergence of fields of pairwise independent identically distributed random variables.

Let us recall that the study of the SLLN for random fields was started in [**7, 8**]. The most important continuation may be found in [**2, 3**]. There were many papers concerning dependent random fields; let us only mention [**4, 5**].

2. Non-random case. In the present section, we study properties of *Etemadi's subsequences*. For clarity, let us first consider a single index setting. Let $\alpha > 1$ and, for $k \in \mathbb{N}$, we shall call $e(k) := [\alpha^k] \in \mathcal{E}_\alpha$

Etemadi's numbers. It is clear that the first Etemadi's numbers $e(1),\ldots,e(k)$ for $k \le -\ln\ln\alpha/\ln\alpha$ are the first consecutive natural numbers, the "gaps" appear for $k \ge -\ln\ln\alpha/\ln\alpha$. We define the "closest" Etemadi's numbers: if $n \in \mathcal{E}_\alpha$, we put $l(n) = u(n) = n$, whereas if $n \notin \mathcal{E}_\alpha$ we find $k_n \in \mathbb{N}$ such that $\alpha^{k_n} < n < \alpha^{k_n+1}$, and put $l(n) = [\alpha^{k_n}]$ and $u(n) = [\alpha^{k_n+1}]$. Then $l(n) \le n \le u(n)$ and $u(n)/l(n) = 1$ if $n \in \mathcal{E}_\alpha$ or if $n \notin \mathcal{E}_\alpha$ (in this case $k_n \ge -\ln\ln\alpha/\ln\alpha$) we have

$$\frac{u(n)}{l(n)} \le \frac{\alpha^{k_n+1}}{\alpha^{k_n}-1} = \frac{\alpha}{1-1/\alpha^{k_n}} \le \frac{\alpha}{1-\ln\alpha},$$

since $\alpha^{k_n} \ge \alpha^{-\ln\ln\alpha/\ln\alpha} = 1/\ln\alpha$. Furthermore $\alpha/(1-\ln\alpha) \ge 1$. Concluding the above estimates, for any $n \in \mathbb{N}$, we have defined $u(n), l(n) \in \mathcal{E}_\alpha$ in such a way that $l(n) \le n \le u(n)$ and

$$1 \le \frac{u(n)}{l(n)} \le \frac{\alpha}{1-\ln\alpha} \longrightarrow 1, \quad \text{as } \alpha \to 1^+.$$

For any $\underline{n} \in \mathbb{N}^d$, we extend the above definition to $l(\underline{n}) \in \mathcal{E}_\alpha$ and $u(\underline{n}) \in \mathcal{E}_\alpha$ in an obvious manner, by setting coordinatewise $l(\underline{n}) = l(n_1,\ldots,n_d) = (l(n_1),\ldots,l(n_d))$ and similarly $u(\underline{n})$. We easily see that

$$1 \le \frac{|u(\underline{n})|}{|l(\underline{n})|} \le \left(\frac{\alpha}{1-\ln\alpha}\right)^d \longrightarrow 1, \quad \text{as } \alpha \to 1^+.$$

With such preparations done, we are now ready to state the main result of this section, which is interesting for its own sense, but shall also be the main tool in the proof of the SLLN for pairwise independent random fields.

Theorem 2.1. *Let* $(a(\underline{n}))_{\underline{n}\in\mathbb{N}^d}$, $(b(\underline{n}))_{\underline{n}\in\mathbb{N}^d}$ *be two fields of nonnegative numbers such that* $\sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n}) < \infty$. *Assume that, for each* $\alpha \in \mathcal{A}$ *satisfying* (1.2),

$$\frac{1}{|\underline{n}|}\left(\sum_{\underline{k}\le\underline{n}} a(\underline{k}) - \sum_{\underline{k}\le\underline{n}} b(\underline{k})\right) \longrightarrow_{\mathcal{E}_\alpha,\max} 0,$$

then

$$\frac{1}{|\underline{n}|}\left(\sum_{\underline{k}\le\underline{n}} a(\underline{k}) - \sum_{\underline{k}\le\underline{n}} b(\underline{k})\right) \longrightarrow_{\max} 0.$$

Proof. Set $\varepsilon > 0$ and note that, since α may be arbitrarily close to 1, then $(\alpha/(1-\ln\alpha))^d$ may be also arbitrarily close to 1. Therefore, we choose $\alpha > 1$ small enough, that is, such that $(\alpha/(1-\ln\alpha))^d - 1 < \varepsilon$. By the definition of convergence, there exists $\underline{n}' \in \mathcal{E}_\alpha$ such that, for every $\underline{n} \nleq \underline{n}'$, $\underline{n} \in \mathcal{E}_\alpha$, the following estimate holds true:

$$\left| \frac{1}{|\underline{n}|} \left(\sum_{\underline{k} \leq \underline{n}} a(\underline{k}) - \sum_{\underline{k} \leq \underline{n}} b(\underline{k}) \right) \right| < \varepsilon.$$

Now, for any $\underline{n} \nleq \underline{n}'$, we define $l(\underline{n}), u(\underline{n}) \in \mathcal{E}_\alpha$ as above. Since $\underline{n}' \in \mathcal{E}_\alpha$, then also $l(\underline{n}) \nless \underline{n}'$. By the nonnegativity of $(a(\underline{n}))_{\underline{n}\in\mathbb{N}^d}$, $(b(\underline{n}))_{\underline{n}\in\mathbb{N}^d}$ and the triangle inequality, we get

$$\begin{aligned}
&\frac{1}{|\underline{n}|} \left(\sum_{\underline{k} \leq \underline{n}} a(\underline{k}) - \sum_{\underline{k} \leq \underline{n}} b(\underline{k}) \right) \\
&\quad \leq \left| \frac{\sum_{\underline{k} \leq u(\underline{n})} a(\underline{k}) - \sum_{\underline{k} \leq u(\underline{n})} b(\underline{k})}{|u(\underline{n})|} \right| \frac{|u(\underline{n})|}{|\underline{n}|} \\
&\qquad + \left| \frac{\sum_{\underline{k} \leq u(\underline{n})} b(\underline{k}) - \sum_{\underline{k} \leq l(\underline{n})} b(\underline{k})}{|\underline{n}|} \right| \\
&\quad \leq \varepsilon \frac{|u(\underline{n})|}{|l(\underline{n})|} + \frac{|u(\underline{n})| - |l(\underline{n})|}{|l(\underline{n})|} \sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n}) \\
&\quad \leq \varepsilon \left(\frac{\alpha}{1-\ln\alpha} \right)^d + \left(\left(\frac{\alpha}{1-\ln\alpha} \right)^d - 1 \right) \sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n}) \\
&\quad \leq \varepsilon \left(\sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n}) + \left(\frac{\alpha}{1-\ln\alpha} \right)^d \right).
\end{aligned}$$

Similarly, we obtain the lower bound, which is slightly different:

$$\begin{aligned}
\frac{1}{|\underline{n}|} \left(\sum_{\underline{k} \leq \underline{n}} a(\underline{k}) - \sum_{\underline{k} \leq \underline{n}} b(\underline{k}) \right) &\geq - \left| \frac{\sum_{\underline{k} \leq l(\underline{n})} a(\underline{k}) - \sum_{\underline{k} \leq l(\underline{n})} b(\underline{k})}{|l(\underline{n})|} \right| \frac{|l(\underline{n})|}{|\underline{n}|} \\
&\qquad - \left| \frac{\sum_{\underline{k} \leq l(\underline{n})} b(\underline{k}) - \sum_{\underline{k} \leq u(\underline{n})} b(\underline{k})}{|\underline{n}|} \right| \\
&\geq -\varepsilon - \left(\left(\frac{\alpha}{1-\ln\alpha} \right)^d - 1 \right) \sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n})
\end{aligned}$$

$$\geq -\varepsilon\left(1 + \sup_{\underline{n}\in\mathbb{N}^d} b(\underline{n})\right).$$

Since $\varepsilon > 0$ is arbitrary and α may be arbitrarily close to 1, we arrive at the conclusion. □

3. Strong law of large numbers for random fields. In what follows, we shall demonstrate the usefulness of Etemadi's subsequence technique in the proof of the SLLN for pairwise independent random variables.

Theorem 3.1. *Let $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ be a field of pairwise independent and identically distributed random variables. Then the following conditions are equivalent*:

$$\frac{1}{|\underline{n}|}\sum_{\underline{k}\leq\underline{n}} X_{\underline{k}} \to_{\max} c, \tag{3.1}$$

almost surely, for some constant c.

$$E\left|X_{\underline{1}}\right| \log_{+}^{d-1}\left|X_{\underline{1}}\right| < \infty. \tag{3.2}$$

Furthermore, if (3.2) *holds, then* $c = EX_{\underline{1}}$.

The proof will be based on the Borel-Cantelli lemmas for random events with multidimensional indices, which we recall for completeness.

Lemma 3.2. *Set $(\Omega, \mathcal{F}, P)$ to be a probability space and $\{A_{\underline{n}} \in \mathcal{F}, \underline{n} \in \mathbb{N}^d\}$ to be a family of events. Denote $\{A_{\underline{n}}, \text{i.o.}\} = \bigcap_{\underline{n}\in\mathbb{N}^d}\bigcup_{\underline{k}\not\leq n} A_{\underline{k}}$. Then*:

(a) *if $\sum_{\underline{n}\in\mathbb{N}^d} P(A_{\underline{n}}) < \infty$, then $P(A_{\underline{n}}, \text{i.o.}) = 0$,*
(b) *if the events $(A_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ are pairwise independent and*

$$\sum_{\underline{n}\in\mathbb{N}^d} P(A_{\underline{n}}) = \infty,$$

then $P(A_{\underline{n}}, \text{i.o.}) = 1$.

Proof. The proof of (a) is standard, while (b) is a special case of [**5,** Lemma 3.3]. □

With such preparations done, we proceed to the proof of the main result of this section.

Proof of Theorem 2. From (3.1), it follows that $X_{\underline{n}}/|\underline{n}| \to_{\max} 0$, almost surely. Thus, by standard arguments and Lemma 3.2 (b), we get:

$$\sum_{\underline{n}\in\mathbb{N}^d} P(|X_{\underline{n}}| \geq |\underline{n}|) = \sum_{\underline{n}\in\mathbb{N}^d} P(|X_{\underline{1}}| \geq |\underline{n}|) < \infty,$$

which is equivalent to (3.2) by [**2,** Lemma 2.1].

To prove the sufficiency of (3.2), let us note that, in [**1**], it is proved that

$$\frac{1}{|\underline{n}|}\sum_{\underline{k}\leq\underline{n}} X_{\underline{k}} \to_{\min} EX_{\underline{1}}. \tag{3.3}$$

We shall combine this result with our Theorem 2.1 to prove (3.1). For the sake of completeness let us recall some main steps of the proof from [**1**].

We may, and do, assume that $X_{\underline{n}} \geqq 0$; otherwise, one may prove the result for the nonnegative ($X_{\underline{n}}^+ = \max\{X_{\underline{n}}, 0\}$) and nonpositive ($X_{\underline{n}}^- = \max\{-X_{\underline{n}}, 0\}$) parts of the random variables separately. Let us set $Y_{\underline{k}} = X_{\underline{k}}\mathbb{I}[X_{\underline{k}} \leq |\underline{k}|]$. It is easy to see that

$$\sum_{\underline{n}\in\mathbb{N}^d} P(Y_{\underline{n}} \neq X_{\underline{n}}) = \sum_{\underline{n}\in\mathbb{N}^d} P(X_{\underline{1}} > |\underline{n}|) < \infty,$$

since the summability of $\sum_{\underline{n}\in\mathbb{N}^d} P(X_{\underline{1}} > |\underline{n}|)$ is equivalent to $EX_{\underline{1}} \log_+^{d-1} X_{\underline{1}} < \infty$. Thus, by Lemma 3.2, the almost sure limiting behavior of $1/|\underline{n}|\sum_{\underline{k}\leq\underline{n}} X_{\underline{k}}$ is the same as of $1/|\underline{n}|\sum_{\underline{k}\leq\underline{n}} Y_{\underline{k}}$.

Let $\alpha > 1$ be fixed and define an *Etemadi sequence* by $k(\underline{n}) = ([\alpha^{n_1}], \ldots, [\alpha^{n_d}])$, $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. By the standard arguments, for each $\varepsilon > 0$, we get:

$$\sum_{\underline{n}\in\mathbb{N}^d} P\left(\left|\frac{1}{|k(\underline{n})|}\sum_{\underline{i}\leq k(\underline{n})} (Y_{\underline{i}} - EY_{\underline{i}})\right| \geq \varepsilon\right)$$

$$
\begin{aligned}
&\le \frac{1}{\varepsilon^2} \sum_{\underline{n}\in\mathbb{N}^d} \frac{1}{|k(\underline{n})|^2} \sum_{\underline{i}\le k(\underline{n})} EY_{\underline{i}}^2 \\
&\le C \sum_{\underline{i}\in\mathbb{N}^d} EY_{\underline{i}}^2 \sum_{\{\underline{n}\in\mathbb{N}^d : k(\underline{n})\ge \underline{i}\}} \frac{1}{|k(\underline{n})|^2} \\
&\le C \sum_{\underline{i}\in\mathbb{N}^d} \frac{EY_{\underline{i}}^2}{|\underline{i}|^2} \\
&= C \sum_{k=1}^{\infty} \frac{d(k)}{k^2} EX_{\underline{1}}^2 \mathbb{I}\left[X_{\underline{1}} \le k\right] \\
&\le C \sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d(k)}{k^2} \right) EX_{\underline{1}}^2 \mathbb{I}\left[i < X_{\underline{1}} \le i+1\right] \\
&\le C \sum_{i=0}^{\infty} \frac{\log^{d-1}(i+1)}{i+1} EX_{\underline{1}}^2 \mathbb{I}\left[i < X_{\underline{1}} \le i+1\right] \\
&\le C \sum_{i=0}^{\infty} EX_{\underline{1}} \log_+^{d-1} X_{\underline{1}} \mathbb{I}\left[i < X_{\underline{1}} \le i+1\right] \\
&\le C \cdot EX_{\underline{1}} \log_+^{d-1} X_{\underline{1}} < \infty.
\end{aligned}
$$

In the above,

$$d(k) := \mathrm{card}\,\{\underline{n} \in \mathbb{N}^d : |\underline{n}| = k\},$$

and we have used the bound

$$\sum_{k=i+1}^{\infty} \frac{d(k)}{k^2} \le C \cdot \frac{\log^{d-1}(i+1)}{i+1}.$$

Also, note that the constant C may be different in the consecutive inequalities. Therefore, we have proved that, for each $\alpha > 1$

$$\frac{1}{|\underline{n}|} \sum_{\underline{i}\le \underline{n}} \left(Y_{\underline{i}} - EY_{\underline{i}}\right) \longrightarrow_{\mathcal{E}_\alpha,\max} 0, \quad \text{almost surely,} \tag{3.4}$$

i.e., (3.4) holds for $\omega \in \Omega_\alpha$ such that $P(\Omega_\alpha) = 1$. Now, let us take $\mathcal{A}$ to be a countable set of real numbers greater than 1 and such that $\inf \mathcal{A} = 1$. Since, for $\omega \in \bigcap_{\alpha\in\mathcal{A}} \Omega_\alpha$, (3.4) holds, then by Theorem 2.1, we have

$$\frac{1}{|\underline{n}|}\sum_{\underline{i}\le\underline{n}}\big(Y_{\underline{i}}-EY_{\underline{i}}\big)\longrightarrow_{\max}0,\quad\text{almost surely.}\tag{3.5}$$

Moreover, by $1/|\underline{n}|\sum_{\underline{i}\le\underline{n}}EY_{\underline{i}}\to_{\max}EX_{\underline{1}}$ and by the equivalence of $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ and $(Y_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ we get (3.1), and the proof is complete. □

Remark 3.3. From the method of the proof of Theorem 3.1, it follows that the main results in [**4**] hold in the sense $\underline{n}\to_{\max}\infty$, not only as $\underline{n}\to_{\min}\infty$.

Remark 3.4. With some obvious modification, we can replace the assumption of pairwise independence in Theorem 3.1, by pairwise negative quadrant dependence (see [**6**] for the single-index case).

REFERENCES

1. N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahr. Verw. Gebi. **55** (1981), 119–122.

2. A. Gut, *Marcinkiewicz laws and convergence rates in the law of large numbers with multidimensional indices*, Ann. Prob. **6** (1978), 469–482.

3. ______, *Strong laws for independent identically distributed random variables indexed by a sector*, Ann. Prob. **11** (1983), 569–577.

4. M.-H. Ko, T.-S. Kim and H.-C. Kim, *Strong laws of large numbers for asymptotically quadrant independent random variables*, Comm. Kor. Math. Soc. **19** (2004), 765–773.

5. Z.A. Lagodowski and P. Matuła, *On almost sure limiting behavior of weighted sums of random fields*, Acta Math. Hung. **126** (2010), 16–22.

6. P. Matuła, *A note on the almost sure convergence of sums of negatively dependent random variables*, Stat. Prob. Lett. **15** (1992), 209–213.

7. R. Smythe, *Strong laws of large numbers for r-dimensional arrays of random variables*, Ann. Prob. **1** (1973), 164–170.

8. ______, *Sums of independent random variables on partially ordered sets*, Ann. Prob. **2** (1974), 906–917.

Institute of Mathematics, Marie Curie-Skłodowska University, pl. M.C.-Skłodowskiej 1, 20-031 Lublin, Poland

Email address: **matula@hektor.umcs.lublin.pl**

Faculty of Mathematics and Computer Science, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland

Email address: **mseweryn@math.uni.lodz.pl**