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THE SPECTRUM IN $\mathbb{R}$ AND $\mathbb{R}^2$ OF NONLINEAR ELLIPTIC EQUATIONS WITH POSITIVE PARAMETERS

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ABSTRACT. In this paper we study the spectrum in $\mathbb{R}$ and $\mathbb{R}^2$ of nonlinear elliptic equations with positive parameters in their nonlinear part. In order to investigate the spectrum in these specific cases, we introduce the monotone method which is an extension of the upper and lower solution methods. Using the Picard iterative process we prove some existence theorems for nonlinear elliptic boundary value problems. We work with both positive and negative solutions.

1. Introduction. Let Ω be an open, convex and bounded subset of $\mathbb{R}^m$, $m \geq 2$, whose boundary $\partial\Omega$ is smooth. Let L be a partial differential operator of second order which is elliptic, formally self-adjoint, and its coefficients are β-Hölder continuous in $\overline{\Omega}$. We study the following nonlinear parametric problem

$$\begin{cases} Lu(x) + \lambda f(x, u(x)) + \mu g(x, u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where λ and μ are positive real numbers. The nonlinearity $F(x, u(x)) = \lambda f(x,\, u(x)) + \mu g(x, u(x))$ is with respect to u. Moreover, F satisfies the following hypotheses:

(H-0) *F is α-Hölder continuous in $\overline{\Omega} \times \mathbb{R}$.*

(H-1) *F is non-decreasing in the second variable.*

Throughout the paper we assume that the partial differential operator L satisfies the following property.

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(H-M) If $L\psi \leq 0$ in Ω and $\psi \geq 0$ in $\partial\Omega$ where $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$, then it implies that $\psi \geq 0$ in Ω.

We recall that a function $f : A \subset \mathbb{R}^m \to \mathbb{R}$ is α-Hölder continuous in A where $0 < \alpha \leq 1$, if there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^\alpha$ for every $x, y \in A$. When $\alpha = 1$, it is customary to say that f is Lipschitz continuous.

It is important to note that there are a lot of partial differential operators L which satisfy the property (H-M). In fact, from the Hopf maximum principle it follows that L satisfies the (H-M) property if, for instance, the coefficient corresponding to the independent term is not positive [**20**]. It is well known that the (H-M) property is related to the location of the first eigenvalue of L [**4**].

The problem (1.1) arises in a variety of diffusion processes generated by nonlinear sources, in particular, nonlinear heat generation, combustion theory, chemical reactor theory, and population dynamics [**3, 5, 13, 17, 21, 24, 25**].

The investigation of existence and multiplicity of solutions of (1.1) under the presence of one parameter has a long history. Amman and Cohen-Keller were pioneers in the study of this problem at the end of the 20th century. They used the Picard method and the upper and lower solutions to prove the existence of solutions [**2, 9**]. Later, other techniques were adapted to investigate the one parametric problem such as the bifurcation method, computer-assisted method, fixed point method and the mountain pass method. Many substantial and important results have been obtained with these techniques [**8, 10, 11, 22**]. The problem (1.1) with two parameters has recently been studied. Some techniques used in the study of (1.1) with one parameter are being applied in this new context [**6, 7, 15, 16, 26**]. We would like to observe that in [**12**] the fixed point method was extended to the case with two parameters in order to study the spectrum of (1.1). Nowadays, upper and lower solutions are still some of the main tools for investigating the existence of solutions of (1.1).

Most of the authors have concentrated on only positive solutions of (1.1). The investigation of positive solutions is generally divided into three cases: the positone problem $F(0) > 0$, semipositone problem $F(0) < 0$ and $F(0) = 0$ [**8, 18, 19**]. However, the study of negative solutions is also an interesting problem because, if there is a positive solution u of (1.1), then $-u$ is not necessarily a solution. In [**1**], the authors showed the existence of positive and negative solutions of (1.1) under the presence of one parameter.

In this paper, we extend the upper and lower solution method, and we study the spectrum of (1.1) for both positive and negative solutions.

This work is organized as follows. In Section 2, we investigate (1.1) without parameters. For this problem, we introduce the monotone solutions and the Picard iterative process. Using these tools we prove some existence results. In Section 3, we study (1.1) for one parameter, we define the spectrum in $\mathbb{R}$, and we study when $F(0) > 0$ and $F(0) < 0$. In Section 4 we define and study the spectrum in $\mathbb{R}^2$. We define the Picard iterative process of (1.1) for two parameters, and we prove an existence theorem of solutions with this process.

2. Nonlinear elliptic problem. In this section, we study the nonlinear problem

$$\left\{\begin{array}{ll} Lu(x) + f(x, u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{array}\right. \tag{2.1}$$

where f satisfies (H-0) and (H-1).

By a solution u of (2.1), we mean a classical solution, it is $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and u satisfies the two equations of (2.1).

The Picard iterative process for a real function u_0 in Ω is defined as

$$\left\{\begin{array}{ll} Lu_n(x) + f(x, u_{n-1}(x)) = 0, & x \in \Omega, \\ u_n(x) = 0, & x \in \partial\Omega, \end{array}\right. \tag{2.2}$$

for $n = 1, 2, 3, \ldots$.

We observe that in each step of the iteration we solve a linear problem of the form

$$\left\{\begin{array}{ll} Lu(x) + z(x) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{array}\right. \tag{2.3}$$

For this classical linear problem we have the next theorem whose proof can be seen in [**20**].

Theorem 2.1. *Let z be a function in $C^{(0,\alpha)}(\overline{\Omega})$ and $G(x,y)$ a Green function of* (2.3). *A function u is a solution of* (2.3) *if and only if*

$$u(x) = \int_{\Omega} G(x,y)z(y)\,dy, \quad x \in \overline{\Omega}. \tag{2.4}$$

In this work, we suppose that there is a Green function for the linear problem (2.3).

According to Theorem 2.1, if $z \in C^{(0,\alpha)}(\overline{\Omega})$ and u is given by (2.4), then $u \in C^2(\Omega) \cap C(\overline{\Omega})$. By properties of G and the mean value theorem, it is easy to prove that $u \in C^{(0,1)}(\overline{\Omega})$. Therefore, for $u_0 \in C^{(0,1)}(\overline{\Omega})$, the Picard iterative process generates a sequence of functions $u_n \in C^2(\Omega) \cap C^{(0,1)}(\overline{\Omega})$. By (H-M), we note that this sequence is unique.

Definition 2.2. Let η be a function in $C^2(\Omega) \cap C^{(0,1)}(\overline{\Omega})$ and u_1 the first function obtained from (2.2) with $u_0 = \eta$. We say that η is a monotone lower solution of (2.1), if $\eta(x) \leq u_1(x)$ in Ω. We call η a monotone upper solution of (2.1), if it satisfies the opposite inequality with respect to u_1.

We consider that a function is a monotone solution if it is a monotone lower solution or a monotone upper solution. We observe that a solution of (2.1) is a monotone solution.

Lemma 2.3. *Let χ be a function in $C^2(\Omega) \cap C^{(0,1)}(\overline{\Omega})$. If χ satisfies*

$$\begin{cases} L\chi(x) + f(x,\chi(x)) \geq 0, & x \in \Omega, \\ \chi(x) \leq 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Then χ is a monotone lower solution. If χ satisfies the opposite inequalities in (2.5), *then χ is a monotone upper solution.*

Proof. We only prove the first assertion of this lemma. The proof of the other assertion is similar. Let $\chi \in C^2(\Omega) \cap C^{(0,1)}(\overline{\Omega})$ be such that it satisfies (2.5). From (2.2), for $n = 1$ with $u_0 = \chi$ and of (2.5), we obtain $L[u_1 - \chi] \leq 0$ in Ω and $u_1 - \chi \geq 0$ in $\partial\Omega$. Property (H-M) implies that $u_1 \geq \chi$ in Ω. Therefore, χ is a monotone lower solution. $\square$

In the literature, a function χ that satisfies (2.5) is called a *lower solution*, while if χ satisfies the opposite inequality to (2.5), it is called an *upper solution*. Therefore, the set of upper and lower solutions is a subset of the monotone solution set. This contention is strict because there are monotone solutions which are neither lower nor upper solutions of (2.1), as we show in the next example.

We consider the following problem

$$\begin{cases} \Delta u(x) + \arctan(u(x)) = 0, & x \in B, \\ u(x) = 0, & x \in \partial B, \end{cases} \tag{2.6}$$

where B is the open unit ball in $\mathbb{R}^m$.

We try to construct monotone solutions of (2.6). Let k be a positive constant, and define $\phi(x) = k$ and $\eta(x) = -k$ in Ω. According to the existence and uniqueness theorem of the linear problem (2.3), see **[14, page 56]**, we have that $v(x) = (1 - |x|^2)/2m$ is the unique solution of class $C^2(B) \cap C(\overline{B})$ for

$$\begin{aligned} \Delta v(x) + 1 = 0, &\quad x \in B, \\ v(x) = 0, &\quad x \in \partial B. \end{aligned}$$

By Theorem 2.1, $v(x)$ satisfies

$$\frac{1}{2m}(1 - |x|^2) = \int_B G(x, y)\, dy, \quad x \in B. \tag{2.7}$$

Let u_1 be the first function obtained from (2.2) for (2.6) starting with $u_0(x) = k$. By Theorem 2.1, $u_1(x)$ satisfies

$$u_1(x) = \arctan(k) \int_B G(x, y)\, dy, \quad x \in B. \tag{2.8}$$

Since $\arctan(k) < k$, from (2.7) and (2.8), we obtain

$$u_1(x) = \frac{\arctan(k)}{2m}(1 - |x|^2) < k = \phi(x), \quad x \in B.$$

Thus $u_1 \le \phi$ in B. Therefore, ϕ is a monotone upper solution. However, ϕ is neither a lower nor an upper solution of (2.6) because $\Delta\phi + \arctan(\phi) > 0$ in B and $\phi > 0$ in ∂B.

In a similar way for $u_0(x) = -k$ and x in Ω, we obtain

$$u_1(x) = \arctan(-k) \int_B G(x,y)\, dy = -\arctan(k)(1-|x|^2) > -k = \eta(x).$$

Therefore, $u_1 \geq \eta$ in B. As $\Delta\eta + \arctan(\eta) < 0$ in B and $\eta < 0$ in ∂B, we conclude that η is a monotone lower solution, and it is neither a lower nor an upper solution of (2.6).

From this example, we observe that a positive constant can be a monotone upper solution while a positive constant cannot be an upper solution of (2.1). This fact can simplify the problem of looking for a solution of (2.1), as we will see in the Section 3.

We say that a monotone solution is strict if it is not a solution of (2.1).

2.1. Results of existence by monotone methods. We remind the reader that a succession $\{u_n(x)\}$ is uniformly bounded in Ω if there is an $M > 0$ such that $|u_n(x)| \leq M$ for every x in Ω and $n = 0, 1, 2, \dots$.

We say that $\{u_n(x)\}$ is the Picard sequence of (2.1) starting with a given function $u_0 \in C^{(0,1)}(\overline{\Omega})$, if it is the sequence obtained by the Picard iterative process given by (2.2) with u_0. The next theorem is a generalization of a theorem proved by Cohen-Keller in [**9**]. They assumed f is positive in $\Omega \times \mathbb{R}$ and $\{u_n\}$ is the Picard sequence of (2.1) for $u_0 = 0$.

Theorem 2.4. *Let f be such that it satisfies* (H-0) *and* (H-1). *Let $\{u_n\}$ be the Picard sequence of* (2.1) *starting with $u_0 \in C^{(0,1)}(\overline{\Omega})$. If $\{u_n\}$ is uniformly bounded in Ω and it converges to u pointwise in $\overline{\Omega}$, then u is a solution of* (2.1).

Proof. By Theorem 2.1 and the fact that $G = 0$ in $\partial\Omega$ we obtain

$$u_n(x) = \int_{\overline{\Omega}} G(x,y) f(y, u_{n-1}(y))\, dy, \quad x \in \overline{\Omega}$$

for each $n = 1, 2, 3, \dots$. Since $\{u_n\}$ converges to u pointwise in $\overline{\Omega}$, then for each x in Ω, we have

$$u(x) = \lim_{n\to\infty} \int_{\overline{\Omega}} G(x,y) f(y, u_{n-1}(y))\, dy.$$

Thus, we would like to interchange the limit with the integral sign to prove that u has an integral representation. We use the dominated convergence theorem.

Since f is continuous in $\overline{\Omega} \times \mathbb{R}$, the sequence of functions $h_n(y) = G(x,y)\ f(y,u_n(y))$ converges to $h(y) = G(x,y)f(y,u(y))$ (almost everywhere) in $\overline{\Omega}$. As $\{u_n\}$ is bounded in Ω and $u_n = 0$ in $\partial\Omega$, there exists $M > 0$ such that $|u_n| \leq M$ in $\overline{\Omega}$. From the fact that $G(x,y) \geq 0$ in $\overline{\Omega} - \{x\}$ and from (H-1), we obtain

$$G(x,y)f(y,-M) \leq G(x,y)f(y,u_n(y)) \leq G(x,y)f(y,M), \quad y \in \overline{\Omega} - \{x\}.$$

Now, we define $g(y) = \max\{|G(x,y)f(y,-M)|, |G(x,y)f(y,M)|\}$. Since G and f are continuous in $\overline{\Omega} - \{x\}$, then $g \in L^1(\overline{\Omega})$ and $|h_n(y)| \leq g(y)$ in $\overline{\Omega} - \{x\}$. Therefore, the dominated convergence theorem implies

$$u(x) = \int_{\overline{\Omega}} G(x,y)f(y,u(y))\,dy, \quad x \in \overline{\Omega}.$$

Thus, $u(x) = \int_\Omega G(x,y)f(y,u(y))\,dy$ in Ω and $u = 0$ in $\partial\Omega$. Since G and its first partial derivatives are continuous, then u is Lipschitz continuous in $\overline{\Omega}$. As f is α-Hölder continuous in $\overline{\Omega} \times \mathbb{R}$ and u is Lipschitz continuous in $\overline{\Omega}$, we have that $f(x,u(x))$ is Lipschitz continuous in $\overline{\Omega}$. By Theorem 2.1, we conclude that u is a solution of (2.1). $\square$

Next, we prove some results which show that, if the Picard sequence $\{u_n\}$ for a monotone solution is uniformly bounded, then it converges uniformly to a solution of (2.1).

Proposition 2.5. *Let $\{u_n\}$ be the Picard sequence of* (2.1) *starting with $\eta \in C^{(0,1)}(\overline{\Omega})$. If $\eta(x)$ is a monotone lower solution, then $\eta(x) \leq u_1(x) \leq u_2(x) \leq \cdots$ in Ω.*

Proof. We proceed by induction. Since η is a monotone solution, we have $\eta \leq u_1$ in Ω. Assume that $u_{k-1} \leq u_k$ in Ω. From (2.2) for $n = k$ and $n = k+1$, we obtain $L[u_{k+1} - u_k] = -[f(x,u_k) - f(x,u_{k-1})]$ in Ω. From induction hypothesis and from (H-1) we have $L[u_{k+1} - u_k] \leq 0$ in Ω and $u_{k+1} - u_k = 0$ in $\partial\Omega$. Then, the (H-M) property implies that $u_k \leq u_{k+1}$ in Ω. Therefore, $u_n \leq u_{n+1}$ in Ω for $n = 0, 1, 2, \ldots$. $\square$

Proposition 2.6. *Let* $\{u_n\}$ *be the Picard sequence of* (2.1) *starting with* $\phi \in C^{(0,1)}(\overline{\Omega})$. *If* ϕ *is a monotone upper solution, then* $\phi(x) \geq u_1(x) \geq u_2(x) \geq \cdots$ *in* Ω.

Proof. The proof is similar to the proof of Proposition 2.5. □

Proposition 2.7. *Let* χ, ψ *be two functions in* $C^{(0,1)}(\Omega)$, *and let* $\{u_n(x)\}$ *and* $\{v_n(x)\}$ *be the Picard sequence of* (2.1) *starting with* χ *and* ψ, *respectively. If* $\chi \leq \psi$ *in* Ω, *then* $u_n(x) \leq v_n(x)$ *in* Ω *and* $n = 0, 1, 2, \ldots$.

Proof. The proof is similar to the proof of Proposition 2.5. □

We denote by $u_{u_0}(x)$ the solution of (2.1) obtained by the Picard iterative process starting with u_0.

Corollary 2.8. *Let* χ *and* ψ *be two functions in* $C^{(0,1)}(\Omega)$ *such that* $\chi \leq \psi$ *in* Ω. *If there exist* u_χ *and* u_ψ, *then* $u_\chi(x) \leq u_\psi(x)$ *in* Ω.

Proof. It is a consequence of Proposition 2.7. □

We remind the reader that a sequence $\{u_n(x)\}$ is bounded above if there exists a constant M such that $u_n(x) \leq M$ in Ω for $n = 0, 1, 2, \ldots$. The sequence $\{u_n\}$ is bounded below if it satisfies the opposite inequality for some M.

Theorem 2.9. *Let* η *be a monotone lower solution of* (2.1), *and let* $\{u_n(x)\}$ *be the Picard sequence of* (2.1) *starting with* η. *If* $\{u_n(x)\}$ *is bounded above, then it converges uniformly to a solution of* (2.1).

Proof. By hypothesis, there is an M such that $u_n(x) \leq M$ in Ω. According to Proposition 2.5, we have $\eta(x) \leq u_1(x) \leq u_2(x) \leq \cdots \leq M$ in Ω. Since $u_n(x) = 0$ in $\partial\Omega$, for $n \geq 1$, and $\eta(x)$ is a continuous function in $\overline{\Omega}$. Then $\{u_n(x)\}$ is a bounded monotone increasing sequence in $\overline{\Omega}$. Therefore, $\{u_n(x)\}$ converges pointwise to a function $u(x)$ in $\overline{\Omega}$. Theorem 2.4 implies that u is a solution of (2.1).

To prove uniform convergence, we observe that $\{u_n(x)\}$ is a continuous sequence in $\overline{\Omega}$, and it converges monotonously to a continuous function u in $\overline{\Omega}$. Thus, Dini's theorem implies that the convergence is uniform. □

Theorem 2.10. *Let ϕ be a monotone upper solution of* (2.1), *and let $\{u_n(x)\}$ be the Picard sequence of* (2.1) *starting with ϕ. If $\{u_n(x)\}$ is bounded below, then it converges uniformly to a solution of* (2.1).

Proof. The proof is similar to the proof of Theorem 2.9. □

Corollary 2.11. *Let f satisfy* (H-0), (H-1), *and f is bounded in $\overline{\Omega}\times\mathbb{R}$. If u_0 exists as a monotone solution of* (2.1), *then the Picard sequence for u_0 converges uniformly to a solution of* (2.1).

Proof. From (2.4), we can see easily that the Picard sequence $\{u_n(x)\}$ starting with u_0 is uniformly bounded because f is bounded in $\overline{\Omega}\times\mathbb{R}$. Therefore, by Theorems 2.9 and 2.10 we conclude the statement of this corollary. □

The next theorem is the main result of this section.

Theorem 2.12. *Let $\eta(x)$ be a monotone lower solution and $\phi(x)$ a monotone upper solution of* (2.1). *If $\eta(x) \le \phi(x)$ in Ω, then there exist $u_\eta(x)$ and $u_\phi(x)$ solutions of* (2.1) *such that $\eta(x) \le u_\eta(x) \le u_\phi(x) \le \phi(x)$ in Ω.*

Proof. Let $\{u_n(x)\}$ be the Picard sequence for η and $\{v_n(x)\}$ the Picard sequence for ϕ. According to Propositions 2.5 and 2.6, we have $\eta(x) \le u_n(x)$ and $v_n(x) \le \phi(x)$ in Ω for $n = 1, 2, 3, \ldots$. As $\eta \le \phi$ in Ω, Proposition 2.7 implies that $u_n(x) \le v_n(x)$ in Ω. Therefore,

$$\eta(x) \le u_n(x) \le v_n(x) \le \phi(x), \quad x \in \Omega. \tag{2.9}$$

Since η and ϕ are continuous in $\overline{\Omega}$, we have that $u_n(x)$ is bounded above and $v_n(x)$ is bounded below in $\overline{\Omega}$. By Theorems 2.9 and 2.10, there exist $u_\eta(x)$ and $u_\phi(x)$ solutions of (2.1). Finally, taking the limit in (2.9) when n tends to infinity, we obtain $\eta(x) \le u_\eta(x) \le u_\phi(x) \le \phi(x)$ in Ω. □

Theorem 2.12 was also proved by Amman for upper and lower solutions using the fixed point revisited **[2]**.

2.2. Positive and negative solutions. When $f > 0$ in $\Omega \times \mathbb{R}$, $\eta = 0$ is a strict monotone lower solution. This is because $\eta = 0$ satisfies (2.5) and it is not a solution of (2.1). Moreover, if u is a solution, then $Lu = -f(x, u) < 0$ in Ω and $u = 0$ in $\partial\Omega$. Property (H-M) implies that u is positive in Ω. Therefore, all the solutions of (2.1) are positive. From these observations, we have the following result.

Corollary 2.13. *Let f satisfy* (H-0), (H-1) *and $f > 0$ in $\Omega \times \mathbb{R}$. The problem* (2.1) *has a solution if and only if the Picard sequence $\{u_n(x)\}$ for $u_0 = 0$ is bounded above. Furthermore, if this sequence is bounded above, then it converges uniformly to a solution.*

Proof. If $u(x)$ is a solution of (2.1) then u is positive. According to Proposition 2.7 and taking $\chi = 0$ and $\psi = u$ we conclude that $\{u_n(x)\}$ is bounded above by the maximum of u in $\overline{\Omega}$. The other statements of this corollary are consequences of Theorem 2.9 with $\eta = 0$. $\square$

Corollary 2.13 also was proved by Cohen-Keller in **[9]**. We note by Corollary 2.13 if f is positive and there exists a solution u of (2.1), then there is the solution $u_{\min}(x) = u_{u_0}(x)$ with $u_0 = 0$ of (2.1). We also observe that $u_{\min} \leq u$ in $\overline{\Omega}$ for every solution u of (2.1). Therefore, by Theorem 2.12, if there is a monotone lower solution η such that $\eta \leq u_{\min}$ in Ω, then $u_\eta = u_{\min}$ in Ω. This implies that negative monotone lower solutions do not generate another solution different from $u_{\min}$.

When $f < 0$ in $\Omega \times \mathbb{R}$, we observe that $\phi = 0$ is a strict monotone upper solution of (2.1) and all the solutions of (2.1) are negative.

Corollary 2.14. *Let f satisfy* (H-0), (H-1) *and $f < 0$ in $\Omega \times \mathbb{R}$. The problem* (2.1) *has a solution if and only if the Picard sequence $\{u_n(x)\}$ for $u_0 = 0$ is bounded below. Furthermore, if this sequence is bounded below, then it converges uniformly to a solution.*

Proof. The proof of this corollary is similar to the proof of Corollary 2.13. $\square$

We observe from Corollary 2.14 that if f is negative and there is a solution u of (2.1), then there exists the solution $u_{max}(x) = u_{u_0}(x)$ with $u_0 = 0$. We also note that, for every solution u of (2.1), we have

$u \leq u_{\max}$ in $\overline{\Omega}$. Therefore, by Theorem 2.12, if there are $u_{\max}$ and a monotone upper solution ϕ such that $u_{\max} \leq \phi$ in Ω, then $u_\phi = u_{\max}$ in Ω. This implies that positive monotone upper solutions do not generate a different solution from $u_{\max}$.

We conclude this section with the following corollary.

Corollary 2.15. *Let f satisfy* (H-0), (H-1) *and f is bounded in $\overline{\Omega} \times \mathbb{R}$. If $f > 0$ or $f < 0$ in $\Omega \times \mathbb{R}$, then* (2.1) *has a solution.*

Proof. If $f > 0$ in $\Omega \times \mathbb{R}$, then $\eta = 0$ is a monotone lower solution of (2.1). If $f < 0$ in $\Omega \times \mathbb{R}$, then $\phi = 0$ is a monotone upper solution of (2.1). Corollary 2.11 implies that there exists a solution of (2.1) in both cases. $\square$

3. The spectrum in $\mathbb{R}$. In this section, we study the nonlinear problem

$$\begin{cases} Lu + \lambda f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{3.1}$$

for $\lambda > 0$, and f satisfies (H-0), (H-1), and $f(0) \neq 0$.

A function u is a solution of (3.1) if $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and it satisfies the two equations of (3.1) for $\lambda > 0$. We denote by $u(x;\lambda)$ a solution of (3.1) for a given λ.

Definition 3.1. The spectrum S_1 of (3.1) is the set of $\lambda > 0$ for which there exists at least a solution of (3.1) for this λ.

Generally, S_1 is defined as the set of $\lambda > 0$ for which there exists at least a positive solution of (3.1). In our definition, we observe if $\lambda \in S_1$. Then $u(x;\lambda)$ is not necessarily positive. Since $f(0) \neq 0$, we do not consider the zero function as a solution of (3.1).

We note that, for each $\lambda > 0$, (3.1) has the form of (2.1) if we define $F(x,r) = \lambda f(x,r)$ for $x \in \overline{\Omega}$ and $r \in \mathbb{R}$. Thus, the Picard iterative process of (3.1) for u_0 and λ is defined as

$$\begin{cases} Lu_n(x) + \lambda f(x, u_{n-1}(x)) = 0, & x \in \Omega, \\ u_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

for $n = 1, 2, 3, \ldots$. Let $\eta \in C^2(\Omega) \cap C^{(0,1)}(\overline{\Omega})$ and $\lambda > 0$. The function η is a monotone lower solution of (3.1) for λ if $\eta(x) \leq u_1(x)$ in Ω, η is a monotone upper solution of (3.1) for λ if $\eta(x) \geq u_1(x)$ in Ω, where u_1 is the first function obtained from (3.2) for λ and $u_0 = \eta$.

Corollary 3.2. *Let $\eta(x)$ be a monotone lower solution and $\phi(x)$ a monotone upper solution of* (3.1) *for a given λ. If $\eta(x) \leq \phi(x)$ in Ω, then there exists at least a solution of* (3.1) *for this λ.*

Proof. It is an immediate consequence of Theorem 2.12 observing that η and ϕ are monotone lower and upper solutions, respectively, of the following problem

$$\begin{aligned} Lu + F(x,u) &= 0, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned}$$

where $F(x, r) = \lambda f(x, r)$ for $x \in \overline{\Omega}$ and $r \in \mathbb{R}$. □

Proposition 3.3. *Let f be a function that satisfies* (H-0), (H-1) *and $f > 0$ in $\Omega \times \mathbb{R}$. If $\lambda_1 \in S_1$, then $(0, \lambda_1] \subset S_1$.*

Proof. Let $u(x; \lambda_1)$ be a solution of (3.1), and let λ be such that $0 < \lambda < \lambda_1$. Since $f > 0$ in Ω and $\lambda > 0$, we observe that $\eta = 0$ is a monotone lower solution of (3.1) and $u(x; \lambda_1)$ is a positive monotone upper solution of (3.1) for this λ. Corollary 3.2 implies that there is a solution of (3.1) for λ. Thus, $\lambda \in S_1$, and then $(0, l_1] \subset S_1$. □

Proposition 3.4. *Let f be a function that satisfies* (H-0), (H-1) *and $f < 0$ in $\Omega \times \mathbb{R}$. If $\lambda_1 \in S_1$, then $(0, \lambda_1] \subset S_1$.*

Proof. The proof is similar to the proof of Proposition 3.3. □

Corollary 3.5. *Let f be a function that satisfies* (H-0), (H-1) *and $f > 0$ or $f < 0$ in $\Omega \times \mathbb{R}$. If $S_1 \neq \emptyset$, then $S_1 = (0, \lambda^*)$ or $S_1 = (0, \lambda^*]$, where λ^* is the supremum of S_1.*

Proof. It is a consequence of Propositions 3.3 and 3.4. □

Proposition 3.6. *For the nonlinear problem*

$$\begin{cases} Lu(x) + \lambda f(u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

If f satisfies (H-0), (H-1) *and $f(0) > 0$, then there exists $\lambda' > 0$ such that* $(0, \lambda') \subset S_1$.

Proof. We observe that $\eta = 0$ is a monotone lower solution of (3.3) for every $\lambda > 0$ because $f(0) > 0$. We now construct a positive monotone upper solution of (3.3) for some $\lambda > 0$.

Let k be a positive constant, and let $e(x)$ be the solution of the following problem

$$\begin{cases} Lv(x) + 1 = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

Since $e(x)$ is a solution of (3.4), then $e(x)$ is bounded in $\overline{\Omega}$ and, by (H-M), $e(x)$ is positive in Ω. Thus, there exists $M > 0$ such that $e(x) \le M$ in $\overline{\Omega}$. Also, Theorem 2.1 implies that

$$e(x) = \int_\Omega G(x, y)\, dy, \quad x \in \Omega. \tag{3.5}$$

Let u_1 be the first function obtained from the Picard iterative process for (3.3) with $u_0 = k$ and $\lambda > 0$. Theorem 2.1 and (3.5) imply

$$u_1(x) = \lambda f(k) \int_\Omega G(x, y)\, dy = \lambda f(k) e(x), \quad x \in \Omega. \tag{3.6}$$

Since $f(0) > 0$ and k is positive, (H-1) implies that $f(k) > 0$. Now, we define $\lambda' = k/f(k)M$, where M is a positive upper bound of $e(x)$. Let λ be such that $0 \le \lambda < \lambda'$. From (3.6), the fact that $e(x) \le M$ in Ω and $\lambda < \lambda'$, we conclude $u_1(x) \le k$ for x in Ω. Thus, $\eta = 0$ is a monotone lower solution and $\phi = k$ is a monotone upper solution of (3.3) for λ. Corollary 3.2 implies that there is a solution of (3.3) for this λ. Therefore, $\lambda \in S_1$ and then $(0, \lambda') \subset S_1$. □

Proposition 3.7. *Let S_1 be the spectrum of* (3.3). *If f satisfies* (H-0), (H-1) *and $f(0) < 0$. Then there exists $\lambda' > 0$ such that* $(0, \lambda') \subset S_1$.

Proof. The proof is similar to the proof of Proposition 3.6. □

In the following we prove that the solutions of (3.1) obtained by the Picard iterative process are monotonous with respect to λ and with respect to the initial function.

For $\lambda \in S_1$, we denote with $u_{u_0}(x;\lambda)$ the solution of (3.1) which is obtained from the Picard iterative process starting with u_0 and λ. We observe if $f > 0$ in $\Omega \times \mathbb{R}$ and $\lambda \in S_1$. Then, by Theorem 3.2, the solution $u_0(x;\lambda)$ always exists.

Proposition 3.8. *Let f be such that it satisfies* (H-0), (H-1) *and $f > 0$ in $\Omega \times \mathbb{R}$. Let $\lambda_1, \lambda_2 \in S_1$, and let χ and ψ be such that there exist $u_\chi(x;\lambda_1)$ and $u_\psi(x;\lambda_2)$. If $\lambda_1 \leq \lambda_2$ and $\chi \leq \psi$ in Ω, then $u_\chi(x;\lambda_1) \leq u_\psi(x;\lambda_2)$ in Ω.*

Proof. Let $\{u_n(x)\}$ be the sequence generated by (3.2) with $u_0 = \chi$ and λ_1, and let $\{v_n(x)\}$ be the sequence generated by (3.2) with $u_0 = \psi$ and λ_2. By induction, we can prove that

$$u_n(x) \leq v_n(x), \quad x \in \Omega. \tag{3.7}$$

The case $n = 0$ is trivial. Assume that (3.7) is true for $n = k$. From (3.2) for $n = k+1$, χ and ψ, respectively, we obtain

$$L[v_{k+1}(x) - u_{k+1}(x)] = -\lambda_2 f(x, v_k(x)) + \lambda_1 f(x, u_k(x)), \quad x \in \Omega$$

and $v_{k+1} - u_{k+1} = 0$ in $\partial\Omega$. Since $f > 0$ and $\lambda_2 \geq \lambda_1 > 0$, by induction hypothesis and (H-1), we have $L[v_{k+1} - u_{k+1}] \leq 0$ in Ω. Therefore, (H-M) property implies that $v_{k+1} \leq u_{k+1}$ in Ω. Thus, (3.7) is true for $n = 0, 1, 2, \ldots$. Taking limits in (3.7) when n converges to infinity, we conclude the proof of this proposition. □

Corollary 3.9. *Let f be such that it satisfies* (H-0), (H-1) *and $f < 0$ in $\Omega \times \mathbb{R}$. Let $\chi(x)$ and $\psi(x)$ be such that $u_\chi(x;\lambda)$, $u_\psi(x;\lambda)$ and $\chi(x) \leq \psi(x)$ in Ω exist. Then $u_\chi(x;\lambda) \leq u_\psi(x;\lambda)$ for every x in Ω.*

Proof. Fixing λ, the proof is similar to the proof of Corollary 2.8. □

Corollary 3.10. *Let f be a function that satisfies* (H-0), (H-1), *$f < 0$ in $\Omega \times \mathbb{R}$ and $\lambda_1, \lambda_2 \in S_1$. If $\lambda_1 \leq \lambda_2$, then $u_0(x;\lambda_2) \leq u_0(x;\lambda_1)$ in Ω.*

Proof. Let $u_n(x)$ and $v_n(x)$ be the sequences generated for (3.2) with $u_0 = 0$, λ_1 and λ_2, respectively. Since $\lambda_1 \in S_1$, there is a solution u of (3.1) for λ_1. By Corollary 3.2, taking $\eta = u$ and $\phi = 0$ implies that the solution $u_0(x; \lambda_1)$ exists. Analogously, since $\lambda_2 \in S_2$, then the solution $u_0(x; \lambda_2)$ exists.

By induction, we can prove that $u_n(x) \le v_n(x)$ for every x in Ω and $n = 0, 1, 2, \ldots$. The case $n = 0$ is trivial. Suppose that $v_k(x) \ge u_k(x)$. From (3.2), (H-1), $f < 0$, and $\lambda_1 \le \lambda_2$, we obtain

$$\begin{aligned} L[v_{k+1}(x) - u_{k+1}(x)] &= -\lambda_1 f(x, v_k(x)) + \lambda_2 f(x, u_k(x)) \\ &\le -\lambda_2 [f(x, v_k(x)) - f(x, u_k(x))] \\ &\le 0. \end{aligned}$$

Thus, $L[v_{k+1} - u_{k+1}] \le 0$ in Ω and $v_{k+1} - u_{k+1} = 0$ in $\partial\Omega$. From (H-M), we have $v_{k+1} \le u_{k+1}$ in Ω. Therefore,

$$u_n(x) \le v_n(x), \quad x \in \Omega \text{ and } n = 0, 1, 2, \ldots .$$

Taking limits in the last inequality when n converges to infinity, we obtain $u_0(x; \lambda_1) \le u_0(x; \lambda_2)$. $\square$

4. The spectrum in $\mathbb{R}^2$. In this section we study the nonlinear problem

$$\begin{cases} Lu + \lambda f(x, u) + \mu g(x, u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ \lambda, \mu > 0, \end{cases} \tag{4.1}$$

where f and g satisfy (H-0), (H-1), and f, g are two positive functions in $\Omega \times \mathbb{R}$.

Definition 4.1. The spectrum S_2 of (4.1) is the set of (λ, μ) such that exist at least a solution of (4.1) for this pair of positive numbers.

We prove that S_2 is either a rectangle of $\mathbb{R}^2$ or the empty set.

Proposition 4.2. *Let f and g be such that they satisfy* (H-0), (H-1), *$f > 0$ and $g > 0$ in $\Omega \times \mathbb{R}$. If $(\lambda_1, \mu_1) \in S_2$, then $(0, \lambda_1] \times (0, \mu_1] \subset S_2$.*

Proof. Let λ, μ be such that $0 < \lambda \leq \lambda_1$ and $0 < \mu \leq \mu_1$. Since $(\lambda_1, \mu_1) \in S_2$, there is a solution u of (4.1) for λ_1 and μ_1. For $F(x, r) = \lambda f(x, r) + \mu g(x, r)$ in $\overline{\Omega} \times \mathbb{R}$, we define the nonlinear problem

$$\begin{cases} Lv + F(x, v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

From the fact that $\lambda, \mu > 0$ and by the hypotheses of f and g, we have that F satisfies (H-0), (H-1) and $F > 0$ in $\Omega \times \mathbb{R}$. Thus, $\eta = 0$ is a monotone lower solution of (4.2), and u is a positive monotone upper solution of (4.2). By Theorem 2.12, there is a solution v of (4.2). So v is a solution of (4.1) for λ and μ. Therefore, $(\lambda, \mu) \in S_2$, and it implies that $(0, \lambda_1] \times (0, \mu_1] \subset S_2$. $\square$

We denote by (λ^*, μ^*) the supremum of S_2. If we assume that (λ^*, μ^*) is not in S_2, we have the following result.

Corollary 4.3. *Let f and g be two positive functions in $\Omega \times \mathbb{R}$ that satisfy* (H-0) *and* (H-1). *S_2 is either the rectangle $(0, \lambda^*) \times (0, \mu^*)$ or the empty set.*

Proof. If $S_2 \neq \emptyset$, it is easy to prove $S_2 \subset (0, \lambda^*) \times (0, \mu^*)$, and the opposite contention is an immediate consequence of Proposition 4.2. $\square$

Corollary 4.4. *Let f and g be two functions which satisfy* (H-0), (H-1), *$f > 0$, $f + g > 0$, and $g < 0$ in $\Omega \times \mathbb{R}$. For $\lambda_1 > 0$, if v is a solution of the problem*

$$\begin{cases} Lv + \lambda_1 f(x, v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

then $(\lambda, \mu) \in S_2$ for every $0 < \mu \leq \lambda \leq \lambda_1$.

Proof. Let v be a solution of (4.3) for λ_1, and let λ, μ be as in the statement of the corollary. For these λ, μ we define the nonlinear problem

$$\begin{cases} Lu + F(x, u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

where $F(x,r) = \lambda f(x,r) + \mu g(x,r)$ in $\overline{\Omega} \times \mathbb{R}$. Since $\lambda, \mu > 0$ and f, g satisfy (H-0) and (H-1), F also satisfies these properties. From the fact that $\lambda \geq \mu > 0$, $f + g > 0$ and $g < 0$ in $\Omega \times \mathbb{R}$, we have

$$L0 + F(x,0) = \lambda(f+g)(x,0) + (\mu - \lambda)g(x,0) \geq 0, \quad x \in \Omega.$$

Therefore, $\eta = 0$ is a monotone upper solution of (4.4). Since $f > 0$ and $g < 0$ in $\Omega \times \mathbb{R}$, $\lambda \leq \lambda_1$, and $\mu > 0$ we obtain

$$Lv + F(x,v) \leq Lv + \lambda f(x,v) \leq Lv + \lambda_1 f(x,v) = 0, \quad x \in \Omega.$$

and $v = 0$ in $\partial\Omega$. Thus, v is a monotone upper solution of (4.4). By Theorem 2.12, there is a solution u of (4.4). Therefore, u is a solution of (4.1) for λ and μ, so $(\lambda, \mu) \in S_2$. □

Corollary 4.5. *Let f and g be two functions such that f and g satisfy* (H-0)*, f and $f+g$ satisfy* (H-1)*, $f > 0$ and $g > 0$ in $\Omega \times \mathbb{R}$. If $(\lambda_1, \mu_1) \in S_2$ and $\mu_1 \leq \lambda_1$, then $(\lambda, \mu) \in S_2$ for every $0 < \mu \leq \lambda \leq \lambda_1$ and $\mu \leq \mu_1$.*

Proof. The proof is similar to that of Proposition 4.2 observing that $F(x,r) = \lambda f(x,r) + \mu g(x,r)$ satisfy (H-0) and (H-1) in $\overline{\Omega} \times \mathbb{R}$ for every $\mu \leq \lambda$. □

4.1. Picard method revisited. In this subsection, we introduce the Picard iterative processes for (4.1) in order to prove an existence theorem of this problem. We do not assume that f and g are positive in $\Omega \times \mathbb{R}$.

For two given real numbers λ and μ, two fixed functions η and ϕ in $C^{(0,1)}(\overline{\Omega})$, and $u_1 \in C^2(\overline{\Omega})$ such that u_1 satisfies

$$\begin{cases} Lu_1 + \lambda f(x,\eta) + \mu g(x,\phi) \geq 0, & x \in \Omega, \\ u_1 \leq 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

We define the Picard iterative process of (4.1) as

$$\begin{cases} Lu_n + \lambda f(x,u_{n-1}) + \mu g(x,u_{n-2}) = 0, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

with $u_0 = \phi$ and $n \geq 2$.

Proposition 4.6. *Let η, ϕ and u_1 be such that* (4.5) *holds. If $\eta \leq u_1$ and $\phi \leq u_1$ in Ω, then $\eta(x) \leq u_1(x) \leq u_2(x) \leq \cdots$.*

Proof. We prove the inequalities for $n = 1, 2, 3$. For $n \geq 3$, it follows immediately by induction. By hypothesis, $\eta(x) \leq u_1(x)$ in Ω. From (4.5), (4.6), $\lambda > 0$ and (H-1), we obtain

$$L[u_2(x) - u_1(x)] \leq -\lambda[f(x, u_1(x)) - f(x, \eta(x))] \leq 0, \quad x \in \Omega$$

and $u_2 - u_1 \geq 0$ in Ω. The (H-M) property implies that $u_2 \geq u_1$ in Ω. Proceeding as before for $n = 3$ and observing that $\lambda, \mu > 0$, $u_1 \leq u_2$, and $\phi \leq u_1$ in Ω, we have

$$\begin{aligned} L[u_3(x) - u_2(x)] &\leq -\lambda[f(x, u_2(x)) - f(x, u_1(x))] \\ &\quad - \mu[g(x, u_1(x)) - g(x, \phi(x))] \\ &\leq 0, \end{aligned}$$

for every x in Ω and $u_3 - u_2 = 0$ in $\partial\Omega$. Using the (H-M) property we conclude that $u_3 \geq u_2$ in Ω. □

Corollary 4.7. *Let λ, μ be positive. Let f and g be such that they satisfy* (H-0), (H-1) *and $g > 0$ in $\Omega \times \mathbb{R}$. Let η be a lower solution of*

$$\begin{cases} Lv + \lambda f(x, v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{4.7}$$

Let u_1 be the first function obtained by the Picard iterative process of (4.7) *with $u_0 = \eta$, and let $\phi \in C^2(\Omega)$. If $\phi(x) \leq \eta(x)$ in Ω, then $\{u_n\}$ obtained from* (4.6) *with this function is a monotone increasing sequence.*

Proof. Since η is a monotone lower solution of (4.7) by Proposition 2.5 we have $\eta \leq u_1$ in Ω. As $\phi \leq \eta$ in Ω, then $\phi \leq u_1$ in Ω. From the fact that $\mu > 0$, $g > 0$ in Ω, and from the definitions of u_1 and η, we obtain $Lu_1 + \lambda f(x, \eta) + \mu g(x, \phi) \geq \mu g(x, \phi) > 0$ in Ω and $u_1 \leq 0$ in $\partial\Omega$. Therefore, $\chi \leq u_1$, $\phi \leq u_1$ in Ω, and u_1 satisfies (4.5). By Proposition 4.6, we conclude the proof of this corollary. □

Theorem 4.8. *Let $\{u_n\}$ be the sequence generated by* (4.6) *for η, ϕ and u_1. If $\{u_n\}$ is uniformly bounded and it converges to u pointwise in $\overline{\Omega}$, then u is a solution of* (4.1).

Proof. The proof is similar to the proof of Theorem 2.9 beginning with $n = 2$. □

Corollary 4.9. *Let $\{u_n\}$ be the sequence obtained from* (4.6) *for* η, ϕ, u_1. *If $\{u_n\}$ is a monotone increasing sequence and it is bounded above, then $\{u_n\}$ converges uniformly to a solution of* (4.1).

Proof. The proof is a consequence of Theorem 4.8 and Dini's theorem. □

5. Conclusions.

- In this work, we prove a general existence theorem for nonlinear elliptic boundary value problems, Theorem 2.4. The importance of this result is that we can construct a solution of (1.1) without using monotone solutions.
- If there exist η a monotone lower solution and ϕ a monotone upper solution of (2.1) such that $\eta \leq \phi$ in Ω, then there exists at least a solution u of this problem such that $\eta \leq u \leq \phi$ in Ω.
- We give a definition of S_1, the spectrum in $\mathbb{R}$ of (3.1), which involves not only positive solutions. When $f > 0$ or $f < 0$ in Ω, we prove that S_1 is an interval or it is empty. This result is also observed when the spectrum is defined only for positive solutions and $f(0) > 0$ [**9**].
- We define S_2, the spectrum in $\mathbb{R}^2$ of (4.1), for both positive and negative solutions. We prove that S_2 is a rectangle or it is empty. We also define the Picard iterative process of (4.1) to prove an existence theorem for this problem.

Open questions.

- If F satisfies (H-0), (H-1), $F(0) \neq 0$ and there exist $x, y \in \Omega$ such that $F(x) < 0 < F(y)$, are S_1 and S_2 of problem (1.1) a connected set?
- If there exists a negative solution of (1.1) when F changes sign, is there at least a positive solution of this problem? We would like to note that this question is true when the coefficient corresponding to the independent term of L is zero and F is an odd function.

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