ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 3, 2015

ZEROS OF HIGH DERIVATIVES OF THE RIEMANN ZETA FUNCTION

THOMAS BINDER, SEBASTIAN PAULI AND FILIP SAIDAK

ABSTRACT. We describe new zero-free regions for the derivatives $\zeta^{(k)}(s)$ of the Riemann zeta function, which take the form of vertical strips in the right half-plane. We show that the zeros located in the narrow complements of these zero-free regions are simple and exhibit vertical periodicities that enable one to give exact formulas for their number.

1. Introduction. In this paper, we investigate the distribution of zeros of higher derivatives of the Riemann zeta function. In order to put our main results in perspective, we first give a brief summary of some of the most important results and outstanding conjectures in this area.

Let $s = \sigma + it$. For all $k \in \mathbb{N}$, the kth derivative of the Riemann zeta function $\zeta^{(k)}(s)$ is

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}, \quad \text{for } \sigma > 1, \tag{1}$$

and can be extended to a meromorphic function on $\mathbb{C}$, with a single pole (of order k) at the point $s = 1$. However, unlike $\zeta(s)$ itself, the functions $\zeta^{(k)}(s)$ have neither Euler products nor functional equations. Thus, their nontrivial zeros do not lie on a line but appear to be distributed (seemingly at random) to the right of the critical line $\sigma = \frac{1}{2}$. Speiser **[8]** was the first to show, in 1934, that the Riemann hypothesis (RH) is equivalent to the fact that $\zeta'(s)$ has no zeros with $0 < \sigma < \frac{1}{2}$. Levinson and Montgomery **[5]** gave a simpler and more instructive proof of this and also showed that $\zeta'(s)$ can vanish on the critical line only at a multiple zero of $\zeta(s)$, if ever such a zero exists. They also showed, assuming the RH, that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros with $\sigma < \frac{1}{2}$, for $k \geq 1$. For $k = 1$, they

2010 AMS *Mathematics subject classification.* Primary 11M06, 11M26.
Received by the editors on October 29, 2012, and in revised form on March 28, 2013.
DOI:10.1216/RMJ-2015-45-3-903

proved unconditionally that $\zeta'(s)$ has only real zeros in the closed half-plane $\sigma \leq 0$. For $k = 2$ and $k = 3$, Yıldırım [**14**] established, assuming the RH, that $\zeta^{(k)}(s)$ has no zeros with $0 \leq \sigma < \frac{1}{2}$, and, unconditionally, that both $\zeta''(s)$ and $\zeta'''(s)$ have exactly one pair of nontrivial zeros with $\sigma < 0$. Namely, $\zeta''(s)$ has zeros at approximately $s = -0.35508433021 \pm 3.590839324398i$ and $\zeta'''(s)$ at approximately $s = -2.110145792653 \pm 2.58422477204i$.

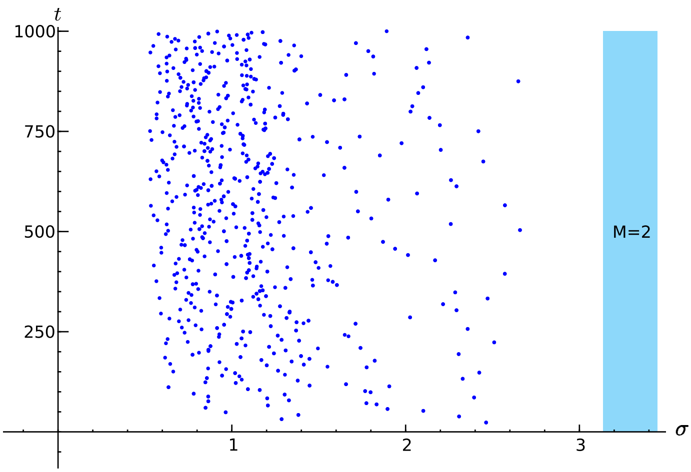

FIGURE 1. Zeros of $\zeta'(s)$ in $\mathbb{C}$, with the zero-free region.

In regions to the right of the critical line, i.e., for $\sigma \geq \frac{1}{2}$, the total number of zeros of $\zeta^{(k)}(s)$ does not differ by much from the number of zeros of $\zeta(s)$. In fact, if we let $N(T)$ and $N_k(T)$ denote the number of such zeros ρ, with $0 \leq \Im(\rho) \leq T$, of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively, then according to a theorem of Berndt [**1**]

$$N_k(T) = N(T) - \frac{T}{2\pi}\log 2 + O_k(\log T), \tag{2}$$

for all $k \geq 1$, where, by the classical Riemann-von Mangoldt formula (see Landau [**4**]),

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

It should also be noted that most nontrivial zeros of $\zeta^{(k)}(s)$ are located relatively close to the line $s = \frac{1}{2} + it$. In fact, in recent years, in a series of improvements, Soundararajan [**7**], Zhang [**15**] and Feng [**2**] succeeded in showing (conditionally) that, for $k = 1$, a positive portion of the zeros ρ of $\zeta'(s)$ satisfies $\Re(\rho) < \frac{1}{2} + c/\log T$. Nevertheless, for all $k \in \mathbb{N}$, many of the zeros of $\zeta^{(k)}(s)$ lie much farther to the right, even though their real parts can still be effectively bounded from above by absolute constants (see Figure 1 for illustration of the bound in the case $k = 1$). For $k \geq 3$, such general upper bounds were first given by Spira [**9**] in 1965, and they were later improved by Verma and Kaur [**13**] (see Table 1):

$$\zeta^{(k)}(\sigma + it) \neq 0 \quad \text{for } \sigma > (1.13588\ldots)k + 2.$$

TABLE 1. Lower real bounds for zero-free regions in the right half-plane.

	ζ	ζ'	ζ''	$\zeta^{(k)}$ for $k \geq 3$
Hadamard [**3**], de la Vallée-Poussin [**12**]	1			
Titchmarsh [**11**]		$E < 3$		
Spira [**9**]				$\frac{7}{4}k + 2$
Verma and Kaur [**13**]				$(1.13588\ldots)k + 2$
Skorokhodov [**6**]		2.93938	4.02853	

In this work, we prove the existence of a sequence of zero-free regions for $\zeta^{(k)}(s)$, between the critical line $\Re(s) = \frac{1}{2}$ and the previously known far-right zero-free region $\Re(s) > (1.13588\ldots)k + 2$ (due to Verma and Kaur [**13**], a bound that happens to be close to best possible). Furthermore, we show that the zeros found in the strips between the new zero free regions are simple and exhibit a vertical periodicity, which also enables us to give exact formulas for their number.

2. Statement of main results. In what follows, we restrict our treatment to the case $k \geq 3$. To state our results precisely, we introduce some notation and definitions. Let

$$Q_n^k(s) := (\log n)^k / n^s$$

denote the nth term of the Dirichlet series (1) for $(-1)^k \zeta^{(k)}(s)$. All the previously known zero-free regions for $\zeta^{(k)}(s)$ have been obtained by

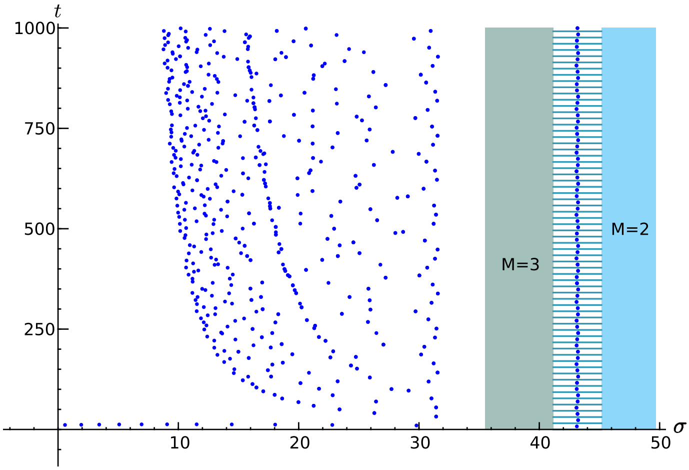

FIGURE 2. Zeros of $\zeta^{(38)}(s)$ in $\mathbb{C}$, with zero-free regions (characterized by the dominance of $Q_M^{38}(s)$ for $M = 2$ and 3).

finding solutions to

$$\left|\zeta^{(k)}(s)\right| = \left|\sum_{n=2}^{\infty} Q_n^k(s)\right| \geq Q_2^k(\sigma) - \sum_{n=3}^{\infty} Q_n^k(\sigma) > 0,$$

or some variation thereof (see **[6, 11, 13]**); that is, by finding the regions of the complex plane where the term $Q_2^k(s)$ dominates all the other terms of the expansion (1) of $\zeta^{(k)}(s)$ (i.e., $Q_2^k(s)$ is greater in modulus than the rest of the terms combined), because then, evidently, $\zeta^{(k)}(s) \neq 0$. However, $Q_2^k(s)$ is not always the dominant term; any other term can not only be the largest in modulus, but takes the dominant role as well. This is clear from the fact that $|Q_n^k(s)| = Q_n^k(\sigma)$, viewed as a function of n, has its global maximum at $n = e^{k/\sigma}$. Using this simple property, one can show the existence of regions where $Q_n^k(s)$ (for any $n \geq 2$) becomes the dominant term of (1), which then provides us with a new zero-free region of $\zeta^{(k)}(s)$, for each $n \in \mathbb{N}$, for every sufficiently large k.

Let us denote by $Q_M^k(s)$ the term of (1) which has the largest modulus. If we fix some such M, then the moduli of the terms of (1) will increase for $m < M$ and decrease for $m > M$, in monotone fashion (see Section 3). Since no term $Q_M^k(s)$ can attain dominance on a line where its absolute value is equal to that of another term (and by the aforesaid property this can only happen when $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ or $Q_M^k(\sigma) = Q_{M-1}^k(\sigma)$), it is reasonable to expect that the zeros of $\zeta^{(k)}(s)$ will be located close to the lines where this equality occurs. Thus, we define

$$q_M := \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}, \tag{3}$$

so that $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ whenever $\sigma = q_M k$. (Note that $q_2 = 1.13588\ldots$, $q_3 = 0.808484\ldots$, $q_4 = 0.668855\ldots$, where q_2 is the constant that appears in Table 1.) In the $k\sigma$-plane, $\sigma = q_M k$ defines a line of slope q_M.

Our first main result describes zero-free regions between these lines for sufficiently large k:

Theorem 2.1. *Let $k \in \mathbb{N}$ and $u \in \mathbb{R}^{>0}$ be a solution of*

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \geq 0.$$

(a) *If $q_3 k + 4\log 3 < q_2 k - 2$, then $\zeta^{(k)}(s) \neq 0$ for*

$$q_3 k + 4\log 3 \leq \sigma \leq q_2 k - 2.$$

(b) *If $M \in \mathbb{N}$, $M > 3$ and $q_M k + (M+1)u \leq q_{M-1}k - Mu$, then $\zeta^{(k)}(s) \neq 0$ for*

$$q_M k + (M+1)u \leq \sigma \leq q_{M-1}k - Mu.$$

Remark 2.2. We have $u = 1.1879426249\ldots$.

Thus, the Mth zero-free region S_M^k of $\zeta^{(k)}(s)$ is the open set

$$S_2^k := \{\sigma + it \mid q_2 k + 2 < \sigma < q_2 k - 2\},$$
$$S_3^k := \{\sigma + it \mid q_3 k - 4u < \sigma < q_3 k + 4\log 3\},$$
$$S_M^k := \{\sigma + it \mid q_M k - (M+1)u < \sigma < q_M k + (M+1)u\}$$

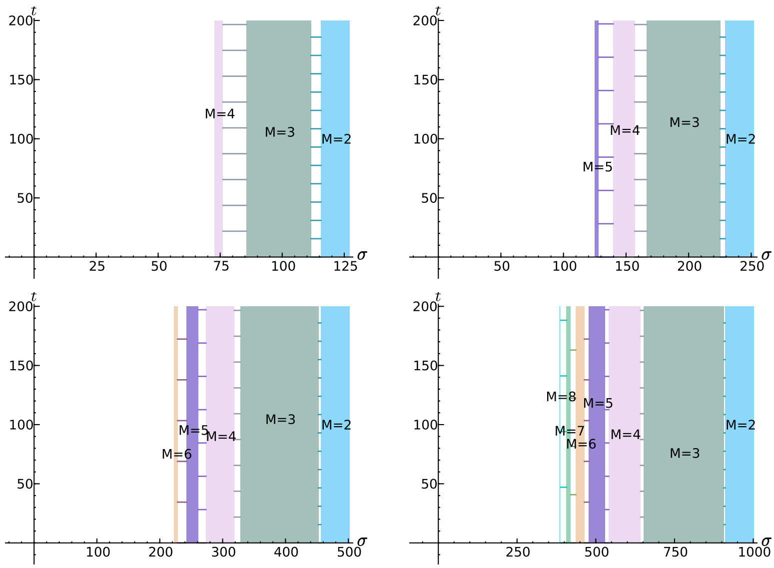

FIGURE 3. Zero-free regions and horizontal zero-free line segments for $\zeta^{(100)}$, $\zeta^{(200)}$, $\zeta^{(400)}$ and $\zeta^{(800)}$.

for $M > 3$. So the zero-free regions are the connected components that remain after one removes S_M^k from the right half-plane.

Another way to visualize the strips S_M^k is to consider them in the $k\sigma$-plane (see Figure 4). In this representation, the wedges correspond to the zero-free regions, i.e., the regions of dominance of the terms $\frac{\log^k M}{M^s}$ (for $M = 2$ this is treated by Verma and Kaur [**13**], for $M \geq 3$ it is new), while the strips S_M^k are the narrow regions centered around the lines that separate the wedges. For $M \geq 3$, the k-coordinates of the tips of the wedges in the $k\sigma$-plane are

$$k_3 = \frac{4\log 3 + 2}{q_2 - q_3} \quad \text{and} \quad k_M = \frac{(2M+1)u}{q_{M-1} - q_M} \quad \text{for } M \geq 4, \tag{4}$$

which immediately implies that the first strips S_2^k can be observed for all $k \geq 20$, the second S_2^k for all $k \geq 77$ and the third S_3^k for all $k \geq 163$, and so on. With some extra work, these values can be improved to $k \geq 19$ for S_2^k, $k \geq 58$ for S_3^k, and $k \geq 123$ for S_4^k (see Remark 4.5).

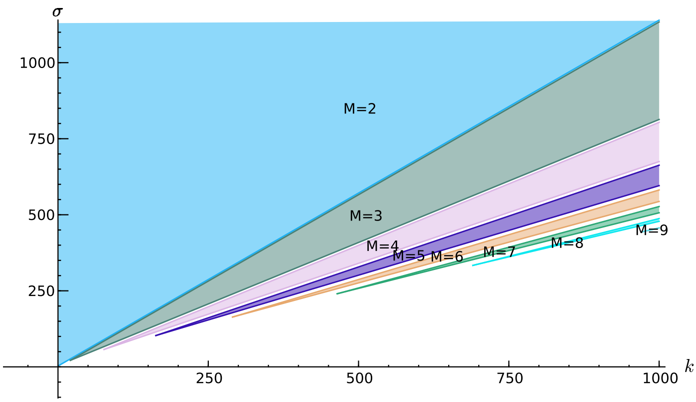

FIGURE 4. Zero-free regions of $\zeta^{(k)}(\sigma+it)$, for $M=2,\dots,9$.

Moreover, if one also considers the imaginary parts of the solutions of $Q_M^k(q_Mk+it)+Q_{M+1}^k(q_Mk+it)=0$, then one obtains

$$t=\frac{\pi(2j+1)}{\log(M+1)-\log(M)} \tag{5}$$

for $j\in\mathbb{Z}$, showing that the location of the zeros ρ inside S_M^k is close to

$$k\cdot q_M+\frac{\pi(2j+1)i}{\log\left(\frac{M+1}{M}\right)} \tag{6}$$

for some $j\in\mathbb{N}$. This suggests a vertical periodicity in the limit of the zeros of $\zeta^{(k)}(s)$. (The computational data confirms that the Mth period equals $\pi/(\log(M+1)-\log(M))$.) With the help of Rouché's theorem, we are able to show that, between every two consecutive lines $s=\sigma+\frac{2\pi ji}{\log(M+1)-\log M}$, which horizontally partition the strip S_M^k (see Figure 3), there is exactly one zero of $\zeta^{(k)}(s)$.

That is our second main result:

Theorem 2.3. *Let $u\in\mathbb{R}^{>0}$ be a solution of*

$$1-\frac{1}{e^u-1}-\frac{1}{e^u}\left(1+\frac{1}{u}\right)\geq 0.$$

Let $M \in \mathbb{N}$, $M > 3$ and $j \in \mathbb{N}$. If there is $k \in \mathbb{N}$ with

$$q_{M+1}k + (M+2)u \leq q_M k - (M+1)u,$$

then each rectangle $R_j \subset S_M^k$, consisting of all $s = \sigma + it$ with

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u$$

and

$$\frac{2\pi j}{\log(M+1) - \log(M)} < t < \frac{2\pi(j+1)}{\log(M+1) - \log(M)},$$

contains exactly one zero of $\zeta^{(k)}(s)$. This zero is simple.

Remark 2.4. The corresponding result also holds for the strips S_2^k and S_3^k.

Clearly, Theorem 2.3 can be converted into an exact formula for the number of zeros of $\zeta^{(k)}(s)$ (for carefully chosen values of T) inside any given strip.

Corollary 2.5. *Let $N_M^k(T)$ denote the number of zeros ρ of $\zeta^{(k)}(s)$ which are inside S_M^k and satisfy $\Im(\rho) \leq T$. Then, for all $j \geq 1$,*

$$N_M^k\left(\frac{2\pi j}{\log(M+1) - \log(M)}\right) = j.$$

Remark 2.6. An immediate consequence is that, for $k \geq 3$ and $T > 0$,

$$N_M^k(T) = \frac{\log(M+1) - \log(M)}{2\pi} T + O(1).$$

This, of course, implies that the total number of zeros contained within any fixed strip is $O(T) = o(N_k(T))$.

Spira **[9]** had already noticed that the zeros of $\zeta'(s)$ and $\zeta''(s)$ seem to come in pairs, where the zero of $\zeta''(s)$ is always located to the right of the zero of $\zeta'(s)$. More recently, with the help of extensive computations, Skorokhodov **[6]** observed this behavior for higher derivatives as well. Our observations support a straightforward one-to-one correspondence between the zeros of $\zeta^{(k)}(s)$ and $\zeta^{(m)}(s)$ for all $k, m \geq 1$ (Figure 5).

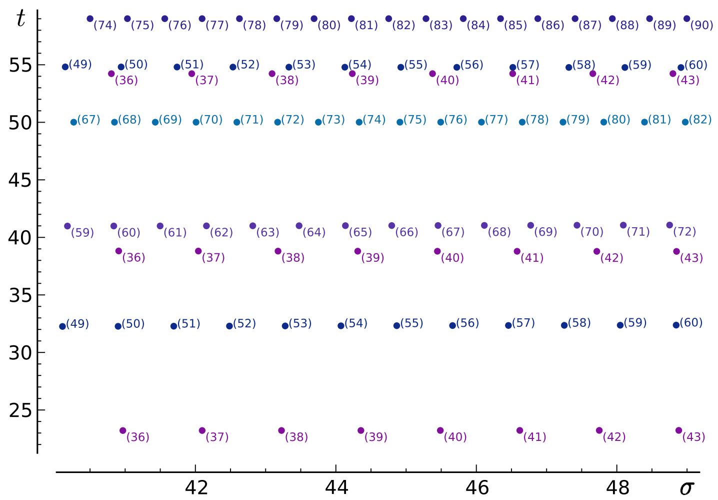

FIGURE 5. The consecutive zeros $\bullet^{(k)}$ of the derivatives of $\zeta^{(k)}(\sigma + it)$ in the region $40 < \sigma < 49$, $20 < t < 60$.

Indeed, for given M and sufficiently large k it follows from Theorem 2.3 that for each zero of $\zeta^{(k)}(s)$ contained in S_M^k there is a corresponding zero of $\zeta^{(k+1)}(s)$ contained in S_M^{k+1}. An approximation of the location of a zero in S_M^k is given by (6).

Remark 2.7. Due to growing density of zeros near the critical line $s = 1/2$, it is difficult to translate this surprising property into quantitative asymptotics with vanishing error terms. However, for a positive integer k, in the right half-plane we observe a certain "dynamical" exactness between the numbers of zeros of different derivatives of the Riemann zeta function; in other words, we find a one-to-one correspondence between the non-trivial zeros of $\zeta^{(k)}(s)$ with $\zeta^{(k+1)}(s)$, for all k, such that the index M for two such corresponding zeros is the same, and their difference is approximately q_M. This correspondence could be established by finding unique, continuous paths which the zeros of fractional derivatives $\zeta^{(k)}(s)$ undergo, as $k \in \mathbb{R}$ runs through the interval $[1, \infty)$.

Remark 2.8. The zero-free regions obtained in Theorem 2.1 may be generalized to a large class of Dirichlet series. Since we only consider the absolute values of the coefficients, it follows that if

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

and $|a_M| \geq |a_n|$ for some $M \geq 3$ and all $n \geq 2$, then $L^{(k)}(s) \neq 0$ for $q_M k + cM \leq \sigma \leq q_{M-1}k - c(M-1)$, for a suitable constant $c \geq 0$. There are technical rather than theoretical obstacles that prevent us from making these general results explicit.

3. An auxiliary lemma. In this section, we prove a technical lemma which will be used in the proof of Theorem 2.1. Let us consider the function $z : \mathbb{R}^{>0} \to \mathbb{R}$,

$$x \longmapsto \frac{\log^k x}{x^\sigma}$$

for fixed $\sigma > 1$ and $k \in \mathbb{N}$. We have

$$z'(x) = \left(\left(\frac{\log x}{x^\sigma}\right)^k\right)' = k\left(\frac{x^{\sigma-1} - \sigma(\log x)x^{\sigma-1}}{x^{2\sigma}}\right)\left(\frac{\log x}{x^\sigma}\right)^{k-1}.$$

Hence, $z'(x) = 0$ if $x^{\sigma-1}(1 - \sigma \log x) = 0$, that is, $x = e^{1/\sigma}$. Since $z'(x) > 0$ for $0 < x < e^{1/\sigma}$ and $z'(x) < 0$ for $x > e^{1/\sigma}$, the function $z(x)$ has its maximum at $x = e^{1/\sigma}$.

As we have chosen q_M such that $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ for $\sigma = q_M k$, the maximum of $z(x)$ (again for $\sigma = q_M k$) lies between $x = M$ and $x = M+1$. As the maximum of $z(x)$ is at $x = e^{1/\sigma}$, the maximum of $z(x)$ for $\sigma > q_M k$ is to the left of the maximum of $z(x)$ for $\sigma = q_M k$. So the value of σ for which $Q_M^k(\sigma)$ is the largest term in the Dirichlet series representation of $\zeta^{(k)}(\sigma)$ is between $\sigma = q_M k$ and $\sigma = q_{M-1}k$. Thus, $Q_M^k(\sigma)$ can dominate $\zeta^{(k)}(\sigma)$ only there.

We will use these monotonicity and dominance considerations implicitly in the proofs of our theorems.

Now, we consider the $k\sigma$-plane interpretation of Theorem 2.1. In general, the wedges in Figure 4 are the sets containing all points (k, σ) that satisfy

$$q_M k + b_1 < \sigma < q_{M-1}k + b_2$$

for some $M \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{R}$. Thus,

$$k \geq \frac{b_1 - b_2}{q_{M-1} - q_M}, \tag{7}$$

with equality holding exactly if $k = k_M$.

The growth properties of q_M play an important role in understanding the strips S_M^k.

Lemma 3.1. *For all $n \geq 3$, we have*

$$\frac{1}{\log n} < q_{n-1} < \frac{1}{\log(n-1)}.$$

Proof. In order to prove the lower bound, we write

$$\begin{aligned}
\alpha_{n-1} &:= \frac{\log(n-1)}{\log n} = 1 + \frac{\log(n-1) - \log n}{\log n} = 1 + \frac{\log(\frac{n-1}{n})}{\log n}, \\
\beta_{n-1} &:= \log(\alpha_{n-1}) \\
&= \log\left(1 + \frac{\log(\frac{n-1}{n})}{\log n}\right) < \frac{\log(\frac{n-1}{n})}{\log n},
\end{aligned}$$

where the last inequality holds because $\log(1+x) < x$ whenever $x > -1$. The desired lower bound now immediately follows from $q_{n-1} = \beta_{n-1}/\log((n-1)/n)$.

In order to prove the upper bound, we write $\theta_n := -\log(\frac{n-1}{n})$. Then we have

$$\begin{aligned}
q_{n-1} &= \frac{\log\left(\frac{\log(n-1)}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} = \frac{\log\left(1 - \frac{-\log\left(\frac{n-1}{n}\right)}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} = \frac{\log\left(1 - \frac{\theta_n}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} \\
&= \frac{1}{\log n} + \frac{\theta_n}{2(\log n)^2} + \frac{\theta_n^2}{3(\log n)^3} + \frac{\theta_n^3}{4(\log n)^4} + \cdots \\
&< \frac{1}{\log n} + \frac{1}{2\log n}\left(\frac{\theta_n}{\log n} + \left(\frac{\theta_n}{\log n}\right)^2 + \left(\frac{\theta_n}{\log n}\right)^3 + \cdots\right) \\
&= \frac{1}{\log n} + \frac{1}{2\log n}\,\frac{\theta_n}{\log n - \theta_n} \\
&= \frac{1}{\log n} + \frac{1}{2\log n}\,\frac{\log(1 + \frac{1}{n-1})}{\log(n-1)}
\end{aligned}$$

$$< \frac{1}{\log n} + \frac{1}{2(\log n)(\log(n-1))(n-1)} < \frac{1}{\log(n-1)},$$

where the last inequality holds if and only if

$$\log n - \log(n-1) > \frac{1}{2(n-1)},$$

which is true by the mean value theorem. □

How many distinct strips of $\zeta^{(k)}(s)$ that contain nontrivial zeros are there inside the region $1/2 \leq \sigma < q_2 k + 2$? Let $c(k)$ denote that number. Then, in view of Lemma 3.1, it seems reasonable to expect that, for all $k \geq 2$, there exist positive constants A and B, such that

$$A \frac{\sqrt{k}}{\log k} < c(k) < B \frac{\sqrt{k}}{\log k}.$$

Upper bounds of the desired order are easier to prove than lower bounds: obviously, one can just count the number of wedges, with their tips located at points described in (4), and then invert the relation. Since the difference $q_{M-1} - q_M$ in the denominator of this fraction can be nicely bounded from above (but not from below), using the estimates in our lemma, effective upper bounds can be obtained.

4. Proof of Theorem 2.1. Now we are ready to prove our first main result. We will show that $\zeta^{(k)}(s)$ has no zeros if (k, σ) in the $k\sigma$-plane lies in one of the wedges given by an inequality of the form

$$q_M k + b_1 \leq \sigma \leq q_{M-1} k + b_2$$

for suitably chosen $b_1, b_2 \in \mathbb{R}$. We choose b_1, b_2 such that these wedges are the regions where $Q_M^k(s) = \frac{\log^k M}{M^s}$ is the dominant term (in the modulus) of $\zeta^{(k)}(s)$. Everywhere hereafter we write $H_M^k(s)$ for the "head" and $T_M^k(s)$ for the "tail" of the series $\zeta^{(k)}(s)$ split by $Q_M^k(s)$:

$$H_M^k(s) := \sum_{n=2}^{M-1} Q_n^k(s) = \sum_{n=2}^{M-1} \frac{\log^k n}{n^s}$$

and

$$T_M^k(s) := \sum_{n=M+1}^{\infty} Q_n^k(s) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^s}.$$

Our goal will be to show that

$$\begin{aligned}|\zeta^{(k)}(s)| &\geq Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma)\\ &= Q_M^k(\sigma)\left(1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{T_M^k}{Q_M^k}(\sigma)\right) > 0\end{aligned}$$

for our choice of b_1 and b_2, keeping in mind that

$$\frac{Q_{M+1}^k}{Q_M^k}(q_M k + b_1) = \left(\frac{M}{M+1}\right)^{b_1}$$

and

$$\frac{Q_{M-1}^k}{Q_M^k}(q_{M-1} k + b_2) = \left(\frac{M}{M-1}\right)^{b_2},$$

as one can easily verify.

The tails. We first find an upper bound for the tails $T_M^k(\sigma)$.

Lemma 4.1. *Fix some integer $M \geq 2$, and assume $k - 1 < (\sigma - 1)\log M$. Then*

$$T_M^k(\sigma) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^\sigma} \leq \int_M^\infty \frac{\log^k x}{x^\sigma}\, dx < Q_M^k(\sigma) R_M^k(\sigma), \tag{8}$$

where

$$R_M^k(\sigma) = \frac{M}{\sigma - 1}\left(1 + \frac{k}{(\sigma - 1)\log M - k + 1}\right).$$

Proof. For $k \in \mathbb{Z}$, the integral in (8) can be written in a closed form. Applying recursively the general formula (for all $b \neq 1$)

$$\int \frac{(\log x)^a}{x^b}\, dx = -\frac{(\log x)^a}{(b-1)x^{b-1}} + \frac{a}{b-1}\int \frac{(\log x)^{a-1}}{x^b}\, dx,$$

we obtain

$$\begin{aligned}\int_M^\infty \frac{\log^k x}{x^\sigma}\, dx &= \frac{\log^k M}{M^\sigma}\frac{M}{\sigma - 1}\sum_{r=0}^{k} \frac{k!}{(k-r)!}\frac{\log^{-r} M}{(\sigma - 1)^r}\\ &\leq Q_M^k(\sigma)\frac{M}{\sigma - 1}\left(1 + \sum_{r=1}^{k} k(k-1)^{r-1}\left(\frac{1}{(\sigma - 1)\log M}\right)^r\right)\end{aligned}$$

$$< Q_M^k(\sigma)\frac{M}{\sigma-1}\left(1+\frac{k}{(\sigma-1)\log M}\sum_{r=0}^{\infty}\left(\frac{k-1}{(\sigma-1)\log M}\right)^r\right)$$
$$= Q_M^k(\sigma)\frac{M}{\sigma-1}\left(1+\frac{k}{(\sigma-1)\log M-k+1}\right),$$

where the convergence of the geometric series is implied by $k-1 < (\sigma-1)\log M$. $\square$

It is clear why estimating $R_M^k(\sigma)$ will be vital for the proofs of our theorems. We note:

Lemma 4.2. *If* $a_1k+b_1 \le \sigma$ *and* $K \le k$, *then*

$$R_M^k(\sigma) \le R_M^k(a_1k+b_1) \le R_M^K(a_1K+b_1), \tag{9}$$

as long as the following two conditions are satisfied:

$$a_1 > \frac{1}{\log M} \quad \text{and} \quad (a_1\log M-1)K+1+(b_1-1)\log M > 0,$$

and, in the case of $b_1 < 1-1/\log M$ *also*

$$K \ge \frac{1}{a_1\log M}\left(-(b_1-1)\log M-1+\sqrt{\frac{|(b_1-1)\log M+1|}{a_1\log M-1}}\right).$$

Proof. The left-hand inequality of (9) is evident from the fact that $R_M^k(\sigma)$ is decreasing when viewed as a function of σ alone. The right-hand inequality of (9) is equivalent to saying that $R_M^k(\sigma)$ is decreasing as a function of k. To see this, we rewrite

$$\frac{1}{M\log M}R_M^k(a_1k+b_1)$$

in the form

$$y(k) = \frac{1}{(c+1)k+d-1}\,\frac{(c+1)k+d}{ck+d},$$

where $c := a_1\log M - 1 > 0$ and $d := 1+(b_1-1)\log M$. Then, clearly,

$$y'(k) = -\frac{c(1+c)^2k^2+2cdk(1+c)+d(1+cd)}{((c+1)k+d-1)^2(ck+d)^2},$$

from which it is easy to see that $y'(k)$ can change sign only if $d<0$ (otherwise it remains non-positive). However, the condition $d<0$

translates to $b_1 < 1 - 1/\log M$, in which case one requires $K \geq z_0$, where

$$z_0 := -\frac{d}{1+c} + \frac{1}{1+c}\sqrt{\frac{|d|}{c}}$$

is the right zero of the numerator of the above expression for $y'(k)$. $\square$

We will use the estimate for $T_M^k(\sigma)$ from Lemma 4.1 in the proof of Theorem 2.1 via the separation

$$\begin{aligned} T_M^k(\sigma) &= Q_{M+1}^k(\sigma) + T_{M+1}^k(\sigma) \\ &\leq Q_{M+1}^k(\sigma)(1 + R_{M+1}^k(\sigma)) \\ &\leq Q_M^k(q_M k + b_1)(1 + R_{M+1}^k(q_M k + b_1)), \end{aligned}$$

since $Q_{M+1}^k(\sigma) \leq Q_M^k(\sigma)$. The series with the remainder $R_{M+1}^k(q_M k + b_1)$ will converge because $q_M > 1/\log(M+1)$ by Lemma 3.1, if b_1 is suitably chosen. Verma and Kaur's bound (see Table 1) follows directly from Lemma 4.1 and Lemma 4.2. We include a proof of their result because it exemplifies several of the important ideas and illustrates key workings of our general method, being the special case of $M = 2$ (representing the dominance of the term $Q_2^k(\sigma)$).

Theorem 4.3 ([**13,** Theorem (A)]). *For all $\sigma \geq q_2 k + 2$, we have $\zeta^{(k)}(s) \neq 0$.*

Proof. First, write

$$\begin{aligned} |\zeta^{(k)}(s)| &\geq \frac{\log^k 2}{s^\sigma} - T_2^k(\sigma) \\ &\geq Q_2^k(\sigma)\left(1 - \frac{Q_3^k}{Q_2^k}(\sigma) - \frac{Q_4^k}{Q_2^k}(\sigma)\left(1 + R_4^k(\sigma)\right)\right). \end{aligned}$$

By Lemma 4.2, we have $R_4^k(\sigma) \leq R_4^k(q_2 k + 2) < 1.57$, for $k \geq 3$. Furthermore,

$$\frac{Q_4^k}{Q_2^k}(\sigma) = 2^{k-\sigma} \leq 2^{k - q_2 k + 2} \leq 2^{3(1-q_2)+2} \leq 0.19.$$

The quotient $\frac{Q_3^k}{Q_2^k}(\sigma)$ is decreasing in σ, and hence

$$\frac{Q_3^k}{Q_2^k}(\sigma) \leq \frac{Q_3^k}{Q_2^k}(q_2k+2) = \frac{4}{9}.$$

So, we obtain

$$1 - \frac{Q_3^k}{Q_2^k}(\sigma) - \frac{Q_4^k}{Q_2^k}(\sigma)\left(1+R_4^k(\sigma)\right) \geq 1 - \frac{4}{9} - 0.19(1+1.57) > 0,$$

which establishes the result. $\square$

Since Theorem 2.1 (a) deals with the next case of $M = 3$ (corresponding to the dominance of the term $Q_3^k(\sigma)$), and only a little bit of extra effort is needed to prove it, we give a proof of it right now.

Proof of Theorem 2.1 (a). For a zero-free region to exist we must have

$$q_3k + 4\log 3 \leq q_2k - 2,$$

which implies $k \geq 20$. Separating the dominant term $Q_3^k(\sigma)$, we get

$$\begin{aligned}|\zeta^{(k)}(s)| &\geq Q_3^k(\sigma) - Q_2^k(\sigma) - T_3^k(\sigma)\\ &\geq Q_3^k(\sigma)\left(1 - \frac{Q_2^k}{Q_3^k}(\sigma) - \frac{Q_4^k}{Q_3^k}(\sigma)\left(1+R_4^k(\sigma)\right)\right).\end{aligned}$$

Therefore, we only need to show that

$$1 - \frac{Q_2^k}{Q_3^k}(\sigma) - \frac{Q_4^k}{Q_3^k}(\sigma)\left(1+R_4^k(\sigma)\right) > 0.$$

By Lemma 4.2, $R_4^k(\sigma) \leq R_4^k(q_3k+4\log 3) \leq R_4^{k_3}(q_3k_3+4\log 3) < 0.72$, for $\sigma \geq q_3k+4\log 3$ and $k \geq k_3 = \frac{4\log 3+2}{q_2-q_3} = 19.5311\ldots$. Also,

$$\frac{Q_4^k}{Q_3^k}(\sigma) \leq \frac{Q_4^k}{Q_3^k}(q_3k+4\log 3) < 0.29$$

and

$$\frac{Q_2^k}{Q_3^k}(\sigma) \leq \frac{Q_2^k}{Q_3^k}(q_2k-2) < 0.45.$$

Hence,

$$1-\frac{Q_2^k}{Q_3^k}(\sigma)-\frac{Q_4^k}{Q_3^k}(\sigma)\left((1+R_4^k(\sigma)\right)>1-0.45-0.29(1+0.72)>0,$$

as desired. $\square$

Theorem 2.1 (b) deals with the dominance of the general term $Q_M^k(\sigma)$ and, consequently, requires knowledge of the behavior of the sum of all the terms preceding it.

The heads. We rewrite the heads of the series (1) in the following form:

(10)

$$H_M^k(\sigma)=Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma)+\frac{Q_{M-2}^k}{Q_M^k}(\sigma)+\cdots+\frac{Q_2^k}{Q_M^k}(\sigma)\right)$$

$$=Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma)\left(1+\frac{Q_{M-2}^k}{Q_{M-1}^k}(\sigma)\left(1+\cdots\left(1+\frac{Q_2^k}{Q_3^k}(\sigma)\right)\cdots\right)\right)\right), \tag{11}$$

and we will find upper bounds for all the above quotients $\frac{Q_{n-1}^k}{Q_n^k}(\sigma)$ of consecutive terms. Clearly,

$$\frac{Q_{n-1}^k}{Q_n^k}(\sigma)=\left(\frac{\log(n-1)}{\log n}\right)^k\left(\frac{n}{n-1}\right)^\sigma,$$

and therefore $\frac{H_M^k}{Q_M^k}(\sigma)$ increases with σ. For $2\le n\le M$ and $\sigma\le q_{M-1}k+b_2$, we get

$$\begin{aligned}\frac{Q_{n-1}^k}{Q_n^k}(\sigma)&\le\frac{Q_{n-1}^k}{Q_n^k}(q_{M-1}k+b_2)\\&\le\frac{Q_{n-1}^k}{Q_n^k}(q_{n-1}k+b_2)=\left(\frac{n}{n-1}\right)^{b_2},\end{aligned}$$

where the second inequality holds because $q_{M-1}<q_n$ for $n\le M$, while the equality holds because $\sigma=q_{n-1}k$ is the solution of $Q_n^k(\sigma)=Q_{n-1}^k(\sigma)$. Thus, in order for $\frac{H_M^k}{Q_M^k}(\sigma)$ to stay bounded, we must choose $b_2<0$.

Lemma 4.4. *Let $c \in \mathbb{R}$ be positive. Then $y(n) = \left(\frac{n-1}{n}\right)^{cn}$ is monotonously increasing with asymptote $1/e^c$.*

Proof. As

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{cn} = e^c,$$

we evidently have $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{cn} = 1/e^c$. Finally,

$$y'(n) = c\cdot y(n)\left(\log\left(1-\frac{1}{n}\right)+\frac{1}{n-1}\right) > 0$$

proves the monotonicity assertion. □

Thus, for $2 \le n \le M$ and $\sigma \le q_{M-1}k - uM$, we have

$$\frac{Q_{n-1}^k}{Q_n^k}(\sigma) \le \left(\frac{n}{n-1}\right)^{-uM} \le \left(\frac{M}{M-1}\right)^{-uM} \le \frac{1}{e^u}.$$

Now (11) yields

$$\frac{H_M^k}{Q_M^k}(\sigma) \le \sum_{n=1}^{\infty}\frac{1}{(e^u)^n} = \frac{1}{1-(1/e^u)} - 1 = \frac{1}{e^u-1}. \tag{12}$$

Proof of Theorem 2.1 (b). Similarly to the proof of Theorem 2.1 (a) we write

$$\begin{aligned}\left|\zeta^{(k)}(s)\right| &\ge Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma)\\ &\ge Q_M^k(\sigma)\left(1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{Q_{M+1}^k}{Q_M^k}(\sigma)\left(1+R_{M+1}^k(\sigma)\right)\right).\end{aligned}$$

Now, notice that

$$R_M^k(\sigma) := \frac{M}{\sigma-1}\left(1+\frac{k}{(\sigma-1)\log M - k + 1}\right) < \frac{1}{u}$$

is equivalent to $(\sigma-1)^2\log M-(\sigma-1)(cM\log M+k-1)-uM>0$, and this quadratic inequality is satisfied whenever

$$\begin{aligned}\sigma &> 1+\frac{(uM\log M+k-1)}{2\log M}+\frac{\sqrt{(uM\log M+k-1)^2+4M\log M}}{2\log M}\\ &> 1+\frac{2(uM\log M+k-1)}{2\log M}\\ &= 1+uM+\frac{k-1}{\log M}.\end{aligned}$$

Thus, by Lemma 4.2, for $\sigma \geq q_M k+u(M+1)$,

$$k \geq k_M = \frac{(2M+1)u}{q_{M-1}-q_M},$$

and $M \geq 4$, we have

$$R^k_{M+1}(\sigma) \leq R^{k_M}_{M+1}(q_M k_M+u(M+1)) < \frac{1}{u}.$$

By Lemma 4.4 we also have

$$\frac{Q^k_{M+1}}{Q^k_M}(q_M k+u(M+1)) = \left(\frac{M}{M+1}\right)^{u(M+1)} < \frac{1}{e^u};$$

thus, with (12), we obtain, for $M \geq 4$ and $q_M k+u(M+1) \leq \sigma \leq q_{M-1}k+uM$,

$$1-\frac{H^k_M}{Q^k_M}(\sigma)-\frac{Q^k_{M+1}}{Q^k_M}(\sigma)\left(1+R^k_M(\sigma)\right) > 1-\frac{1}{e^u-1}-\frac{1}{e^u}\left(1+\frac{1}{u}\right) \geq 0,$$

which proves the theorem. □

Remark 4.5. The zero-free regions we have given are not the largest possible. For example, if one considered the lines $\sigma = \frac{1}{2}((q_M+q_{M-1})k+u)$ through the centers of the wedges and searched for the lowest k for which there were no zeros on those lines, then one would obtain the following values for k_M (which are lower than the values we have for the tips of the wedge-shaped regions):

M	3	4	5	6	7	8	9	10
k_M on line	19	58	123	220	354	529	748	1014
k_M at the tip	20	77	163	291	465	691	971	1313

5. Proof of Theorem 2.3. Because of the property of the quasi-periodicity of the zeros of $\zeta^{(k)}(s)$ inside S_M^k, we are able to count the zeros by individual separation. In order for our approach to work, we first find horizontal, periodically-spaced zero-free line segments within the strips (in Lemma 5.1). Then we show that there is always exactly one zero of $\zeta^{(k)}(s)$ in the rectangles R_j (for $j \in \mathbb{N}$) that are delimited by the vertical edges of two neighboring zero-free regions and two horizontal zero-free lines (see Figure 6).

As already mentioned above, in the strips S_M^k, which are located between two consecutive zero-free regions, where the expansion of $\zeta^{(k)}(s)$ is dominated by the terms $Q_M^k(s)$ and $Q_{M+1}^k(s)$, respectively, one can obtain values of the imaginary parts t of expected zeros by solving the equation $Q_M^k(\sigma + it) = Q_{M+1}^k(\sigma + it)$ (an act of balancing the real and imaginary parts of two largest terms), and then choosing the horizontal lines of separation exactly halfway between them, thus managing to avoid even the most irregular of zeros inside S_M^k. That is exactly what we do below. As a consequence all zeros of $\zeta^{(k)}(s)$ inside S_M^k are simple.

Lemma 5.1. *Let $M \geq 2$ and $k \in \mathbb{N}$. If $s \in S_M^k$, then $\zeta^{(k)}(s) \neq 0$ for*

$$s = \sigma + i \cdot \frac{2\pi j}{\log(M+1) - \log M}.$$

Proof. In the center of the strip S_M^k, that is on the line $\sigma = q_M k$, we have $|Q_M^k(s)| = |Q_{M+1}^k(s)|$. We consider the line segments in S_M^k with

$$q_M k - (M+1)u \leq \sigma \leq q_M k + (M+1)u$$

and

$$t = \frac{2\pi j}{\log(M+1) - \log M}, \quad \text{where } j \in \mathbb{Z},$$

see Figure 6. Our choice of t gives $Q_M^k(q_M k + it) + Q_{M+1}^k(q_M k + it) = 0$ (compare with (5)) and therefore $\cos(t \log M) = \cos(t \log(M+1))$ and $\sin(t \log M) = -\sin(t \log(M+1))$. We set $s = \sigma + it$, with t and σ as above, and consider the real and imaginary parts of

$$\zeta^{(k)}(s) = \sum_{n=2}^{\infty} (\cos(t \log n) - i \cdot \sin(t \log n))\, Q_n^k(\sigma).$$

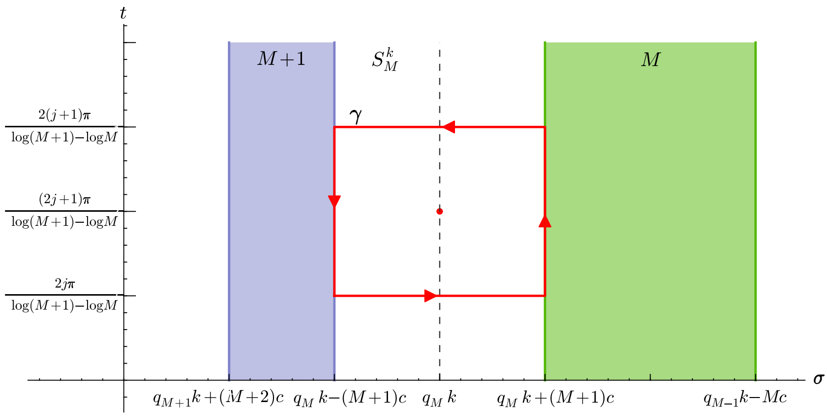

FIGURE 6. The curve γ is the boundary of the rectangle R_j. The point $\bullet$ represents a zero of $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$ on the line $\sigma = q_M k$.

With $|\Im(Q_n^k(s)| \le Q_n^k(\sigma)$ and $|\Re(Q_n^k(s)| \le Q_n^k(\sigma)$, we obtain

$$\begin{aligned}|\Re(\zeta^{(k)}(s))| &\ge |\cos(t\log M)Q_M^k(\sigma) + \cos(t\log(M+1))Q_{M+1}^k(\sigma)| \\ &\quad - H_M^k(\sigma) - T_{M+1}^k(\sigma), \\ |\Im(\zeta^{(k)}(s))| &\ge |\sin(t\log M)Q_M^k(\sigma) + \sin(t\log(M+1))Q_{M+1}^k(\sigma)| \\ &\quad - H_M^k(\sigma) - T_{M+1}^k(\sigma).\end{aligned}$$

If $t = 0$, the situation is trivial. If $t \ne 0$, then we either have $|\sin(t\log M)| \ge \sin(\pi/4) = 1/\sqrt{2}$ or $|\cos(t\log M)| \ge \cos(\pi/4) = 1/\sqrt{2}$. Because $|\zeta^{(k)}(s)| \ge |\Re(\zeta^{(k)}(s))|$ and $|\zeta^{(k)}(s)| \ge |\Im(\zeta^{(k)}(s))|$, we get:

$$\begin{aligned}|\zeta^{(k)}(s)| &\ge \frac{1}{\sqrt{2}}\left(Q_M^k(\sigma) + Q_{M+1}^k(\sigma)\right) - H_M^k(\sigma) - T_{M+1}^k(\sigma) \\ &= Q_M^k(\sigma)\Bigg(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{Q_{M+1}^k}{Q_M^k}(\sigma) - \frac{H_M^k}{Q_M^k}(\sigma) \\ &\qquad\qquad - \frac{Q_{M+2}^k}{Q_M^k}(\sigma) - \frac{T_{M+2}^k}{Q_M^k}(\sigma)\Bigg)\end{aligned}$$

$$= Q_M^k(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(\sigma) + \frac{Q_{M+1}^k}{Q_M^k}(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma) - \frac{T_{M+2}^k}{Q_{M+1}^k}(\sigma)\right)\right)$$

From the proof of Theorem 2.1 (b) we know that for $\sigma \geq q_{M+1}k + (M+2)u$ and $u = 1.1879426249\ldots$ (see Remark 2.2)

$$\frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma) - \frac{T_{M+2}^k}{Q_{M+1}^k}(\sigma) \geq \frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma)\,(1 + R_{M+2}(\sigma)) \geq \frac{1}{\sqrt{2}} - \frac{1}{e^u}\left(1 + \frac{1}{c}\right) > 0.$$

Similarly, since $\frac{H_M^k}{Q_M^k}(\sigma)$ is increasing in σ (see (11)) and because $\sigma < q_{M-1}k - Mu$, we get with (12) that

$$\frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(\sigma) \geq \frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(q_{M-1}k - Mu) \geq \frac{1}{\sqrt{2}} - \frac{1}{e^u - 1} > 0,$$

which concludes the proof of the lemma. $\square$

Proof of Theorem 2.3. Let $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$. It is easy to check that the function $Z(s)$ has exactly one (simple) zero in R_j, namely,

$$s = q_M k + i \cdot \frac{(2j+1)\pi}{\log(M+1) - \log M}.$$

In order to be able to apply Rouché's theorem we need to show that $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ for all s on R_j.

The vertical sides of R_j are in the zero free regions for M and $M+1$. As shown in the proof of Theorem 2.1 the term $Q_M^k(s)$ dominates $\zeta^{(k)}(s)$ on the right vertical side of R_j, and the term $Q_{M+1}^k(s)$ dominates $\zeta^{(k)}(s)$ on the left vertical side of R_j. Thus, $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ on the vertical sides of R_j. Furthermore, we have seen in the proof of Lemma 5.1 that $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$ dominates $\zeta^{(k)}(s)$ on the horizontal sides of R_j. Hence, $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ on the horizontal sides of R_j.

Therefore, by Rouché's theorem, $Z(s)$ and $\zeta^{(k)}(s)$ have the same number of zeros inside R_j, for every $j \in \mathbb{N}$. This proves both the simplicity of all zeros of $\zeta^{(k)}(s)$ inside S_M^k, and the sharp formula for $N_M^k(T)$, as given in Corollary 2.5. $\square$

Acknowledgments. The research presented in this work was supported in part by a New Faculty Grant from the University of North Carolina at Greensboro held by S. Pauli and F. Saidak. Most of the investigations were conducted while T. Binder was a visiting researcher at the University of North Carolina at Greensboro, during the Fall 2008 semester. The authors would like to thank Garry J. Tee from the University of Auckland, David W. Farmer from AIM, and the referee, whose many comments and remarks have helped to improve this paper considerably. All computations and plots were done with the computer algebra system Sage [**10**].

REFERENCES

1. B. Berndt, *The number of zeros for* $\zeta^{(k)}(s)$, J. Lond. Math. Soc. **2** (1970), 577–580.

2. S. Feng, *A note on the zeros of the derivative of the Riemann zeta function near the critical line*, Acta Arith. **120** (2005), 59-68.

3. J. Hadamard, *Théorème sur les séries entières*, Acta. Math. **22** (1899), 55–63.

4. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Chelsea, New York, 1974.

5. N. Levinson and H.L. Montgomery, *Zeros of the derivatives of the Riemann zeta function*, Acta Math. **133** (1974), 49–65.

6. A.L. Skorokhodov, *Padé approximants and numerical analysis of the Riemann zeta function*, Zh. Vych. Mat. Fiz. **43** (2003), 1330–1352.

7. K. Soundararajan, *The horizontal distribution of zeros of* $\zeta'(s)$, Duke Math. J. **91** (1998), 33–59.

8. A. Speiser, *Geometrisches zur Riemannschen Zetafunktion*, Math. Ann. **110** (1934), 514–521.

9. R. Spira, *Zero-free region for* $\zeta^{(k)}(s)$, J. Lond. Math. Soc. **40** (1965), 677–682.

10. W. Stein et al., *Sage*: *Open source mathematics software*, 2009, `http://www.sagemath.org`.

11. E. Titchmarsh, *The theory of the Riemann Zeta-function*, 2nd Edition, Oxford University Press, Oxford, 1986.

12. C.J. de la Vallée-Poussin, *Recherches analytiques sur la théorie des nombres* (*premiér partie*), Ann. Soc. Sci. Bruxelles **20** (1896), 183–256, 281–397.

13. D.P. Verma and A. Kaur, *Zero-free regions of derivatives of Riemann zeta function*, Proc. Ind. Acad. Sci. Math. **91** (1982), 217–221.

14. C.Y. Yıldırım, *Zeros of* $\zeta''(s)$ *and* $\zeta'''(s)$ *in* $\sigma < 1/2$, Turkish J. Math. **24** (2000), 89–108.

15. Y. Zhang, *On the zeros of* $\zeta'(s)$ *near the critical line*, Duke Math. J. **110** (2001), 555–572.

University of Lübeck, Institute of Mathematics, Wallstrasse 40, 23560 Lübeck, Germany
Email address: **thmsbinder@gmail.com**

Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27402
Email address: **s_pauli@uncg.edu**

Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27402
Email address: **f_saidak@uncg.edu**