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# TOPOLOGICAL PROPERTIES OF PATH CONNECTED COMPONENTS IN SPACES OF WEIGHTED COMPOSITION OPERATORS INTO $L^\infty$

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ABSTRACT. This paper demonstrates equivalence amongst the topological structures of path connected components in the spaces of weighted composition operators from $L^\infty$, $H^\infty$ and the disk algebra into $L^\infty$ on the unit circle.

**1. Introduction.** Let $\mathbb{D}$ be the open unit disk in the complex plane and $\partial\mathbb{D}$ its boundary. Let $\mathcal{S}(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. For an analytic function $u$ on $\mathbb{D}$ and $\varphi \in \mathcal{S}(\mathbb{D})$, we define the weighted composition operator $M_uC_\varphi$ as the product of multiplication and composition operators by $(M_uC_\varphi)f(z) = u(z)f(\varphi(z))$ for analytic functions $f$ on $\mathbb{D}$ and $z \in \mathbb{D}$. The properties of (weighted) composition operators have been extensively studied over the past few decades. See **[6, 23]** for an overview of these results.

Some of the most long-standing open questions are related to the topological structure of the space of (weighted) composition operators on the Banach space of analytic functions on $\mathbb{D}$ endowed with the operator norm and the essential operator norm, which was originally considered on the classical Hardy space $H^2$. In 1981, Berkson **[2]** first studied the component structure of the space of all composition operators on $H^2$ in the operator norm topology, and MacCluer **[16]** continued. Shapiro and Sundberg **[24]** further investigated and raised

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the problems on the component structure in the operator and the essential operator norm topologies.

Then, MacCluer, Zhao and the third author **[17]** considered these problems on $H^\infty$ (also see **[11]**), where $H^\infty$ is the Banach space of bounded analytic functions on $\mathbb{D}$ with the supremum norm. The first and third authors together with Hosokawa investigated the component structure in the space of weighted composition operators on $H^\infty$ and determined path connected components (**[10,** Theorem 4.1]). Refer to **[1, 3, 4, 7, 18, 19]** for results on various analytic function spaces.

On the other hand, by Sarason **[22]** $C_\varphi$ can be viewed as an integral operator acting on $\partial\mathbb{D}$ via Poisson extension. Let $m$ be the normalized Lebesgue measure on $\partial\mathbb{D}$. For $f \in L^p = L^p(\partial\mathbb{D}, dm)$ $(1 \le p \le \infty)$, let $P_z[f]$ be the Poisson extension of $f$ onto $\mathbb{D}$. Then $P_z[f]\circ\varphi$ is a harmonic function and has a radial limit $(P_z[f] \circ \varphi)^*$ almost everywhere on $\partial\mathbb{D}$. We have $(P_z[f] \circ \varphi)^* \in L^p$. Hence, we may define the composition operator $C_\varphi$ on $L^p$ by

$$C_\varphi f = (P_z[f] \circ \varphi)^*.$$

Let $L^\infty = L^\infty(\partial\mathbb{D})$ be the Banach space of all bounded measurable functions $f$ on $\partial\mathbb{D}$ with the essential supremum norm $\|f\|_\infty$. For $u \in L^\infty$, we may define the weighted composition operator $M_uC_\varphi$ on $L^\infty$. For $f \in L^\infty$, let $f^\#$ be the function on $\overline{\mathbb{D}}$ that takes the value of $P_z[f]$ in $\mathbb{D}$ and the value of $f$ on $\partial\mathbb{D}$. Then $M_uC_\varphi f = u(f^\# \circ \varphi^*)$ almost everywhere on $\partial\mathbb{D}$. The authors have extended the investigation of (weighted) composition operators on $L^\infty$ (**[12, 13, 14]** and see **[20, 25]** also).

Let $A = A(\overline{\mathbb{D}})$ be the space of continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. Usually $A(\overline{\mathbb{D}})$ is called the disk algebra. For each $f \in H^\infty$, we identify $f$ with its radial limit function $f^*$ on $\partial\mathbb{D}$. We may consider that $A(\overline{\mathbb{D}}) \subset H^\infty \subset L^\infty$. We denote by $\mathcal{C}_w(L^\infty, L^\infty)$ the space of nonzero weighted composition operators on $L^\infty$, that is,

$$\mathcal{C}_w(L^\infty, L^\infty) = \{M_uC_\varphi : u \in L^\infty, u \neq 0, \varphi \in \mathcal{S}(\mathbb{D})\}.$$

For $M_uC_\varphi \in \mathcal{C}_w(L^\infty, L^\infty)$, we denote by $\|M_uC_\varphi\|_{(L^\infty,L^\infty)}$ its operator norm. Restricting the operator $M_uC_\varphi$ on $H^\infty$ and $A(\overline{\mathbb{D}})$, we may consider that $M_uC_\varphi$ are bounded linear mappings from $H^\infty$ and $A(\overline{\mathbb{D}})$ into $L^\infty$. For these operator norms, we write $\|M_uC_\varphi\|_{(H^\infty,L^\infty)}$ and

$\|M_uC_\varphi\|_{(A,L^\infty)}$, and we have the spaces $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$.

We note that, as sets,

$$\mathcal{C}_w(L^\infty, L^\infty) = \mathcal{C}_w(H^\infty, L^\infty) = \mathcal{C}_w(A, L^\infty).$$

Naturally, the question occurs as to whether the topological structures in $\mathcal{C}_w(L^\infty, L^\infty)$, $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$ are the same. The topologies in these spaces deeply depend on the norms of differences of the two weighted composition operators on them.

Trivially, we have

$$\begin{aligned}\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty)} &\le \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty)}\\ &\le \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty)}.\end{aligned}$$

However, we note that generally the inequality

$$\|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty)} \le \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty)}$$

is strict. For example, see [**12,** Theorem 4.1] and [**15,** Theorem 4.1]. Also refer to [**5,** page 172] and [**17,** Proposition 4] in the unweighted case.

So the main theme of this paper is to consider the question whether the topological structures of path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$, $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$ are the same. In Section 2, we shall show that

$$\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty)} = \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty)}.$$

So path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ are path connected sets in $\mathcal{C}_w(H^\infty, L^\infty)$, and the topological structures of path connected components in $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$ are the same. Moreover, we shall also show that if $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(H^\infty,L^\infty)} \to 0$, then $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(L^\infty,L^\infty)} \to 0$. This fact shows that the structures of path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ and $\mathcal{C}_w(H^\infty, L^\infty)$ are the same as sets. But it is unclear whether the topological properties of path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ and $\mathcal{C}_w(H^\infty, L^\infty)$ are the same (is an open and closed path connected component in $\mathcal{C}_w(L^\infty, L^\infty)$ open and closed in $\mathcal{C}_w(H^\infty, L^\infty)$?).

We denote by $\mathcal{C}_{w,0}(L^\infty, L^\infty)$ the space of operators in $\mathcal{C}_w(L^\infty, L^\infty)$ which are not compact. Similarly we have the spaces $\mathcal{C}_{w,0}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0}(A, L^\infty)$. In [**13**], the authors determined the structures of path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ and $\mathcal{C}_{w,0}(L^\infty, L^\infty)$. In Section 2, we shall show that the topological structures of path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ and $\mathcal{C}_w(H^\infty, L^\infty)$ are the same. We shall also prove that $\mathcal{C}_{w,0}(L^\infty, L^\infty) = \mathcal{C}_{w,0}(H^\infty, L^\infty) = \mathcal{C}_{w,0}(A, L^\infty)$ and topological properties of path connected components in them are the same.

Let $\mathcal{H} = L^\infty$ or $H^\infty$ or $A(\overline{\mathbb{D}})$. We denote by ball$(\mathcal{H})$ the closed unit ball of $\mathcal{H}$. For a bounded linear operator $T$ from $\mathcal{H}$ to $L^\infty$, let $\|T\|_{(\mathcal{H},L^\infty,e)} = \inf_K \|T-K\|_{(\mathcal{H},L^\infty)}$, where $K$ moves in the space $\mathcal{K}(\mathcal{H}, L^\infty)$ of all compact operators from $\mathcal{H}$ into $L^\infty$. Usually $\|T|_{(\mathcal{H},L^\infty,e)}$ is called the essential operator norm of $T$. We denote by $\mathcal{C}_{w,0,e}(\mathcal{H}, L^\infty)$ the space $\mathcal{C}_{w,0}(\mathcal{H}, L^\infty)$ with the essential operator norm. Since

$$\mathcal{K}(L^\infty, L^\infty)|_{H^\infty} \subset \mathcal{K}(H^\infty, L^\infty) \quad \text{and} \quad \mathcal{K}(H^\infty, L^\infty)|_A \subset \mathcal{K}(A, L^\infty),$$

we have

$$(1.1) \qquad \begin{aligned} \|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} &\le \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty,e)} \\ &\le \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty,e)}. \end{aligned}$$

So it is also unclear whether the topological structures of path connected components in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$, $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0,e}(A, L^\infty)$ are the same. In **[14]**, the authors determined the structure of path connected components in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$. In Section 3, we shall prove that the topological structures of path connected components in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$, $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0,e}(A, L^\infty)$ are the same. The authors think that equalities hold in (1.1), but at this moment it is unclear.

Let

$$\mathcal{C}_w(H^\infty, H^\infty) = \big\{M_uC_\varphi : u \in H^\infty, u \ne 0, \varphi \in \mathcal{S}(\mathbb{D})\big\}.$$

Similarly, we have the spaces $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ and $\mathcal{C}_{w,0,e}(A, H^\infty)$. As sets, we have $\mathcal{C}_{w,0,e}(H^\infty, H^\infty) = \mathcal{C}_{w,0,e}(A, H^\infty)$. Since $\mathcal{K}(H^\infty, H^\infty)|_A \subset \mathcal{K}(A, H^\infty)$, we have

$$\|M_uC_\varphi - M_vC_\psi\|_{(A,H^\infty,e)} \le \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,H^\infty,e)}.$$

The authors determined the structure of path connected components of $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ in **[14]**. In Section 4, we shall prove that the topological structures of path connected components in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ and $\mathcal{C}_{w,0,e}(A, H^\infty)$ are the same.

**2. Path connected components.** Let $C = C(\partial\mathbb{D})$ be the space of continuous functions on $\partial\mathbb{D}$. Similarly, we have the space $\mathcal{C}_w(C, L^\infty) = \mathcal{C}_w(L^\infty, L^\infty)|_C$ and

$$\|M_uC_\varphi - M_vC_\psi\|_{(C,L^\infty)} \le \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty)}.$$

**Lemma 2.1.**

(i) $\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty)} = \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty)}$.
(ii) $\|M_uC_\varphi - M_vC_\psi\|_{(C,L^\infty)} = \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty)}$.

*Proof.* (i) Let $\alpha = \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty)}$. For $\varepsilon > 0$, there is a function $f \in \mathrm{ball}\,(H^\infty)$ such that $\alpha - \varepsilon < \|uC_\varphi f - vC_\psi f\|_\infty$. Then there is a function $F \in \mathrm{ball}\,(L^1)$ such that

$$\alpha - \varepsilon < \Big|\int_{\partial\mathbb{D}} (u(f\circ\varphi)^* - v(f\circ\psi)^*)F\,dm\Big|.$$

By Lindelöf's theorem, we have $(f\circ\varphi)^* = f^\# \circ \varphi^*$ almost everywhere on $\partial\mathbb{D}$ (see **[6,** page 31], **[12, 21]**). For $0 < r < 1$ and $z \in \overline{\mathbb{D}}$, let $f_r(z) = f(rz)$. Then it is easy to check that $f_r\circ\varphi^* \to f^\#\circ\varphi^*$, $f_r\circ\psi^* \to f^\#\circ\psi^*$ almost everywhere on $\partial\mathbb{D}$ as $r \to 1$. By the Lebesgue dominated convergence theorem,

$$\alpha - \varepsilon < \Big|\int_{\partial\mathbb{D}} (u(f_r\circ\varphi^*) - v(f_r\circ\psi^*))F\,dm\Big|$$

for $r$ sufficiently close to 1. Since $f_r \in \mathrm{ball}\,(A)$, $\alpha - \varepsilon < \|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty)}$. Thus, we get (i).

(ii) Let $\beta = \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty)}$. For $\varepsilon > 0$, there is a function $f \in \mathrm{ball}\,(L^\infty)$ such that $\beta - \varepsilon < \|uC_\varphi f - vC_\psi f\|_\infty$. Then there is a function $F \in \mathrm{ball}\,(L^1)$ such that

$$\beta - \varepsilon < \Big|\int_{\partial\mathbb{D}} (u(f^\#\circ\varphi^*) - v(f^\#\circ\psi^*))F\,dm\Big|.$$

Since $P_z[f]_r\circ\varphi^* \to f^\#\circ\varphi^*$ as $r \to 1$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. In the same way as (i), we get (ii).

$\square$

By Lemma 2.1 (i), we have the following.

**Corollary 2.2.** *The topological structures of path connected components in $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$ are the same.*

For $(z, w) \in \overline{\mathbb{D}}^2$, let

$$\rho(z,w) = \begin{cases} 1, & (z,w) \in \overline{\mathbb{D}}^2 \setminus \mathbb{D}^2, z \neq w \\ \left|\dfrac{z-w}{1-\overline{w}z}\right|, & (z,w) \in \mathbb{D}^2, z \neq w \\ 0, & z = w. \end{cases}$$

For $e^{i\theta} \in \partial\mathbb{D}$ such that $\varphi^*(e^{i\theta})$ and $\psi^*(e^{i\theta})$ exist, we define

$$d_A(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) = \sup_{f \in \mathrm{ball}\,(A)} |f(\varphi^*(e^{i\theta})) - f(\psi^*(e^{i\theta}))|$$

and

$$d_C(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) = \sup_{f \in \mathrm{ball}\,(C)} |f^\#(\varphi^*(e^{i\theta})) - f^\#(\psi^*(e^{i\theta}))|.$$

The following is a known fact (see [**5, 17**]).

**Lemma 2.3.** *We have that*

$$\begin{aligned}\rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \le d_A(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) &\le d_C(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta})) \\ &\le 2\rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta}))\end{aligned}$$

*almost everywhere on $\partial\mathbb{D}$.*

**Lemma 2.4.** *If $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(H^\infty, L^\infty)} \to 0$, then $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(L^\infty, L^\infty)} \to 0$.*

By Lemma 2.1, $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,L^\infty)} \to 0$. Hence, $\|u_n - v\|_\infty \to 0$ and

$$\|M_{u_n-v}C_{\varphi_n}\|_{(A,L^\infty)} = \|M_{u_n-v}C_{\varphi_n}\|_{(C,L^\infty)} \to 0.$$

Since

$$\begin{aligned}\|M_v(C_{\varphi_n} - C_\psi)\|_{(A,L^\infty)} \le \|M_{u_n}C_{\varphi_n} &- M_vC_\psi\|_{(A,L^\infty)} \\ &+ \|M_{u_n-v}C_{\varphi_n}\|_{(A,L^\infty)},\end{aligned}$$

we have $\|M_v(C_{\varphi_n} - C_\psi)\|_{(A,L^\infty)} \to 0$. Hence,

$$
\begin{aligned}
&\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(L^\infty,L^\infty)} \\
&= \|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(C,L^\infty)} \qquad \text{by Lemma 2.1} \\
&\le \|M_{u_n-v}C_{\varphi_n}\|_{(C,L^\infty)} + \|M_v(C_{\varphi_n} - C_\psi)\|_{(C,L^\infty)} \\
&= \|M_{u_n-v}C_{\varphi_n}\|_{(C,L^\infty)} + \operatorname{ess\,sup}_{e^{i\theta}\in\partial\mathbb{D}}|v(e^{i\theta})|d_C(\varphi_n^*(e^{i\theta}), \psi^*(e^{i\theta})) \\
&\le \|M_{u_n-v}C_{\varphi_n}\|_{(C,L^\infty)} + 2\operatorname{ess\,sup}_{e^{i\theta}\in\partial\mathbb{D}}|v(e^{i\theta})|d_A(\varphi_n^*(e^{i\theta}), \psi^*(e^{i\theta})) \\
&\qquad\qquad \text{by Lemma 2.3} \\
&= \|M_{u_n-v}C_{\varphi_n}\|_{(C,L^\infty)} + 2\|M_v(C_{\varphi_n} - C_\psi)\|_{(A,L^\infty)} \\
&\longrightarrow 0 \qquad \text{as } n \to \infty. \qquad \square
\end{aligned}
$$

**Corollary 2.5.** *The structures of path connected components in* $\mathcal{C}_w(L^\infty, L^\infty)$, $\mathcal{C}_w(H^\infty, L^\infty)$ *and* $\mathcal{C}_w(A, L^\infty)$ *are the same as sets.*

We shall study topological properties of path connected components in $\mathcal{C}_w(A, L^\infty)$. Let $M(H^\infty)$ and $M(L^\infty)$ be the maximal ideal spaces of $H^\infty$ and $L^\infty$, respectively. We denote the Gelfand transform of a function $f$ in $H^\infty$ (and $L^\infty$) by $\widehat{f}$. We may think of $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of $H^\infty$. Then, for the normalized Lebesgue measure $m$ on $\partial\mathbb{D}$, there exists the probability measure $\widehat{m}$ on $M(L^\infty)$ such that

$$\int_{\partial\mathbb{D}} f\,dm = \int_{M(L^\infty)} \widehat{f}\,d\widehat{m}$$

for every $f \in L^\infty$. Refer to **[8, 9]** for properties of the maximal ideal spaces of $H^\infty$ and $L^\infty$.

Let $\varphi \in \mathcal{S}(\mathbb{D})$. For each $x \in M(H^\infty)$, the mapping $H^\infty \ni f \to \widehat{f\circ\varphi}(x)$ is a nonzero multiplicative linear functional on $H^\infty$. Hence, there is a point $\widetilde{\varphi}(x) \in M(H^\infty)$ such that $\widehat{f\circ\varphi}(x) = \widehat{f}(\widetilde{\varphi}(x))$ for every $f \in H^\infty$. It is easy to show that $\widetilde{\varphi} : M(H^\infty) \to M(H^\infty)$ is a continuous map (see **[10,** page 514]). Considering $f(z) = z$, we have $\widehat{\varphi}(x) = \widehat{z}(\widetilde{\varphi}(x))$. Hence, if $|\widehat{\varphi}(x)| < 1$, then $\widetilde{\varphi}(x) = \widehat{\varphi}(x) \in \mathbb{D}$. One easily checks the following.

**Lemma 2.6.** *For each $\varphi \in \mathcal{S}(\mathbb{D})$ and $f \in A(\overline{\mathbb{D}})$, $\widehat{f \circ \varphi}(x) = f(\widehat{\varphi}(x))$ for every $x \in M(L^\infty)$.*

For $\varphi \in \mathcal{S}(\mathbb{D})$, let

$$E(\varphi) = \{x \in M(L^\infty) : |\widehat{\varphi}(x)| = 1\}$$

and $E^o(\varphi)$ be the interior of $E(\varphi)$ in $M(L^\infty)$. By [**8,** page 18], $E^o(\varphi)$ is an open and closed subset of $M(L^\infty)$. For $0 < r < 1$, we write

$$\{|\widehat{\varphi}| > r\} = \{x \in M(L^\infty) : |\widehat{\varphi}(x)| > r\}$$

and

$$\{r < |\widehat{\varphi}| < 1\} = \{x \in M(L^\infty) : r < |\widehat{\varphi}(x)| < 1\}.$$

**Lemma 2.7.** *For $u, v \in L^\infty$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ with $\varphi \neq \psi$, we have*

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,L^\infty)} \geq \max_{x \in E^o(\varphi)} |\widehat{u}(x)|.$$

*Proof.* We may assume that $E^o(\varphi) \neq \emptyset$. We have $\widehat{m}(E^o(\varphi)) > 0$. Since $\varphi \neq \psi$, $\widehat{m}(\{x \in M(L^\infty) : \widehat{\varphi}(x) = \widehat{\psi}(x)\}) = 0$. Hence,

$$\widehat{m}(\{x \in E^o(\varphi) : \widehat{\varphi}(x) \neq \widehat{\psi}(x)\}) = \widehat{m}(E^o(\varphi)).$$

Let $x \in E^o(\varphi)$ such that $\widehat{\varphi}(x) \neq \widehat{\psi}(x)$. We have $|\widehat{\varphi}(x)| = 1$. Since $\widehat{\varphi}(x) \in \partial\mathbb{D}$ is a peak point for $A(\overline{\mathbb{D}})$, there is a function $g \in A(\overline{\mathbb{D}})$ such that $\|g\|_\infty = 1$, $g(\widehat{\varphi}(x)) = 1$ and $g(\widehat{\psi}(x)) = 0$. By Lemma 2.6, we have

$$\begin{aligned}\|M_u C_\varphi - M_v C_\psi\|_{(A,L^\infty)} &\geq \|u(g \circ \varphi)^* - v(g \circ \psi)^*\|_\infty \\ &\geq |\widehat{u}(x) g(\widehat{\varphi}(x)) - \widehat{v}(x) g(\widehat{\psi}(x))| \\ &= |\widehat{u}(x)|.\end{aligned}$$

Hence,

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,L^\infty)} \geq \sup_{x \in E^o(\varphi); \widehat{\varphi}(x) \neq \widehat{\psi}(x)} |\widehat{u}(x)|.$$

Since $\{x \in M(L^\infty) : \widehat{\varphi}(x) \neq \widehat{\psi}(x)\}$ is dense in $M(L^\infty)$, we get the assertion. $\square$

**Lemma 2.8.** *Let $\varphi \in \mathcal{S}(\mathbb{D})$. Then $\{M_u C_\varphi \in \mathcal{C}_w(A, L^\infty) : u \in L^\infty\}$ is closed in $\mathcal{C}_w(A, L^\infty)$.*

*Proof.* Let $\{u_n\}_n$ be a sequence of nonzero functions in $L^\infty$ such that $M_{u_n}C_\varphi \to M_vC_\psi \in \mathcal{C}_w(A, L^\infty)$ as $n \to \infty$. Then $\|u_n - v\|_\infty \to 0$ and $\|u_n\varphi^* - v\psi^*\|_\infty \to 0$. Hence, $v(\varphi^* - \psi^*) = 0$, so $\varphi^* = \psi^*$ almost everywhere on $\{e^{i\theta} \in \partial\mathbb{D} : v(e^{i\theta}) \neq 0\}$. Since $v \neq 0$, by Jensen's inequality (see [**9,** page 51]) we have $\varphi = \psi$. Thus, we get the assertion. □

For $\varphi \in \mathcal{S}(\mathbb{D})$, we write $\{|\varphi^*| = 1\} = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi^*(e^{i\theta})| = 1\}$. Similarly, we may define $\{r < |\varphi^*|\}$ and $\{r < |\varphi^*| < 1\}$ for every $0 < r < 1$. The following is given in [**13,** Theorem 3.6].

**Lemma 2.9.**

(i) *If* $\varphi \in \mathcal{S}(\mathbb{D})$ *and* $m(\{|\varphi^*| = 1\}) = 1$, *then* $\{M_uC_\varphi \in \mathcal{C}_w(L^\infty, L^\infty) : u \in L^\infty\}$ *is open and closed, and a path connected component in* $\mathcal{C}_w(L^\infty, L^\infty)$.

(ii) *The set*

$$\big\{M_uC_\psi \in \mathcal{C}_w(L^\infty, L^\infty) : u \in L^\infty, \psi \in \mathcal{S}(\mathbb{D}), m(\{|\psi^*| = 1\}) < 1\big\}$$

*is open and closed, and a path connected component in* $\mathcal{C}_w(L^\infty, L^\infty)$.

**Theorem 2.10.** *The topological structures of path connected components in* $\mathcal{C}_w(L^\infty, L^\infty)$, $\mathcal{C}_w(H^\infty, L^\infty)$ *and* $\mathcal{C}_w(A, L^\infty)$ *are the same.*

*Proof.* By Corollary 2.2, the topological structures of path connected components in $\mathcal{C}_w(H^\infty, L^\infty)$ and $\mathcal{C}_w(A, L^\infty)$ are the same. As mentioned in the introduction, path connected components in $\mathcal{C}_w(L^\infty, L^\infty)$ are path connected sets in $\mathcal{C}_w(A, L^\infty)$. To show the assertion, it is sufficient to prove that each path connected component in $\mathcal{C}_w(L^\infty, L^\infty)$ is open and closed in $\mathcal{C}_w(A, L^i)$.

Let $\varphi \in \mathcal{S}(\mathbb{D})$ satisfy $m(\{|\varphi^*| = 1\}) = 1$. Then $E(\varphi) = E^o(\varphi) = M(L^\infty)$. Let $u \in L^\infty$ with $u \neq 0$ and $M_vC_\psi \in \mathcal{C}_w(A, L^\infty)$ with $\varphi \neq \psi$. By Lemma 2.7,

$$\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty)} \geq \max_{x \in M(L^\infty)} |\widehat{u}(x)| = \|u\|_\infty > 0.$$

This shows that $\{M_uC_\varphi \in \mathcal{C}_w(A, L^\infty) : u \in L^\infty\}$ is open in $\mathcal{C}_w(A, L^\infty)$. By Lemma 2.8, $\{M_uC_\varphi \in \mathcal{C}_w(A, L^\infty) : u \in L^\infty\}$ is closed in $\mathcal{C}_w(A, L^\infty)$.

Next, we shall show that

$$X := \{M_u C_\varphi \in \mathcal{C}_w(A, L^\infty) : u \in L^\infty, \varphi \in \mathcal{S}(\mathbb{D}), m(\{|\varphi^*| = 1\}) = 1\}$$

is open and closed in $\mathcal{C}_w(A, L^\infty)$. By the last paragraph, $X$ is open in $\mathcal{C}_w(A, L^\infty)$. Let $\{M_{u_n} C_{\varphi_n}\}_n$ be a sequence in $X$ such that $M_{u_n} C_{\varphi_n} \to M_v C_\psi \in \mathcal{C}_w(A, L^\infty)$. Then $\|u_n - v\|_\infty \to 0$. If $M_v C_\psi \notin X$, then $\varphi_n \neq \psi$ for every $n \geq 1$. Hence, by Lemma 2.7, we have $\|u_n\|_\infty \to 0$, so $v = 0$. This is a contradiction. Thus, $X$ is closed in $\mathcal{C}_w(A, L^\infty)$.

By the above facts,

$$\{M_u C_\psi \in \mathcal{C}_w(A, L^\infty) : u \in L^\infty, \psi \in \mathcal{S}(\mathbb{D}), m(\{|\psi^*| = 1\}) < 1\}$$

is open and closed in $\mathcal{C}_w(A, L^\infty)$. By Lemma 2.9, we get the assertion. □

We shall give the equivalence of the compactness of weighted composition operators from $L^\infty, H^\infty$ and $A(\overline{\mathbb{D}})$ to $L^\infty$.

**Lemma 2.11.** *Let $u \in L^\infty$ with $u \neq 0$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then the following conditions are equivalent.*

(i) *$M_u C_\varphi : L^\infty \to L^\infty$ is compact.*
(ii) *$M_u C_\varphi : H^\infty \to L^\infty$ is compact.*
(iii) *$M_u C_\varphi : A \to L^\infty$ is compact.*
(iv) *$\|u\chi_{\{|\varphi^*|>r\}}\|_\infty \to 0$ as $r \to 1$.*

*Proof.* It is trivial that (i) implies (ii) and (ii) implies (iii). By [**12,** Theorem 3.2], the equivalence of (i) and (iv) holds. To show the implication (iii) ⇒ (iv), suppose that $\|u\chi_{\{|\varphi^*|>r\}}\|_\infty > \delta_1 > 0$ for every $r$ with $0 < r < 1$.

First, assume that $\|u\chi_{\{|\varphi^*|=1\}}\|_\infty = 0$. We have $(M_u C_\varphi)z^n = u(\varphi^*)^n \to 0$ almost everywhere on $\partial\mathbb{D}$ as $n \to \infty$. By (iii), $\|u(\varphi^*)^n\|_\infty \to 0$. Hence, there is a positive integer $n$ such that $\|u(\varphi^*)^n\|_\infty < \delta_1/2$. Take $1/2 < R < 1$. We have

$$\begin{aligned} \delta_1/2 > \|u(\varphi^*)^n\|_\infty &\geq R\|u\chi_{\{|\varphi^*|^n>R\}}\|_\infty \\ &= R\|u\chi_{\{|\varphi^*|>\sqrt[n]{R}\}}\|_\infty > R\delta_1. \end{aligned}$$

This is a contradiction.

Next, assume that $\|u\chi_{\{|\varphi^*|=1\}}\|_\infty > \delta_2 > 0$. Then $m(\{|\varphi^*| = 1\}) > 0$, so $\widehat{m}(E(\varphi)) > 0$. Since $\varphi \in H^\infty$, $\widehat{\varphi}(E(\varphi))$ is an uncountable set. Hence, there is a sequence $\{x_n\}_n$ in $E(\varphi)$ such that $\widehat{\varphi}(x_n) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$, $\widehat{\varphi}(x_n) \neq \alpha$ and $|\widehat{u}(x_n)| > \delta_2$ for every $n \geq 1$. Since $|\widehat{\varphi}(x_n)| = 1$ and $\widehat{\varphi}(x_n)$ is a peak point for $A(\overline{\mathbb{D}})$, there is a function $f_n \in A(\overline{\mathbb{D}})$ such that $\|f_n\|_\infty = 1$, $f_n(\widehat{\varphi}(x_n)) = 1$ and $|f_n| < |\widehat{\varphi}(x_n) - \alpha|$ on the set

$$\{e^{i\theta} \in \partial\mathbb{D} : |e^{i\theta} - \widehat{\varphi}(x_n)| \geq |\widehat{\varphi}(x_n) - \alpha|\}.$$

Then $f_n(z) \to 0$ as $n \to \infty$ for every $z \in \overline{\mathbb{D}}$. Hence, $f_n \to 0$ weakly in $A(\overline{\mathbb{D}})$. By (iii), $\|M_u C_\varphi f_n\|_\infty \to 0$. We have

$$\begin{aligned} |\widehat{M_u C_\varphi f_n}(x_n)| &= |\widehat{u}(x_n)\widehat{f_n \circ \varphi}(x_n)| \\ &= |\widehat{u}(x_n) f_n(\widehat{\varphi}(x_n))| \quad \text{by Lemma 2.6} \\ &= |\widehat{u}(x_n)| > \delta_2. \end{aligned}$$

This shows that $\|M_u C_\varphi f_n\|_\infty > \delta_2$ for every $n \geq 1$. This is a contradiction. □

By Lemma 2.11, as sets we have

$$\mathcal{C}_{w,0}(L^\infty, L^\infty) = \mathcal{C}_{w,0}(H^\infty, L^\infty) = \mathcal{C}_{w,0}(A, L^\infty).$$

Let $\Lambda$ be the set of $\varphi \in \mathcal{S}(\mathbb{D})$ satisfying

$$0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r\})$$

for $0 < r < 1$ sufficiently close to 1. Then $\varphi \in \Lambda$ if and only if $E(\varphi) = E^o(\varphi) \neq \emptyset$. The following is proved in [**13,** Theorem 3.11].

**Lemma 2.12.**

(i) *If $\varphi \in \Lambda$, then $\{M_u C_\varphi \in \mathcal{C}_{w,0}(L^\infty, L^\infty) : u \in L^\infty\}$ is open and closed, and a path connected component in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$.*

(ii) *The set*

$$\begin{aligned} &\{M_u C_\varphi \in \mathcal{C}_{w,0}(L^\infty, L^\infty) : u \in L^\infty, \varphi \in \mathcal{S}(\mathbb{D}), \\ &\qquad m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\}) \quad \textit{for any } r, 0 < r < 1\} \end{aligned}$$

*is open and closed, and a path connected component in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$.*

Now we shall study the topological structures of path connected components in $\mathcal{C}_{w,0}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0}(A, L^\infty)$.

**Theorem 2.13.** *The topological structures of path connected components in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$, $\mathcal{C}_{w,0}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0}(A, L^\infty)$ are the same.*

*Proof.* As the proof of Theorem 2.10, path connected components in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$ are path connected sets in $\mathcal{C}_{w,0}(H^\infty, L^\infty)$, and the topological structures of path connected components in $\mathcal{C}_{w,0}(H^\infty, L^\infty)$ and $\mathcal{C}_{w,0}(A, L^\infty)$ are the same. In Lemma 2.12, path connected components in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$ are given. We shall show that each path connected component in $\mathcal{C}_{w,0}(L^\infty, L^\infty)$ is open and closed in $\mathcal{C}_{w,0}(A, L^\infty)$.

Let $\varphi \in \Lambda$. Then $E(\varphi) = E^o(\varphi)$. Let $M_u C_\varphi \in \mathcal{C}_{w,0}(A, L^\infty)$. By Lemma 2.11, $\|u\chi_{\{|\varphi^*|=1\}}\|_\infty > 0$. For $M_v C_\psi \in \mathcal{C}_{w,0}(A, L^\infty)$ with $\psi \neq \varphi$, by Lemma 2.7, we have

$$\begin{aligned}\|M_u C_\varphi - M_v C_\psi\|_{(A,L^\infty)} &\geq \max_{x \in E(\varphi)} |\widehat{u}(x)| \\ &= \|u\chi_{\{|\varphi^*|=1\}}\|_\infty > 0.\end{aligned}$$

This shows that $\{M_u C_\varphi \in \mathcal{C}_{w,0}(A, L^\infty) : u \in L^\infty\}$ is open in $\mathcal{C}_{w,0}(A, L^\infty)$. By Lemma 2.8, $\{M_u C_\varphi \in \mathcal{C}_{w,0}(A, L^\infty) : u \in L^\infty\}$ is closed in $\mathcal{C}_{w,0}(A, L^\infty)$.

Let

$$X = \{M_u C_\varphi \in \mathcal{C}_{w,0}(A, L^\infty) : u \in L^\infty, \varphi \in \Lambda\}.$$

To prove the rest of the assertion, it is sufficient to show that $X$ is closed in $\mathcal{C}_{w,0}(A, L^\infty)$. Suppose that $\{M_{u_n} C_{\varphi_n}\}_n$ is a sequence in $X$ and $M_{u_n} C_{\varphi_n} \to M_v C_\psi \in \mathcal{C}_{w,0}(A, L^\infty) \setminus X$. We shall show a contradiction. Since $M_{u_n} C_{\varphi_n} \in X$, $E(\varphi_n) = E^o(\varphi_n) \neq \emptyset$. Since $\varphi_n \neq \psi$ for every $n \geq 1$, by Lemma 2.7, we have

$$\max_{x \in E(\varphi_n)} |\widehat{u}_n(x)| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{2.1}$$

We also have

$$\|u_n - v\|_\infty \longrightarrow 0 \quad \text{as } n \to \infty, \tag{2.2}$$

so $\|M_{u_n} C_{\varphi_n} - M_v C_{\varphi_n}\|_{(A,L^\infty)} \to 0$. Hence,

$$\|M_v(C_{\varphi_n} - C_\psi)\|_{(A,L^\infty)} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{2.3}$$

Since $M_v C_\psi : A \to L^\infty$ is not compact, by Lemma 2.11, there is a positive number $\delta$ such that $\|v\chi_{\{|\psi^*|>r\}}\|_\infty > \delta$ for every $r$ with $0 < r < 1$. This is equivalent to

$$\sup_{x\in\{|\widehat{\psi}|>r\}} |\widehat{v}(x)| > \delta \quad (0 < r < 1).$$

By (2.1) and (2.2), we may assume that

$$|\widehat{v}| < \delta/2 \quad \text{on} \quad E(\varphi_n) \quad (n \ge 1). \tag{2.4}$$

By (2.3) and Lemma 2.7, we have $\widehat{v} = 0$ on $E^o(\psi)$. Since $M_v C_\psi$ does not belong to $X$, $\psi$ does not belong to $\Lambda$. Hence, we have $\widehat{m}(\{r < |\widehat{\psi}| < 1\}) > 0$ for every $0 < r < 1$ and

$$\sup_{x\in\{r<|\widehat{\psi}|<1\}} |\widehat{v}(x)| > \delta \quad (0 < r < 1).$$

Therefore, there is a sequence $\{x_k\}_k$ in $\{0 < |\widehat{\psi}| < 1\}$ such that

$$|\widehat{\psi}(x_k)| \longrightarrow 1 \quad \text{as } k \to \infty \tag{2.5}$$

and $|\widehat{v}(x_k)| > \delta$ for every $k \ge 1$. By (2.4), we have $|\widehat{\varphi}_n(x_k)| < 1$ for every $n, k \ge 1$. Since $\varphi_n \in \Lambda$,

$$\sigma_n := \sup_{k\ge1} |\widehat{\varphi}_n(x_k)| < 1 \quad (n \ge 1). \tag{2.6}$$

Then, for each fixed $n$, we have

$$\begin{aligned}
\|M_v(C_{\varphi_n} - C_\psi)\|_{(A,L^\infty)} &= \sup_{g\in \text{ball}\,(A)} \|v((g\circ\varphi_n)^* - (g\circ\psi)^*)\|_\infty \\
&\ge \sup_{g\in \text{ball}\,(A)} |\widehat{v}(x_k)(g(\widehat{\varphi}_n(x_k)) - g(\widehat{\psi}(x_k)))| \\
&\ge \delta \sup_{g\in \text{ball}\,(A)} |g(\widehat{\varphi}_n(x_k)) - g(\widehat{\psi}(x_k))| \\
&\longrightarrow 2\delta \quad \text{as } k \to \infty \text{ by (2.5) and (2.6).}
\end{aligned}$$

This contradicts with (2.3). Thus, $X$ is open and closed in $\mathcal{C}_{w,0}(A, L^\infty)$. □

**3. The essential operator norm topology.** To study the topological properties of path connected components in the essential operator norm topology, we need the following lemma.

**Lemma 3.1.** *For $M_uC_\varphi$, $M_vC_\psi \in \mathcal{C}_{w,0}(A, L^\infty)$ with $\varphi \neq \psi$, we have*

$$\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} \geq \max_{x\in E^o(\varphi)} |\widehat{u}(x)|.$$

*Proof.* We may assume that $\widehat{m}(E^o(\varphi)) > 0$. Since $\widehat{m}(\{\widehat{\varphi} = \lambda\}) = 0$ for every $\lambda \in \partial\mathbb{D}$, there is a sequence $\{x_n\}_n$ in $E^o(\varphi)$ such that $\widehat{\varphi}(x_n) \to \alpha \in \partial\mathbb{D}$ and $\widehat{\varphi}(x_n) \neq \alpha$ for every $n \geq 1$. Since $\varphi \neq \psi$, we may assume that $\widehat{\varphi}(x_n) \neq \widehat{\psi}(x_n)$ for every $n \geq 1$. Moreover, we may assume that

$$|\widehat{u}(x_n)| \longrightarrow \max_{x\in E^o(\varphi)} |\widehat{u}(x)|.$$

Since $\widehat{\varphi}(x_n)$ is a peak point for $A(\overline{\mathbb{D}})$, there is a sequence $\{g_n\}_n$ in $A(\overline{\mathbb{D}})$ such that $\|g_n\|_\infty = 1$, $g_n(\widehat{\varphi}(x_n)) = 1$, $g_n(\widehat{\psi}(x_n)) = 0$ and

$$|g_n(e^{i\theta})| \leq |\widehat{\varphi}(x_n) - \alpha|$$

for any $e^{i\theta} \in \partial\mathbb{D}$ with $|e^{i\theta} - \widehat{\varphi}(x_n)| \geq |\widehat{\varphi}(x_n) - \alpha|$ and $n \geq 1$. Then $g_n \to 0$ weakly in $A(\overline{\mathbb{D}})$.

Let $\varepsilon > 0$. Then there is a compact operator $K : A \to L^\infty$ such that

$$\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} + \varepsilon \geq \|M_uC_\varphi - M_vC_\psi - K\|_{(A,L^\infty)}.$$

It is well known that $\|Kg_n\|_\infty \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}
&\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} + \varepsilon \\
&\geq \limsup_{n\to\infty} \|M_uC_\varphi g_n - M_vC_\psi g_n\|_\infty \\
&\geq \limsup_{n\to\infty} |\widehat{u}(x_n)g_n(\widehat{\varphi}(x_n)) - \widehat{v}(x_n)g_n(\widehat{\psi}(x_n))| \\
&= \limsup_{n\to\infty} |\widehat{u}(x_n)| \\
&= \max_{x\in E^o(\varphi)} |\widehat{u}(x)|.
\end{aligned}$$

Thus, we get the assertion. □

The following is given in [**14,** Theorem 3.11].

**Lemma 3.2.**

(i) *Let $\varphi \in \Lambda$. Then $\{M_uC_\varphi \in \mathcal{C}_{w,0}(L^\infty, L^\infty) : u \in L^\infty(\partial\mathbb{D})\}$ is open and closed, and a path connected component in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$.*

(ii) *The set* $\{M_uC_\varphi \in \mathcal{C}_{w,0}(L^\infty, L^\infty) : u \in L^\infty, \varphi \in \Lambda\}$ *is open and closed in* $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$.

(iii) *The set*

$$\big\{M_uC_\varphi \in \mathcal{C}_{w,0}(L^\infty, L^\infty) : u \in L^\infty, \varphi \in \mathcal{S}(D),\\ m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\}) \quad \text{for any } r, 0 < r < 1\big\}$$

*is open and closed, and a path connected component in* $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$.

**Theorem 3.3.** *The topological structures of path connected components in* $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$, $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$ *and* $\mathcal{C}_{w,0,e}(A, L^\infty)$ *are the same.*

*Proof.* As mentioned in the introduction, we have

$$\begin{aligned}\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} &\le \|M_uC_\varphi - M_vC_\psi\|_{(H^\infty,L^\infty,e)}\\ &\le \|M_uC_\varphi - M_vC_\psi\|_{(L^\infty,L^\infty,e)}.\end{aligned}$$

Hence, path connected components in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$ are path connected sets in $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$, and also path connected components in $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$ are path connected sets in $\mathcal{C}_{w,0,e}(A, L^\infty)$. In Lemma 3.2, path connected components in $\mathcal{C}_{w,0,e}(L^\infty, L^\infty)$ are given. We shall show that each path connected component in $\mathcal{C}_{w,0,e}(H^\infty, L^\infty)$ is open and closed in $\mathcal{C}_{w,0,e}(A, L^\infty)$.

Let $\varphi \in \Lambda$. By Lemmas 2.11 and 3.1, for $\varphi \ne \psi$, we have

$$\begin{aligned}\|M_uC_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} &\ge \sup_{x\in E(\varphi)} |\widehat{u}(x)|\\ &= \|u\chi_{\{|\varphi^*|=1\}}\|_\infty > 0.\end{aligned}$$

This shows that $\{M_uC_\varphi \in \mathcal{C}_{w,0}(A, L^\infty) : u \in L^\infty\}$ is open in $\mathcal{C}_{w,0,e}(A, L^\infty)$. To prove the closedness, let $\{M_{u_n}C_\varphi\}_n$ be a sequence in $\mathcal{C}_{w,0}(A, L^\infty)$ such that $\|M_{u_n}C_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} \to 0$ for some $M_vC_\psi \in \mathcal{C}_{w,0,e}(A, L^\infty)$. Suppose that $\psi \ne \varphi$. By Lemma 3.1, $\|u_n\chi_{\{|\varphi^*|=1\}}\|_\infty \to 0$. Let

$$p_n(e^{i\theta}) = \begin{cases} 0, & e^{i\theta} \in \{|\varphi^*| = 1\}\\ u_n(e^{i\theta}), & e^{i\theta} \notin \{|\varphi^*| = 1\}.\end{cases}$$

Then $p_n \in L^\infty$. Since $\varphi \in \Lambda$, by Lemma 2.11 $M_{p_n}C_\varphi : A \to L^\infty$ is compact, so

$$\|M_{(u_n-p_n)}C_\varphi - M_vC_\psi\|_{(A,L^\infty,e)} \longrightarrow 0.$$

Since $\|u_n - p_n\|_\infty \to 0$, we have $\|M_{(u_n-p_n)}C_\varphi\|_{(A,L^\infty,e)} \to 0$. Then $\|M_vC_\psi\|_{(A,L^\infty,e)} = 0$. Hence, $M_vC_\psi : A \to L^\infty$ is compact, and this contradicts the fact that $M_vC_\psi \in \mathcal{C}_{w,0}(A,L^\infty)$. Thus, $\{M_uC_\varphi \in \mathcal{C}_{w,0}(A,L^\infty) : u \in L^\infty\}$ is closed in $\mathcal{C}_{w,0,e}(A,L^\infty)$.

Let $X = \{M_uC_\varphi \in \mathcal{C}_{w,0}(A,L^\infty) : u \in L^\infty, \varphi \in \Lambda\}$. To prove the assertion, it is sufficient to show that $X$ is closed in $\mathcal{C}_{w,0,e}(A,L^\infty)$. Let $\{M_{u_n}C_{\varphi_n}\}_n$ be a sequence in $X$ such that $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,L^\infty,e)} \to 0$ for some $M_vC_\psi \in \mathcal{C}_{w,0}(A,L^\infty)$. Suppose that $M_vC_\psi \notin X$. Then $\psi \neq \varphi_n$ for every $n \geq 1$. By Lemma 3.1, we have $\|u_n\chi_{\{|\varphi_n^*|=1\}}\|_\infty \to 0$. Let

$$q_n(e^{i\theta}) = \begin{cases} 0, & e^{i\theta} \in \{|\varphi_n^*| = 1\} \\ u_n(e^{i\theta}), & e^{i\theta} \notin \{|\varphi_n^*| = 1\}. \end{cases}$$

Then $q_n \in L^\infty$. Since $\varphi_n \in \Lambda$, by Lemma 2.11, $M_{q_n}C_{\varphi_n} : A \to L^\infty$ is compact. Hence, we have

$$\begin{aligned} \|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,L^\infty,e)} &\geq \|M_vC_\psi\|_{(A,L^\infty,e)} - \|M_{(u_n-q_n)}C_{\varphi_n}\|_{(A,L^\infty,e)} \\ &\geq \|M_vC_\psi\|_{(A,L^\infty,e)} - \|u_n - q_n\|_\infty \\ &= \|M_vC_\psi\|_{(A,L^\infty,e)} - \|u_n\chi_{\{|\varphi_n^*|=1\}}\|_\infty. \end{aligned}$$

Letting $n \to \infty$, we have $\|M_vC_\psi\|_{(A,L^\infty,e)} = 0$, but this contradicts the fact that $M_vC_\psi \in \mathcal{C}_{w,0}(A,L^\infty)$. Thus, $X$ is closed in $\mathcal{C}_{w,0,e}(A,L^\infty)$. This completes the proof. □

**4. Spaces of analytic functions.** By Lemma 2.11, we have the following.

**Lemma 4.1.** *Let $u \in H^\infty$ with $u \neq 0$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then the following conditions are equivalent.*

(i) $M_uC_\varphi : H^\infty \to H^\infty$ *is compact.*
(ii) $M_uC_\varphi : A \to H^\infty$ *is compact.*
(iii) $\|u\chi_{\{|\varphi^*|>r\}}\|_\infty \to 0$ *as* $r \to 1$.
(iv) $\max_{x\in\{|\widehat{\varphi}|>r\}} |\widehat{u}(x)| \to 0$ *as* $r \to 1$.

By this lemma, we have $C_{w,0}(H^\infty, H^\infty) = C_{w,0}(A, H^\infty)$ as sets. By Lemma 2.1, we have

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,H^\infty)} = \|M_u C_\varphi - M_v C_\psi\|_{(H^\infty,H^\infty)}.$$

Hence, the topological structures of path connected components in $\mathcal{C}_{w,0}(H^\infty, H^\infty)$ and $\mathcal{C}_{w,0}(A, H^\infty)$ are the same. In the same way as the proof of Lemma 3.1, we have the following.

**Lemma 4.2.** *For $M_u C_\varphi$, $M_v C_\psi \in \mathcal{C}_{w,0}(A, H^\infty)$ with $\varphi \neq \psi$, we have*

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,H^\infty,e)} \geq \max_{x \in E^o(\varphi)} |\widehat{u}(x)|.$$

Recall that $\Lambda$ is the set of $\varphi \in \mathcal{S}(\mathbb{D})$ satisfying

$$0 < m(\{|\varphi^*| = 1\}) = m(\{|\varphi^*| > r\})$$

for $0 < r < 1$ sufficiently close to 1. In **[14,** Theorem 4.9], the authors proved the following.

**Lemma 4.3.**

(i) *Let $\varphi \in \Lambda$. Then $\{M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty, H^\infty) : u \in H^\infty\}$ is open and closed, and a path connected component in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$.*
(ii) *The set $\{M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty, H^\infty) : u \in H^\infty, \varphi \in \Lambda\}$ is open and closed in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$.*
(iii) *The set*

$$\big\{M_u C_\varphi \in \mathcal{C}_{w,0}(H^\infty, H^\infty) : u \in H^\infty, \varphi \in \mathcal{S}(\mathbb{D}),$$
$$m(\{|\varphi^*| = 1\}) < m(\{|\varphi^*| > r\}) \quad \text{for any } r, 0 < r < 1\big\}$$

*is open and closed, and a path connected component in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$.*

**Theorem 4.4.** *The topological structures of path connected components in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ and $\mathcal{C}_{w,0,e}(A, H^\infty)$ are the same.*

*Proof.* Since

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,H^\infty,e)} \leq \|M_u C_\varphi - M_v C_\psi\|_{(H^\infty,H^\infty,e)},$$

path connected components in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ are path connected sets in $\mathcal{C}_{w,0,e}(A, H^\infty)$. In Lemma 4.3, path connected components in $\mathcal{C}_{w,0,e}(H^\infty, H^\infty)$ are given.

Let $\varphi \in \Lambda$. We shall show that $\{M_u C_\varphi \in \mathcal{C}_{w,0}(A, H^\infty) : u \in H^\infty\}$ is open and closed in $\mathcal{C}_{w,0,e}(A, H^\infty)$. We have $\widehat{m}(E^o(\varphi)) > 0$. By Lemma 4.2, for $\varphi \neq \psi$ we have

$$\|M_u C_\varphi - M_v C_\psi\|_{(A,H^\infty,e)} \geq \max_{x\in E^o(\varphi)} |\widehat{u}(x)| > 0.$$

This shows that $\{M_u C_\varphi \in \mathcal{C}_{w,0}(A, H^\infty) : u \in H^\infty\}$ is open in $\mathcal{C}_{w,0,e}(A, H^\infty)$.

To prove the closedness, let $\{M_{u_n} C_\varphi\}_n$ be a sequence in $\mathcal{C}_{w,0}(A, H^\infty)$ such that $\|M_{u_n} C_\varphi - M_v C_\psi\|_{(A,H^\infty,e)} \to 0$ for some $M_v C_\psi \in \mathcal{C}_{w,0}(A, H^\infty)$. To show $\psi = \varphi$, suppose that $\psi \neq \varphi$. By Lemma 4.2,

$$\max_{x\in E^o(\psi)} |\widehat{v}(x)| = 0.$$

Since $v \in H^\infty$ and $v \neq 0$, this shows that $E^o(\psi) = \emptyset$. Since $M_v C_\psi \in \mathcal{C}_{w,0}(A, H^\infty)$, we have

$$\widehat{m}(\{r < |\widehat{\psi}| < 1\}) = \widehat{m}(\{r < |\widehat{\psi}| \leq 1\}) \neq 0$$

for every $r$ with $0 < r < 1$. By Lemma 4.1, there is a positive constant $\delta$ such that

$$\delta < \sup_{x\in\{r<|\widehat{\psi}|<1\}} |\widehat{v}(x)|$$

for every $r$ with $0 < r < 1$. Then there is a sequence $\{x_k\}_k$ in $M(L^\infty)$ such that $0 < |\widehat{\psi}(x_k)| < 1$ and $|\widehat{v}(x_k)| > \delta$ for every $k \geq 1$, and $|\widehat{\psi}(x_k)| \to 1$ as $k \to \infty$. We may assume that $\widehat{\psi}(x_k) \to \alpha \in \partial\mathbb{D}$. One may show that there is a sequence $\{g_k\}_k$ in ball$\,(A)$ such that $g_k \to 0$ weakly in $A(\overline{\mathbb{D}})$ and $g_k(\widehat{\psi}(x_k)) \to 1$ as $k \to \infty$ (see [**14,** Lemma 4.8]). Since $\varphi \in \Lambda$, there exists a constant $R$, $0 < R < 1$, such that $0 < \widehat{m}(\{|\widehat{\varphi}| = 1\}) = m(\{|\widehat{\varphi}| > R\})$. Hence, we may assume that either $|\widehat{\varphi}(x_k)| = 1$ for every $k \geq 1$ or $|\widehat{\varphi}(x_k)| \leq R$ for every $k \geq 1$. For each $n$, we have

$$\|M_{u_n} C_\varphi - M_v C_\psi\|_{(A,H^\infty,e)} \geq \limsup_{k\to\infty} \left\|u_n(g_k \circ \varphi)^* - v(g_k \circ \psi)^*\right\|_\infty.$$

First, we assume that $|\widehat{\varphi}(x_k)| = 1$ for every $k \geq 1$. Then we have

$$\begin{aligned}
&\|M_{u_n}C_\varphi - M_vC_\psi\|_{(A,H^\infty,e)} \\
&\qquad \geq \limsup_{k\to\infty} \left|\widehat{u}_n(x_k)g_k(\widehat{\varphi}(x_k)) - \widehat{v}(x_k)g_k(\widehat{\psi}(x_k))\right| \\
&\qquad \geq \limsup_{k\to\infty} |\widehat{v}(x_k)g_k(\widehat{\psi}(x_k))| - |\widehat{u}_n(x_k)g_k(\widehat{\varphi}(x_k))| \\
&\qquad \geq \delta - \sup_{x\in E(\varphi)} |\widehat{u}_n(x)|.
\end{aligned}$$

Since $\varphi \in \Lambda$, $E^o(\varphi) = E(\varphi)$. By Lemma 4.2, we have

$$\sup_{x\in E(\varphi)} |\widehat{u}_n(x)| \to 0 \quad \text{as } n \to \infty.$$

Therefore, we get

$$0 = \lim_{n\to\infty} \|M_{u_n}C_\varphi - M_vC_\psi\|_{(A,H^\infty,e)} \geq \delta.$$

This is a contradiction.

Next, we assume that $|\widehat{\varphi}(x_k)| \leq R$ for every $k \geq 1$. We also have

$$\|M_{u_n}C_\varphi - M_vC_\psi\|_{(A,H^\infty,e)} \geq \delta - \|u_n\|_\infty \sup_{|z|\leq R} |g_k(z)|.$$

Since $g_k \to 0$ weakly in $A(\overline{\mathbb{D}})$, letting $k \to \infty$, we have

$$\|M_{u_n}C_\varphi - M_vC_\psi\|_{(A,H^\infty,e)} \geq \delta.$$

This also leads to a contradiction. Thus, we get $\psi = \varphi$. Therefore, $\{M_uC_\varphi \in \mathcal{C}_{w,0}(A,H^\infty) : u \in H^\infty\}$ is open and closed in $\mathcal{C}_{w,0,e}(A,H^\infty)$.

Let

$$X = \left\{M_uC_\varphi \in \mathcal{C}_{w,0}(A,H^\infty) : u \in H^\infty, \varphi \in \Lambda\right\}.$$

We shall prove that $X$ is open and closed in $\mathcal{C}_{w,0,e}(A,H^\infty)$. We have already proved that $X$ is open. We shall show that $X$ is closed in $\mathcal{C}_{w,0,e}(A,H^\infty)$. Let $\{M_{u_n}C_{\varphi_n}\}_n$ be a sequence in $X$ such that $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} \to 0$ as $n \to \infty$ for some $M_vC_\psi \in \mathcal{C}_{w,0}(A,H^\infty)$. We assume that $M_vC_\psi \notin X$. Hence, $\varphi_n \neq \psi$ for every $n \geq 1$. By Lemma 4.2, we have

$$\max_{x\in E(\varphi_n)} |\widehat{u}_n(x)| \longrightarrow 0 \quad \text{as } n \to \infty \tag{4.1}$$

and

$$\max_{x\in E^o(\psi)} |\widehat{v}(x)| = 0. \tag{4.2}$$

Since $\psi \notin \Lambda$, $\{r < |\widehat{\psi}| < 1\} \neq \emptyset$ for every $r$ with $0 < r < 1$. Since $M_vC_\psi \in \mathcal{C}_{w,0}(A, H^\infty)$, by (4.2) and Lemma 4.1 there is a positive constant $\delta$ such that

$$\delta < \sup_{x\in\{r<|\widehat{\psi}|<1\}} |\widehat{v}(x)|$$

for every $r$ with $0 < r < 1$. Then there is a sequence $\{x_k\}_k$ in $M(L^\infty)$ such that $0 < |\widehat{\psi}(x_k)| < 1$, $|\widehat{v}(x_k)| > \delta$ for every $k \geq 1$ and $|\widehat{\psi}(x_k)| \to 1$ as $k \to \infty$. We may assume that $\widehat{\psi}(x_k) \to \alpha \in \partial\mathbb{D}$. One may take a sequence $\{g_k\}_k$ in ball$(A)$ such that $g_k \to 0$ weakly in $A(\overline{\mathbb{D}})$ and $g_k(\widehat{\psi}(x_k)) \to 1$ as $k \to \infty$ (see **[14,** Lemma 4.8]).

For each fixed positive integer $n$, since $\varphi_n \in \Lambda$ there exists a constant $R_n$, $0 < R_n < 1$, such that $0 < \widehat{m}(\{|\widehat{\varphi}_n| = 1\}) = \widehat{m}(\{|\widehat{\varphi}_n| > R_n\})$. Hence, there is a subsequence $\{x_{k_{n,j}}\}_j$ of $\{x_k\}_k$ satisfying that either $|\widehat{\varphi}_n(x_{k_{n,j}})| = 1$ for every $j \geq 1$ or $|\widehat{\varphi}_n(x_{k_{n,j}})| \leq R_n$ for every $j \geq 1$. We have

$$\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} \geq \limsup_{j\to\infty} \left\|u_n(g_{k_{n,j}}\circ\varphi_n)^* - v(g_{k_{n,j}}\circ\psi)^*\right\|_\infty.$$

First, we assume that $|\widehat{\varphi}_n(x_{k_{n,j}})| = 1$ for every $j \geq 1$. Then we have

$$\begin{aligned}\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} &\\ \geq \limsup_{j\to\infty} |\widehat{v}(x_{k_{n,j}})g_{k_{n,j}}(\widehat{\psi}(x_{k_{n,j}}))| - |\widehat{u}_n(x_{k_{n,j}})g_{k_{n,j}}(\widehat{\varphi}_n(x_{k_{n,j}}))|.\end{aligned}$$

Since $|\widehat{v}(x_{k_{n,j}})| \geq \delta$, by (4.1) we have

$$\lim_{n\to\infty} \|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} \geq \lim_{n\to\infty} \Big(\delta - \sup_{x\in E(\varphi_n)} |\widehat{u}_n(x)|\Big) = \delta.$$

This contradicts the fact that $\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} \to 0$.

Next, assume that $|\widehat{\varphi}_n(x_{k_{n,j}})| \leq R_n$ for every $j \geq 1$. We also have

$$\|M_{u_n}C_{\varphi_n} - M_vC_\psi\|_{(A,H^\infty,e)} \geq \limsup_{j\to\infty} \Big(\delta - \|u_n\|_\infty \sup_{|z|\leq R_n} |g_{k_{n,j}}(z)|\Big).$$

Since $g_{k_{n,j}} \to 0$ weakly in $A(\overline{\mathbb{D}})$ as $j \to \infty$, we have

$$\|M_{u_n} C_{\varphi_n} - M_v C_\psi\|_{(A,H^\infty,e)} \geq \delta.$$

This contradicts the fact that $\|M_{u_n} C_{\varphi_n} - M_v C_\psi\|_{(A,H^\infty,e)} \to 0$. Hence, $X$ is closed in $\mathcal{C}_{w,0,e}(A, H^\infty)$. This completes the proof. □

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