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# SURVEY ARTICLE: SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS AND THEIR SPECTRUM

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*Dedicated to the memory of W.N. (Norrie) Everitt and to Zhijiang Cao on the occasion of his 80th birthday.*

ABSTRACT. We survey the theory of ordinary self-adjoint differential operators in Hilbert space and their spectrum. Such an operator is generated by a symmetric differential expression and a boundary condition. We discuss the very general modern theory of these symmetric expressions which enlarges the class of these expressions by many dimensions and eliminates the smoothness assumptions required in the classical case as given, e.g., in the celebrated books by Coddington and Levinson and Dunford and Schwartz. The boundary conditions are characterized in terms of square-integrable solutions for a real value of the spectral parameter, and this characterization is used to obtain information about the spectrum. Many of these characterizations are quite recent and widely scattered in the literature, some are new. A comprehensive review of the deficiency index (which determines the number of independent boundary conditions required in the singular case) is also given for an expression $M$ and for its powers. Using the modern theory mentioned above, these powers can be constructed without any smoothness conditions on the coefficients.

**1. Introduction.** In this paper we survey the theory of self-adjoint ordinary differential operators in Hilbert space and their spectrum. Many of the results are recent and widely scattered in the literature. For a general discussion of self-adjoint ordinary differential operators in symplectic spaces, see the 1999 monograph [**40**] by Everitt and Markus.

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**1.1. John von Neumann.** *" ... when America's National Academy of Science asked shortly before his death what he thought were his three greatest achievements ... Johnny replied to the academy that he considered his most important contributions to have been on the theory of self-adjoint operators in Hilbert space, and on the mathematical foundations of quantum theory and the ergotic theorem."*
*Macrae's biography of John von Neumann.*

**1.2. Applications.** *"From the point of view of applications, the most important single class of operators are the differential operators. The study of these operators is complicated by the fact that they are necessarily unbounded. Consequently, the problem of choosing a domain for a differential operator is by no means trivial; ... for unbounded operators the choice of domains can be quite crucial."*
*Dunford Schwartz* v. II ([**24,** page 1278]).

A self-adjoint ordinary differential operator in Hilbert space is generated by two things:

(1) *A symmetric* (*formally self-adjoint*) *differential expression.*
(2) *A boundary condition.*

*Given such a self-adjoint differential operator, a basic question is*: *What is its spectrum?*

These are the three things we discuss in this paper: (i) symmetric ordinary differential expressions, (ii) boundary conditions which determine self-adjoint differential operators and (iii) spectral properties of these self-adjoint operators.

**Notation 1.** Let $\mathbb{R}$ denote the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}_2 = \{2, 3, 4, \ldots\}$, $J = (a, b)$ for $-\infty \leq a < b \leq \infty$, $M_{nk}(X)$ the $n \times k$ matrices with entries from $X$, $M_n(X) = M_{nk}(X)$ when $n = k$, $M_{n1}(X)$ is also denoted by $X^n$; $L(J, \mathbb{R})$ and $L(J, \mathbb{C})$ the Lebesgue integrable real and complex valued functions on $J$, respectively, $L_{\mathrm{loc}}(J, \mathbb{R})$ and $L_{\mathrm{loc}}(J, \mathbb{C})$ the real and complex valued functions which are Lebesgue integrable on all compact subintervals of $J$, respectively. We also use $L_{\mathrm{loc}}(J) = L_{\mathrm{loc}}(J, \mathbb{C})$ and $L(J) = L(J, \mathbb{C})$. $AC_{\mathrm{loc}}(J)$ denotes the complex valued functions which are absolutely continuous on compact subintervals of $J$ and $AC(J)$ denotes the absolutely continuous functions on $J$, $C^j(J)$ denotes the complex functions on $J$ which have $j$ continuous derivatives. $D(A)$ denotes the domain of the operator $A$.

**Definition 1.** For $w \in L_{\text{loc}}(J, \mathbb{R})$, $w > 0$ almost everywhere in $J$, $L^2(J, w)$ denotes the Hilbert space of functions $f : J \to \mathbb{C}$ satisfying $\int_J |f|^2 w < \infty$ with inner product $(f, g)_w = \int_J f\,\overline{g}\,w$.

Such a $w$ is called a *weight function.*

**2. Classical symmetric expressions.** Given a classical differential expression $M$ of the form

$$My = p_n y^{(n)} + p_{n-1}\, y^{(n-1)} + \cdots + p_1\, y' + p_0, \quad y \text{ on } J, \tag{2.1}$$

the expression $M^+$ given by

$$M^+ y = (-1)^n (\overline{p_n} y)^{(n)} + (-1)^{n-1} (\overline{p}_{n-1} y)^{(n-1)} + \cdots + (-1)\overline{p}_1 y' + \overline{p}_0 \quad y \text{ on } J, \tag{2.2}$$

is called the adjoint expression of $M$. And $M$ is called symmetric (formally self-adjoint) if $M^+ = M$.

Thus, to check an expression (2.1) for symmetry, one must write (2.2) in the same form as (2.1) and compare coefficients. To do this, one must assume that the coefficients $p_j$ are sufficiently smooth, i.e., $p_j \in C^j(J)$.

It is well known [**24,** pages 1285–1289] that, if $n > 1$, $M = M^+$ and $p_j \in C^j(J)$ implies that $M$ given by (2.1) can be expressed in the form

$$\begin{aligned} M\,y = &\sum_{j=0}^{[n/2]} (-1)^j (a_j y^{(j)})^{(j)} \\ &+ i \sum_{j=0}^{[(n-1)/2]} (-1)^j [(b_j y^{(j)})^{(j+1)} + (b_j y^{(j+1)})^{(j)}] \quad \text{on } J, \end{aligned} \tag{2.3}$$

where $a_j$ and $b_j$ are real, $i$ is the complex number $\sqrt{-1}$ and $[x]$ denotes the greatest integer $\leq x$; and that $M$ given by (2.3) is equal to its adjoint expression $M^+$. Thus, (2.3) is a closed form for all symmetric expressions (2.1) with sufficiently smooth coefficients. (In **[24]** it is assumed that the $p_j$ are in $C^\infty(J)$, but the proof given there clearly is valid for $p_j \in C^j(J)$.)

Note that, if the coefficients $p_j$ in (2.1) are all real, then the complex second term in (2.3) vanishes. Hence, a real symmetric expression $M$ given by (2.1) with sufficiently smooth coefficients $p_j$ must be of even order $n = 2k$ and have the form

$$M\,y = \sum_{j=0}^{k} (-1)^j (a_j y^{(j)})^{(j)} \tag{2.4}$$

with $a_j$ real, $j = 0, 1, 2, \ldots, k$.

For real coefficients (2.4) is the familiar Sturm-Liouville form

$$My = -(a_1 y')' + a_0 y \tag{2.5}$$

when $n = 2$, and for $n = 4$, we have

$$My = (a_2 y'')'' - (a_1 y')' + a_0 y. \tag{2.6}$$

In the odd order case, $n = 3$ (2.3) takes the form:

$$My = -(a_1 y')' + a_0 y + i\{[-(b_1 y')'' + (b_1 y'')'] + [(b_0 y)' + (b_0 y')]\} \tag{2.7}$$

with $a_j, b_j$ real, $j = 0, 1$.

If the coefficients $a_j, b_j$ are not sufficiently smooth, then the form (2.3) does not reduce to the form (2.1). Nevertheless, as we shall see below, analogues of (2.3) are "symmetric" without any smoothness assumption on the coefficients at all. Thus, if one wishes to study general symmetric differential expressions with nonsmooth coefficients, one is forced to consider so-called quasi-differential expressions. In (2.5), $(a_1 y')$ is called the quasi-derivative of $y$, and the reason for the parenthesis in $(a_1 y')$ is that it follows from the general theory of linear differential equations **[131]** that the product $a_1 y'$ is continuous at all points of the underlying open interval $J$ but the separated terms $a_1(t) y'(t)$ may not exist for all $t$ in $J$.

It turns out, as we will see below, that there exist much more general quasi-differential symmetric expressions than those analogous to (2.3) or (2.4). These will be identified below.

Very general quasi-differential expressions, in particular symmetric ones, have been considered by Shin [**96**]–[**99**]. They were rediscovered by Zettl [**45, 124**] in a slightly different but equivalent form. Special cases of these have been used by many authors, including Barrett [**6**], Glazman [**54**], Hinton [**64**], Kogan and Rofe-Beketov [**71**], Naimark [**84**], Reid [**91**], Stone [**100**], Walker [**105**] and Weyl [**117**]. The forms given in [**45, 124**] for general $n \in \mathbb{N}_2$ were motivated to some extent by the forms used by Barrett [**6**] for the cases when $n = 3$ and $n = 4$.

The development of the theory of symmetric differential operators in the books by Naimark [**84**] and Akhieser and Glazman [**1**] is based on the real symmetric form analogous to (2.4). Although these authors refer to Shin's more general symmetric expressions, they do not use them. In [**128**], Zettl showed that the techniques in these books, based largely on the work of Glazman, can be applied to a much larger class of symmetric operators generated by the very general symmetric expressions identified below.

**3. Quasi-derivative formulation of the classical expressions.** Before introducing general quasi-derivatives we discuss the quasi-derivative formulation of the classical real expression (2.4) and make a comment about its misuse in the literature.

It is well known [**131**] that the classical Sturm-Liouville theory, including the operators it generates in the Hilbert space $L^2(J, w)$, applies to the equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J, \tag{3.1}$$

with coefficients satisfying

$$\frac{1}{p}, q, w \in L_{\text{loc}}(J, \mathbb{R}), \quad w > 0, \text{ almost everywhere on } J. \tag{3.2}$$

The local integrability conditions of (3.2) (without the positivity condition on $w$) are necessary and sufficient for every initial value problem to have a unique solution defined on the whole interval $J$ [**43**]. In this sense, the local integrability conditions (3.2) are minimal conditions for the classical modern Sturm-Liouville theory (with Caratheodory solutions), including the operators it generates in $L^2(J, w)$, to hold. We note for later reference that there is no positivity condition on $p$ in (3.2).

In the fourth order case

$$My = (py'')'' - (ry')' + qy = \lambda wy \quad \text{on } J, \tag{3.3}$$

the conditions $p \in C^2$, $r \in C^1$, $q, w \in C$, $p > 0$ and $w > 0$ can be weakened to:

$$\frac{1}{p}, r,\ q, w \in L_{\text{loc}}(J, \mathbb{R}), \quad w > 0, \text{ almost everywhere on } J, \tag{3.4}$$

provided the expression $M$ in (2.4) is modified to

$$My = [(py'')' - (ry')]' + qy = \lambda wy \quad \text{on } J. \tag{3.5}$$

This modification, i.e., the use of the extra bracket [ ] and the quasi-derivative $[(py'')' - (ry')]$ seem to be a small price to pay for weakening the smoothness conditions to just local integrability. In particular, this allows the coefficients to be piece-wise constant which is important for both the theoretical and numerical approximations of the equation.

Similarly, in the higher order cases $n = 2k$, $k > 2$, the smoothness conditions on the coefficients of the classical equation (2.4) can be replaced by the corresponding local integrability conditions of the type (3.4), provided the classical equation is replaced by its quasi-differential analogue containing an appropriate number of parentheses. For $k = 3$, this requires the introduction of two additional parentheses (brackets) rather than just one as in case $k = 2$. See below for details.

**Remark 1.** In his well-known book [**84**], Naimark uses the classical form (2.4) with just local integrability assumptions on the coefficients but neglects to use the required additional parentheses (brackets) as indicated above. This oversight has been repeated in the literature by many authors. See Everitt and Zettl [**45**] for more details.

Next we construct much more general symmetric expressions than those analogous to the smooth forms (2.3) in the complex case and (2.4) in the real case. But first we discuss the connection between first order systems and higher order scalar, not necessarily symmetric, equations.

**4. First order systems and higher order scalar equations.** In this section, we construct general quasi-differential expressions of even and odd order with real or complex coefficients and discuss some of their basic properties. For a more comprehensive discussion of quasi-differential expressions, the reader is referred to [**45, 124**] in the scalar coefficient case and to [**48, 82, 116**] for matrix coefficients.

Just as the second order quasi-differential expression $-(py')' + qy$ enjoys many advantages over the more classical $py'' + ry' + qy$, so one can formulate quasi-differential expressions of higher order to replace the classical expression:

$$My = p_n y^{(n)} + p_{n-1}\, y^{(n-1)} + \cdots + p_1\, y' + p_0\, y \quad \text{on } J. \tag{4.1}$$

Among the advantages of these quasi-differential expressions over the classical ones are the following:

(1) They are more general.
(2) An adjoint expression can be defined which has the same form as the original–in contrast to the classical case. (See below for details.)
(3) The Lagrange identity is much simpler. It involves a sesquilinear form with constant coefficients in contrast to the classical form which depends on the coefficients in a complicated way.
(4) The fact that the adjoint of the adjoint is the original is immediately clear.
(5) Powers of expressions can be formed in a natural way without any smoothness or other additional conditions on the coefficients [**129**].

**Definition 2.** For $n > 1$, let

$$\begin{aligned} Z_n(J) := \{A = (a_{rs})_{r,s=1}^n \in M_n(L_{\text{loc}}(J)), \quad & a_{r,r+1} \neq 0 \text{ a.e. on } J,\\ & a_{r,r+1}^{-1} \in L_{\text{loc}}(J), \quad 1 \le r \le n-1,\\ & a_{rs} = 0 \text{ a.e. on } J, \quad 2 \le r+1 < s \le n;\\ & a_{rs} \in L_{\text{loc}}(J), \quad s \neq r+1,\ 1 \le r \le n-1\}. \end{aligned} \tag{4.2}$$

For $A \in Z_n(J)$, we define

$$V_0 := \{y : J \longrightarrow \mathbb{C},\ y \text{ is measurable}\} \tag{4.3}$$

and

$$y^{[0]} = y \quad (y \in V_0). \tag{4.4}$$

Inductively, for $r = 1, \dots, n$, we define

$$V_r = \{y \in V_{r-1} : y^{[r-1]} \in (AC_{\mathrm{loc}}(J))\}, \tag{4.5}$$

$$y^{[r]} = a_{r,r+1}^{-1}\Big\{y^{[r-1]'} - \sum_{s=1}^{r} a_{rs} y^{[s-1]}\Big\} \quad (y \in V_r), \tag{4.6}$$

where $a_{n,n+1} := 1$. Finally, we set

$$M\, y = M_A\, y = i^n\, y^{[n]} \quad (y \in V_n). \tag{4.7}$$

The expression $M = M_A$ is called the quasi-differential expression associated with or generated by $A$. For $V_n$, we also use the notations $D(A)$ and $V(M)$. The function $y^{[r]}$ $(0 \le r \le n)$ is called the $r$-th quasi-derivative of $y$. Since the quasi-derivative depends on $A$, we sometimes write $y_A^{[r]}$ instead of $y^{[r]}$.

**Definition 3.** In Definition 2, if $a_{rs} \in \mathbb{R}$, $1 \le r,\ s \le n$, we use the notation $A \in Z_n(J, \mathbb{R})$.

**Definition 4.** When $n = 2k$, the coefficient $a_{k,k+1}$ is called the leading coefficient of $M_A$. We say that the leading coefficient $a_{k,k+1}$ changes sign on $J$ if it assumes positive values and negative values, each on a subset of $J$ which has positive Lebesgue measure. (The sign of the leading coefficient $a_{k,k+1}$ plays an important role in the semiboundedness of even order operators generated by $M_A$ as we will see below.)

**Remark 2.** The subclass of matrices $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \ge 1$, which generate the symmetric quasi-differential expressions studied by Naimark in **[84]**, see Section 3 above, have the form:

$$A = \begin{bmatrix} & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & a_{3,4} & 0 & 0 \\ & & a_{43} & & 1 & 0 \\ & a_{52} & & & & 1 \\ a_{61} & & & & & \end{bmatrix}$$

when $n = 6$ and similar forms for other even $n$. Below we refer to these as matrices of GN (Glazman-Naimark) type.

**Definition 5** (Regular endpoint)**.** Let $A \in Z_n(J)$, $J = (a, b)$. The expression $M = M_A$ is said to be regular (R) at $a$ if, for some $c$, $a < c < b$, we have

$$\begin{aligned} a_{r,r+1}^{-1} &\in L(a, c), \quad r = 1, \ldots, n-1; \\ a_{rs} &\in L(a, c), \quad 1 \le r,\ s \le n,\ s \ne r+1. \end{aligned}$$

Similarly the endpoint $b$ is regular if, for some $c$, $a < c < b$, we have:

$$\begin{aligned} a_{r,r+1}^{-1} &\in L(c, b), \quad r = 1, \ldots, n-1; \\ a_{rs} &\in L(c, b), \quad 1 \le r,\ s \le n,\ s \ne r+1. \end{aligned}$$

Note that, from (4.2), it follows that if the above hold for some $c \in J$, then they hold for any $c \in J$. We say that $M$ is regular on $J$, or just $M$ is regular, if $M$ is regular at both endpoints.

**Remark 3.** For a given $A$, $M = M_A$ is determined by Definition 2. However, $M_A$ does not determine $A$ uniquely, in general, i.e., there may be a $B \in Z_n(J)$ such that $M_A = M_B$. See the proof of Theorem 10 in Section 9 for an example. In this example, $M_B$ is regular at both endpoints and $M_A$ is singular at the endpoint $a$.

**Remark 4.** In much of the literature when an endpoint of $J$ is infinite, the problem is automatically classified as singular; note that, in Definition 5, $a = -\infty$ and $b = \infty$ are allowed. For any $J$, observe that $M$ is regular on any compact subinterval of $J$.

To illustrate the use of quasi-derivatives we give a simple example.

**Example 1.** Let $A = (a_{rs}) \in Z_2(J)$. Then $y^{[0]} = y$, $y^{[1]} = a_{12}^{-1}(y' - a_{11}y)$,

$$y^{[2]} = [a_{12}^{-1}(y' - a_{11}y)]' - a_{21}y - a_{12}^{-1}a_{22}(y' - a_{11}y) \tag{4.8}$$

and $M = M_A$ is given by

$$My = i^2 y^{[2]} = -[a_{12}^{-1}(y' - a_{11}y)]' + a_{21}y + a_{12}^{-1}a_{22}(y' - a_{11}y). \tag{4.9}$$

Note that, if $a_{11} = 0 = a_{22}$, then (4.9) reduces to the familiar Sturm-Liouville form

$$My = -(py')' + qy$$

with the notation $p = a_{12}^{-1}$, $q = a_{21}$, but without the assumption that $p$ and $q$ are real valued and, if $p$ is real valued, without the hypothesis that $p$ is of one sign as long as $1/p \in L_{\mathrm{loc}}(J)$.

The first-order vector system $Y' = AY + F$ and the quasi-differential equation $y_A^{[n]} = f$ are equivalent in the following sense:

**Proposition 1.** *Let $A \in Z_n(J)$ and $f \in L_{\mathrm{loc}}(J)$. Set $M = M_A$*

$$F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

(i) *If $Y \in (AC_{\mathrm{loc}}(J))^n$ is a solution of*

$$Y' = AY + F, \tag{4.10}$$

*then there is a unique $y \in D(M)$ such that*

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix} \tag{4.11}$$

*and*

$$y^{[n]} = f; \tag{4.12}$$

(ii) *If $y \in D(M)$ is a solution of* (4.12), *then*

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix} \in AC_{\mathrm{loc}}(J)^n \tag{4.13}$$

*and*

$$Y' = AY + F. \tag{4.14}$$

*Proof.* This follows from a straightforward computation; for details, see [**82,** Section 2]. □

The initial value problem associated with $Y' = AY + F$ has a unique solution:

**Proposition 2.** *For each $F \in (L_{\mathrm{loc}}(J))^n$, each $\alpha$ in $J$ and each $C \in \mathbb{C}^n$, there is a unique $Y \in (AC_{\mathrm{loc}}(J))^n$ such that*

$$Y' = AY + F \quad \text{on } J \text{ and } Y(\alpha) = C. \tag{4.15}$$

*Proof.* See [**131,** Chapter 1]. □

From Proposition 2, we immediately infer the next corollary:

**Corollary 1.** *For each $f \in L_{\mathrm{loc}}(J)$, each $\alpha \in J$ and $c_0, \ldots, c_{n-1} \in \mathbb{C}$, there is a unique $y \in D(A)$ such that*:

$$y^{[n]} = f \quad \text{and} \quad y^{[r]}(\alpha) = c_r \quad (r = 0, \ldots, n-1). \tag{4.16}$$

*If $f \in L(J)$, $J$ is bounded and all components of $A$ are in $L(J)$, then $y \in AC(J)$.*

**5. Maximal and minimal operators and their domains.** In this section, we construct the maximal and minimal operators for general quasi-differential expressions and discuss their basic properties.

**Definition 6.** Let $A \in Z_n(J)$, let $w$ be a weight function, and let $H = L^2(J, w)$. The maximal operator $S_{\max}(A, J)$ with domain $D_{\max}(A, J)$ is defined by:

$$\begin{gathered} D_{\max}(A, J) = \{y \in H : y \in D(A),\ w^{-1} M_A\, y \in H\} \\ S_{\max}(A, J)\, y = w^{-1} M_A\, y,\ y \in D_{\max}(A, J). \end{gathered} \tag{5.1}$$

We will use the next theorem to define the minimal operator.

**Theorem 1.** *Let $A \in Z_n(J)$, let $w$ be a weight function, and let*

$$A^+ = -E^{-1}A^*E, \text{ where } E = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{5.2}$$

*Then*

(i) $A^+ = (a_{ij}^+) \in Z_n(J)$.

(ii) $D_{\max}(A,J)$ *is dense in $H$. Let $S_{\min}(A,J) = S_{\max}^*(A,J)$, and let $D_{\min}(A,J)$ denote the domain of $S_{\min}(A,J)$.*

(iii) $S_{\min}(A,J)$ *is a closed operator in $H$ with dense domain, and we have*

$$S_{\min}^*(A,J) = S_{\max}(A^+,J), \ S_{\min}(A,J) = S_{\max}^*(A^+,J). \tag{5.3}$$

(iv) *If $A^+ = A$, then $S_{\min}(A,J)$ is a closed symmetric operator in $H$ with dense domain and*

$$S_{\min}^*(A,J) = S_{\max}(A,J), \ S_{\min}(A,J) = S_{\max}^*(A,J). \tag{5.4}$$

*Proof.* Part (1) follows directly from the definition. The method of Naimark [**84,** Chapter V] can be adapted to prove this theorem with minor modifications. See also [**45, 82**]. □

**Notation 2.** Below we will also use minimal and maximal domain functions and their restrictions on subintervals $(\alpha,\beta)$ of $J = (a,b)$, particularly for $(\alpha,\beta) = (a,c)$ and $(\alpha,\beta) = (c,b)$ for $c \in (a,b)$. Since $A \in Z_n(J)$ implies that $A \in Z_n((\alpha,\beta))$, Definition 6 and Theorem 1 can be applied in the Hilbert space $L^2((\alpha,\beta),w)$ with $J$ replaced by $(\alpha,\beta)$. Below when we use the notation $D_{\max}(\alpha,\beta)$, $D_{\min}(\alpha,\beta)$, it is understood that we use Definition 6 and Theorem 1 with $J$ replaced by $(\alpha,\beta)$, $A$ and $w$ replaced by their restrictions to $(\alpha,\beta)$ and $L^2(J,w)$ replaced by $L^2((\alpha,\beta),w)$. Below, $A$, $J$, $(\alpha,\beta)$, as well as the Hilbert space, may be omitted when these are clear from the context.

**Remark 5.** We comment on the denseness of the minimal domain $D_{\min}(A,J)$. As mentioned in the proof of Theorem 1 above, this fact is proven in the well known book by Naimark [**84,** page 68] for a special subclass of the general Lagrange symmetric matrices $A$ discussed here– the subclass we call the GN matrices. It was shown by Zettl [**124**] that Naimark's method can be applied to prove that $D_{\min}(A)$ is dense in $H$ for the Lagrange symmetric matrices $A$ discussed here, see also Everitt-Zettl [**45**] and Möller-Zettl [**82**]. In general, under the local integrability assumptions used here, $C_0^\infty(J)$ may not be contained in $D_{\min}(A,J)$. Moreover, it may not be easy to find explicit functions in $D_{\min}(A,J)$ other than the zero function. Nevertheless, the proof given in [**84,** page 68], with minor modifications, can be used to prove that $D_{\min}(A,J)$ is dense under the local integrability assumptions used here.

In much of the current literature in mathematics and mathematical physics, even for the expression $My = -y'' + qy$, many authors assume that $q \in L^2_{\mathrm{loc}}(J)$ because this implies that $C_0^\infty(J) \subset D_{\min}$, and therefore $D_{\min}$ is dense. For $q \in L_{\mathrm{loc}}(J)$, $C_0^\infty(J)$ may not be contained in $D_{\min}$ so this proof is not valid, but Naimark's proof does 'work,' see [**45, 82**].

**6. The Lagrange identity.** Fundamental to the study of boundary value problems is the Lagrange identity which is given in the next theorem.

**Theorem 2.** *Let $A \in Z_n(J)$, and let*

$$B = -E^{-1}A^*E, \quad \text{where } E = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n, \text{ i.e.,} \tag{6.1}$$

$$b_{rs} = (-1)^{r+s-1}\overline{a}_{n+1-s,n+1-r}, \quad 1 \le r,s \le n. \tag{6.2}$$

*Then $B \in Z_n(J)$ and, for any $y \in D(M_A)$, $z \in D(M_B)$, we have*

$$\overline{z}\, M_A\, y - y\, \overline{M_B\, z} = [y,z]', \tag{6.3}$$

*where*

$$[y,z] = i^n \sum_{r=0}^{n-1} (-1)^{n+1-r} \overline{z}_B^{[n-r-1]} y_A^{[r]}. \tag{6.4}$$

*Proof.* See Möller-Zettl [**82**]. The proof given in [**82**] is based on a method of Everitt and Neuman [**42**]; it greatly simplifies the earlier proofs of Zettl [**124**] and Weidmann [**116**]. □

**Definition 7** (Symplectic matrix $E$)**.** The symplectic matrix $E$ in (6.1) plays an important role in the theory of Lagrange symmetric matrices and in boundary value problems. For $k \in \mathbb{N}_2$, let $E_k$ be defined by

$$E_k = ((-1)^r \delta_{r,k+1-s})_{r,s=1}^n,$$

where $\delta_{i,j}$ is the Kronecker $\delta$.

Motivated by the Lagrange identity (6.4), we make the following definition:

**Definition 8.** Let $A \in Z_n(J)$, and let $A^+ = B$ and $M_A^+ = M_B$ where $B$ is given by (6.1). We call $A^+$ the Lagrange adjoint or L-adjoint matrix of $A$ and $M_A^+$ the Lagrange adjoint or L-adjoint expression of $M_A$. (In Theorem 2, we used the notation $B = A^+$ and $M_B = M_A^+$ for simplicity of exposition.) Note that $A^{++} = A$ and $M_A^{++} = M_A$ follows immediately from the definition. The bracket $[y, z]$ or just the bracket $[\cdot, \cdot]$ is called the Lagrange bracket of $M$.

Next, we give some useful characterizations of the minimal domain.

**Theorem 3.** *Let $A \in Z_n(J)$, let $w$ be a weight function and let $A = A^+$. Then*

(i)

$$D_{\min} = \{y \in D_{\max} : [y, z](b) - [y, z](a) = 0 \quad \text{for all } z \in D_{\max}\}, \tag{6.5}$$

*where $[y, z]$ denotes the Lagrange bracket.*

(ii) *If $a$ is regular, then $y^{[j]}(a) = 0$ for $j = 0, 1, \ldots, n-1$ and similarly for $b$.*

We illustrate the construction of L-adjoint matrices with some simple examples.

**Example 2.** $n = 2$. Let $A = (a_{rs}) \in Z_2(J)$. Then

$$A^+ = \begin{bmatrix} a_{11} & \overline{a}_{12} \\ \overline{a}_{21} & -\overline{a}_{11} \end{bmatrix}. \tag{6.6}$$

Note that, just as in an Hermitian matrix, three out of the four components of $A$ are independent.

**Example 3.** $n = 4$. Let $A = (a_{rs}) \in Z_4(J)$. Then

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & \overline{a}_{23} & 0 \\ a_{31} & \overline{a}_{32} & -\overline{a}_{22} & \overline{a}_{12} \\ \overline{a}_{41} & -\overline{a}_{31} & \overline{a}_{21} & -\overline{a}_{11} \end{bmatrix}. \tag{6.7}$$

Note that, of the 13 nonzero components of $A$, 8 are independent. (The reason for the three zeros in the upper right corner is that we use these matrices to construct scalar differential expressions. Clearly we could define $A^+$ for any $A \in M_n(\mathbb{C})$.)

**Example 4.** $n = 3$. Let $A = (a_{rs}) \in Z_3(J)$. Then

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & -\overline{a}_{22} & \overline{a}_{12} \\ -\overline{a}_{31} & \overline{a}_{21} & -\overline{a}_{11} \end{bmatrix}. \tag{6.8}$$

Here, five of the eight nonzero components are independent.

**Remark 6.** We comment on the comparison of the quasi-differential Lagrange identity (6.3) and (6.4) with the classical one as given in Coddington-Levinson [**21,** Chapter 11] and Dunford and Schwartz [**24,** Chapter XIII]. In (6.4), the sesquilinear form $[y, z]$ is very simple with constant coefficients, whereas in the classical case it depends on the coefficients of the differential equation in a rather complicated way. In (6.4), this coefficient dependence is subsumed in the definition of the quasi-derivatives. Also, as remarked above, the derivation of (6.4) requires only local integrability of the coefficients in contrast with the derivation in the classical case which requires strong smoothness assumptions on the coefficients. In particular, the coefficients in the quasi-differential case can be step functions.

For the rest of this section we assume that $A = A^+$. Let $M = M_A$. Then $M = M^+$.

**Lemma 1.** *For any $y, z$ in $D_{\max}$, we have*

$$\int_{c_1}^{c} \{\overline{z}My - y\overline{Mz}\} = [y, z](c) - [y, z](c_1), \tag{6.9}$$

*for any $c, c_1 \in J = (a, b)$.*

*Proof.* This follows from the Lagrange identity, i.e., Theorem 2 and integration. □

**Lemma 2.** *For any $y$, $z$ in $D_{\max}$, the limits*

$$\lim_{t\to b^-}[y,z](t), \qquad \lim_{t\to a^+}[y,z](t),$$

*exist and are finite, and*

$$\int_a^b \{\overline{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a).$$

*Proof.* This follows from (6.9) by taking limits as $c_1 \to a$, $c \to b$. That the limits exist and are finite can be seen from the definition of $D_{\max}$, see (5.1) above. $\square$

The finite limits of Lemma 2 will play a critical role below in the characterization of self-adjoint domains at singular endpoints.

**Corollary 2.** *If $M\,y = \lambda w\,y$ and $M\,z = \overline{\lambda} w\,z$ on some interval $(\alpha,\beta) \subset (a,b)$, then $[y,z]$ is constant on $(\alpha,\beta)$. In particular, if $\lambda$ is real and $M\,y = \lambda w\,y$, $Mz = \lambda wz$ on some interval $(\alpha,\beta) \in (a,b)$. Then $[y,z]$ is constant on $(\alpha,\beta)$.*

*Proof.* This follows directly from (6.9). Since the coefficients are real, the last statement follows from the observation that if $M\,y = \lambda w\,y$ then $M\overline{y} = \lambda w\overline{y}$. $\square$

**Remark 7.** In much of the literature when an endpoint of the underlying interval is infinite, the problem is automatically classified as singular; note that, in Definition 5, $a = -\infty$ and $b = \infty$ are allowed. For any $J$, observe that $M$ is regular on any compact subinterval of $J$.

**Lemma 3.** *Suppose $M$ is regular at $c$. Then, for any $y \in D_{\max}$, the limits*

$$y^{[r]}(c) = \lim_{t\to c} y^{[r]}(c)$$

*exist and are finite, $r = 0,\ldots,n-1$. In particular, this holds at any regular endpoint and at each interior point of $J$. At an endpoint, the limit is the appropriate one sided limit.*

*Proof.* See [**84, 115**]. Although our result is more general than those stated in these references the same method of proof can be used here. □

Following Everitt and Zettl [**45**], we call the next lemma the *Naimark patching lemma* or just the *Patching lemma.* Our version of it is more general than that given by Naimark [**84,** Lemma 2, page 63], but the method of proof is basically the same.

**Lemma 4.** (Naimark patching lemma). *Let $A \in Z_n(J,\mathbb{R})$, and assume that $M$ is regular on $J$. Let $\alpha_0,\dots,\alpha_{n-1}$, $\beta_0,\dots,\beta_{n-1} \in \mathbb{C}$. Then there is a function $y \in D_{\max}$ such that*

$$y^{[r]}(a) = \alpha_r, \qquad y^{[r]}(b) = \beta_r \quad (r = 0,\dots,n-1).$$

**Corollary 3.** *Let $a < c < d < b$ and $\alpha_0,\dots,\alpha_{n-1},\beta_0,\dots,\beta_{n-1} \in \mathbb{C}$. Then there is a $y \in D_{\max}$ such that $y$ has compact support in $J$ and satisfies*:

$$y^{[r]}(c) = \alpha_r, \qquad y^{[r]}(d) = \beta_r \quad (r = 0,\dots,n-1).$$

*Proof.* The Patching lemma gives a function $y_1$ on $[c,d]$ with the desired properties. Let $c_1, d_1$ with $a < c_1 < c < d < d_1 < b$. Then use the Patching lemma again to find $y_2$ on $(c_1,c)$ and $y_3$ on $(d,d_1)$ such that

$$\begin{gathered} y_2^{[r]}(c_1) = 0, \qquad y_2^{[r]}(c) = \alpha_r, \\ y_3^{[r]}(d) = \beta_r, \qquad y_3^{[r]}(d_1) = 0 \\ (r = 0,\dots,n-1). \end{gathered}$$

Now set

$$y(x) := \begin{cases} y_1(x) & \text{for} \quad x \in [c,d] \\ y_2(x) & \text{for} \quad x \in (c_1,c) \\ y_3(x) & \text{for} \quad x \in (d,d_1) \\ 0 & \text{for} \quad x \in I \setminus (c_1,d_1). \end{cases}$$

Clearly, $y$ has compact support in $J$. Since the quasi-derivatives at $c_1, c, d, d_1$ coincide on both sides, $y \in D_{\max}$ follows. □

**Corollary 4.** *Let $a_1 < \cdots < a_k \in J$, where $a_1$ and $a_k$ can also be regular endpoints. Let $\alpha_{jr} \in \mathbb{C}$, $(j = 1,\dots,k;\ r = 0,\dots,n-1)$. Then* *there is a* $y \in D_{\max}$ *such that*

$$y^{[r]}(a_j) = \alpha_{jr} \qquad (j = 1, \dots, k;\ r = 0, \dots, n-1).$$

*Proof.* This follows from repeated applications of Corollary (3). $\square$

**7. Symmetric expressions.** In this section, we define and study Lagrange symmetric expressions $M = M_A$. We call these expressions Lagrange symmetric, or L-symmetric for short, because they generate symmetric operators in Hilbert space. It is the characterization of the self-adjoint extensions of these symmetric (minimal) operators and the spectrum of these extensions which are our main interest in this paper. The Lagrange identity plays a critical role in the characterization of the self-adjoint extensions.

The development of the theory of symmetric differential operators in the books by Naimark **[84]** and Akhieser and Glazman **[1]** is based on the real symmetric form analogous to (2.4). Although these authors refer to Shin's more general symmetric expressions in a footnote they do not use them. In **[128]**, Zettl showed that the techniques in these books, based largely on the work of Glazman, can be applied to a much larger class of symmetric operators generated by very general symmetric expressions. This larger class is discussed here.

**Definition 9.** Let $A \in Z_n(J)$, and suppose that $A$ satisfies

$$A = -E^{-1}A^{*}E, \text{ where } E = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^{n}, \text{ i.e.,}$$

$$a_{rs} = (-1)^{r+s-1}\overline{a}_{n+1-s,n+1-r}, \quad 1 \le r, s \le n. \tag{7.1}$$

Then $A$ is called a Lagrange symmetric or L-symmetric matrix and the expression $M = M_A$ defined above is called a Lagrange symmetric, or just a symmetric, differential expression.

The next proposition gives a 'visual' interpretation of the L-symmetry condition (7.1). It follows immediately from (7.1) by inspection.

**Proposition 3.** *Suppose* $A = (a_{rs}) \in Z_n(J)$ *is Lagrange symmetric. Then* $A$ *is invariant under the composition of the following three operations*:

(i) *Flipping the elements* $a_{rs}$ *about the secondary diagonal.*
(ii) *Replacing* $a_{rs}$ *by its conjugate* $\overline{a_{rs}}$.
(iii) *Changing the sign of* $a_{rs}$ *if* $r+s$ *is even.*

*Note that these three operations commute with each other and each one is idempotent.*

To illustrate Definition 9, we specialize Examples 2, 3 and 4 to the symmetric case and compute their symmetric differential expressions.

**Example 5.** Assume $A=(a_{rs})\in Z_2(J)$ is Lagrange symmetric. Then $a_{12}$ and $a_{21}$ are both real and $a_{11}=-\overline{a_{22}}$ implies that $a_{11}=ib_{11}=a_{22}$ with $b_{11}$ real. Thus, $M=M_A$ is given by

$$\begin{aligned} My &= -[a_{12}^{-1}(y'-a_{11}y)]'+a_{21}y+a_{12}^{-1}a_{22}(y'-a_{11}y) \\ &= -[a_{12}^{-1}(y'-ib_{11}y)]'+a_{21}y+a_{12}^{-1}ib_{11}y'+a_{12}^{-1}b_{11}y). \end{aligned} \tag{7.2}$$

If $b_{11}=0$, then (7.2) reduces to

$$My=-(a_{12}^{-1}\,y')'+a_{21}\,y,$$

which is the classical Sturm-Liouville form

$$My=-(p\,y')'+q\,y \tag{7.3}$$

with the notation $p=a_{12}^{-1}$ and $q=a_{21}$.

**Remark 8.** Note that, for (7.2), our assumptions on $a_{rs}$ are that $a_{12}\neq 0$ almost everywhere on $J$, $a_{12}$ and $a_{21}$ are both real, $a_{11}=ib_{11}=a_{22}$ with $b_{11}$ real, and

$$a_{12}^{-1},\, a_{21}, b_{11}\in L_{\mathrm{loc}}(J). \tag{7.4}$$

**Remark 9.** We make the following observations:

(i) When $b_{11}\neq 0$, (7.2) is a second order symmetric expression with nonreal coefficients in contrast with the classical case ($b_{11}=0$) where the coefficients of second order symmetric expressions must be real.
(ii) Although $a_{12}\neq 0$ almost everywhere on $J$, $p=a_{12}^{-1}$ may change sign, may be 0 at any number of points in $J$, and may even be

identically zero on one or more subintervals of $J$. (For an elaboration of the latter statement see the paper by Kong, Volkmer and Zettl **[72]** where a class of self-adjoint Sturm-Liouville problems is identified which are equivalent to (finite-dimensional) matrix problems.) But we don't discuss the degenerate case when $a_{12}^{-1}$ is zero on the entire interval $J$.

Since the general pattern for even order Lagrange symmetric matrices is not evident from the second order case $n = 2$, the next example is for $n = 4$. It is followed by $n = 3$ to illustrate the odd order case which has some features significantly different from the even order cases.

**Example 6.** Assume $A = (a_{rs}) \in Z_4(J)$ is Lagrange symmetric. Then, from (7.1), it follows that $A$ has the form

$$A = \begin{bmatrix} ia_{11} & a_{12} & 0 & 0 \\ a_{21} & ia_{22} & a_{23} & 0 \\ ia_{31} & a_{32} & ia_{22} & a_{12} \\ a_{41} & ia_{31} & a_{21} & ia_{11} \end{bmatrix}, \tag{7.5}$$

where the $a_{rs}$ are real.

Here $y^{[0]} = y$,

$$\begin{aligned} y^{[1]} &= a_{12}^{-1}(y' - ia_{11}y), \quad y \in V_1 \\ y^{[2]} &= a_{23}^{-1}\{y^{[1]\prime} - a_{21}y - ia_{22}y^{[1]}\}, \quad y \in V_2 \\ y^{[3]} &= a_{12}^{-1}\{y^{[2]\prime} - ia_{31}y - a_{32}y^{[1]} - ia_{22}y^{[2]}\},\ y \in V_3 \\ y^{[4]} &= \{y^{[3]\prime} - a_{41}y - ia_{31}y^{[1]} - a_{21}y^{[2]} - ia_{11}y^{[3]}, \quad y \in V_4. \end{aligned} \tag{7.6}$$

and $M = M_A$ is given by

$$My = i^4 y^{[4]} = y^{[4]}, \quad y \in V_4 = D(A). \tag{7.7}$$

If $a_{11} = 0 = a_{22} = a_{31} = a_{21}$ and $a_{12} = 1$, then (7.7) reduces to the modified Naimark form (3.5)

$$My = [(a_{23}^{-1}y'')' - a_{32}y']' - a_{41}y, \quad y \in V_4 = D(A), \tag{7.8}$$

with the notation $p = a_{23}^{-1}$, $r = a_{32}$, $q = -a_{41}$.

**Remark 10.** Without the extra assumptions that $a_{21} = 0$ and $a_{12} = 1$ (7.7) is a fourth order symmetric expression *with real coefficients* which has two more independent components than the Naimark form, namely, $a_{21}$ and $a_{12}$. Removing the restriction that $a_{11} = 0 = a_{22} = a_{31}$ produces symmetric quasi-differential expressions $M_A$ with three additional but complex coefficients.

**Example 7.** Assume $A = (a_{rs}) \in Z_3(J)$ is Lagrange symmetric. Then, from (7.1), it follows that $A$ has the form

$$A = \begin{bmatrix} ia_{11} & a_{12} & 0 \\ a_{21} & ia_{22} & a_{12} \\ ia_{31} & a_{21} & ia_{11} \end{bmatrix} \tag{7.9}$$

with $a_{rs}$ real.

Here $y^{[0]} = y$ and

$$\begin{aligned} y^{[1]} &= a_{12}^{-1}(y' - ia_{11}y), \quad y \in V_1 \\ y^{[2]} &= a_{12}^{-1}\{y^{[1]\prime} - a_{21}y - ia_{22}y^{[1]}\}, \quad y \in V_2 \\ y^{[3]} &= \{y^{[2]\prime} - ia_{31}y - a_{21}y^{[1]} - ia_{11}y^{[2]}\}, \quad y \in V_3. \end{aligned} \tag{7.10}$$

Then $M = M_A$ is given by

$$My = i^3 y^{[3]} = -iy^{[3]}. \tag{7.11}$$

The special case when $a_{11} = 0$, $a_{12} = -1/p$, $a_{21} = b_0/p$, $a_{22} = -ia_1/p^2$, $a_{31} = -ia_0$ and $p^2 = 2b_1$ reduces to the classical form (2.3) for $n = 3$ when $a_0, a_1$, $b_0, b_1$ and $y$ are sufficiently differentiable. This follows from a direct computation and the observation that

$$(p(py')')' = (b_1y')'' + (b_1y'')'.$$

Note that in this special case our coefficient hypotheses are that

$$a_{12}, a_{21}, a_{22}, a_{31} \in L_{\mathrm{loc}}(J, \mathbb{R})$$

and the relationship $p^2 = 2b_1$ imposes the sign restriction $b_1 > 0$ almost everywhere on $J$ in contrast with the even order case where there is no sign restriction on the leading coefficient. See **[45]** for additional comments about this point.

**8. Boundedness below the minimal operator.** In [**116,** page 109], Weidmann says:

*Operators of even order with positive coefficient of highest order have turned out to be semibounded from below if the lower order terms are sufficiently small. On the other hand, operators of odd order are usually expected to be not semibounded* (*from below or above*). *We shall see this in many examples.... Anyhow we do not know of a general result which assures this for every operator of odd order.*

For a class of operators larger than that studied by Weidmann, Möller and Zettl [**82**] proved–*without any smoothness or smallness assumptions on the coefficients*–that:

(1) Odd order operators, regular or singular, and regardless of the sign of the leading coefficient, are unbounded above and below.
(2) Regular operators of even order with positive leading coefficient are bounded below.
(3) Even order regular or singular operators with leading coefficient that changes sign are unbounded above and below.

In this section, we present a summary of these results, make some comments and mention open problems.

Recall that, if $A$ is L-symmetric, $M = M_A$ and $w$ is a weight function, then $M$ is a symmetric expression and $S_{\min}(M)$ is a symmetric operator in $L^2(J, w)$.

The following results were established in [**82**].

**Theorem 4** (Möller-Zettl). *Assume $A \in Z_n(J)$ satisfies* (7.1)*, and let $M = M_A$ be the symmetric expression generated by $A$ as above. Then*

(i) *If $n = 2k+1$, $k > 0$, then $S_{\min}(A)$ is unbounded above and below in $H = L^2(J, w)$ for any weight function $w$.*
(ii) *If $n = 2k$, $k > 0$, and $M$ is regular with positive leading coefficient, then $S_{\min}(A)$ is bounded below and unbounded above in $H = L^2(J, w)$ for any weight function $w$.*
(iii) *If $n = 2k$, $k > 0$, and the leading coefficient of $M$ changes sign on $J$, then $S_{\min}(A)$ is unbounded above and unbounded below in $H = L^2(J, w)$ for any weight function $w$. Note that $M_A$ may be regular or singular.*

*Proof.* See [**82**]. □

The hypothesis that $M$ is regular with positive leading coefficient cannot be removed in Theorem 4 (ii). For $k = 1$, the hypothesis '$M$ is regular with positive leading coefficient' can be extended to '$M$ has a positive leading coefficient and is regular or LCNO at each endpoint.' (See Definition 10 below for LCNO or see [**131**]). What is the corresponding extension for $k > 1$? In [**78**], Marletta and Zettl prove the following result:

**Theorem 5** ([**78**]). *Suppose $A = (a_{rs}) \in Z_n(J, \mathbb{R})$ with $n = 2k$, $k > 1$ is Lagrange symmetric, and assume that $A$ is of* GN *type, i.e., $a_{r,r+1} = 1$, for $1 \le r < k$ and $k < r < n$; $a_{k,k+1} > 0$ on $J$ and all other $a_{rs} = 0$ except $a_{r,n+1-r}$ for $r = 1, \ldots, k$. If the left endpoint $a$ is regular and the right endpoint $b$ is disconjugate in the sense of Reid, then $S_{\min}(A)$ is bounded below in $H = L^2(J, w)$ for any weight function $w$.*

*Proof.* See [**78**]. This proof uses methods from the theory of Hamiltonian systems and the theory of disconjugacy as developed by Reid [**91**]. Disconjugacy theory is closely related to nonoscillation theory. □

Although the next theorem is a special case of a well-known abstract result [**115**] we state it here for the sake of completeness even though it uses a definition and result from Sections 11 and 13 below. See Section 11 for a definition of deficiency index. The GKN theorem in Section 13 implies that every self-adjoint extension of the minimal operator $S_{\min}(A)$ is a finite dimensional extension.

**Theorem 6.** *Assume $A \in Z_n(J)$, $n = 2k$, $k \ge 1$, is Lagrange symmetric with equal deficiency indices $d = d^+ = d^-$. If $S_{\min}(A)$ is bounded below, then every self-adjoint extension of $S_{\min}(A)$ is bounded below.*

*Proof.* By [**115,** page 247, Corollary 2], we have: If $S_{\min}$ is bounded below with lower bound $c$ and $S$ is a self-adjoint extension of $S_{\min}$ then there exist at most a finite number of eigenvalues $\lambda_0$, $\lambda_1, \ldots$ of $S$ less than $c$. Thus, $S$ is bounded below by $\lambda_0$ or by $c$. □

Next, we discuss the semi-boundedness results for $S_{\min}$ in the second order case and then comment on the comparison with the higher order case.

Consider the equation

$$(8.1) \quad My = -(py')' + qy = \lambda wy \quad \text{on } J = (a,b); \qquad \frac{1}{p}, \quad q, w \in L_{\text{loc}}(J, \mathbb{R}), \ w > 0.$$

Note that $M = M_A$ where $A$ is a Lagrange symmetric matrix of the form

$$(8.2) \qquad A = \begin{bmatrix} 0 & \frac{1}{p} \\ -q & 0 \end{bmatrix}.$$

We briefly recall the needed definitions and basic facts for equation (8.1).

**Definition 10.** Let (8.1) hold and assume, in addition, that $p > 0$ on $J$. Recall that the endpoint $a$ of equation (8.1) is said to be in the limit-circle (LC) case if all solutions of (8.1) are in $L^2((a,c), w)$ for some $c$ in $J$ and some $\lambda \in \mathbb{C}$. Otherwise, $a$ is said to be in the limit-point (LP) case. The endpoint $a$ is classified as oscillatory (O) if there is a nontrivial solution with an infinite number of zeros in $(a,c)$ for some $c \in J$ and nonoscillatory (NO) otherwise. This classification is independent of $\lambda$ if $a$ is LC. (But not if $a$ is (LP).) Similar definitions are made at $b$. Each regular endpoint (when $p > 0$ on $J$) is limit-circle and nonoscillatory (LCNO). A singular endpoint may be LCNO or LCO–limit-circle and oscillatory. See the book [**131**] for details.

The next theorem summarizes well-known results for the second order case [**131**]:

**Theorem 7.** *Let* (8.1) *hold. Let* $H = L^2(J, w)$.

(i) *If $p$ changes sign on $J$, then $S_{\min}(A)$ is unbounded below and unbounded above in $H$.*
(ii) *Suppose $p > 0$ almost everywhere on $J$. If each endpoint is either regular or* LCNO, *then $S_{\min}(A)$ is bounded below, but not above, in $H$.*

(iii) *Suppose* $p > 0$ *almost everywhere on* $J$. *If one endpoint is* LCO, *then* $S_{\min}(A)$ *is unbounded below and unbounded above in* $H$.

*Proof.* Part (i) was proven by Möller in [**80**]. For parts (ii) and (iii) see Möller [**80**] or Niessen and Zettl [**87**]; see also [**131**]. □

In the next remark, we comment on the comparison between the second order results of Theorem 7 and the corresponding higher order results of Theorems 4 and 7.

**Remark 11.** Comparison between $n = 2$ and $n > 2$.

(i) Theorem 4 (iii) extends Theorem 7 (i) to the higher order case $n$ for any $n > 2$ even or odd.
(ii) Theorem 4 (ii) extends Theorem 7 (ii) to the even higher order case $n = 2k$, $k > 1$, *but only for regular problems.* (*Regular endpoints can be considered* LCNO.)
(iii) Regarding the extension of Theorem 7 (ii) from $n = 2$ to $n = 2k$ with $k > 1$ given by *Theorem* 5, *there are a number of natural questions*: (a) Is there a condition more transparent, i.e., easier to check than disconjugacy in the sense of Reid? (b) What if both endpoints are singular?
(iv) How does Theorem 7 (iii) extend from $n = 2$ to $n = 2k$ with $k > 1$? What is the 'appropriate' definition for LCO when the deficiency index $d = n$? When $d < n$?

**9. The Friedrichs extension.** Let $H$ be a Hilbert space with inner product $(\cdot,\cdot)$, and let $S$ be a closed densely defined bounded below symmetric operator with domain $D(S) \subset H$. In a celebrated paper [**49**], Friedrichs constructed a self-adjoint extension of $S$ in $H$ with the same lower bound as $S$. It has come to be known as the Friedrichs extension, and we denote it by $S_F$. In general, there are other self-adjoint extensions of $S$ which have the same lower bound as $S$ (see [**131**] for examples.)

The theorems in Section 8 identified classes of the densely defined symmetric minimal operators $S_{\min}(A)$ which are bounded below. In general, as we will see below, these minimal operators have an uncountable number of self-adjoint extensions. Which boundary condition determines the Friedrichs extension? This question is discussed in this section.

Friedrichs himself asked this question [**50**] and proved that, for the expression $-y'' + qy$ on $J = (a, b)$ with regular endpoints and $q$ real and continuous the Dirichlet condition

$$y(a) = 0 = y(b)$$

determines the Friedrichs extension. This result has been generalized by many authors for regular and singular problems. In the regular case, there is a very general result due to Niessen-Zettl [**86**] and Möller-Zettl [**82**]:

**Theorem 8.** *Suppose $A = (a_{rs}) \in Z_n(J, \mathbb{R})$ with $n = 2k$ $k \geq 1$ is Lagrange symmetric and each endpoint is regular. Then, for any weight function $w$, the Friedrichs extension of the minimal operator $S_{\min}(A)$ is determined by the Dirichlet boundary condition:*

$$y^{[j]}(a) = 0 = y^{[j]}(b), \quad j = 0, 1, \ldots, k-1.$$

*Proof.* See [**82, 86**]. □

Below, we will give a precise meaning to "determined by the Dirichlet boundary condition."

Next we discuss the second order singular case. Consider the equation

$$(9.1) \qquad -(py')' + qy = \lambda wy \quad \text{on } J, \ \frac{1}{p}, \ q, w \in L_{\text{loc}}(J, \mathbb{R}),$$
$$p > 0, \quad w > 0, \ \lambda \in \mathbb{R}.$$

Note that we have added the condition that $p > 0$ since the minimal operator is not bounded below when $p$ changes sign. Let

$$A = \begin{bmatrix} 0 & \frac{1}{p} \\ -q & 0 \end{bmatrix},$$

and let $M = M_A$. Then $A \in Z_2(J, \mathbb{R})$, $M$ is Lagrange symmetric and $S_{\min} = S_{\min}(A)$ is a symmetric operator in $H = L^2(J, w)$ for any weight function $w$. The next theorem uses the notion of principal solution; see [**87**] for a definition and for a proof. Also see [**131**].

**Theorem 9.** *Suppose each endpoint is either regular or* LCNO, $u_a$ *is the principal solution at* $a$ *and* $u_b$ *is the principal solution at* $b$. *Then the minimal operator* $S_{\min}$ *is bounded below, and its Friedrichs extension is given by the self-adjoint boundary condition:*

$$[y, u_a](a) = 0 = [y, u_b](b). \tag{9.2}$$

*Recall that the principal solution at an endpoint is unique up to constant real multiples so that* (9.2) *does not depend on the choice of principal solutions. Also recall that, if* $a$ *is regular* $[y, u_a](a) = 0$ *reduces to* $y(a) = 0$ *and similarly for* $b$. *Also recall that* $[y, z]$ *is the Lagrange sesquilinear form* $[y, z] = y(p\overline{z}') - \overline{z}(py')$.

What is a natural extension of Theorem 9 to the higher order case $n = 2k$, $k > 1$? (In the odd order case the minimal operator is not bounded below so there is no Friedrichs extension.) There are two major obstacles to extending Theorem 9 to $k > 1$:

1. What is the appropriate definition of 'principal solution'?

2. At a singular endpoint the number of boundary conditions depends on the deficiency index $d$, and $d$ can have any value in the range: $k \leq d \leq 2k = n$. In the case $n = 2$, $d = 1$ which is known as the LP case or $d = 2-$ the LC case. For $k > 1$, there are the 'intermediate' cases when $k < d < 2k = n$.

These two obstacles are 'connected': How do you define principal solution so that the correct number of boundary conditions are obtained?

In [**78**], Marletta and Zettl characterize the Friedrichs extension for Lagrange symmetric expressions $A$ of GN type. They obtain this characterization using methods from the theory of Hamiltonian systems, the theory of disconjugacy near a singular endpoint as developed by Reid [**91**] and a limiting process using Theorem 8 for regular problems. Disconjugacy theory is closely related to oscillation theory or, more accurately, nonoscillation theory.

Before the appearance of [**78**] few results were known for singular problems when $n > 2$. An exception is given by a result of Zettl [**130**]. This result is for a narrow class of problems and does not follow in any apparent way from the much more general result in [**78**]. It circumvents the two obstacles mentioned above and has a number of other interesting special features which we will comment on below in this section.

**Theorem 10.** *For $n = 2k$, $k > 1$ and $w$ a weight function, consider*

$$My = (-1)^k y^{(n)} + qy = \lambda wy \quad \text{on } J = (a, b), \tag{9.3}$$

*where $q$ and $w$ satisfy*

$$q \in L_{\text{loc}}(J, \mathbb{R}), \qquad Q = \int q \in L(J, \mathbb{R}), \quad w \in L(J) \tag{9.4}$$

*where $Q$ is an anti-derivative of $q$. Then*

(i) *The minimal operator $S_{\min}(M)$ is bounded below in $H = L^2(J, w)$.*
(ii) *For all $y \in D_{\max}(M)$, the limits*

$$\lim_{t \to a^+} y^{[j]}(t), \quad \text{and} \quad \lim_{t \to b^-} y^{[j]}(t)$$

*exist and are finite for $j = 0, 1, \ldots, n-2$ (but not for $j = n-1$ in general).*
(iii) *The domain $D_F$ of the Friedrichs extension of $S_{\min}(M)$ is given by the self-adjoint boundary conditions:*

$$y^{(j)}(a) = 0 = y^{(j)}(b), \quad j = 0, 1, \ldots, k-1. \tag{9.5}$$

*Proof.* Under hypothesis (9.4), the endpoints may be singular. In general, the limits (9.5) do not exist at a singular endpoint, so the existence of these limits is part of the theorem. Let

$$A = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -q & & & & \end{bmatrix}, \quad B = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ -Q & & & & 1 \\ & Q & & & \end{bmatrix}, \tag{9.6}$$

where all the nonspecified entries are zero. Note that $A$ and $B$ are in $Z_n(J, \mathbb{R})$ and both are Lagrange symmetric. We compute the quasi-derivatives and note that $y_A^{[j]} = y^{(j)} = y_B^{[j]}$, $j = 0, 1, \ldots, n-2$; and

$$y_A^{[n-1]} = y^{(n-1)} \quad \text{but} \quad y_B^{[n-1]} = (y^{(n-1)} + Qy).$$

Hence, $M_A\, y = i^4 y_A^{[n]} = y^{(n)} + qy$, and

$$M_B\, y = i^4 y_B^{[n]} = \{(y_B^{[n-1]}) + Qy\}' - Qy' = y^{(n)} + qy.$$

Therefore, $D(A) = D(B)$ and $M_A = M_B$. By hypothesis, (9.4) $M_B$ is regular on $J$, and therefore the conclusions follow from Theorem 8. $\square$

Next we give an example to illustrate Theorem 10.

**Example 8.** For $n = 2k$, $k > 1$, consider

$$\begin{aligned} &My = (-1)^k y^{(n)} + qy = \lambda wy \quad \text{on } J = (0,1),\\ &q(t) = t^r, \quad t \in J,\ r \in (-2,1] \end{aligned} \tag{9.7}$$

Note that $w = 1$, the endpoint $a = 0$ is singular, the endpoint $b = 1$ is regular and the hypothesis (9.4) holds. Hence, the Friedrichs extension is given by the self-adjoint boundary conditions

$$y^{(j)}(0) = 0 = y^{(j)}(1), \quad j = 0, 1, \dots, k-1. \tag{9.8}$$

**Remark 12.** In this example, by Theorem 10, we have that for all $y \in D_{\max}(M)$, the limits

$$\lim_{t\to 0^+} y^{[j]}(t), \quad \text{and} \quad \lim_{t\to 1^-} y^{[j]}(t) \tag{9.9}$$

exist and are finite for $j = 0, 1, \dots, n-2$ .What about $j = n-1$? The limit $\lim_{t\to 1^-} y^{[n-1]}(t)$ exists since the endpoint $b = 1$ is regular. In this case, the limit $\lim_{t\to 0^+} y^{[n-1]}(t)$ does not exist because if it did then that and the fact that $y_B^{[n-1]} = (y^{(n-1)} + Qy)$ has a finite limit at 0 would imply that $Q$ has a finite limit at 0, which is not the case by inspection. For this reason, the quasi-derivative $y_B^{[n-1]}$ is used in the formulation of the general self-adjoint boundary conditions for equation (9.7). Only those self-adjoint boundary conditions can be expressed solely in terms of classical derivatives which involve derivative orders strictly less than $n-1$. The Friedrichs extension is one of these since it involves only derivatives up to order $k-1$, and these are all classical in this case.

**Remark 13.** Since $M = M_A = M_B$, we can give the characterization of the singular self-adjoint boundary conditions for $M_A$ by using the regular representation $M_B$. Thus, we have: all self-adjoint realizations of the problems (9.3) and (9.4) are given by

$$A\begin{pmatrix} y(a) \\ y^{[1]}(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B\begin{pmatrix} y(b) \\ y^{[1]}(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{9.10}$$

where $y^{[j]} = y^{(j)}$, $j = 0, 1, \dots, n-2$, and $y^{[n-1]} = (y^{(n-1)} + Qy)$, and the $n \times n$ complex matrices $A, B$ satisfy

$$\operatorname{rank}(A : B) = n; \tag{9.11}$$

and

$$A\,E\,A^* = B\,E\,B^*, \qquad E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{9.12}$$

**10. The deficiency index.** Above we showed that every Lagrange symmetric matrix $A$ generates a symmetric expression $M_A$ and that each such expression $M_A$ generates a minimal operator $S_{\min}(A)$ and a maximal operator $S_{\max}(A)$ in a Hilbert space $H = L^2(J, w)$ for any weight function $w$. The operator $S_{\min}(A)$ is densely defined closed and symmetric in $H$. It is well known that $S_{\min}(A)$ has self-adjoint extensions in $H$ if and only if its deficiency indices are equal: $d(A) = d^+(A) = d^-(A)$. Any such self-adjoint extension of $S_{\min}(A)$ is also a self-adjoint restriction of the maximal operator $S_{\max}(A)$. These restrictions are determined by boundary conditions on the maximal domain $D_{\max}$. The number of independent boundary conditions required is given by the deficiency index $d(A)$. The characterization of these self-adjoint boundary conditions is discussed in Sections 12–18.

In this section we define the deficiency indices $d^+(A)$ and $d^-(A)$ of symmetric differential operators $M_A$ generated by Lagrange symmetric matrices $A$ and state the basic classification results for them. We also introduce the deficiency index problem, comment on its history, and introduce some of its literature.

**Definition 11.** Assume that $A \in Z_n(J)$ is Lagrange symmetric. Let $M = M_A$ be as constructed above, let $w$ be a weight function, and let $H = L^2(J, w)$.

(1) For $\lambda \in \mathbb{C}$, let $d(\lambda)$ denote the number of linearly independent solutions of the equation

$$M\,y = \lambda w \quad y \text{ on } J \tag{10.1}$$

which lie in $H$.
(2) For $\lambda \in \mathbb{R}$, let $r(\lambda) = d(\lambda)$.
(3) Let $d^+ = d(i)$ and $d^- = d(-i)$.
(4) $d^+$ and $d^-$ are called the deficiency indices of $(M, w)$ on $H = L^2(J, w)$. When $d^+ = d^-$, we speak of the deficiency index $d = d^+ = d^-$ of $(M, w)$ on $H = L^2(J, w)$.

**Theorem 11.** *Let $A \in Z_n(J)$ be Lagrange symmetric, let $M = M_A$, and let $w$ be a weight function. Then $d(\lambda_1) = d(\lambda_2)$ if $\mathrm{Im}\,(\lambda_j) > 0$ or if $\mathrm{Im}\,(\lambda_j) < 0$ for $j = 1, 2$.*

*Proof.* This follows from a well-known abstract theorem [**84,** page 33, Theorem 5]. □

**Remark 14.** There is a sharp contrast between the behavior of $d(\lambda)$ for $\lambda$ nonreal and $\lambda$ real; $d(\lambda)$ is constant in the upper complex plane and in the lower complex plane. For $A \in Z_n(J, \mathbb{R})$, these constants are the same, but for $A \in Z_n(J, \mathbb{C})$ these constants may be different. This was first shown by McLeod [**79**]. Since the spectrum of a self-adjoint operator in Hilbert space is real, $r(\lambda)$ contains spectral information about the self-adjoint realizations of the equation (10.1) as we will see below. This is our motivation for introducing the notation $r(\lambda) = d(\lambda)$ for $\lambda \in \mathbb{R}$.

Next we discuss deficiency index inequalities.

**Notation 3.** Since $d^+$and $d^-$depend on $(M, w)$ and on the interval $J$ we indicate this dependence by using the notation $d^+ = d^+(M, w, J)$, $d^- = d^-(M, w, J)$. Observe that the deficiency indices are well defined when $J$ is replaced by any subinterval of $J$. We are particularly interested in the subintervals $(a, c)$ and $(c, b)$ of $J$, but for any subinterval $(\alpha, \beta)$ of $J$, we use the notation $d^+ = d^+(M, w, (\alpha, \beta))$, $d^- = d^-(M, w, (\alpha, \beta))$. (In this case, $M$ and $w$ are replaced by their restrictions to the subinterval.) When we study the dependence on $M$ with $w$ and $J$ fixed, we abbreviate the notation to $d^+(M)$, $d^-(M)$ and similarly for $d^+(w)$, $d^-(w)$; $d^+(J)$, $d^-(J)$. Below, we will use the notation $d_a^+ = d^+(M, w, (a, c))$, $d_a^- = d^-(M, w, (a, c)$, $d_b^+ = d^+(M, w, (c, b)$, $d_b^- = d^-(M, w, (c, b)$. Note that this is independent of $c$ for $a < c < b$, and so when $d_a^+ = d_a^-$, we use the notation $d_a = d_a^+ = d_a^+$; similarly for $b$. The deficiency index problem is the problem of determining $d^+$ and $d^-$ (or $d$ if $d^+ = d^-$) in terms of the coefficients and weight function: $d^\pm = d^\pm(a_{ij}, w;\ 1 \le i, j \le n)$. Here $d^+ = d^+(M, w, (\alpha, \beta))$, $d^- = d^-(M, w, (\alpha, \beta))$ for any open subinterval $(\alpha, \beta)$ of $J$ where $M = M_A$ is constructed as above and $M$ and $w$ are the restrictions of $M$ and $w$ on $(\alpha, \beta)$.

The subintervals $(a, c)$ and $(c, b)$ where $c$ is in $(a, b)$ are of special interest. One reason for that is the following result which is sometimes referred to as Kodaira's formula:

**Theorem 12.** *Let $A \in Z_n(J)$ be Lagrange symmetric, let $M = M_A$, let $w$ be a weight function, and let $c \in J = (a, b)$. Then*

$$d^\pm(M, w, J) = d_a^\pm(M, w) + d_b^\pm(M, w) - n. \tag{10.2}$$

*Proof.* The proof in [**84,** page 72] can readily be adapted to this more general result. $\square$

Since any $c \in J = (a, b)$ is a regular endpoint for both intervals $(a, c)$ and $(c, b)$, Theorem 12 reduces the problem of determining $d^+$ and $d^-$ when both endpoints are singular to the case when one endpoint is regular.

So, for the rest of this section, we present the deficiency index classification for the case when the endpoint $a$ is regular and $b$ is singular. The case when $a$ is singular and $b$ is regular is similar.

**Theorem 13** (Deficiency index classification)**.** *Let $A \in Z_n(J)$ be Lagrange symmetric, $J = (a, b)$, and assume that $a$ is regular. Let $M = M_A$, and let $w$ be a weight function. Then the deficiency indices $d^+ = d^+(M, w, J)$ and $d^- = d^-(M, w, J)$ satisfy the following inequalities*:

(i) *If* $n = 2k$, *then*

$$k \le d^+,\, d^- \le n. \tag{10.3}$$

(ii) *If* $n = 2k+1$, $k \ge 1$, *and the leading coefficient of* $M$ *is positive on* $J$, *then*

$$k \le d^+ \le n; \quad k-1 \le d^- \le n. \tag{10.4}$$

(iii) *If* $n = 2k+1$, $k \ge 1$, *and the leading coefficient of* $M$ *is negative on* $J$, *then*

$$k-1 \le d^+ \le n; \quad k \le d^- \le n. \tag{10.5}$$

*These inequalities are best possible.*

*Proof.* The right inequalities are clear since there cannot be more than $n$ linearly independent solutions. For the inequalities on the left in (10.3) see the operator-theoretic proof of Kogan and Rofe-Beketov [**71**] or the more classical proof of Everitt [**32, 33, 34**]. For the inequalities on the left of (10.4) and (10.5) the methods given in [**32, 34**] can be adapted; see also [**69, 71**]. □

In general, $d^+ \ne d^-$, but if one of these is maximal then so is the other one. In fact, there is an even stronger result given by the next theorem.

**Theorem 14.** *Let* $A = (a_{ij}) \in Z_n(J)$ *be Lagrange symmetric,* $J = (a,b)$, *let* $M = M_A$ *and let* $w$ *be a weight function. If, for some* $\lambda \in \mathbb{C}$, *all solutions of*

$$My = \lambda w y \quad \text{on } J$$

*are in* $L^2(J,w)$, *then this is true for every* $\lambda \in \mathbb{C}$. (*Note that real values of* $\lambda$ *are included here.*)

*Proof.* See [**30,** Theorem 9.1]. □

**Definition 12.** We say that the symmetric expression $M = M_A$ is in the limit-circle case, or LC for short, if its deficiency indices are maximal. Note that, by Theorem 14, in the maximal case we have $d^+ = d^- = n$. We say that the symmetric expression $M = M_A$ is in the limit-point case, or LP for short, if its deficiency indices according to Theorem 13 are minimal. Thus, for $n = 2k$ and one regular endpoint, this means $d^+ = d^- = k$, for $n = 2k + 1$ and one regular endpoint this means that $d^+ = k$ and $d^- = k - 1$ if the leading coefficient is positive and $d^+ = k - 1$ and $d^- = k$ if the leading coefficient is negative. This limit-point and limit-circle terminology has its roots in the seminal paper of Weyl [**117**] in which, for $n = 2$, he constructed a sequence of concentric circles in the complex plane which converged either to a circle or a point. Although most authors now use a Hilbert space approach pioneered by Glazman [**54**], especially for $n > 2$, the Weyl terminology, abbreviated here as LC and LP, is widely used even when $n > 2$.

**Remark 15.** We comment on the possible values of $d^+$, $d^-$. In view of Theorem 12, we can restrict ourselves to the case when one endpoint is regular and the other singular, say $a$ is regular, $b$ singular.

(1) If all $a_{ij}$ are real, then $My = \lambda wy$ if and only if $M\overline{y} = \overline{\lambda}w\overline{y}$. From this and the fact that $y \in L^2(J, w)$ implies that $\overline{y} \in L^2(J, w)$, it follows that $d^+ = d^-$.

For any $n = 2k$, Glazman [**54**] has shown that there exist symmetric expressions whose deficiency index $d = r$ for any $r$, $k \leq r \leq n = 2k$. Later Orlov and Read [**90**] constructed simpler examples. See also [**69**], and see the next remark for a historical comment.

(2) The lower bounds in (10.4) and (10.5) are achieved by simple constant coefficient expressions. McLeod [**79**] was the first to show that $d^+$, $d^-$ can be different in the even order case when the coefficients are complex. In [**71**], Kogan and Rofe-Beketov showed that all possibilities not ruled out by Theorems 13 and 14 and by

$$|d^+ - d^-| \leq 1$$

actually occur. Gilbert in [**52, 53**] has shown that there exist symmetric differential expressions $M$ such that

$$|d^+(M) - d^-(M)| = p$$

for any positive integer $p$ provided the order of $M$ is large enough.

**Remark 16.** It seems that all possibilities for $d^+$, $d^-$ not ruled out by Theorems 13 and 14 actually occur. But this is still an open problem.

In particular, (i) there is no symmetric expression $M$ of order 5 known to the authors with $d^+(M) = 2$ and $d^-(M) = 4$ (or $d^+(M) = 4$ and $d^-(M) = 2$); (ii) there is no known example with $n = 6$, $d^+ = 5$, $d^- = 3$ (or $n = 6$, $d^+ = 3$, $d^- = 5$), (iii) $n = 8$, $d^+ = 7$, $d^- = 5$, or $n = 8$, $d^+ = 6$, $d^- = 4$, etc. Note that, by Theorem 14, if one of $d^+$, $d^-$ is maximal, then so is the other, i.e., $d^+ = n$ if and only if $d^- = n$ and, more generally, if $d(\lambda) = n$ for some $\lambda \in \mathbb{C}$ then $d(\lambda) = n$ for every $\lambda \in \mathbb{C}$ (including the real values of $\lambda$).

In honor of the work of Glazman **[54]** and Naimark **[84]** we call the matrices $A$ these authors used in their development of the theory of symmetric and self-adjoint differential operator matrices of GN type.

**Definition 13.** A Lagrange symmetric matrix $A = (a_{ij}) \in Z_n(J)$ is of GN type if $n = 2k$, $k = 1, 2, 3, \ldots$; all $a_{ij}$ are real, and all $a_{ij} = 0$ except for the following:

(1) $a_{i,i+1} = 1$ for all $i = 1, \ldots, n-1$, except when $i = k$, then $a_{k,k+1}^{-1} \in L_{\mathrm{loc}}(J, \mathbb{R})$; and
(2) $a_{n,1}, a_{n-1,2}, \ldots, a_{k,k+1} \in L_{\mathrm{loc}}(J)$.

Thus, for $n = 2, 4, 6$, a GN matrix $A$ has the following form:

$$A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 1 \\ a_{41} & 0 & 0 & 0 \end{bmatrix}, \tag{10.6}$$

$$A = \begin{bmatrix} & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & a_{34} & 0 & 0 \\ & & a_{43} & & 1 & 0 \\ & a_{52} & & & & 1 \\ a_{61} & & & & & \end{bmatrix}. \tag{10.7}$$

Recall that the matrices of GN type generate the quasi-derivative forms of the real classical symmetric expressions (2.4), see Section 3 above.

For symmetric expressions $M$ generated by matrices $A$ of GN type there is a vast literature studying the dependence of $d(M) = d(a_{n,1}, a_{n-1,2}, \dots, a_{k,k+1}; w) = d(a_0, a_1, \dots, a_k; w)$ on these coefficients when $w = 1$. Not much is known for general $w$, but see [**105**] for an exception. (Here the notation $a_{n,1}, a_{n-1,2}, \dots, a_{k,k+1}$ corresponds to $a_0, a_1, \dots, a_k$ used in (2.4).)

**Remark 17.** $n = 2$. This case has a voluminous literature dating back at least to the seminal 1910 paper of Weyl [**117**] discussing the dependence of $d$ on $a_1$ and $a_0$. (The standard notation for this case is $p = a_1$, $q = a_0$.) In this case, with one regular and one singular endpoint, we have either $d = 1$ (LP) or $d = 2$ (LC). Many sufficient and some necessary conditions are known for LP. There is also a necessary and sufficient condition known [**69**] but it cannot be checked in every case. So in general, despite the vast literature on this problem (partly because of its interest in quantum mechanics where $a_1 = 1$ and $a_0$ is the potential function for the one dimensional Schrödinger equation), the problem is still open. See Kauffman, Read and Zettl [**69**] for an extensive but not comprehensive (and not up to date) discussion of the case $n = 2$. Also see [**21, 24, 131**].

**Remark 18.** $n = 2k$, $k > 1$. This case has an interesting history and also an extensive literature. The classification (10.3) for real coefficients when $d^+ = d^- = d : k \leq d \leq n$ was established by Weyl [**117**] in 1910 for the case $n = 2$ and by Glazman in 1950 [**54**] for general $n$. Glazman showed that all values of $d$ are realized. Simpler examples were later found by Orlov [**88**] and by Read [**90**], also see [**69**]. In this 40 year interval Windau [**118**] in 1921 and Shin in 1936–1940 claimed to have established that only the two cases $d(M) = k$ (LP) and $d(M) = 2k$ (LC) are possible. In the papers [**96**]–[**99**], Shin also discovered the general Lagrange symmetric quasi-differential expressions discussed here which were rediscovered, in a somewhat different but equivalent form, by Zettl [**123, 124**].

**Remark 19.** Assume that one endpoint is regular. Most of the known conditions on the coefficients in the even order case are for $d = k$, see [**29**], as well as Kauffman, Read and Zettl [**69**]. The few results known for the even order intermediate cases $k < d < 2k$ and the odd order case are based on the asymptotic form of solutions, see [**24, 84**], Gilbert [**52, 53**], the book by Rofe-Beketov and Kholkin [**93**] and its references. There are some exceptions. For the fourth order case see [**25**] where it is shown that $d \neq 2$ and $d \neq 4$, and therefore $d$ must be 3.

**Remark 20** (Delicate nature of deficiency index)**.** Assume one endpoint is regular and $A \in Z_n(J, \mathbb{R})$ so that, for $M = M_A$, we have $d(M) = d^+(M) = d^-(M)$.

For $n = 2$ and $My = -(py')' + qy$, it is known that when $d(M) = 2$, given any positive $\varepsilon$, it is possible to change the coefficients $p, q$ on a set of Lebesgue measure less than $\varepsilon$ such that, for the changed $M$, we have $d(M) = 1$. It is possible to find sufficient conditions for $d(M) = 1$ which are given only on a sequence of intervals. See [**69,** Chapter III].

For $n = 2k$, $k > 1$, and one regular endpoint, $k \leq d(M) \leq 2k$. For any $d(M) > k$, it is possible to change the coefficients of $M$ only on a sequence of intervals such that, for the changed $M$, we have $d(M) = k$.

Thus, for $n = 2$ and $n = 2k$, $k > 1$, there are strong 'interval-type' sufficient conditions for the LP case $d(M) = k$ but not for any of the cases $d(M) > k$. Thus, the LP case is 'special' and, for all the other cases $d(M) > k$, including the LC case, the dependence of $d(M)$ on the coefficient is delicate. There are no 'interval-type' sufficient conditions known for the cases $d(M) > k$. See [**69**], especially page 45, for details.

**11. Powers of differential expressions and their deficiency index.** In this section, for any Lagrange symmetric matrix $A = (a_{ij}) \in Z_n(J)$, we study the relationship between the deficiency indices of the symmetric expression $M = M_A$ and its powers $M^s$, $s = 1, 2, 3, \ldots$. These powers $M^s$ of $M$ are defined naturally by induction: $M^2(y) = M(My)$, $M^3(y) = M^2(My), \ldots$ and are symmetric expressions generated by Lagrange symmetric matrices $A^{[s]} \in Z_{ns}(J)$ to be constructed below. In sharp contrast with the classical case (2.3) and (2.4) this construction requires no smoothness assumptions on the coefficients, nor any other additional assumptions. Moreover, this construction applies to polynomials $p(M)$, where $p$ is any polynomial with real coefficients. Thus, the coefficients $a_{ij}$ and a given weight function $w$ determine sequences $d^+(M^s)$, $d^-(M^s)$, $s = 1, 2, 3, \ldots$, of deficiency indices. Here we give the deficiency index classification theory for these powers $M^s$, i.e., the analogue of Theorem 13 for powers $M^s$. And we give a complete description of the possible such sequences for $M = M_A$ where $A \in Z_{2k}(J, \mathbb{R})$ is Lagrange symmetric. In this case, the deficiency indices are equal so there is only one sequence $d(M)$, $d(M^2)$, $d(M^3), \ldots$, and this entire sequence is determined by the coefficients $a_{ij}$ and $w$.

Our construction of the matrices $A^{[s]}$ which generate the powers $M^s = M_{A^{[s]}}$ is based on a method of Zettl [**124**] and will be given below. Our deficiency index results are based on the Kauffman, Read and Zettl monograph [**69**]. Although these authors studied powers $M^s$ of the classical symmetric expressions $M$ given by (2.3) and (2.4) on the interval $J = (0, \infty)$ with 0 a regular endpoint and with weight function $w = 1$ on $J$, their proofs readily extend not only to general $J$ and $w$ but also, when combined with the extension of the Glazman-Naimark theory given above, to $M = M_A$ and its powers $M^s = M_{A^{[s]}}$. In particular, the proofs given in [**69**] make no use whatsoever of the strong smoothness assumptions on the coefficients other than for the construction of the powers of the classical expressions $M^s$ (so that these also have the form (2.3) and (2.4)). Thus, we state our results in this section for general symmetric expressions as generated above and their powers and refer to [**69**] for most of the proofs.

Given

$$My = -(py')' + qy \quad \text{on } J, \tag{11.1}$$

how are the deficiency indices of $M^2$ related to those of $M$? Is it possible to deduce the deficiency indices of $M^2$ from those of $M$ or vice versa? Chaudhury and Everitt [**20**] investigated these questions and showed that, for sufficiently smooth real coefficients, $d(M^2) = 4$ if and only if $d(M) = 2$ in $L^2(J, 1)$ for $J = [0, \infty)$ where 0 is a regular endpoint and they found examples to show that when $d(M) = 1$, then $d(M^2)$ may be either 2 or 3. This subject then received great impetus from a paper of Everitt and Giertz [**37**] in which they introduced the concept of partial separation and gave sufficient conditions on the coefficients of a second order $M$ which guarantees that all powers $M^s$ are partially separated, and they then show that all powers are LP. In [**122**], Zettl extended some of the results in [**37**] to $M$ of arbitrary order and showed that they hold also for $p(M)$ for any polynomial $p$ with real coefficients.

We start with the construction of powers of quasi-differential expressions given in [**129**].

**Theorem 15.** *Assume that* $A = (a_{ij}) \in Z_n(J)$ *is a Lagrange symmetric matrix. Let* $M = M_A$ *be the symmetric differential expression as generated above, and define* $M^2$ *by* $M^2 y = M(My), \dots, M^s y = M(M^{s-1}y)$. *Let* $A^{[1]} = A$ *and, for* $s \in \mathbb{N}_2$, *let* $A^{[s]}$ *denote the block diagonal matrix*

$$A^{[s]} = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} \tag{11.2}$$

*where there are* $s$ *matrices* $A$ *on the diagonal and all other entries in this* $sn \times sn$ *matrix are zero except for the entries in positions* $(n, n+1), (2n, 2n+1), \dots, ((s-1)n, (s-1)n+1)$ *which are all equal to* 1. *Then the matrices* $A^{[s]}$ *are in* $Z_{sn}(J)$, *are Lagrange symmetric, and the symmetric differential expression* $M^s$ *is given by*

$$M^s = M_{A^{[s]}}, \quad s \in \mathbb{N}. \tag{11.3}$$

*Proof.* The Lagrange symmetry follows from the characterization

$$a_{ij} = (-1)^{i+j-1}\overline{a}_{n+1-j,n=1-i}$$

applied to (11.2). The construction follows from a direct computation using the construction in Section 4. Note that, for the construction of $M^2$, $y$ is replaced by $My$ due to the 1 in the $(n, n+1)$ position to get $M^2$. Then $My$ is replaced by $M^2y$, and the construction is again repeated using a 1 in the $(2n, 2n+1)$ position to obtain $M^3$, etc. □

**Remark 21.** It is interesting to observe that, if, in the above construction, $A$ is of GN type, the matrices $A^{[s]}$ for $s > 1$ *are not of* GN *type*. Thus, the theory developed by Naimark [**84**] and by Glazman [**54**] for matrices $A$ of GN type does not directly apply to $M^s = M_{A^{[s]}}$ for $s > 1$, but the extension of this theory developed by Zettl [**124**] and by Everitt and Zettl [**45**] does apply. But it should be mentioned that the extended theory in [**45, 124**], uses some of the abstract Hilbert space results of Glazman [**54, 55**] which were developed by Glazman in extending the Weyl limit-point, limit-circle theory from $n = 2$ to general even order $n = 2k$ using Hilbert space methods rather than extending the Weyl concentric circles approach.

**Remark 22.** We also note that, in the classical theory, the minimal domain contains the $C_0^\infty$ functions and is therefore dense. With just local integrability assumptions on the coefficients, the minimal domain may not contain the $C_0^\infty$ functions. With only local integrability assumptions on the coefficients it is, in general, not easy to find non zero functions in the minimal domain. Nevertheless, Naimark [**84**] contains a proof that the domain of the minimal operator is dense under only local integrability conditions on the coefficients for expressions of GN type, and Zettl [**124**] has shown that Naimark's proof extends to the expressions used here.

**Remark 23.** Many authors who have studied products and powers of differential expressions have placed strong differentiability assumptions on the coefficients; in many cases, it was assumed that the coefficients are $C^\infty$, e.g., Dunford and Schwarz [**24**]. These assumptions were then used only to construct these powers and products in the classical way. It follows from our development based on $Z_n(J)$ and the associated quasi-derivatives that (i) the results of these authors are valid with just the local integrability assumptions used here, and (ii) these results hold for the much more general differential expressions developed here. In particular, these conclusions apply to the following papers: [**20, 26, 29, 30, 35, 37, 44, 67, 68, 69**].

There are not many sufficient conditions on the coefficients known for the maximal deficiency case. Next we strengthen Theorem 14. This gives a simple way to construct higher order equations with maximal deficiency index. The elementary proof is based on two lemmas which we establish first.

**Lemma 5.** *Let $M = M_A$ with $A$ a Lagrange symmetric matrix in $Z_n(J)$, $\lambda$ a complex number, $w$ a weight function and $p$ a real polynomial. If $My = \lambda wy$, then $p(M)y = p(\lambda)wy$.*

*Proof.* Let $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, $a_j$ real. Then:

$$\begin{aligned} p(M)y &= (a_k M^k + a_{k-1}M^{k-1} + \cdots + a_1 M + a_0)y \\ &= (a_k\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0)y \\ &= p(\lambda)y. \end{aligned}$$

□

**Lemma 6.** *Let the hypotheses and notation of Lemma* 5 *hold. If* $\lambda_1, \dots, \lambda_k$ *are distinct complex numbers and, for each* $j$, $z_q^j$ *with* $q = 1, \dots, m$ *are linearly independent solutions of* $My = \lambda_j wy$, *then the set of functions* $\{z_q^j,\ j = 1, \dots, k; q = 1, \dots, m\}$ *is linearly independent.*

*Proof.* Suppose there exist complex numbers such that $c_q^j$, $j = 1, \dots, k$; $q = 1, \dots, m$ such that $y_1 + \cdots + y_k = 0$, where $y_j = c_1^j z_1^j + c_2^j z_2^j + \cdots + c_m^j z_m^j$ for $j = 1, \dots, k$. Then $M(y_1 + \cdots + y_k) = (\lambda_1 y_1 + \cdots + \lambda_k y_k) = 0$. Repeated applications of $M$ yield

$$(\lambda_1)^r y_1 + \cdots + (\lambda_k)^r y_k = 0, \quad r = 0, \dots, k-1. \tag{11.4}$$

The $k \times k$ coefficient of the homogeneous system (11.4) is a Vandermonde matrix whose determinant is not zero. Hence, $y_i = 0$ for $i = 1, \dots, k$.

Therefore, by the linear independence of $(z_1^j, z_2^j, \dots, z_k^j)$, it follows that $c_q^j = 0$, $j = 1, \dots, k$; $q = 1, \dots, m$. □

**Theorem 16.** *Let* $A$ *in* $Z_n(J)$ *be a Lagrange symmetric matrix, and let* $M = M_A$. *If, for some* $\lambda \in \mathbb{C}$, *all solutions of*

$$My = \lambda wy \quad \text{on } J$$

*are in* $L^2(J, w)$, *then this is true for all solutions of*

$$M^s y = \lambda wy \quad \text{on } J$$

*for every* $\lambda \in \mathbb{C}$ *and every* $s \in \mathbb{N}$.

*Proof.* By Theorem 14, if all solutions of an equation $My = \lambda wy$ are in $L^2(J, w)$ for some $\lambda \in \mathbb{C}$, then this is true for all $\lambda \in \mathbb{C}$. (This can be proved using the variation of parameters formula.) Choose a complex number $C$ such that the roots, say $\lambda_1, \dots, \lambda_k$, of the real polynomial equation $p(x) = C$ are distinct. For each $j \in \{1, \dots, k\}$, let $z_1^j, z_2^j, \dots, z_m^j$ be linearly independent solutions of $My = \lambda_j wy$. By Lemma 5, each $z_q^j$ is a solution of

$$p(M)y = Cwy$$

and, by Lemma 6, the $z_q^j$, $j = 1, \dots, k$; $q = 1, \dots, m$ form a fundamental set of solutions of $p(M)y = Cwy$. Hence, $p(M)$ is LC since each $z_q^j$ is in $L^2(J, w)$.

On the other hand, if $p(M)$ is LC, choose $\lambda_j$ as above and conclude that all solutions of $My = \lambda_1 wy$ are in $L^2(J, w)$, i.e., $M$ is LC in $L^2(J, w)$. $\square$

Next, we extend the general classification result given by Theorem 13 for $M$ to powers $M^s$. Recall that, by Theorem 12, the case when both endpoints are singular reduces to the case when one endpoint is regular. So we state the results for the case when $a$ is regular; the case when $b$ is regular is entirely similar.

**Theorem 17.** *Let $A \in Z_n(J)$ be* L*-symmetric, $M = M_A$, let $w$ be a weight function, and assume that the endpoint $a$ is regular. Let $M^s = M_{A^{[s]}}$, $s \in \mathbb{N}_2$, be constructed as above. Then, for any polynomial $p(x) = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0$ with $a_s \neq 0$ with real coefficients $a_j$, we have*

(i) *For $s = 2r$, $r > 0$,*

$$d^+(p(M)), \quad d^-(p(M)) \geq r\,[d^+(M) + d^-(M)].$$

(ii) *For $s = 2r + 1$, $r > 0$,*

$$d^+((M)) \geq (r+1)d^+(M) + rd^-(M)$$
$$d^-(p(M)) \geq rd^+(M) + (r+1)d^-(M)$$

*Strict inequality can occur in each of these inequalities.*

*Proof.* See [**122, 124**]; see, also, [**69**]. $\square$

**Corollary 5.** *If, for some $s > 1$, one of $d^+(M^s)$ or $d^-(M^s)$ take on the minimum value possible according to the general classification Theorem* 13*, then both $d^+(M)$ and $d^-(M)$ are minimal, i.e., $M$ is* LP*. In particular, if some power $M^s$ is* LP*, then $M$ is* LP*.*

Recall that the converse of Corollary 5 is false in general, as mentioned above. Next we take up the question of when does the converse hold?

**Definition 14** (Partial separation)**.** Let the hypotheses and notation of Theorem 17 hold, and let $H = L^2(J, w)$, $s \in \mathbb{N}_2$. Let $p(x) = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0$ with $a_s \neq 0$ be a polynomial with real coefficients $a_j$. We say that $p(M)$ is partially separated if $f \in H$, $M^s f \in H$ together imply that $M^r f \in H$, for $r = 1, 2, \dots, s-1$. (Note that $p(M)$ is partially separated if and only if $M^s$ is partially separated.)

**Theorem 18.** *Let the hypotheses and notation of Corollary* 5 *hold, and assume that $M^s$ is partially separated. Then*:

(i) *If $s = 2r$, then*

$$d^+(M^s) = d^-(M^s) = r\,[d^+(M) + d^-(M)].$$

(ii) *If $s = 2r+1$, $r > 0$, then*

$$d^+(M^s) = (r+1)d^+(M) + rd^-(M) \text{ and}$$
$$d^-(M^s) = rd^+(M) + (r+1)d^-(M).$$

*Conversely,*

(iii) *If, for $s = 2r$, even, either $d^+(M^s) = (r+1)d^+(M) + rd^-(M)$ or $d^-(M^s) = rd^+(M) + (r+1)d^-(M)$, Then $M^s$ is partially separated.*

(iv) *If, for $s = 2r+1$, $r > 0$, odd, either $d^+(M^s) = (r+1)d^+(M) + rd^-(M)$ or $d^-(M^s) = rd^+(M) + (r+1)d^-(M)$. Then $M^s$ is partially separated.*

*Proof.* See [**122, 124**]; see also [**69**]. □

Next we state an important corollary.

**Corollary 6.** *Let $A \in Z_n(J)$ be L-symmetric, $M = M_A$, let $w$ be a weight function, and assume that the endpoint $a$ is regular. Let $M^s = M_{A^{[s]}}$, $s \in \mathbb{N}_2$, be constructed as above. If $d^+(M) = d^-(M) = d(M)$, as is always the case when $A \in Z_n(J, \mathbb{R})$, then $d(M^s) = s\,d(M)$ if and only if $M^s$ is partially separated.*

What are the possible sequences $d^+(M^s)$, $d^-(M^s)$, $s = 1, 2, 3, \dots$? The next theorem answers this question for $A \in Z_{2k}(J, \mathbb{R})$.

**Theorem 19.** *Assume $A \in Z_n(J,\mathbb{R})$, $n = 2k$, $M = M_A$, $A$ is Lagrange symmetric and $a$ is a regular endpoint. Let $M^m = M_{A^{[m]}}$, $r_0 = 0$, $r_m = d(M^m)$, $m = 1, 2, 3, \dots$. Then:*

(i)
$$k\,m \le r_m \le 2\,k\,m, \quad m = 0, 1, 2, 3, \dots .$$

(ii) *The sequence*
$$\{s_m = r_m - r_{m-1}\}_{m=1}^{\infty}$$
*is nondecreasing.*

(iii) *Given any sequence of integers $\{r_m\}_{m=0}^{\infty}$ satisfying* (i) *and* (ii), *there exists an $M = M_A$ with $A \in Z_{2k}(J,\mathbb{R})$ such that $r_m = d(M^m)$, $m = 1, 2, 3, \dots$.*

*Proof.* Part (i) follows from Theorem 13 applied to $M^m$. The proof for part (ii) given in **[69]** for the classical case with smooth coefficients readily adapts to the quasi-differential expressions discussed here; it is based on a method of Kauffman **[68]**. Examples for part (iii) are also constructed in **[69]**. □

Next, we give two well-known examples to illustrate some of the above results and make some comments about some of their other interesting features. See **[69]** for many other examples.

Let

$$My = -y'' - qy, \; q(t) = t^{\alpha}, \quad t \in J = (1, \infty). \tag{11.5}$$

We state the next results as theorems even though they are corollaries of either well-known theorems or of theorems from above and then comment on some of their interesting features.

**Theorem 20.** *Consider equation* (11.5).

(i) *If $\alpha \le 2$, then $d(M) = 1$ in $L^2(J,1)$, i.e., $M$ is* LP *at $\infty$.*
(ii) *If $\alpha > 2$, then $d(M) = 2$ in $L^2(J,1)$, i.e., $M$ is* LC *at $\infty$.*
(iii) *If $\alpha \le 2$, then $d(M^s) = s$ in $L^2(J,1)$, i.e., $M^s$ is* LP *at $\infty$ for all $s = 1, 2, 3, \dots$.*
(iv) *If $\alpha > 2$, then $d(M^s) = 2\,s$ in $L^2(J,1)$, i.e., $M^s$ is* LC *at $\infty$ for all $s = 1, 2, 3, \dots$.*

*Proof.* We give references for the proofs and make comments for each part.

(i) This is a special case of the well-known Levinson LP condition. Together with part (ii), it shows that $\alpha = 2$ is the critical exponent which separates the LP and LC cases. Evans and Zettl in **[30]** extended the well-known Levinson condition for $d(M) = 1$ and showed that the extended version is sufficient for all powers of $M$ to be in the LP case.
(ii) This is well known; see **[69]** for more information.
(iii) This is established by Evans and Zettl in **[30]**; see also **[69]**.
(iv) This follows from Theorem 16. □

Consider

$$My = y^{(4)} + qy, \qquad q(t) = t^{\alpha}, \quad t \in J = (1, \infty). \tag{11.6}$$

**Theorem 21.** *For equation* (11.6):

(i) $d(M) = 2$ *in* $L^2(J, 1)$, *i.e.,* $M$ *is* LP *at* $\infty$ *if and only if* $\alpha \le 4/3$.
(ii) *If* $\alpha \le 4/3$, *then all powers of* $M$ *are* LP *at* $\infty$.

*Proof.* We give references for the proofs and make comments for each part.

(i) The critical constant was found by Hinton **[65]**, see **[69]** for a general discussion including extensions of various kinds.
(ii) This is established by Evans and Zettl **[29]**, see **[69]** for a general discussion including extensions of various kinds. □

**12. Sturm-Liouville operators.** In this section, we briefly review the second order case, see the book **[131]** for more details. We discuss this case separately for two reasons: (i) because it is of wide interest and (ii) it helps to 'set the stage' for the higher order cases discussed below.

Consider the equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J = (a, b); \tag{12.1}$$
$$\frac{1}{p}, \quad q, w \in L_{\text{loc}}(J, \mathbb{R}), \;\; w > 0.$$

Note that $M = M_A$ where $A$ is a Lagrange symmetric matrix of the form

$$A = \begin{bmatrix} 0 & \frac{1}{p} \\ -q & 0 \end{bmatrix}.$$

Note that there is no sign condition on $p$.

Every self-adjoint extension $S$ of the minimal operator $S_{\min}(A)$ in the space $L^2(J, w)$ satisfies (see [**131**] or Section 15 below)

$$S_{\min} = S_{\min}(A) \subset S = S^* \subset S_{\max}(A) = S_{\max}. \tag{12.2}$$

Thus, every such operator $S$ is an extension of the minimal operator $S_{\min}(A)$ and a restriction of the maximal operator $S_{\max}$. These restrictions are on the maximal domain and are called *self-adjoint boundary conditions*. We review these characterizations in this section.

Everitt and Race [**43**] showed that the local integrability condition on the coefficients in (12.1) is necessary and sufficient for every initial value problem to have a unique solution defined on (all of) $J$. A solution is defined as a function $y \in AC_{\mathrm{loc}}(J)$ such that $(py') \in AC_{\mathrm{loc}}(J)$ and (12.1) holds almost everywhere on $J$. Note that $y'(t)$ may not exist for all $t \in J$, but the product $(py')$ exists and is continuous on $J$, hence the notation $(py')$ since this product may not be 'expandable' to $p(t)y'(t)$ for all $t \in J$. If the coefficients are integrable on the whole interval, i.e., $1/p, q, w \in L(J, \mathbb{R})$, then the limits

$$\lim_{t\to a^+} y(t), \quad \lim_{t\to a^+} (py')(t), \quad \lim_{t\to b^-} y(t), \quad \lim_{t\to b^-} (py')(t)$$

exist and are finite regardless of whether the endpoints $a, b$ are finite or infinite. This is the motivation for calling $a$ regular if $1/p, q, w \in L((a, c), \mathbb{R})$ for some (and hence any) $c \in J$, and similarly for $b$. Equation (12.1) is regular on $J$ if both endpoints are regular; in this case, $1/p, q, w \in L(J, \mathbb{R})$. Equation (12.1) is singular at $a$ if it is not regular at $a$, and similarly for $b$. There are two mutually exclusive subclasses of the singular case at each endpoint: limit-circle (LC) and limit-point (LP). (This terminology dates back to the seminal paper of Weyl [**117**] where he made this definition motivated by his proof using concentric circles in the complex plane which converged either to a circle or a point.) We base our definition on the Weyl alternative or Weyl dichotomy result, which we state next.

**Theorem 22** (Weyl dichotomy). *Consider equation* (12.1) *on a subinterval* $(\alpha, \beta)$ *of* $J$. (*We are primarily interested in subintervals of the form* $(a, c)$ *or* $(c, b)$.) *If all solutions of* (12.1) *are in* $L^2((\alpha, \beta), w)$ *for some* $\lambda$ *in* $\mathbb{C}$, *then this is true for all* $\lambda$ *in* $\mathbb{C}$.

Based on Theorem 22, we make the following definition.

**Definition 15** (LC/LP). Let $c \in (a, b)$, and consider equation (12.1) on $(a, c)$. If, for some $\lambda$ in $\mathbb{C}$, all solutions are in $L^2((a, c), w)$, then $a$ is in the limit-circle (LC) case; otherwise, $a$ is in the limit-point (LP) case. This classification is independent of $c$ and of $\lambda$ (by Theorem 22). A similar definition is made at $b$.

The next theorems characterize the self-adjoint extensions of the minimal operator for all endpoint classifications.

**Theorem 23** (R/R). *Assume each endpoint is regular, i.e.,* $1/p, q, w \in L(J, \mathbb{R})$. *Suppose* $A$, $B \in M_2(\mathbb{C})$ *satisfy*

$$AEA^* = BEB^*, \qquad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \operatorname{rank}(A : B) = 2. \tag{12.3}$$

*Then the linear submanifold* $D(S)$ *of* $D_{\max}$ *defined by*

$$D(S) = \left\{ y \in D_{\max} : AY(a) + BY(b) = 0, \qquad Y = \begin{pmatrix} y \\ (py') \end{pmatrix} \right\} \tag{12.4}$$

*is the domain of a self-adjoint extension of* $S_{\min}$, *and every self-adjoint extension of* $S_{\min}$ *is determined this way.*

*The boundary conditions* (12.3) *and* (12.4) *can be categorized into two mutually exclusive classes*: *separated and coupled, and these have the canonical forms*

$$\begin{aligned} &\cos(\alpha)\, y(a) + \sin(\alpha)\, (py')(a) = 0, \quad 0 \le \alpha < \pi; \\ &\cos(\beta)\, y(b) + \sin(\beta)\, (py')(b) = 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{12.5}$$

$$Y(b) = e^{i\theta} K\, Y(a), \quad -\pi < \theta \le \pi, \ \det(K) = 1. \tag{12.6}$$

*If $p > 0$ almost everywhere on $J$, then the minimal operator $S_{\min}$ and all of its self-adjoint extensions are bounded below and the Dirichlet boundary condition*

$$y(a) = 0 = y(b)$$

*determines the Friedrichs extension. If $p$ changes sign on $J$, then $S_{\min}$ is not bounded below.*

The next theorem is for the case when both endpoints are singular and in the limit-circle (LC) case. As mentioned above, in this case the boundary conditions (12.5) and (12.6) are not well defined since, for solutions and maximal domain functions, $y$ and $(py')$ do not have finite limits at the endpoints $a, b$, in general.

This obstacle is overcome by using the Lagrange sesquilinear form:

$$[y, z] = y(p\overline{z}') - \overline{z}(py'), \quad y, z \in D_{\max}. \tag{12.7}$$

It is well known **[131,** page 172, Lemma 10.2.3] that this form has finite limits at each endpoint for all $y, z \in D_{\max}$. Below we use the notation $[y, z]$ and call this the Lagrange sesquilinear form or the Lagrange bracket for any$y, z \in D_{\max}$.

To state the characterizations for singular endpoints it is convenient to make the following definition:

**Definition 16.** We consider several cases.

(i) Suppose $a$ is LC. Choose $c \in (a, b)$. Let $u_1$ and $u_2$ be two real-valued linearly independent solutions of (12.1) on $(a, c)$ for some real $\lambda$ normalized to satisfy $[u_1, u_2](a) = 1$. We call $(u_1, u_2)$ a boundary condition basis at $a$.

(ii) Suppose $b$ is LC. Choose $c \in (a, b)$. Let $v_1$ and $v_2$ be two linearly independent real-valued solutions of (12.1) on $(c, b)$ for some real $\lambda$ normalized to satisfy $[v_1, v_2](b) = 1$. We call $(v_1, v_2)$ a boundary condition basis at $b$.

(iii) If neither endpoint is LP, let $u_1$, $u_2$ be two real-valued linearly independent solutions of (12.1) on $(a, b)$ for some real $\lambda$ normalized to satisfy $[u_1, u_2](t) = 1$ for $t \in (a, b)$; in this case, we call $(u_1, u_2)$ a boundary condition basis at $a$ and at $b$ or a boundary basis on $(a, b)$. See the next remark.

**Remark 24.** We comment on Definition 16. Note that $[u_1, u_2]$ is the Wronskian, which is a non zero constant for linearly independent solutions; hence, the normalization is possible for each case. The choice of $c$ is arbitrary–it merely serves to get a boundary condition basis consisting of linearly independent solutions. For each case, the boundary condition basis depends on $\lambda$. The boundary conditions (12.10) and (12.11) below change with each change of $\lambda$, but once $\lambda$ and the basis is chosen all self-adjoint realizations $S$ of the equation (12.1) are obtained by varying the boundary condition constants $(K, \gamma)$ or $\alpha, \beta$.

The next theorem extends Theorem 23 to singular LC endpoints. We state it for the case when both endpoints are LC, but it reduces to the case when one or both are regular.

**Theorem 24** (LC/LC)**.** *Let* (12.3) *hold. Assume that both endpoints are singular and in the limit-circle* (LC) *case. Suppose* $u_1, u_2$ *are linearly independent real solutions of* (12.1) *on* $(a, b)$ *for some* real $\lambda$ *normalized to satisfy* $[u_1, u_2] = 1$. *Then the linear submanifold* $D(S)$ *of* $D_{\max}$ *defined by*

$$D(S) = \{y \in D_{\max} : AY(a) + BY(b) = 0\}, \tag{12.8}$$

$$Y(a) = \begin{pmatrix} [y, u_1](a) \\ [y, u_2](a) \end{pmatrix}, \qquad Y(b) = \begin{pmatrix} [y, u_1](b) \\ [y, u_2](b) \end{pmatrix}. \tag{12.9}$$

*is the domain of a self-adjoint extension of* $S_{\min}$, *and every self-adjoint extension of* $S_{\min}$ *is determined this way.*

*The boundary conditions* (12.3) *and* (12.8) *can be categorized into two mutually exclusive classes*: *separated and coupled, and these have the canonical forms*:

$$\begin{aligned} &\cos(\alpha)[y, u_1](a) + \sin(\alpha)[y, u_2](a) = 0, \quad 0 \le \alpha < \pi; \\ &\cos(\beta)[y, u_1](b) + \sin(\beta)[y, u_2](b) = 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{12.10}$$

$$Y(b) = e^{i\gamma} KY(a), \quad -\pi < \gamma \le \pi, \ \det(K) = 1. \tag{12.11}$$

*If* $p > 0$ *almost everywhere on* $J$ *and both endpoints are* LCNO, *then the minimal operator* $S_{\min}$ *is bounded below. If* $u_a$ *is the principal* *solution at $a$ and $u_b$ is the principal solution at $b$, then the boundary condition*

$$[y, u_1](a) = 0 = [y, v_1](b) \tag{12.12}$$

*determines the Friedrichs extension $S_F$ of $S_{\min}$. (The principal solution at each endpoint is unique up to real constant multiples, but the principal solution at $a$ may not be the principal solution at $b$, see* **[131]***.)*

*If one or both endpoints are* LCO*, then $S_{\min}$ is not bounded below in $L^2(J, w)$, and so there is no Friedrichs extension.*

**Remark 25.** There is no canonical form for self-adjoint boundary conditions comparable to (12.10) and (12.11) known in the higher order case. Hao, Sun and Zettl **[61, 62]** found canonical forms for regular and singular problems of order $n = 4$ but these are not 'comparable' to (12.10) and (12.11) in the sense that they involve many more cases.

**Theorem 25** (LP/LP)**.** *If both endpoints are* LP*, then $S_{\min}$ is self-adjoint and has no proper self-adjoint extension*; *it may or may not be bounded below.*

**Theorem 26** (R/LP, LP/R)**.** *If $a$ is regular and $b$ is* LP*, then for any $\alpha \in [0, \pi)$, the boundary condition*

$$\cos(\alpha)\, y(a) + \sin(\alpha)\, (py')(a) = 0, \quad 0 \le \alpha < \pi \tag{12.13}$$

*determines a self-adjoint extension of $S_{\min}$, and every self-adjoint extension is determined by* (12.13)*.*

*If $b$ is regular and $a$ is* LP*, then for any $\beta \in [0, \pi)$, the boundary condition*

$$\cos(\beta)\, y(b) + \sin(\beta)\, (py')(b) = 0, \quad 0 < \beta \le \pi \tag{12.14}$$

*determines a self-adjoint extension, and every self-adjoint extension is determined by* (12.14)*. The minimal operator may or may not be bounded below.*

The different normalizations for $\alpha$ and $\beta$ are customary and are convenient for use of the Prüfer transformation in studying theoretical and numerical properties of eigenvalues and eigenfunctions of Sturm-Liouville problems. Although we make no direct use of these different normalizations in this section, we follow this well-established custom here.

**Theorem 27** (R/LC, LC/R). *If $a$ is regular and $b$ is* LC, *then* (12.8), (12.10) *and* (12.11) *hold with* $[y, u_1](a)$ *replaced by* $y(a)$ *and* $[y, u_2](a)$ *replaced by* $(py')(a)$.

*If $b$ is* LCNO *and $u_b$ is the principal solution at $b$, then $S_{\min}$ is bounded below and the Friedrichs extension is determined by the boundary condition*

$$y(a) = 0 = [y, u_b](b). \tag{12.15}$$

*If $b$ is* LCO, *then $S_{\min}$ is not bounded below (even when $p > 0$ on $J$). There is an entirely similar result when $a$ is* LC *and $b$ is regular.*

**Theorem 28** (LC/LP, LP/LC). *If $a$ is* LC, *$(u_1, u_2)$ is a boundary condition basis at $a$ and $b$ is* LP, *then*

$$\cos(\alpha)[y, u_1](a) + \sin(\alpha)[y, u_2](a) = 0, \quad 0 \le \alpha < \pi \tag{12.16}$$

*determines all self-adjoint extensions of $S_{\min}$.*

*If $b$ is* LC, *$a$ is* LP *and $(v_1, v_2)$ is a boundary condition basis at $b$. Then*

$$\cos(\beta)[y, v_1](b) + \sin(\beta)[y, v_2](b) = 0, \quad 0 < \beta \le \pi \tag{12.17}$$

*determines all self-adjoint extensions of $S_{\min}$.*

As mentioned in the introduction to this section, one reason we surveyed the second order results here is to 'set the stage' for the presentation of the results for the higher order case. How do the theorems of this section extend to the higher order case when $n = 2k$ and $k > 1$? This will be discussed in the next two sections.

**13. The GKN theorem.** In this section, we discuss the GKN theorem and the extensions of Theorems 23 and 24 to the general higher order case. The GKN theorem was so named by Everitt and Zettl in [**47**] in honor of the work of Glazman, Krein and Naimark for the reasons given in [**47,** Section 9]. Today this theorem and its extensions are widely used by this name to study self-adjoint differential operators, difference operators, Hamiltonian systems, etc. For ordinary differential operators, the GKN theorem has been extended from the classical one interval case studied by Glazman, Krein and Naimark to any finite or infinite number of intervals [**47**].

The next theorem uses elements of the maximal domain to characterizes all self-adjoint extensions of $S_{\min}$.

**Theorem 29** (GKN). *Let $A = (a_{ij})$ be a Lagrange symmetric matrix in $Z_n(J)$, let $M = M_A$, let $w$ be a weight function, let $S_{\min} = S_{\min}(M)$ and $S_{\max} = S_{\max}(M)$ be the minimal and maximal operators of $M$ with domains $D_{\max} = D(S_{\max})$, $D_{\min} = D(S_{\min})$, respectively. Assume that $d^+(M) = d^-(M) = d(M) = d$, say. Let $[\cdot,\cdot]$ denote the Lagrange bracket of $M$. Then a linear submanifold $D(S) \subset D_{\max}$ is the domain of a self-adjoint extension of $S_{\min}$ in $L^2(J,w)$ if and only if there exist functions $u_1,\dots,u_d$ in $D_{\max}$ such that*

(i) *$u_1,\dots,u_d$ are linearly independent modulo $D_{\min}$, i.e., no nontrivial linear combination of $u_1,\dots,u_d$ is in $D_{\min}$.*
(ii) *$[u_i,u_j](b) - [u_i,u_j](a) = 0$, $i,j = 1,\dots,d$;*
(iii) *$D(S) = \{y \in D_{\max} : [y,u_j](b) - [y,u_j](a) = 0,\ j = 1,\dots,d\}$.*

*Recall from Section* 6 *that, for any $y,z \in D_{\max}$, the Lagrange bracket $[y,z]$ has finite limits at each endpoint $a$ and $b$.*

*Proof.* The proof in Naimark [**84,** page 75] can readily be adapted to this generality. $\square$

Condition (i) gives the number of linearly independent conditions required, (ii) gives the conditions that these $d$ conditions must satisfy and (iii) gives the boundary conditions which determine the self-adjoint domains. Note that the deficiency indices are equal if $A \in Z_n(J,\mathbb{R})$ since, in this case, $\overline{y} \in H$ whenever $y \in H$ for any solution $y$ of the equation $My = \lambda wy$.

**Definition 17.** We call a set of functions $\{u_1,\dots,u_d\}$ satisfying conditions (i) and (ii) of the GKN theorem a GKN-set. These sets depend on the coefficients $(a_{ij})$ and on $w$, and this dependence is implicit and complicated.

Can we find a GKN set $\{u_1, \ldots, u_d\}$ of solutions? This question is discussed later in this section for $d = n$ and in the next section for general $d < n$.

The next theorem shows that the dependence on the GKN set of maximal domain functions can be eliminated if both endpoints are regular. In this case, an explicit characterization can be given in terms of two-point boundary conditions involving only solutions and their quasi-derivatives at the endpoints.

When each endpoint is either regular or singular LC, Theorem 16 together with the Lagrange identity shows that any *real* solution basis for any *real* value of $\lambda$ is a GKN set. In this case $d = n$. For $d < n$, the situation is much more complicated as we will see in the next sections. In this case, if there are $d$ linearly independent solutions for some *real* $\lambda$ (there may not be) we must find a way to identify which of these can be used to form part of a GKN set. This we do in the next few sections.

**Theorem 30.** *Let $A$ be a Lagrange symmetric matrix in $Z_n(J)$, let $M = M_A$, let $w$ be a weight function on $J$, and assume that each endpoint is regular. Then a linear submanifold $D(S) \subset D_{\max}(M)$ is the domain of a self-adjoint extension of $S_{\min}(M)$ in $L^2(J, w)$ if and only if there exist matrices $A, B \in M_n(\mathbb{C})$ such that*

$$\operatorname{rank}(A : B) = n; \tag{13.1}$$

$$A\,E\,A^* = B\,E\,B^*, \qquad E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n; \tag{13.2}$$

*and*

$$A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{13.3}$$

*Proof.* See [**124**]. □

Theorem 30 is a natural extension of Theorem 23 to general $n$ even or odd. At a singular endpoint, such a characterization is not possible since solutions and their quasi-derivatives do not, in general, have finite limits at that endpoint. Also, in that case, $d$ is less than $n$, in general.

Next, we specialize to the even order case.

**Remark 26.** Suppose $A \in Z_n(J,\mathbb{R})$, $n = 2k$, $k > 1$. If each endpoint is either regular or LC, then $d = n$ and there is a natural extension of Theorem 24. In this extension, the roles of $y^{[r]}(a)$ and $y^{[r]}(b)$ are played by the Lagrange brackets $[y, u_{r+1}](a)$ and $[y, u_{r+1}](b)$, $r = 0, 1, \ldots, n-1$, where $u_1, \ldots, u_n$ are a basis of real-valued solutions of (10.1) for an arbitrary (but fixed) $\lambda = \lambda_0 \in \mathbb{R}$.

**Theorem 31.** *Let $A$ be a Lagrange symmetric matrix in $Z_n(J,\mathbb{R})$, $n = 2k$, $k > 1$, let $M = M_A$, let $w$ be a weight function on $J$, and suppose that each endpoint is* LC *in $L^2(J,w)$. Assume $u_1, \ldots, u_n$ are linearly independent real valued solutions of $My = \lambda w y$ on $(a,b)$ for some real $\lambda_0$. Then a linear submanifold $D(S) \subset D_{\max}$ is the domain of a self-adjoint extension of $S_{\min}$ in $L^2(J,w)$ if and only if there exist matrices $A, B \in M_n(\mathbb{C})$ satisfying the conditions*:

$$\text{rank}\,(A : B) = n; \tag{13.4}$$

$$A\,E\,A^* = B\,E\,B^*, \qquad E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n; \tag{13.5}$$

*such that $D(S)$ consists of all $y \in D_{\max}$ satisfying*

$$A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix} + B \begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_n](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{13.6}$$

Here $[y, u]$ denotes the Lagrange bracket which has finite limits at each endpoint.

*Proof.* See **[124]**. $\square$

**Corollary 7.** *In Theorem* 31*, if the endpoint $a$ is regular, then*

$$A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix}$$

*can be replaced by*

$$A\begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix};$$

*similarly if $b$ is regular, then change $A$ and $a$ to $B$ and $b$ in the above replacement. Thus, if both endpoints are regular, then Theorem* 31 *reduces to Theorem* 30.

*Proof.* See [**58**]. □

**Remark 27.** It is interesting to compare Theorems 30 and 31. The regular self-adjoint boundary conditions (13.3) are expressed in terms of the quasi-derivatives $y^{[r]}(a)$ and $y^{[r]}(b)$, whereas in (13.6), the singular self-adjoint conditions are expressed in terms of the Lagrange brackets $[y, u_{r+1}](a)$ and $[y, u_{r+1}](b)$. These exist as finite limits for every $y \in D_{\max}$ by the Lagrange identity. This in spite of the fact some of the terms in the definition of $[y, u_{r+1}]$ may blow up at $a$ or $b$, but each of these blow ups is "canceled" by another interior term $[y, u_{r+1}]$. For example, in the second order case $[y, u] = y(pu') - u(py')$, if $y(pu')$ blows up, then $u(py')$ blows up in such a way that the difference has a finite limit at the endpoint. Since the interval $(a, b)$ is open, the quasi-derivatives $y^{[r]}$ are also defined as finite limits but the quasi-derivatives $y^{[r]}$ are continuous at a regular endpoint. At a singular endpoint the quasi-derivatives $y^{[r]}$ do not exist at the endpoints, in general. At a singular LC endpoint the limits of $[y, u_{r+1}]$ do exist for all maximal domain functions $y$. So when each maximal domain function $y$ is 'matched' with the corresponding $u_{r+1}$ then the limit $[y, u_{r+1}]$ exists and is finite. The blowups or wild oscillations of $y$ and $u_{r+1}$ cancel to produce a finite limit for $[y, u_{r+1}]$ at each LC endpoint.

**Remark 28.** When $d = n = 2k$, Theorem 31 extends the GKN theorem by showing that any solution basis for any real $\lambda$ is a GKN set. If each endpoint is regular or LC, then $d = n$. If one endpoint is not regular or LC, then $d < n$. This case is much more complicated and will be discussed in the next sections.

**14. Decomposition of the maximal domain.** If deficiency index $d = n$, then all solutions of

$$My = \lambda wy \quad \text{on } J = (a, b), \quad -\infty \le a < b \le \infty, \tag{14.1}$$

are in $H = L^2(J, w)$ for any $\lambda \in \mathbb{C}$. And, by Theorem 31 for any *real* $\lambda$, every solution basis is a GKN set. When the deficiency indices satisfy $d^+ = d^- = d < n$, there are exactly $d$ linearly independent square-integrable solutions of (14.1) for any $\lambda$ with $\operatorname{Im}(\lambda) \neq 0$. But such a solution basis does not form a GKN set because condition (ii) of the GKN theorem is not satisfied. Sun [**101**] constructed a GKN set of solutions near the endpoints for this case. This construction is based on a new (in [**101**]) decomposition of the maximal domain $D_{\max}(J)$.

In this section, we present the Sun [**101**] decomposition and its proof. We give a detailed proof here because our result is more general than that given in [**101**], but the method of proof is basically the same.

Given a closed symmetric densely defined operator $T$ in a Hilbert space $H$, the well-known von Neumann formula [**84**]

$$D(T^*) = D(T) \dotplus N_\lambda \dotplus N_{\overline{\lambda}}, \quad \operatorname{Im}(\lambda) \neq 0, \tag{14.2}$$

characterizes the domain of its adjoint in terms of its deficiency spaces $N_\lambda$, $N_{\overline{\lambda}}$.

When applied to the minimal operator $S_{\min} = S_{\min}(A)$ where $A \in Z_n(J)$ is Lagrange symmetric, using Theorem 1, the von Neumann formula yields

$$D(S_{\max}) = D(S_{\min}) \dotplus N_\lambda \dotplus N_{\overline{\lambda}},\ \operatorname{Im}(\lambda) \neq 0, \tag{14.3}$$

where

$$N_\lambda = \{y \in D(S_{\max}) : M_A\, y = \lambda w\, y,\ \operatorname{Im}(\lambda) \neq 0\}. \tag{14.4}$$

The deficiency spaces $N_\lambda$ and $N_{\overline{\lambda}}$ in the von Neumann formula (14.3) and (14.4) consist of solution bases of the equation $M_A\, y = \lambda w\, y$ on the whole interval $J = (a, b)$. When $d < n$, the behavior of the solutions may be very different near the two endpoints of the underlying interval $J$. (When $d = n$, each endpoint is either regular or LC; in the regular case, the solutions and their quasi-derivatives are continuous at that endpoint, in the singular LC case the solutions are asymptotically similar at that endpoint [**131**].) To take this different behavior into account, the Sun decomposition, see Theorem 32 below, replaces the direct sum $N_\lambda \dotplus N_{\overline{\lambda}}$ in (14.3) with

$$\text{span}\,\{u_1, \dots, u_{m_a}\} \dotplus \text{span}\,\{v_1, \dots, v_{m_b}\}, \tag{14.5}$$

where the $u_j$ are solutions near $a$ and the $v_j$ are solutions near $b$. This decomposition is then used to characterize all self-adjoint domains in terms of boundary conditions specified only at the endpoints $a$ and $b$ for regular or singular endpoints and for any deficiency index $d$.

Next we state the Sun decomposition theorem.

**Theorem 32.** *Let $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $1 < k$, $M = M_A$, assume that $A$ is Lagrange symmetric and $w$ is a weight function. Let $c \in (a, b)$, $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) \neq 0$. Then the deficiency indices of $M$ are equal on $(a, b)$, $(a, c)$ and on $(c, b)$: $d^+(a, b) = d^-(a, b) = d(a, b) = d$, $d^+(a, c) = d^-(a, c) = d(a, c) = d_a$, $d^+(c, b) = d^-(c, b) = d(c, b) = d_b$. Let $m_a = 2d_a - n$, $m_b = 2d_b - n$. Then there exist $v_j \in D_{\max}(a, b)$, $j = 1, 2, \dots, m_b$ and $u_j \in D_{\max}(a, b)$, $j = 1, 2, \dots, m_a$, such that*

- (i) (a) *$u_1, \dots, u_{m_a}$ are linearly independent solutions of $M_A\, y = \lambda w\, y$ on $(a, c)$;*
  - (b) *the $m_a \times m_a$ matrix $U = [u_i, u_j](a)$, $1 \leq i, j \leq m_a$, is nonsingular.*
  - (c) *$u_j$ is identically zero in a neighborhood of $b$;*
  - (d) *$u_1, \dots, u_{m_a}$ are linearly independent modulo $D_{\min}$.*
- (ii) (a) *$v_1, \dots, v_{m_b}$ are linearly independent solutions of $M_A\, y = \lambda w\, y$ on $(c, b)$;*
  - (b) *the $m_b \times m_b$ matrix $V = [v_i, v_j](b)$, $1 \leq i, j \leq m_b$, is nonsingular;*
  - (c) *$v_j$ is identically zero in a neighborhood of $a$;*
  - (d) *$v_1, \dots, v_{m_b}$ are linearly independent modulo $D_{\min}$.*
- (iii)

$$D_{\max}(a, b) = D_{\min}(a, b) \dotplus \text{span}\,\{u_1, \dots, u_{m_a}\} \dotplus \text{span}\,\{v_1, \dots, v_{m_b}\}. \tag{14.6}$$

*Proof.* Part (c) follows from (a) and Lemma 4 (Naimark patching lemma); part (d) follows from (b) and (c). The proof of parts (a) and (b) is long and technical and will be given below; it depends on several lemmas.

First we observe that (iii) follows from (i) and (ii). By Von Neumann's formula (14.3), $\dim D_{\max}(a,b)/D_{\min}(a,b) = 2d$. Since $m_a + m_b = 2(d_a + d_b - n) = 2d$, from parts (i) and (ii) it follows that $D_{\max}(a,b)/\ D_{\min}(a,b) \geq 2d$, completing the proof of (iii). Clearly, we need to prove only part (ii) since the proof of part (i) is entirely similar. ☐

Next, we observe that, for any $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,(\lambda) \neq 0$, we have the following unique representation for all $y \in D_{\max}(c,b)$:

$$y = y_0 + \sum_{i=1}^{d_b} c_i z_i(\cdot,\lambda) + \sum_{i=1}^{d_b} \widetilde{c}_i z_i(\cdot,\overline{\lambda}), \tag{14.7}$$

where $y_0 \in D_{\min}(c,b)$, $z_i(\cdot,\lambda)$, $i = 1,\dots,d_b$, are linearly independent solutions of $M(y) = \lambda w y$ on $(c,b)$ and $c_i, \widetilde{c}_i \in \mathbb{C}$. This follows from the Von Neumann formula applied to the interval $(c,b)$.

Let

$$\begin{aligned} x_1 &= z_1(\cdot,\lambda),\dots,x_{d_b} = z_{d_b}(\cdot,\lambda),\\ x_{d_b+1} &= z_1(\cdot,\overline{\lambda}),\dots,x_{2d_b} = z_{d_b}(\cdot,\overline{\lambda}). \end{aligned} \tag{14.8}$$

Thus, if $y \in D_{\max}(c,b)$, we have

$$y = y_0 + \sum_{i=1}^{2d_b} a_i x_i, \quad y_0 \in D_{\min}(c,b),\ a_i \in \mathbb{C}, \tag{14.9}$$

and this representation is unique.

**Lemma 7.** *Define the matrix $X$ by*

$$X = ([x_i, x_j](b))_{\substack{1\leq i\\ j\leq 2d,}} \tag{14.10}$$

*where $[x_i, x_j]$ is the Lagrange bracket. Then*

$$\mathrm{rank}\,(X) = m_b = 2d_b - n.$$

*Proof.* Let $g_i$ be a set of functions in $D_{\max}(c,b)$ which satisfy the following conditions:

$$g_i^{[k-1]}(c) = \delta_{ik}, \quad (k, i = 1,\dots,n), \tag{14.11}$$

and are identically zero in a neighborhood of $b$, $j = 1, \dots, n$. By Lemma 4, there exist such functions in $D_{\max}(c, b)$.

By (14.9), we have

$$g_i = y_{i_0} + \sum_{j=1}^{2d_b} a_{ij} x_j, \quad (i = 1, \dots, n), \tag{14.12}$$

where $y_{i_0} \in D_{\min}(c, b)$. Hence,

$$g_i^{[k-1]} = y_{i_0}^{[k-1]} + \sum_{j=1}^{2d_b} a_{ij} x_j^{[k-1]}, \quad (i, k = 1, \dots, n).$$

Since $y_{i_0} \in D_{\min}(c, b)$ and $c$ is a regular point, it follows that $y_{i_0}^{[k-1]}(c) = 0$, $(i, k = 1, \dots, n)$, and therefore we have:

$$\begin{pmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$
$$= \begin{pmatrix} a_{11} & \cdots & a_{1\ 2d_b} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\ 2d_b} \end{pmatrix}_{n\times 2d_b} \begin{pmatrix} x_1(c) & \cdots & x_1^{[n-1]}(c) \\ \cdots & \cdots & \cdots \\ x_{2d_b}(c) & \cdots & x_{2d_b}^{[n-1]}(c) \end{pmatrix}_{2d_b\times n}$$

Noting that $2d_b \geq 2k = n$, we have

$$\operatorname{rank}(a_{ij})_{n\times 2d_b} = n, \quad \operatorname{rank}(x_j^{[k-1]}(c))_{2d_b\times n} = n. \tag{14.13}$$

By (14.9), we have

$$[g_i, x_s](b) = [y_{i_0}, x_s](b) + \sum_{j=1}^{2d_b} a_{ij} [x_j, x_s](b),$$
$$(i = 1, \dots, n; \quad s = 1, \dots, 2d_b).$$

Since $g_i(b) = 0$ and $y_{i_0} \in D_{\min}(c, b)$, therefore $[y_{i0} x_s](b) = 0$, it follows that

$$0_{n\times 2d_b} = (a_{ij})_{n\times 2d_b}([x_j, x_s](b))_{2d_b\times 2d_b}. \tag{14.14}$$

From this, we obtain

$$\operatorname{rank}([x_j, x_s](b))_{2d_b\times 2d_b} \leq 2d_b - n = m_b. \tag{14.15}$$

Because $k \le d_b \le n$, we have

$$0 \le m_b = 2d_b - n \le d_b. \tag{14.16}$$

By Lemma 2 and Corollary 2, for any $t \in (c, b)$,

$$[z(t,\lambda),\, z(t,\overline{\lambda})](t) = [z(t,\lambda),\, z(t,\overline{\lambda})](c);$$

hence,

$$[z(t,\lambda),\, z(t,\overline{\lambda})](b) = [z(t,\lambda),\, z(t,\overline{\lambda})](c).$$

By (14.8), we have

$$\begin{aligned} ([x_i, x_j](b))^T_{1\le i\le d_b, d_b+1\le j\le 2d_b} &= ([z_k(t,\lambda),\, z_s(t,\bar{\lambda})](b))^T_{1\le k,s\le d_b} \\ &= ([z_k(t,\lambda), z_s(t,\overline{\lambda})](c))^T_{1\le k,s\le d_b} \\ &= Z^*(c,\overline{\lambda}) E_{d_b} Z(c,\lambda). \end{aligned}$$

Since $\operatorname{rank} Z(c,\lambda) = n$, $\operatorname{rank} Z(c,\overline{\lambda}) = n$ and $\operatorname{rank} E_{d_b} = d_b$, it follows that

$$\operatorname{rank}\left([x_i, x_j](b)\right)_{\substack{1\le i\le d_b \\ d_b+1\le j\le 2d_b}} \ge 2d_b - n = m_b. \tag{14.17}$$

Thus,

$$\operatorname{rank}\left([x_i, x_j](b)\right)_{\substack{1\le i \\ j\le 2d_b}} \ge 2d_b - n = m_b.$$

Combining this with inequality (14.15), the conclusion follows. $\square$

**Lemma 8.** *Letting*

$$F = \left([x_i, x_j](b)\right)_{\substack{1\le i\le d_b \\ 1\le j\le 2d_b}}, \tag{14.18}$$

*then* $\operatorname{rank} F = m_b = 2d_b - n$.

*Proof.* This follows from inequalities (14.15) and (14.17) in the proof of Lemma 7. $\square$

By Lemma 8, we can assume, without loss of generality, that the first $m_b$ rows of $F$ are linearly independent. Let

$$F_1 = \Big(\ [x_i, x_j](b)\ \Big)_{\substack{1\le i\le m_b \\ 1\le j\le 2d_b}}, \tag{14.19}$$

then

$$\operatorname{rank}(F_1) = m_b. \tag{14.20}$$

**Lemma 9.** *Let* $\{x_i\}$ *be defined by* (14.8)*, which satisfies* (14.20)*. Then, each* $x_i$ $(i = m_b + 1, \ldots, 2d)$ *has a unique representation*

$$x_i = \widetilde{y}_{i_0} + \sum_{j=1}^{n} g_j + \sum_{s=1}^{m_b} b_{is} x_s \tag{14.21}$$
$$(i = m_b + 1, \ldots, 2d_b), \quad b_{is} \in \mathbb{C},$$

*where* $m_b = 2d_b - n$, $\widetilde{y}_{i_0} \in D_{\min}(c, b)$ *and* $g_j$ $(j = 1, \ldots, n)$ *satisfy* (14.11)*.*

*Proof.* By (14.12), we have

$$g_i = y_{i_0} + \sum_{j=1}^{2d_b} a_{ij} x_j, \quad (i = 1, \ldots, n). \tag{14.22}$$

Let

$$(a_{ij})_{n \times 2d_b} = (C_{n \times m_b} D_{n \times n}),$$

where

$$C = \begin{pmatrix} a_{11} & \cdots & a_{1\ m_b} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\ m_b} \end{pmatrix}; \qquad D = \begin{pmatrix} a_{1\ m_b+1} & \cdots & a_{1\ 2d_b} \\ \cdots & \cdots & \cdots \\ a_{n\ m_b+1} & \cdots & a_{n\ 2d_b} \end{pmatrix}.$$

$$X = ([x_i, x_j](b))_{\substack{1 \le i \\ j \le 2d_b}} = \begin{pmatrix} F_{1\ m_b \times 2d_b} \\ F_{2\ n \times 2d_b} \end{pmatrix}.$$

By (14.14),

$$0_{n \times 2d_b} = (a_{ij})_{n \times 2d_b} ([x_i, x_j](b))_{2d_b \times 2d_b},$$

and we have

$$CF_1 + DF_2 = 0_{n \times 2d_b}.$$

If $\operatorname{rank}(D) < n$, then there exists a non-singular matrix of order $n$, say

$G$, such that

$$GD = \begin{pmatrix} a'_{1\ m_b+1} & \cdots & a'_{1\ 2d_b} \\ \cdots & \cdots & \cdots \\ a'_{n-1\ m_b+1} & \cdots & a'_{n-1\ 2d_b} \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$GC = \begin{pmatrix} a'_{11} & \cdots & a'_{1\ m_b} \\ \cdots & \cdots & \cdots \\ a'_{n1} & \cdots & a'_{n\ m_b} \end{pmatrix}.$$

Since $GCF_1 + GDF_2 = 0_{n\times 2d_b}$, we have

$$(a'_{n1} \cdots a'_{n\ m_b})F_1 = (0 \cdots 0)_{1\times 2d_b}.$$

From $\operatorname{rank}(F_1) = m_b$, it follows that

$$a'_{n1} = \cdots = a'_{n\ m_b} = 0.$$

This contradicts (14.13) that $\operatorname{rank}(a_{ij})_{n\times 2d_b} = n$.

Hence, we have $\operatorname{rank} D = n$. Recall that $D$ is composed of the last $n$ columns of $(a_{ij})$, so we solve $x_i$ $(i = m_b + 1, \dots, 2d)$ from equation (14.22)

$$x_i = \widetilde{y}_{i_0} + \sum_{j=1}^{n} c_{ij} g_j + \sum_{s=1}^{m_b} b_{is} x_s \quad (i = m_b + 1, \dots, 2d_b),$$

and the representation is unique, where $y_{i_0} \in D_{\min}(c, b)$. The lemma is proved. □

**Lemma 10.** *Let* $x_i$ $(i = 1, \dots, 2d_b)$*,* $z_i(\cdot, \lambda)$ $(i = 1, \dots, d_b)$ *and* $z_i(\cdot, \overline{\lambda})$ $(i = 1, \dots, d_b)$ *be defined by* (14.8)*, and let*

$$Z = ([x_i, x_j](b))_{\substack{1\le i\\ j\le m_b}} = ([z_i(\lambda), z_j(\overline{\lambda})](b))_{\substack{1\le i\\ j\le m_b}}. \tag{14.23}$$

*Then* $\operatorname{rank}(Z) = m_b$*, where* $m_b = 2d_b - n$*.*

*Proof.* If $\operatorname{rank}(Z) < m_b = 2d_b - n$, without loss of generality, we can assume that the last row of $V$ has all zeros, i.e.,

$$[x_{m_b}, x_i](b) = 0, \quad (i = 1, 2, \dots, m_b). \tag{14.24}$$

By Lemma 9,

$$x_i = \widetilde{y}_{i_0} + \sum_{j=1}^{n} g_j + \sum_{s=1}^{m_b} b_{is} x_s \quad (i = m_b + 1, \ldots, 2d),$$

where $\widetilde{y}_{i_0} \in D_{\min}(c, b)$ and $g_j$ $(j = 1, \ldots, n)$ satisfy (14.11). Combining this with (14.24), we have:

$$[x_{m_b}, x_i](b) = 0, \quad (i = m_b + 1, \ldots, 2d).$$

This contradicts (14.19); therefore, the rank of the matrix

$$F_1 = ([x_i, x_j](b))_{\substack{1 \le i \le m_b \\ 1 \le j \le 2d_b}}$$

is $m_b = 2d_b - n$. □

Now choose $v_j = x(\cdot, \lambda)$, $j = 1, \ldots, m_b$. This completes the proof of Theorem 32.

If one endpoint is regular, then the decomposition (14.6) can be simplified by using solutions defined by initial conditions at that endpoint. We state this as a corollary.

**Corollary 8.** *Let the hypotheses and notation of Theorem* 32 *hold.*

(i) *If the endpoint $a$ is regular, then in* (14.6) *the functions $u_1, \ldots, u_{m_a}$ can be replaced by solutions on the interval $(a, c)$ defined by the initial conditions*:

$$y_j^{[k-1]}(a) = \delta_{j,k}, \quad 1 \le j, k \le n$$

*where $\delta_{j,k}$ is the Kronecker $\delta$.*

(ii) *If the endpoint $b$ is regular, then in* (14.6) *the functions $v_1, \ldots, v_{m_b}$ can be replaced by solutions on the interval $(c, b)$ defined by the initial conditions*:

$$y_j^{[k-1]}(b) = \delta_{j,k}, \quad 1 \le j, k \le n,$$

*where $\delta_{j,k}$ is the Kronecker $\delta$.*

*Proof.* (i) Since $a$ is a regular point, there are $n$ linear independent solutions lying in $L^2((a, c), w)$, $a < c < b$, say $y_1, \ldots, y_n$, which are determined by the following initial conditions:

$$y_j^{[k-1]}(a) = \delta_{j,k}, \quad 1 \le j, k \le n,$$

where $\delta_{j,k}$ is the Kronecker $\delta$. And the solutions $y_1, \dots, y_n$ can be extended to $(a, b)$ such that the extended functions, also denoted by $y_1, \dots, y_n$, satisfy $y_j \in D_{\max}(a, b)$ and $y_j$ is identically zero in a left neighborhood of $b$, $j = 1, \dots, n$. These functions $y_i, i = 1, \dots, n$ satisfy the conditions of the $g_i$ in the proof of Lemma 7 with $c$ replaced by $a$.

Any $y \in D_{\max}$ can be uniquely written as

$$y = \widetilde{y}_0 + \sum_{s=1}^{2d} a_s x_s,$$

where $\widetilde{y}_0 \in D_{\min}$. Combining this with Lemma 9, ($g_i$, $i = 1, \dots, n$ are replaced by $y_i$, $i = 1, \dots, n$; $x_i$, $i = 1, \dots, m_b$ are replaced by $v_i$, $i = 1, \dots, m_b$), and we have

$$y = \widetilde{y}_0 + \sum_{s=1}^{m_b} a_s x_s + \sum_{r=m_b+1}^{2d} a_r \left[ \widetilde{y}_{r_0} + \sum_{j=1}^{n} c_{rj} y_j + \sum_{s=1}^{m_b} b_{rs} v_s \right]. \tag{14.25}$$

Let

$$\widetilde{y}_0 + \sum_{r=m_b+1}^{2d} a_r \widetilde{y}_{r_0} = y_0 \in D_{\min},$$

$$\sum_{r=m_b+1}^{2d} a_r c_{rj} = d_j, \qquad a_s + \sum_{r=m_b+1}^{2d} a_r b_{rj} = \tau_s.$$

Then we obtain

$$y = y_0 + \sum_{k=1}^{n} d_j y_j + \sum_{s=1}^{m_b} \tau_s v_s. \tag{14.26}$$

This implies that

$$D_{\max}(a, b) \subset D_{\min}(a, b) \dot{+} \operatorname{span}\{y_1, \dots, y_n\} \dot{+} \operatorname{span}\{v_1, \dots, v_{m_b}\}.$$

So the conclusion follows from the fact that

$$D_{\max}(a, b) \supset D_{\min}(a, b) \dot{+} \operatorname{span}\{y_1, \dots, y_n\} \dot{+} \operatorname{span}\{v_1, \dots, v_{m_b}\}.$$

The proof of Corollary 8 (ii) is similar and hence omitted. $\square$

Theorem 32 can be used to characterize the self-adjoint domains.

**Theorem 33.** *Let the hypotheses and notation of Theorem* 32 *hold. Then a linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2(J,w)$ if and only if there exists a complex matrix $A_{d\times m_a}$ and a complex matrix $B_{d\times m_b}$ such that the following three conditions hold*:

(i) $\operatorname{rank}(A:B)=d$,
(ii) $AUA^*=BVB^*$,
(iii)
(14.27)

$$D(S)=\left\{y\in D_{\max}:A\begin{pmatrix}[y,u_1](a)\\ \vdots\\ [y,u_{m_a}](a)\end{pmatrix}+B\begin{pmatrix}[y,v_1](b)\\ \vdots\\ [y,v_{m_b}](b)\end{pmatrix}=0\right\},$$

*where $d=d_a+d_b-n$. The Lagrange brackets in* (14.27) *have finite limits.*

*Proof.* See **[101]** for a proof when one endpoint is regular. The method of proof for the two singular endpoint case is similar to the method used in the proof of Theorem 37 in Section 16 below. The fact that the Lagrange brackets in (14.27) have finite limits as $x\to a$ and $x\to b$ follows from the Lagrange identity. □

**Remark 29.** When $n=2k$ and $d_b=k$, then $m_b=0$, and this corresponds to $B=0$ in (ii) and (14.27); in this case, all the conditions of the self-adjoint boundary condition (14.27) are specified at the endpoint $a$ only with no condition required or allowed at $b$. Similarly, when $n=2k$ and $d_a=k$, then $m_a=0$, and this corresponds to $A=0$ in (14.27); in this case, all the conditions of the self-adjoint boundary condition (14.27) are specified at the endpoint $b$ only with no condition required or allowed at $a$. When $n=2k$, $d_a=k$ and $d_b=k$ the minimal operator $S_{\min}$ is self-adjoint and has no proper self-adjoint extension in $L^2(J,w)$.

In view of the wide interest in the case when one endpoint is regular we state these two cases next.

**Theorem 34.** *Let the hypotheses and notation of Theorem 32 hold.*

(i) *Assume that $a$ is regular. Then a linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2(J,w)$ if and only if there exists a complex matrix $A_{d_b\times n}$ and a complex matrix $B_{d_b\times m_b}$ such that the following three conditions hold:*
(a) $\operatorname{rank}(A:B)=d$,
(b) $AEA^*=BVB^*$,
(c)

(14.28)
$$D(S)=\left\{y\in D_{\max}:A\begin{pmatrix} y(a)\\ \vdots \\ y^{(n-1)}(a)\end{pmatrix}+B\begin{pmatrix} [y,v_1](b)\\ \vdots \\ [y,v_{m_b}](b)\end{pmatrix}=0\right\}.$$

(ii) *Assume that $b$ is regular. Then a linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2(J,w)$ if and only if there exists a complex matrix $A_{d_b\times n}$ and a complex matrix $B_{d_b\times n}$ such that the following three conditions hold:*
(a) $\operatorname{rank}(A:B)=d$,
(b) $AUA^*=BEB^*$,
(c)

(14.29)
$$D(S)=\left\{y\in D_{\max}:A\begin{pmatrix} [y,u_1](a)\\ \vdots \\ [y,u_{m_a}](a)\end{pmatrix}+B\begin{pmatrix} y(b)\\ \vdots \\ y^{(n-1)}(b)\end{pmatrix}=0\right\}.$$

*The Lagrange brackets in* (14.28) *and* (14.29) *have finite limits.*

*Proof.* See [**101**]. □

**Remark 30.** When $n=2k$, $d_b=k$, then $m_b=0$, and this corresponds to $B=0$ in (14.28). When $n=2k$ and $d_a=k$, then $m_a=0$ and this corresponds to $A=0$ in (14.29).

**Remark 31.** The hypothesis that $\lambda$ is not real is important in the abstract von Neumann formula (14.2) and for the Sun ordinary differential equations formula (14.6) which leads to the characterization of the self-adjoint domains given by Theorem 33. Condition (ii) of this characterization involves the matrices $U$ and $V$ which depend on the solutions of the differential equation (14.1) near the singular endpoints. In Section 15 below we establish, under a mild additional hypothesis, a refinement of the decomposition (14.6) and then show, in Section 16, that this refinement leads to a characterization of the self-adjoint domains which does not depend on the matrices $U, V$. This refinement uses solutions from a *real value of the spectral parameter* $\lambda$ and replaces the matrices $U, V$ with the simple constant symplectic matrix $E$ of appropriate dimension. This *real* $\lambda$ characterization is then used in Sections 20, 21 and 22 to obtain information about the spectrum of the self-adjoint realizations of equation (14.1).

**Remark 32.** Recall that, for $A \in Z_n(J,\mathbb{R})$, $d(\lambda) = d$ for all $\lambda$ with $\mathrm{Im}\,(\lambda) \neq 0$ but for $\lambda \in \mathbb{R}$, $d(\lambda) \neq d$ in general. We will see below that if there are $d$ linearly independent solutions in $L^2$, then, in contrast to the complex case, not all of these contribute to the characterization of the self-adjoint domains. Those which do contribute we call limit-circle (LC) solutions and those which don't contribute we call limit-point (LP) solutions in analogy with the Sturm-Liouville case. We will identify the LC and LP solutions in Section 15 below.

For papers related to some of the results in this section, see [**11, 12, 13, 14, 15, 16, 17, 18, 19, 51, 74, 75, 94, 95, 103**].

**15. Real parameter LC solutions and the decomposition of the maximal domain.** In this section, under a mild additional hypothesis (see RS below), we establish a refinement of the Sun decomposition (14.6) of the maximal domain in terms of solutions for *real* values of the spectral parameter $\lambda$. In contrast to the complex case discussed in Section 14, in this case not all $L^2$ solutions contribute to the decomposition of the maximal domain. We will identify those solutions which do contribute and call them LC solutions in analogy with the second order case. In the next section we show that these LC solutions can be used to characterize all self-adjoint extensions of the minimal operator $S_{\min}$. In this characterization, the matrices $U, V$ in Theorem 32 which depend on solutions near the singular endpoints are replaced by the simple constant coefficient symplectic matrix $E$. In Sections 20, 21 and

22, this real $\lambda$ characterization is used to obtain information about the spectrum of these self-adjoint operators.

Throughout this section, we assume that $A \in Z_n(J)$ is Lagrange symmetric, has equal deficiency indices, $M = M_A$, $d^+(M) = d^-(M) = d(M) = d$, say,

$$n = 2k, \quad k \in \mathbb{N}_2 = \{2, 3, 4 \ldots\}. \tag{15.1}$$

Let $d_a$ and $d_b$ denote the deficiency indices at $a$ and $b$, respectively, and the following hypothesis (RS) holds:

**(RS)** Let $a < c < b$, and assume that equation (14.1) on $(a, c)$ has $d_a$ linearly independent solutions in $L^2((a, c), w)$ for some real $\lambda = \lambda_a$ and that (14.1) has $d_b$ linearly independent solutions in $L^2((c, b), w)$ for some real $\lambda = \lambda_b$.

Note that $d_a$ and $d_b$ are independent of $c$ and that, if there exist $d_a$ linearly independent solutions on $(a, c)$, then there exist $d_a$ linearly independent *real* solutions on $(a, c)$, and similarly for the endpoint $b$.

**Remark 33.** We comment on hypothesis **RS**. Recall that $r_a(\lambda)$ denotes the number of linearly independent solutions of (14.1) on $(a, c)$ which lie in $L^2((a, c), w)$ for real $\lambda$. For any real $\lambda$, it is known **[107]** that $r_a(\lambda) \leq d_a$ and, if $r_a(\lambda) < d_a$, then $\lambda$ is in the essential spectrum of every self-adjoint extension of $S_{\min}(a, c)$ and of $S_{\min}(a, b)$. Thus, if there does not exist a real $\lambda_a$ such that the equation (14.1) on $(a, c)$ has $d_a$ linearly independent solutions in $L^2((a, c), w)$, then the essential spectrum of all self-adjoint extensions $S_{\min}(a, c)$ and of $S_{\min}(a, b)$ covers the whole real line and similarly for the endpoint $b$. If the essential spectrum of every self-adjoint realization of (14.1) in $L^2((a, b), w)$ covers the whole real line then any eigenvalue, if there is one, is embedded in the essential spectrum. In this case, the dependence of such eigenvalues on the boundary condition seems to be 'coincidental' and nothing seems to be known, aside from examples, about this dependence.

First, some preliminary results. Note that, if $A \in Z_n(J)$ for $J = (a, b)$, then $A \in Z_n((\alpha, \beta))$ for any subinterval $(\alpha, \beta)$ of $J$. We are particularly interested in the subintervals $(a, c)$ and $(c, b)$ for $c \in (a, b)$ but state the next lemma for general subintervals. We will use the notations $D_{\max}(\alpha, \beta)$, $D_{\min}(\alpha, \beta)$, $d(\alpha, \beta)$, etc., to denote the dependence of these quantities on the interval $(\alpha, \beta)$.

**Lemma 11.** *Assume that $A \in Z_n(J, \mathbb{R})$, $J = (a, b)$, $-\infty \leq a < b \leq \infty$, is Lagrange symmetric and $w$ is a weight function on $J$. Let $a \leq \alpha < \beta \leq b$. Then $A$ has equal deficiency indices $d$, $d_a$, $d_b$ and*

(i) (*the restriction of*) *$A$ is in $Z_n((\alpha, \beta))$, is Lagrange symmetric, and $M_A$ on $(\alpha, \beta)$ is a symmetric differential expression.*
(ii) *If $a < \alpha < \beta < b$, then $M_A$ is regular on $(\alpha, \beta)$.*
(iii) *If one endpoint of $(\alpha, \beta)$ is regular, then*

$$k \leq d(\alpha, \beta) \leq 2k = n. \tag{15.2}$$

(iv) *If one endpoint of $(\alpha, \beta)$ is regular, then for all $\lambda \in \mathbb{R}$, we have*

$$r(\lambda) \leq d(\alpha, \beta). \tag{15.3}$$

(v) *If $c \in (a, b) = J$, then*

$$d = d_a + d_b - n. \tag{15.4}$$

(vi) *If both endpoints are singular, then $r(\lambda)$ may be greater than $d$, less than $d$, or equal to $d$. All three possibilities are realized.*

*Proof.* Parts (i)–(iv) are observations which follow directly from the definition of $Z_n(J)$. A proof of the inequalities (15.2) is given in [**45, 124**]; also the proof given in [**84,** pages 71–72] for the special case of $A$ considered there can be readily adapted to this more general case.

Now we prove that, if one endpoint of $(c, d)$ is regular, then for $\lambda \in \mathbb{R}$, the number of linearly independent solutions of $My = \lambda wy$ on $(c, d)$ which lie in $H = L^2((c, d), w)$ is less than or equal to $d$. Assume that $v_1, \dots, v_r$ are linearly independent solutions on $(c, d)$ in $H$ and that $r > d$. First observe that then there exist $r$ linearly independent *real valued solutions in $H$*. This follows from the observation that, since the real and imaginary parts of a complex solution are real solutions, each $v_j$ is a linear combination of real solutions. Let $u_j$, $j = 1, \dots, r$, be linearly independent real solutions in $H$. Define

$$D = D_{\min} \dotplus \operatorname{span}\{u_j, \quad j = 1, \dots, d\}$$
$$D_r = D_{\min} \dotplus \operatorname{span}\{u_j, \quad j = 1, \dots, r\}.$$

Because one endpoint of $(c,d)$ is regular, then $D$ is a self-adjoint domain by the GKN theorem, and it follows from the fact that the $u_j$, $j=1,\dots,r$, are real that $D_r$ is a symmetric domain. Note that the dimensions $\operatorname{mod} D_{\min}$, of $D$ and $D_r$ are $d$ and $r$, respectively. Let $S$ and $T$ be the restrictions of the maximal operator $S_{\max}$ to $D$ and $D_r$, respectively. Then $T$ is a proper symmetric extension of the self-adjoint operator $S$. But this is impossible since

$$S \subset T \subset T^* \subset S^* = S$$

implies that $T = S$. This contradiction proves that the number of linearly independent solutions of $My = \lambda wy$ on $(\alpha, \beta)$ lying in $L^2((\alpha,\beta), w)$ is less than or equal to the deficiency index $d$ for any *real* $\lambda$. For a proof of formula (15.4), see [**84,** page 72]; the proof given there for a special case of $A$ can be readily adapted to this more general case. For part (vi), see the recent paper [**60**]. □

The next theorem constructs LC solutions at each endpoint. For the case when one endpoint is regular and the other singular, these solutions were first constructed by Wang-Sun-Zettl in [**107**].

**Theorem 35.** *Assume that $A \in Z_n(J,\mathbb{R})$, $n = 2k$, $k > 1$, $J = (a,b)m$ $-\infty \le a < b \le \infty$, is Lagrange symmetric, $M = M_A$ and $w$ is a weight function. Let $a < c < b$. Consider the equation*

$$My = \lambda wy \quad \text{on } J. \tag{15.5}$$

*Let $d_a$ denote the deficiency index of* (15.5) *on $(a,c)$ and $d_b$ the deficiency index of* (15.5) *on $(c,b)$. Assume that, for some $\lambda = \lambda_a \in \mathbb{R}$,* (15.5) *has $d_a$ linearly independent solutions $u_1,\dots,u_{d_a}$ on $(a,c)$ which lie in $L^2((a,c),w)$ and that, for some $\lambda = \lambda_b \in \mathbb{R}$,* (15.5) *has $d_b$ linearly independent solutions $v_1,\dots,v_{d_b}$ on $(c,b)$ which lie in $L^2((c,b),w)$. Then*

(i) *There exist $d_a$ linearly independent real-valued solutions $u_1,\dots,u_{d_a}$ on $(a,c)$ which lie in $L^2((a,c),w)$.*
(ii) *There exist $d_b$ linearly independent real-valued solutions $v_1,\dots,v_{d_b}$ on $(c,b)$ which lie in $L^2((c,b),w)$.*
(iii) *For $m_a = 2d_a - 2k$, the solutions $u_1,\dots,u_{d_a}$ can be ordered such that the $m_a \times m_a$ matrix $U = ([u_i,u_j](c))$, $1 \le i,j \le m_a$, is given*

*by*

$$U = (-1)^{k+1} E_{m_a} \tag{15.6}$$

*and is therefore nonsingular.*

(iv) *For $m_b = 2d_b - 2k$, the solutions $v_1, \ldots, v_{d_b}$ on $(c, b)$ can be ordered such that the $m_b \times m_b$ matrix $V = ([v_i, v_j](c))$, $1 \le i, j \le m_b$, is given by*

$$V = (-1)^{k+1} E_{m_b} \tag{15.7}$$

*and is therefore nonsingular.*

(v) *For every $y \in D_{\max}(a, b)$, we have*

$$[y, u_j](a) = 0, \quad \text{for } j = m_a + 1, \ldots, d_a. \tag{15.8}$$

(vi) *For every $y \in D_{\max}(a, b)$ we have*

$$[y, v_j](b) = 0, \quad \text{for } j = m_b + 1, \ldots, d_b. \tag{15.9}$$

(vii) *For $1 \le i,\ j \le d_a$, we have*

$$[u_i, u_j](a) = [u_i, u_j](c). \tag{15.10}$$

(viii) *For $1 \le i,\ j \le d_b$, we have*

$$[v_i, v_j](b) = [v_i, v_j](c). \tag{15.11}$$

(ix) *The solutions $u_1, \ldots, u_{d_a}$ can be extended to $(a, b)$ such that the extended functions, also denoted by $u_1, \ldots, u_{d_a}$, satisfy $u_j \in D_{\max}(a, b)$ and $u_j$ is identically zero in a left neighborhood of $b$, $j = 1, \ldots, d_a$.*

(x) *The solutions $v_1, \ldots, v_{d_b}$ can be extended to $(a, b)$ such that the extended functions, also denoted by $v_1, \ldots, v_{d_b}$, satisfy $v_j \in D_{\max}(a, b)$ and $v_j$ is identically zero in a right neighborhood of $a$, $j = 1, \ldots, d_b$.*

*Proof.* See [**58,** Theorem 4.1]. □

**Definition 18.** We call the solutions $u_1, \ldots, u_{m_a}$ and $v_1, \ldots, v_{m_b}$ LC solutions at $a$ and $b$, respectively. The solutions $u_{m_{a+1}}, \ldots, u_{d_a}$ and $v_{m_b+1}, \ldots, v_{d_b}$ are called LP solutions at $a$ and $b$, respectively. Recall that $k \le d_a \le 2k$ and $k \le d_b \le 2k$ since $c$ is a regular endpoint for both intervals $(a, c)$ and $(c, b)$. If $d_a = k$, then $m_a = 0$, and there are no LC solutions at $a$, i.e., all solutions $u_1, \ldots, u_{d_a}$ are LP solutions at $a$ and (15.6) is vacuous. If $d_a = 2k$, then $m_a = 2k$, all solutions $u_1, \ldots, u_{d_a}$ are LC solutions at $a$ and (15.8) is vacuous. In the intermediate deficiency cases $k < d_a < 2k$, $m_a = 2d_a - 2k$ and $u_1, \ldots, u_{m_a}$ are LC solutions at $a$ and $u_{m_a+1}, \ldots, u_{d_a}$ are LP solutions at $a$. For example, for $n = 4$ and $d_a = 3$, there are two LC solutions and one $LP$ solution; for $n = 6$ and $d_a = 4$ there are two LC and two LP solutions, for $n = 6$ and $d_a = 5$, there are four LC solutions and one LP solution. The other solutions of a solution basis of $My = \lambda_a wy$ on $(a, c)$ are not in $L^2((a, c), w)$. Similar remarks apply for the endpoint $b$. Below we will see that the LC solutions contribute to the determination of the boundary conditions, and the LP solutions do not contribute due to (15.8) and (15.9). (The solutions not in $L^2$ do not play any role in the maximal domain decomposition nor in the characterization of the self-adjoint domains.)

**Remark 34.** Observe that, by Theorem 35, the LC solutions are determined by initial conditions at the regular point $c \in J$, i.e.,

$$E_{m_a} = (-1)^{(k+1)}([u_i, u_j](c)), \quad (i, j = 1, \ldots, m_a),$$
$$E_{m_b} = (-1)^{(k+1)}([v_i, v_j](c)), \quad (i, j = 1, \ldots, m_b).$$

It is interesting to observe that this characterization does not depend on the behavior of the solutions at a singular endpoint. This is an essential difference between Theorems 32 and 33 based on nonreal $\lambda$ on the one hand and Theorems 35, 36 and 37 based on solutions for *real* $\lambda$.

The next theorem gives a decomposition of the maximal domain which we believe is of independent interest. Although it can be considered a corollary of Theorem 35, we state it as:

**Theorem 36.** *Let the notation and hypotheses of Theorem* 35 *hold. Then*:
(15.12)
$$D_{\max}(a, b) = D_{\min}(a, b) \dotplus \operatorname{span}\{u_1, \ldots, u_{m_a}\} \dotplus \operatorname{span}\{v_1, \ldots, v_{m_b}\}.$$

*Proof.* By Von Neumann's formula, $\dim D_{\max}(a, b)/D_{\min}(a, b) \leq 2d$. From Theorem 35 parts (vii), (viii), (ix) and (x) and the observation that the matrices $U$ and $V$ are nonsingular it follows that $u_1, \ldots, u_{m_a}$

and $v_1, \dots, v_{m_b}$ are linearly independent mod $(D_{\min}(a,b))$, since $m_a + m_b = 2(d_a + d_b - n) = 2d$. Therefore, $\dim D_{\max}(a,b)/\ D_{\min}(a,b) \geq 2d$, completing the proof. $\square$

**16. Self-adjoint domains.** Based on Theorems 35 and 36, we can now give a complete characterization of the self-adjoint domains of $(M, w)$ in $H = L^2(J, w)$ for any $M = M_A$ with $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k > 1$. (The case $k = 1$ was discussed previously in Section 12.) Recall that, for this case, the deficiency indices are equal $d^+(A) = d^-(A) = d(A) = d$, and it is well known that there are self-adjoint extensions in this case for any $d$.

The next theorem gives the characterization of the self-adjoint domains in terms of LC solutions, for real values of the spectral parameter $\lambda$, $u_1, \dots, u_{m_a}$; $v_1, \dots, v_{m_b}$, defined in Section 15. For the case when one endpoint is regular and the other singular, these LC solutions at the singular endpoint were first constructed by Wang-Sun-Zettl in **[107]** and used there to characterize the self-adjoint domains. Hao, et al. **[58]** then used this construction to obtain the characterization for two singular endpoints given by the next theorem. This result reduces to the case when one or both endpoints are regular.

**Theorem 37.** *Let the hypotheses and notation of Theorem* 35 *hold. Let $d = d_a + d_b - n$. Then $d$ is the deficiency index of* (15.5) *on* $(a, b)$. *A linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2(J, w)$ if and only if there exists a complex $d \times m_a$ matrix $A$ and a complex $d \times m_b$ matrix $B$ such that the following three conditions hold*:

(i)
$$\operatorname{rank}(A : B) = d;$$

(ii)
$$AE_{m_a}A^* = BE_{m_b}B^*;$$

(iii)
(16.1)
$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_a}](a) \end{pmatrix} + B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_b}](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

*The Lagrange brackets in* (16.1) *have finite limits. In* (16.1)*, the solutions* $u_1, \dots, u_{d_a}$ *and* $v_1, \dots, v_{d_b}$ *have been ordered so that* (14.8) *and* (14.23) *hold.*

*Proof. Sufficiency.* Let the matrices $A$ and $B$ satisfy conditions (i) and (ii) of Theorem 37. We prove that $D(S)$ defined by condition (iii) is the domain of a self-adjoint extension $S$ of $S_{\min}$.

Let

$$A = -(\overline{a}_{ij})_{d\times m_a}, \qquad B = (\overline{b}_{ij})_{d\times m_b}. \tag{16.2}$$

$$w_i = \sum_{j=1}^{m_a} a_{ij}\, u_j + \sum_{j=1}^{m_a} b_{ij} v_j, \quad i = 1, \dots, d. \tag{16.3}$$

Then, for $y \in D_{\max}(a,b)$, we have

$$-A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_a} a_{1j} u_j](a) \\ \vdots \\ [y, \sum_{j=1}^{m_a} a_{dj} u_j](a) \end{pmatrix} = \begin{pmatrix} [y,w_1](a) \\ \vdots \\ [y,w_d](a) \end{pmatrix}.$$

$$B \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_b} b_{1j} v_j](b) \\ \vdots \\ [y, \sum_{j=1}^{m_b} b_{dj} v_j](b) \end{pmatrix} = \begin{pmatrix} [y,w_1](b) \\ \vdots \\ [y,w_d](b) \end{pmatrix}.$$

Therefore, the boundary condition (iii) of Theorem 37 becomes boundary condition (iii) of the GKN theorem, i.e.,

$$[y,w_i](b) - [y,w_i](a) = 0, \quad i = 1, \dots, d. \tag{16.4}$$

It remains to show that $w_i$, $i = 1, \dots, d$, satisfy conditions (i) and (ii) of the GKN theorem.

To show that condition (i) holds, assume that it does not hold. Then there exist constants $c_1, \dots, c_d$, not all zero, such that

$$\gamma = \sum_{i=1}^{d} c_i w_i \in D_{\min}.$$

Hence, we have $[\gamma, y](a) = 0$ for any $y \in D_{\max}$ by Theorem 3. So, using the notation $U$ from Theorem 35, we have

$$(0,\ldots,0) = \left(\left[\sum_{j=1}^{d} c_j w_j, u_1\right](a), \ldots, \left[\sum_{j=1}^{d} c_j w_j, u_{m_a}\right](a)\right)$$
$$= (c_1,\ldots,c_d)(a_{ij})_{d\times m_a} U.$$

Since $U$ is nonsingular, we have $(\overline{c}_1 \ldots \overline{c}_d)\, A = 0$. Similarly, we have $(\overline{c}_1 \ldots \overline{c}_d)\, B = 0$. Hence,

$$(\overline{c}_1 \cdots \overline{c}_d)\,(A : B) = 0.$$

This contradicts the fact that rank $(A : B) = d$.

Next we show that (ii) holds. We have

$$[w_i, w_j](a) = \left[\sum_{l=1}^{m_a} a_{il} u_l, \sum_{s=1}^{m_a} a_{js} u_s\right](a)$$
$$= \sum_{l=1}^{m_a} \sum_{s=1}^{m_a} a_{il} \overline{a}_{js} [u_l, u_s](a),$$

From Theorem 35, we get

$$([w_i, w_j](a))^T_{d\times d} = AU^T A^* = (-1)^k A E_{m_a} A^*.$$

Similarly,

$$(([w_i, w_j](b))^T_{d\times d} = (-1)^k B E_{m_b} B^*.$$

Therefore,

$$([w_i, w_j]_a^b)^T = (-1)^k B E_{m_b} B^* - (-1)^k A E_{m_a} A^* = 0.$$

It follows from the GKN theorem that $D(S)$ is a self-adjoint domain.

*Necessity.* Let $D(S)$ be the domain of a self-adjoint extension $S$ of $S_{\min}$. By the GKN theorem, there exist $w_1, \ldots, w_d \in D_{\max}$ satisfying conditions (i), (ii) and (iii) of this theorem. By Theorem 36, each $w_i$ can be uniquely written as:

$$w_i = \widehat{y}_{i0} + \sum_{j=1}^{m_a} a_{ij} u_j + \sum_{j=1}^{m_b} b_{ij} v_j, \tag{16.5}$$

where $\widehat{y}_{i0} \in D_{\min}$, $a_{ij}, b_{ij} \in \mathbb{C}$.

Let

$$A = -(\overline{a}_{ij})_{d\times m_a}, \qquad B = (\bar{b}_{ij})_{d\times m_b}.$$

Then

$$\begin{pmatrix} [y,w_1](a) \\ \vdots \\ [y,w_d](a) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_a} a_{1j}u_j](a) \\ \vdots \\ [y, \sum_{j=1}^{m_a} a_{dj}u_j](a) \end{pmatrix} = -A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix},$$

$$\begin{pmatrix} [y,w_1](b) \\ \vdots \\ [y,w_d](b) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_b} b_{1j}v_j](b) \\ \vdots \\ [y, \sum_{j=1}^{m_b} b_{dj}v_j](b) \end{pmatrix} = B \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix}.$$

Hence, boundary condition (iii) of the GKN theorem is equivalent to part Theorem 37 (iii).

Next we prove that $A$ and $B$ satisfy conditions (i) and (ii) of Theorem 37.

Clearly, rank $(A : B) \le d$. If rank $(A : B) < d$, then there exist constants $c_1, \dots, c_d$, not all zero, such that

$$(c_1 \cdots c_d)(A : B) = 0. \tag{16.6}$$

Letting $g = \sum_{i=1}^{d} \overline{c}_i w_i$, from (16.5) we obtain

$$g = \sum_{i=1}^{d} \overline{c}_i \widehat{y}_{i0} + \sum_{i=1}^{d}\sum_{j=1}^{m_a} \overline{c}_i a_{ij} u_j + \sum_{i=1}^{d}\sum_{j=1}^{m_b} \overline{c}_i b_{ij} v_j.$$

By (16.6), we have $(c_1 \cdots c_d)A = (c_1 \cdots c_d)B = 0$. Hence,

$$g = \sum_{i=1}^{d} \overline{c}_i \widehat{y}_{i0}.$$

So we have $g \in D_{\min}$. This contradicts the fact that the functions $w_1, w_2, \dots, w_d$ are linearly independent modulo $D_{\min}$. Therefore, rank $(A : B) = d$.

Now we verify part (ii) of Theorem GKN. By (16.5), we have

$$[w_i, w_j](a) = \left[\sum_{l=1}^{m_a} a_{il}u_l, \sum_{s=1}^{m_a} a_{js}u_s\right](a)$$
$$= \sum_{l=1}^{m_a}\sum_{s=1}^{m_a} a_{il}\overline{a}_{js}[u_l, u_s](a), \quad (i, j = 1, \ldots, d).$$

From Theorem 35, we obtain

$$([w_i, w_j](a))^T_{d\times d} = AU^T A^* = (-1)^k AE_{m_a}A^*,$$

and, similarly, we have

$$([w_i, w_j](b))^T_{d\times d} = BV^T B^* = (-1)^k BE_{m_b}B^*.$$

Hence, condition (ii) of Theorem GKN becomes

$$AE_{m_a}A^* = BE_{m_b}B^*.$$

This completes the proof. □

**Remark 35** (LC and LP solutions). Note that, for $\lambda = \lambda_a$, there are $d_a$ linearly independent real-valued solutions on $(a, c)$ which can be ordered such that the first $u_1, u_2, \ldots, u_{m_a}$ with $m_a = 2d_a - 2k$ contribute to the self-adjoint boundary conditions (16.1) and $u_{m_a+1}, \ldots, u_{d_a}$ make no contribution to the boundary conditions (16.1). By (15.8) of Theorem 35, $[y, u_j](a) = 0$ for every $y \in D_{\max}(a, b)$, $j = m_a + 1, \ldots, d_a$. If $u_1, u_2, \ldots, u_{d_a}$ is completed to a full basis $u_1, u_2, \ldots, u_{d_a}, \ldots, u_n$ of solutions of $My = \lambda_a wy$ on $(a, c)$, then no nontrivial linear combination of $u_{d_a+1}, \ldots, u_n$ is in the Hilbert space $L^2((a, c), w)$, and thus these solutions play no role in the formulation of the self-adjoint boundary conditions. For this reason, we call $u_1, u_2, \ldots, u_{m_a}$ LC solutions at $a$ and $u_{m_a+1}, \ldots, u_{d_a}$ LP solutions at $a$. In the Sturm-Liouville case on $(a, c)$, $n = 2$, $d_a = 1$, or $d_a = 2$ corresponding to the celebrated Weyl limit-point (LP) or limit-circle (LC) cases, respectively. When $n > 2$, we have $k \le d_a \le 2k$ and all values of $d_a$ in this range are realized. In the intermediate deficiency cases $k < d_a < 2k$, Theorem 35 characterizes the $m_a = 2d_a - 2k$ LC solutions $u_j$, $j = 1, \ldots, m_a$, and the LP solutions $u_j$, $j = m_a + 1, \ldots, d_a$. Similar remarks apply at the endpoint $b$. In particular, we call $v_1, \ldots, v_{m_b}$ the LC solutions at $b$ and $v_{m_b+1}, \ldots, v_{d_b}$ the LP solutions at $b$. If $d_a = k$, there are no LC solutions at $a$, and there is no self-adjoint boundary condition at $a$, i.e., the term multiplied by $A$ is zero. If $d_a = 2k$, there are no LP solutions at $a$, and similarly at $b$.

All endpoint classifications follow from Theorem 37. The case when both endpoints are regular is covered by Theorem 30. Next we give explicit statements for the other cases.

**Theorem 38.** *Let $A \in Z_n(J)$ be Lagrange symmetric, $M = M_A$, $c \in (a,b)$, let $w$ be a weight function, and assume that $M$ has equal deficiency indices $d$ on $(a,b)$ and the endpoint $a$ is regular. Then $d = d_b$ and $k \le d \le 2k$. Let $m = 2d-2k$, and let $v_1, \dots, v_m$ be* LC *solutions on $(c,b)$ as constructed by Theorem* 35. *Then a linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2((a,b),w)$ if and only if there exists a complex $d \times n$ matrix $A$ and a complex $d \times m$ matrix $B$ such that the following three conditions hold*:

(i) *The* rank $(A : B) = d$;
(ii) $AE_nA^* = BE_mB^*$;
(iii)
(16.7)

$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_m](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

*Proof.* See [**107**]. □

In the next theorem the endpoint $b$ is regular and $a$ may be singular.

**Theorem 39.** *Let $A \in Z_n(J)$ be Lagrange symmetric, $M = M_A$, $c \in J$, let $w$ be a weight function, and assume that $M$ has equal deficiency indices $d$ on $(a,b)$ and the endpoint $b$ is regular. Then $d = d_a$ and $k \le d \le 2k$. Let $m = 2d-2k$, and let $u_1, \dots, u_m$ be* LC *solutions on $(a,c)$ as constructed by Theorem* 35. *Then a linear submanifold $D(S)$ of $D_{\max}(a,b)$ is the domain of a self-adjoint extension $S$ of $S_{\min}(a,b)$ in $L^2((a,b),w)$ if and only if there exists a complex $d \times n$ matrix $A$ and a complex $d \times m$ matrix $B$ such that the following three conditions hold*:

(i) *The* rank $(A : B) = d$;

(ii) $AE_mA^* = BE_nB^*$;
(iii)
(16.8)
$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_m](a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

*Proof.* See [**107**]. □

**Remark 36.** In Theorem 39, when $d_a = k$, then $m = 0$ and $A = 0$. Thus, (16.7) reduces to a separated boundary condition at the regular endpoint $b$:
$$B \begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
where the $d \times n$ complex matrix $B$ satisfies rank $(B) = d$ and $BE_nB^* = 0$. In this case, there is no condition required or allowed at the endpoint $a$.

Similarly, in the minimal deficiency case $d_b = k$, then $m = 0$ and $B = 0$. Thus, the term involving $B$ in (16.7) vanishes and Theorem 38 reduces to the self-adjoint boundary conditions at the regular endpoint $a$:

$$A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{16.9}$$

where the $d \times n$ complex matrix $A$ satisfies rank $(A) = d$ and $AE_nA^* = 0$. In this case, there is no condition required or allowed at the endpoint $b$.

**Remark 37** (Clarification of Everitt-Markus comment)**.** In this remark, we clarify a point raised by Everitt and Markus in their 1999 monograph [**40**]. They state:

> We provide an affirmative answer ... to a long standing open question concerning the existence of real differential expressions of even order $\geq 4$, for which there are non-real

> self-adjoint differential operators specified by strictly separated boundary conditions.... This is somewhat surprising because it is well known that for order $n = 2$ strictly separated conditions can produce only real operators (that is, any given such complex conditions can always be replaced by corresponding real boundary conditions.)

It is clear from Theorems 37, 38 and 39 that such conditions occur naturally and explicitly for regular and singular problems for all $n = 2k$, $k > 1$. Furthermore, the analysis of Wang, Sun and Zettl [**109**] shows that it is not the order of the equation which is the relevant factor for the existence of non-real self-adjoint conditions but the *number* of boundary conditions. If there is only one, regular or singular, separated boundary condition at a given endpoint as must be the case for $n = 2$, then it can always be replaced by an equivalent real condition. On the other hand, if there are two or more separated conditions at a given regular or singular endpoint, then some of these are not equivalent to real conditions.

**17. Strictly separated conditions.** At first glance, each of the $n$ equations of the regular self-adjoint boundary condition

$$(17.1) \qquad A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

may seem to 'couple' the endpoints $a$ and $b$ with each other. This is not the case. We will see below that every equation of (17.1) may 'connect' $a$ with $b$, i.e., be 'coupled,' every equation (17.1) may be separated, i.e., involve one endpoint only (but different endpoints for different equations–they cannot all involve the same endpoint), and some self-adjoint conditions (17.1) are 'mixed,' i.e., involve both separated and coupled equations. Similar remarks apply to the general singular characterization (16.1) of Theorem 37.

In this and the next sections we investigate the number of separated, coupled, and mixed conditions possible for the self-adjoint regular and singular boundary conditions given by Theorem 37. We start by establishing two linear algebra lemmas which may be of independent interest. These lemmas will be used below.

**Lemma 12.** *Let $h$ be any positive integer $\geq 2$, and let $C$ be an $r \times h$ complex matrix with* $\operatorname{rank}(C) = r$. *Assume that*

$$CE_hC^* = 0,$$

*where $E_h$ is the symplectic matrix of order $h$. Then $r \leq h/2$.*

*Proof.* Since $E_h$ is invertible, $\operatorname{rank} E_hC^* = r$. Then $CE_hC^* = 0$ implies that $h - r = \operatorname{null} C \geq r$, so that $2r \leq h$. □

**Lemma 13.** *Let the hypotheses and notation of Lemma* 12 *hold, and assume that $h$ is an even positive integer. Then there exist $\alpha_1, \ldots, \alpha_{h/2}$ in $\mathbb{R}^h$ such that $CE_hC^* = 0$, where $C = (\alpha_1, \alpha_2, \ldots, \alpha_{h/2})^T$.*

*Proof.* From $E_h^* = E_h^{-1} = -E_h$, it follows that $(iE_h)^* = -iE_h^* = iE_h$ and $(iE_h)^{-1} = -iE_h^{-1} = iE_h$. Hence, $iE_h$ is self-adjoint and unitary, so that $\sigma(E_h) \subset \{i, -i\}$. Since the diagonal elements of $E_h$ are 0, the trace of $E_h$ is 0, and therefore the eigenvalues $i$ and $-i$ have equal multiplicity $r = h/2$. Letting $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ be orthogonal bases of the eigenspaces of $E_h$ with respect to the eigenvalues $i$ and $-i$, respectively, it follows that

$$E_h(u_j + v_j) = i(u_j - v_j) = 0, \quad j = 1, \ldots, r.$$

The orthogonality yields

$$\langle u_k + v_k,\, E_h\,(u_j + v_j)\rangle = 0, \quad \text{if } j \neq k$$

and

$$\langle u_j + v_j),\, E_h\,(u_j + v_j)\rangle = \langle u_j, i\,u_j\rangle - \langle v_j, i\,v_j\rangle = 0.$$

Putting $C^* = (u_1 + v_1, \ldots, u_r + v_r)$ gives an $r \times h$ matrix $C$ with $\operatorname{rank} C = r$ and $CE_hC^* = 0$. □

**Remark 38.** These two lemmas show that the rank of matrices $C$ satisfying $CE_hC^* = 0$ is at most $h/2$ and that there are some such matrices $C$ for which it is $h/2$.

The next theorem gives a construction for strictly separated self-adjoint boundary conditions.

**Theorem 40.** *Let the hypotheses and notation of Theorem* 37 *hold. Suppose*

$$A = \begin{pmatrix} C_{l\times m_a} \\ 0_{(d-l)\times m_a} \end{pmatrix}, \qquad B = \begin{pmatrix} 0_{l\times m_b} \\ D_{(d-l)\times m_b} \end{pmatrix},$$

*and assume that* $\operatorname{rank}(C) = l$ *and* $\operatorname{rank}(D) = d - l$. *Then* $\operatorname{rank}(A : B) = d$, *and the boundary conditions*

$$A \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_a}](a) \end{pmatrix} + B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_b}](b) \end{pmatrix} = 0$$

*are self-adjoint if and only if*

$$CE_{m_a}C^* = 0 \quad \textit{and} \quad DE_{m_b}D^* = 0. \tag{17.2}$$

*Furthermore,* $l = d_a - k$ *in this case, and so* $d - l = d_b - k$.

*Proof.* It is clear that $\operatorname{rank}(A : B) = d$. Note that

$$AE_{m_a}A^* = \begin{pmatrix} CE_{m_a}C^* & 0_{l\times(d-l)} \\ 0_{(d-l)\times l} & 0_{(d-l)\times(d-l)} \end{pmatrix},$$
$$BE_{m_b}B^* = \begin{pmatrix} 0_{l\times l} & 0_{l\times(d-l)} \\ 0_{(d-l)\times l} & DE_{m_b}D^* \end{pmatrix},$$

and (4.4) follows. The latter cases follows from Lemmas 12 and 13. By Lemma 12, we have $l \le m_a/2 = d_a - k$ and $d - l \le mb/2 = d_b - k$, i.e., $l \ge d - d_b + k = d_a + d_b - 2k - d_b + k = d_a - k$. Therefore, $l = d_a - k$. $\square$

**Remark 39.** Although we do not give a technical definition of strictly separated boundary conditions until the next section, it is intuitively clear that the construction given by Theorem 40 yields exactly $d_a - k$ separated conditions at the endpoint $a$ and exactly $d_b - k$ separated boundary conditions at the endpoint $b$. Moreover, if each of the $d$ equations of a self-adjoint boundary condition is specified at one endpoint only, then the boundary condition can be put into the form of Theorem 40 by elementary matrix transformations. We state this as the next corollary.

**Corollary 9.** *Assume that* $d_b - k$ *rows of the matrix* $A$ *are zero and* $\operatorname{rank}(A) = d_a - k$, *and suppose that the complementary* $d_a - k$ *rows of* $B$ *are zero and rank*$(B) = d_b - k$. *Let* $C$ *denote the* $(d_a - k) \times m_a$ *submatrix of* $A$ *consisting of the nonzero rows of* $A$. *Similarly, let* $D$ *denote the submatrix of* $B$ *consisting of the nonzero rows of* $B$. *Then* $\operatorname{rank}(A : B) = d$ *and the boundary conditions* (16.1) *are self-adjoint if and only if* (17.2) *holds.*

*Proof.* This follows from the observation that the condition $AE_{m_a}A^* = BE_{m_b}B^*$ is invariant under multiplication on the left by any nonsingular $d \times d$ matrix $G$. Note that

$$(GA)E_{m_a}(GA)^* = (GB)E_{m_a}(GB)^*$$

for any nonsingular $d \times d$ matrix $G$. By choosing elementary matrices $G$, the zero rows of $A$ and of $B$ can be interchanged. $\square$

**Corollary 10.** *Assume that* $d = 2k = n$, *i.e., the maximal deficiency case holds. This occurs if and only if* $d_a = n$ *and* $d_b = n$. *So* $l = k$, $d - l = k$ *and* $m_a = m_b = n$. *Let*

$$A = \begin{pmatrix} C_{k\times n} \\ 0_{k\times n} \end{pmatrix}, \qquad B = \begin{pmatrix} 0_{k\times n} \\ D_{k\times n} \end{pmatrix},$$

*and suppose that* $\operatorname{rank}(C) = \operatorname{rank}(D) = k$. *Then, by Theorem* 40, *the strictly separated boundary conditions*

$$C \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix} = 0 \quad \textit{and} \quad D \begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_n](b) \end{pmatrix} = 0 \tag{17.3}$$

*are self-adjoint if and only if* $CE_nC^* = DE_nD^* = 0$.

*Proof.* It is clear that $\operatorname{rank}(A : B) = n$ and, by computation,

$$A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix} + B \begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_n](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

is equivalent to (17.3). Note that

$$AE_nA^* = \begin{pmatrix} CE_nC^* & 0_{k\times k} \\ 0_{k\times k} & 0_{k\times k} \end{pmatrix}$$
$$BE_nB^* = \begin{pmatrix} 0_{k\times k} & 0_{k\times k} \\ 0_{k\times k} & DE_nD^* \end{pmatrix}.$$

By Corollary 9, we obtain that (17.3) is a self-adjoint boundary condition if and only if $CE_nC^* = DE_nD^* = 0$. □

**Remark 40.** Let $d_a = k$ and $d_b = n$. Then $d = d_a + d_b - n = k$, $m_a = 0$, $m_b = n$, $d_a - k = 0$ and $d_b - k = k$. In this case, by Theorem 40, all self-adjoint boundary conditions are strictly separated. By Corollary 9, we may, without loss of generality, let

$$A = \begin{pmatrix} C_{(d_a-k)\times m_a} \\ 0_{(d_b-k)\times m_a} \end{pmatrix} = 0_{d\times 0},$$
$$B = \begin{pmatrix} 0_{(d_a-k)\times m_b} \\ D_{(d_b-k)\times m_b} \end{pmatrix} = D_{k\times n}$$

with $\operatorname{rank}(D) = k$. Then $\operatorname{rank}(A : B) = k$, and

$$D\begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_n](b) \end{pmatrix} = 0$$

are self-adjoint if and only if $DE_nD^* = 0$.

Similarly, let $d_a = n$ and $d_b = k$. Then $d = k$, $m_a = n$, $m_b = 0$, $d_a - k = k$ and $d_b - k = 0$. In this case, by Theorem 40, all self-adjoint boundary conditions are strictly separated. By Corollary 9, we may, without loss of generality, let

$$A = \begin{pmatrix} C_{(d_a-k)\times m_a} \\ 0_{(d_b-k)\times m_a} \end{pmatrix} = C_{k\times n},$$
$$B = \begin{pmatrix} 0_{(d_a-k)\times m_b} \\ D_{(d_b-k)\times m_b} \end{pmatrix} = 0_{k\times 0}$$

with $\operatorname{rank}(C) = k$. Then $\operatorname{rank}(A : B) = k$ and

$$C \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_n](a) \end{pmatrix} = 0 \tag{17.4}$$

are self-adjoint if and only if $CE_nC^* = 0$.

**18. Classification of self-adjoint conditions.** In this section, we classify the self-adjoint boundary conditions given by Theorem 37 into different types depending on how many of the conditions (16.1) are coupled. Our classification depends on Theorem 41 given below.

Recall that

$$d = d_a + d_b - n, \quad n = 2k,$$

and the following inequalities hold:

$$0 \le d \le n, \quad k \le d_a,\ d_b \le n = 2k.$$

Recall from Section 10 that all values within these ranges are realized.

**Theorem 41.** *Let the hypotheses and notation of Theorem* 37 *hold, and let* $k < d_a \le 2k$; $k < d_b \le 2k$; $d = d_a + d_b - 2k$. *Assume the complex* $d \times m_a$ *matrix* $A$ *and the complex* $d \times m_b$ *matrix* $B$ *satisfy*

$$AE_{m_a}A^* = BE_{m_b}B^*, \qquad \operatorname{rank}(A : B) = d. \tag{18.1}$$

(i) *Suppose* $d_a \ge d_b$. *Then*

(a)

$$d_a - k \le \operatorname{rank}(A) \le d, \quad d_b - k \le \operatorname{rank}(B) \le m_b = 2(d_b - k);$$

(b) *For any* $r$ *satisfying* $0 \le r \le d_b - k$, *if*

$$\operatorname{rank}(A) = d_a - k + r \tag{18.2}$$

*then*

$$\operatorname{rank}(B) = d_b - k + r. \tag{18.3}$$

*Furthermore, for any* $r$ *satisfying* $0 \le r \le d_b - k$, *there exist matrices* $A, B$ *satisfying* (18.1) *such that* (18.2) *and* (18.3) *hold.*

(ii) *Suppose* $d_a < d_b$. *Then*

(a)

$$d_a - k \le \operatorname{rank}(A) \le m_a = 2(d_a - k), \tag{18.4}$$
$$d_b - k \le \operatorname{rank}(B) \le d;$$

(b) *For any $r$ satisfying $0 \le r \le d_a - k$, if*

$$\operatorname{rank}(B) = d_b - k + r, \tag{18.5}$$

*then*

$$\operatorname{rank}(A) = d_a - k + r. \tag{18.6}$$

*Furthermore, for any $r$ satisfying $0 \le r \le d_a - k$, there exist matrices $A, B$ satisfying* (18.1) *such that* (18.4), (18.5) *and* (18.6) *hold.*

*Proof.* We only prove part (i) since the proof of part (ii) is similar.

(a) From the proof of Theorem 40, we have

$$d_a - k = \frac{m_a}{2} \le \operatorname{rank}(A) \le d,$$
$$dd_b - k = \frac{m_b}{2} \le \operatorname{rank}(B) \le m_b.$$

(ii) When $r = 0$, then $\operatorname{rank}(A) = d_a - k$. By Theorem 40, we have $\operatorname{rank}(B) = d_b - k$. In this case, the equations (16.1) determine strictly separated self-adjoint boundary conditions.

Assume that $\operatorname{rank}(A) = d_a - k + r$, $1 \le r \le d_b - k$ and $\operatorname{rank}(B) = d_b - k + h$, $0 \le h \le d_b - k$. Then, by multiplying the boundary conditions (16.1) by a nonsingular matrix and interchanging rows, if necessary, we may assume that the first $d_a - k + r$ rows of $A$ are linearly independent and all other rows are identically zero. By Theorem 37 (i), we may also assume that the last $d_b - k + h$ rows of $B$ are linearly independent and all other rows are identically zero. For simplicity, we set $s_i = d_i - k$, $i = 1, 2$.

Let

$$A = \left(\ \alpha_1, \ldots, \alpha_{s_1}, \alpha_{s_1+1}, \ldots, \alpha_{s_1+r}, 0, \ldots, 0\ \right)^T,$$

and let

$$B = \left(\ 0, \ldots, 0, \beta_{s_2+h}, \ldots, \beta_{s_2+1}, \beta_{s_2}, \ldots, \beta_1\ \right)^T.$$

Then we compute

$$AE_{m_a}A^* = \begin{pmatrix} \alpha_1 E_{ma}\alpha_1^* & \cdots & \alpha_1 E_{m_a}\alpha_{s_1+r}^* & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{s_1+r}E_{m_a}\alpha_1^* & \cdots & \alpha_{s_1+r}E_{m_a}\alpha_{s_1+r}^* & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$BE_{m_b}B^* = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_{s_2+h}E_{m_b}\beta_{s_2+h}^* & \cdots & \beta_{s_2+h}E_{m_b}\beta_1^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \beta_1 E_{mb}\beta_{s_2+h}^* & \cdots & \beta_1 E_{m_b}\beta_1^* \end{pmatrix}.$$

Then, from $AE_{m_a}A^* = BE_{m_b}B^*$, we get

$$\begin{pmatrix} \alpha_1 E_{m_a}\alpha_1^* & \cdots & \alpha_1 E_{m_a}\alpha_{s_1+r}^* \\ \cdots & \cdots & \cdots \\ \alpha_{d-(s_2+h)}E_{m_a}\alpha_1^* & \cdots & \alpha_{d-(s_2+h)}E_{m_a}\alpha_{s_1+r}^* \end{pmatrix} = 0_{(s_1-h)\times(s_1+r)},$$

$$\begin{pmatrix} \beta_{d-s_1-r}E_{m_b}\beta_{s_2+h}^* & \cdots & \beta_{d-s_1-r}E_{m_b}\beta_1^* \\ \cdots & \cdots & \cdots \\ \beta_1 E_{m_b}\beta_{s_2+h}^* & \cdots & \beta_1 E_{m_b}\beta_1^* \end{pmatrix} = 0_{(s_2-r)\times(s_2+h)}. \tag{18.7}$$

Equations (6.9) are equivalent to

$$\alpha_i \in \{\alpha_1 E_{m_a}, \alpha_2 E_{m_a}, \ldots, \alpha_{s_1+r}E_{m_a}\}^\perp, \quad i = 1, 2, \ldots, s_1 - h \tag{18.8}$$

and (18.7) is equivalent to

$$\beta_i \in \{\beta_1 E_{m_b}, \beta_2 E_{m_b}, \ldots, \beta_{s_2+h}E_{m_b}\}^\perp, \quad i = 1, 2, \ldots, s_2 - r. \tag{18.9}$$

Since

$$\dim\{\alpha_1 E_{m_a}, \alpha_2 E_{m_a}, \ldots, \alpha_{s_1+r}E_{m_a}\}^\perp = d_a - k - r$$

in $\mathbb{C}^{m_a}$, by (18.8) we have $s_1 - h \le d_a - k - r$, i.e., $h \ge r$.

Since

$$\dim\{\beta_1 E_{m_b}, \beta_2 E_{m_b}, \ldots, \beta_{s_2+h} E_{m_b}\}^\perp = d_b - k - h,$$

in $\mathbb{C}^{m_b}$, by (18.9) we have $s_2 - r \le d_b - k - h$, i.e., $h \le r$. Therefore, $h = r$.

This shows that, if $0 \le r \le d_b - k$ and $\operatorname{rank}(A) = d_a - k + r$, then $\operatorname{rank}(B) = d_b - k + r$. For the latter part, the construction of matrices $A, B$ satisfying these three conditions is routine and therefore omitted. □

The value of the parameter $r$ in Theorem 41 determines the number of coupled boundary conditions. We expand on this point with the following corollaries.

**Corollary 11.** *Let the notation and hypotheses of Theorem* 41 *hold. If $r = 0$, then the conditions* (16.1) *are strictly separated with exactly $d_a - k$ conditions at $a$ and exactly $d_b - k$ conditions at $b$.*

*Proof.* This follows directly from Theorem 41. □

**Corollary 12.** *Let the notation and hypotheses of Theorem* 41 (i) *hold. If $d_a > d_b$ and $r > 0$, then* (16.1) *has exactly $2r$ coupled boundary conditions and $d_a - k - r$ separated conditions at $a$ and $d_b - k - r$ separated conditions at $b$.*

*Proof.* This follows directly from Theorem 41. □

**Corollary 13.** *Let the notation and hypotheses of Theorem* 41 (i) *hold. If $d_a = d_b$ and $0 < r < d_b - k$, then* (16.1) *has exactly $2r$ coupled boundary conditions and $d_a - k - r$ separated conditions at $a$ and $d_b - k - r$ separated conditions at $b$. If $d_a = d_b$ and $r = d_b - k$, then all conditions of* (16.1) *are coupled. Note that $d = 2r$ in this case and that all conditions of* (16.1) *can be coupled only when $d_a = d_b$.*

*Proof.* This follows directly from Theorem 41. □

**Corollary 14.** *Let the notation and hypotheses of Theorem* 41 (ii) *hold. If $r > 0$, then* (16.1) *has exactly $2r$ coupled boundary conditions and*

*$d_b - k - r$ separated conditions at $b$ and $d_a - k - r$ separated conditions at $a$.*

*Proof.* This follows directly from Theorem 41. $\square$

**Remark 41.** If $0 \leq r \leq \min\{d_a - k, d_b - k\}$, then there are exactly $2r$ coupled conditions in (16.1). Thus, we can classify the self-adjoint boundary conditions (16.1) into $\min\{d_a - k, d_b - k\} + 1$ 'types' such that each type has the same number of coupled conditions. When $r = 0$, all conditions are separated, but note that all conditions can be coupled only when $d_a = d_b$. If $d_a = d_b$ and $r$ assumes its maximum value $r = d_b - k$, then all conditions of (16.1) are coupled.

**Remark 42.** We comment on the three remaining cases: (For these cases, we don't need Theorem 41 since they follow directly from Theorem 37.)

(i) $d_a = k = d_b$. In this case the minimal operator $S_{\min}$ is itself a self-adjoint operator and has no proper self-adjoint extension in $H$. Thus, there are no boundary conditions required or allowed, i.e., in Theorem 37 $A = 0 = B$ and (16.1) is vacuous. This case occurs if and only if $d_a = k$ and $d_b = k$.

(ii) $d_a = k < d_b$. Then $m_a = 0$ and $m_b = 2d_b - n$. In this case, $A = 0$, and all the self-adjoint boundary conditions are given by:

$$B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_b}](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{18.10}$$

where the $d \times m_b$ complex matrix $B$ satisfies $\operatorname{rank}(B) = d$ and $BE_{m_b}B^* = 0$. In this case, every one of the conditions (16.1) is specified at the endpoint $b$ only, and thus (16.1) is strictly separated. If $d_b = n$, the endpoint $b$ is either LC or regular, and $d = k$, $m_b = n$. If $b$ is LC, Theorem 37 reduces to the self-adjoint boundary conditions at the endpoint $b$:

$$B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_n](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

And, if $b$ is regular, (18.10) reduces to

$$B\begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(iii) $k < d_a \leq n$ and $d_b = k$. Then $m_a = 2d_a - n$ and $m_b = 0$. This case is the same as Case 2 with the endpoints $a, b$ interchanged: in Case (2), replace $b$ by $a$, $m_b$ by $m_a$, $v_j$ by $u_j$ and $B$ by $A$.

It is interesting to note that Theorem 41 can be used to give a more rigorous definition of strictly separated self-adjoint boundary conditions.

**Definition 19.** Under the conditions of Theorem 41, we say that the self-adjoint boundary conditions (16.1) of Theorem 37 are strictly separated if $\operatorname{rank}(A) = d_a - k$. In this case, by Theorem 40, $\operatorname{rank}(B) = d_b - k$. Note that this is case $r = 0$ of Theorem 41.

**19. Construction of all types of conditions.** In this section, we give a construction to show that self-adjoint boundary conditions characterized by Theorem 41 in terms of the parameter $r$ can be realized for any $r$, $0 \leq r \leq \min\{d_a - k,\, d_b - k\}$. Also, we construct separated non-real self-adjoint boundary conditions as mentioned above in Remark 37 in connection with the clarification of the Everitt and Markus comment. Examples are given to illustrate the above results. We start with an example to construct conditions for all values of parameter $r$ of Theorem 41 (i). The construction for part (ii) is similar and hence omitted.

**Example 9.** Let the notation and hypotheses of Theorem 41 (i) hold. If $0 \leq r \leq d_b - k$ and $\operatorname{rank}(A) = d_a - k + r$, then by Theorem 41, we have $\operatorname{rank}(B) = d_b - k + r$.

Let

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)_{1\times m_a}.$$

where 1 is located at the $i$th column. Let

$$v_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)_{1\times m_b},$$

where 1 is located at the $i$th column.

The self-adjointness conditions $AE_{m_a}A^* = BE_{m_b}B^*$ given by Theorem 37 can be written as

$$(A : B)\begin{pmatrix} E_{m_a} & 0 \\ 0 & -E_{m_b} \end{pmatrix}(A : B)^* = 0.$$

(i) If $r = 0$, i.e., in the case of strictly separated boundary conditions, we can choose

$$A_{d\times m_a} = (e_1^T, e_2^T, \dots, e_{d_a-k}^T, 0_{1\times m_a}^T, \dots, 0_{1\times m_a}^T)^T,$$

and

$$B_{d\times m_b} = (0_{1\times m_b}^T, \dots, 0_{1\times m_b}^T, v_1^T, v_2^T, \dots, v_{d_b-k}^T)^T.$$

Through computation, we get that Theorem 37 (i) and (ii) hold, and therefore (16.1) is a self-adjoint boundary condition.

(ii) If $0 < r \leq d_b - k$, we can choose

$$\begin{aligned} A = (e_1^T, e_2^T, \dots, e_{d_a-k}^T, (e_1E_{m_a})^T, \dots,& \\ (e_rE_{m_a})^T, 0_{1\times m_a}^T, \dots, 0_{1\times m_a}^T)^T&, \\ B = (-(v_1E_{m_b})^T, -(v_2E_{m_b})^T, \dots, -(v_rE_{m_b})^T,& \\ 0_{1\times m_b}^T, \dots, 0_{1\times m_b}^T, v_1^T, \dots, v_{d_b-k}^T)^T&. \end{aligned}$$

Then Theorem 37 (i) and (ii) hold, and therefore (16.1) is a self-adjoint boundary condition.

(iii) In particular, when $r = d_b - k$ and $d_a = d_b$, then the self-adjoint boundary conditions are coupled. In this case, $d = 2(d_a - k) = m_a = m_b$, and we can choose the matrices $A = B = I_{m_a}$, where $I_{m_a}$ denotes the identity matrix.

As mentioned in Remark 37 above, in the second order case, there are no separated complex boundary conditions in the sense that every such condition is equivalent to a separated real condition. For real coefficient differential expressions of any even order $n = 2k \geq 4$ which satisfy the assumption that $My = \lambda_a wy$ has $d_a$ linearly independent solutions on $(a, c)$ which lie in $L^2((a, c), w)$ for some real $\lambda_b$ and $My = \lambda_b wy$ has $d_b$ linearly independent solutions on $(c, b)$ in $L^2((c, b), w)$ for some real $\lambda_b$, we discuss all the cases $k \leq d_a, d_b \leq n$ and then illustrate whether they have strictly separated non-real self-adjoint conditions. If they have, we construct some examples.

We start with the case $n = 4$.

**Example 10.** Let

$$My = [(p_2 y'')' + p_1 y']' + qy = \lambda wy$$
$$\text{on } J = (a, b),\ -\infty \le a < b \le \infty,$$

where $1/p_2, p_1, q, w \in L_{\mathrm{loc}}(J, \mathbb{R})$, $w > 0$ on $J$.

Here we assume that $d_a = k = 2$ and $2 < d_b \le 4$. Then $m_a = 0$ and $m_b = 2d_b - n = 2d_b - 4$. In this case, we have no condition at $a$ and self-adjoint separated conditions at $b$.

If $d_b = 3$, we have $m_b = 2d_b - 4 = 2$ and $d = d_a + d_b - n = 1$. Then the self-adjoint boundary condition is separated, and there is only one condition at $b$. This condition can be replaced by an equivalent real self-adjoint condition. This will be explained in the following.

If $d_b = 4$, we have $m_b = 4$ and $d = 2$. We choose

$$B_{2\times 4} = \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}.$$

Then $\operatorname{rank}(B) = 2$ and $BE_{m_b}B^* = 0$. By Theorem 37, these separated conditions are self-adjoint, and thus we have the non-real self-adjoint boundary conditions:

$$[y, v_1](b) + i[y, v_2](b) = 0, \qquad [y, v_3](b) - i[y, v_4](b) = 0. \tag{19.1}$$

**Example 11.** For $n = 2k = 6$, we let

$$My = [[(p_3 y''')' + (p_2 y'')']' + p_1 y'\}' + qy = \lambda wy \quad \text{on } J = (a, b),$$
$$-\infty \le a < b \le \infty,$$

where $1/p_3$, $p_2, p_1$, $q$, $w \in L_{\mathrm{loc}}(J, \mathbb{R})$, $w > 0$ on $J$.

We assume that $d_a = 3$ and $3 < d_b \le 6$. Then $m_a = 0$ and $m_b = 2d_b - 6$.

(i) When $d_b = 4$, then $m_b = 2$, $d = 1$, and every non real self-adjoint condition is equivalent to a real self-adjoint condition.
(ii) When $d_b = 5$, then $m_b = 4$, $d = 2$, and we can use (19.1).

(iii) When $d_b = 6$, then $m_b = 6$, $d = 3$. We choose

$$B_{3\times 6} = \begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{rank}(B) = 3$ and $BE_{m_b}B^* = 0$. By Theorem 37, the following condition is self-adjoint:

$$[y, v_1](b) + i[y, v_2](b) = 0,$$
$$[y, v_5](b) - i[y, v_6](b) = 0,$$
$$[y, v_3](b) = 0.$$

**Remark 43.** Let $d_a = k$ and $d_b = k + 1$. Then $d = 1$ and, by Lemma 14, any complex self-adjoint boundary condition at $b$ can be replaced by an equivalent real self-adjoint condition. This can occur only when $d_b = k + 1$.

**Lemma 14.** *When all self-adjoint conditions are separated and there is only one condition at a given endpoint, then this condition can always be replaced by an equivalent real condition.*

*Proof.* The proof given in [**107,** Corollary 5] can be readily adapted to this case. □

**Theorem 42.** *Let $d_a = k$ and $k < d_b \leq n$.*

(i) *If $d_b = k + 1$, then $m_b = 2$, $d = 1$, and the self-adjoint boundary condition has only one condition at $b$. This condition can be replaced by an equivalent real self-adjoint condition.*
(ii) *If $d_b = k + 2$, then $m_b = 4, d = 2$, and we may construct self-adjoint differential operators specified by non-real strictly separated boundary conditions as in* (19.1).
(iii) *If $d_b = k + r$ $(2 < r \leq k)$, then $d = r$ and $m_b = 2r$. Let*

$$e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)_{1\times m_b},$$

*where one is in the $i$ position. Then*

$$e_1 E_{m_b} = (0, 0, \dots, 0, -1)_{1\times 2r},$$
$$e_2 E_{m_b} = (0, 0, \dots, 0, 1, 0)_{1\times 2r}.$$

*Choose*

$$B = \begin{pmatrix} e_1 + ie_2 \\ ie_1 E_{m_b} + e_2 E_{m_b} \\ e_3 \\ e_4 \\ \vdots \\ e_r \end{pmatrix}.$$

*Then* $\operatorname{rank}(B) = d$ *and* $BE_{m_b}B^* = 0$. *By Theorem* 37, *the following are separated non-real self-adjoint boundary conditions*:

$$\begin{gathered} [y, v_1](b) + i[y, v_2](b) = 0, \\ [y, v_{m_b-1}](b) - i[y, v_{m_b}](b) = 0, \\ [y, v_3](b) = 0, \\ [y, v_4](b) = 0, \\ \cdots\cdots, \\ [y, v_r](b) = 0. \end{gathered}$$

Similar to Theorem 42, we can construct separated non real self-adjoint boundary conditions for the case when $k < d_a \le n$, $d_b = k$.

Next, we construct non-real strictly separated boundary conditions for the general case when neither deficiency index is minimal. For the cases $d_a = n$ and $d_b = n$, non-real strictly separated self-adjoint boundary conditions have been constructed in [**107,** Theorem 5].

**Theorem 43.** *Assume that* $k < d_a \le n$, $k < d_b \le n$ *and* $d_a \ge d_b$.

*Let* $k < d_a \le n$, $k < d_b < n$ *and* $d_a > d_b$. *Then* $d_a - k \ge 2$, *and there are at least two separated conditions at the endpoint* $a$. *So, for this case, we can always construct non-real strictly separated self-adjoint boundary conditions. Notice that, when* $d_b = k+1$, *there is only one separated condition at* $b$, *and this condition can always be replaced by an equivalent real condition.*

(i) *Let* $d_b = k+r$ *and* $1 \le r \le k-1$. *Then* $k+r < d_a \le n$, $d_b - k = r$, $m_b = 2(d_b - k) = 2r$, $m_b = 2(d_b - k)$ *and* $d = d_a - k + r$.

(a) *If* $d_a - k = 2$ *and* $d_a > d_b$, *we have* $d_b - k = 1$, $m_a = 4$, $m_b = 2$ *and* $d = 3$. *Choose*

$$A_{3\times 4} = \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

*Then*

$$\operatorname{rank}(A : B) = 3,$$

*and*

$$AE_4A^* = BE_2B^* = 0.$$

*Therefore, we have the following separated non-real self-adjoint boundary conditions at* $a$:

$$[y, u_1](a) + i[y, u_2](a) = 0,$$
$$[y, u_3](a) - i[y, u_4](a) = 0,$$

*and the separated condition at* $b$:

$$[y, v_1](b) = 0.$$

(b) *If* $d_a - k \geq 3$, *we let*

$$e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)_{1\times m_a},$$

*where one is in the* $i$ *position.*
*Set*

$$A = \begin{pmatrix} \widetilde{A}_{(d_1-k)\times m_a} \\ 0_{r\times m_a} \end{pmatrix},$$

*where*

$$\widetilde{A} = \begin{pmatrix} e_1 + ie_2 \\ ie_1E_{m_a} + e_2E_{m_a} \\ e_3 \\ \vdots \\ e_{m_a/2} \end{pmatrix}.$$

*And set*

$$B = \begin{pmatrix} 0_{(d_a-k)\times 2r} \\ \widetilde{B}_{r\times 2r} \end{pmatrix}$$

*where*

$$\widetilde{B}_{r\times 2r} = (I_{r\times r}, \quad 0_{r\times r}).$$

*Then*

$$\operatorname{rank}(A, B) = d$$

*and*

$$\widetilde{A}E_{m_a}\widetilde{A}^* = 0, \qquad \widetilde{B}E_{m_b}\widetilde{B}^* = 0.$$

*Therefore,*

$$AE_{m_a}A^* = 0 = BE_{m_b}B^*.$$

*By Theorem* 37*, these separated conditions are non real self-adjoint boundary conditions*:

$$\begin{gathered}
[y, u_1](a) + i[y, u_2](a) = 0, \\
[y, u_{m_a-1}](a) - i[y, u_{m_a}](a) = 0, \\
[y, u_3](a) = 0, \\
\cdots\cdots \\
[y, u_{m_a/2}](a) = 0, \\
[y, v_1](b) = 0, \\
[y, v_2](b) = 0, \\
\cdots\cdots, \\
[y, v_r](b) = 0.
\end{gathered}$$

(c) *Let $k < d_a < n$, $k < d_b < n$ and $d_a = d_b$.*

(ii) *When $d_1 - k = d_2 - k = 1$, then $d = 2$ and $m_1 = m_2 = 2$. By Theorem* 40*, for the case of strictly separated self-adjoint boundary conditions, there is only one separated condition at $a$ and only one separated condition at $b$. By Lemma* 14*, these conditions can always be replaced by equivalent real conditions. In this case, there are no non-real separated self-adjoint boundary conditions.*

(iii) *Assume that $d_a = d_b = k + r$ $(r \geq 2)$. According to the method of* (b)*, we can construct non-real separated self-adjoint boundary conditions.*

Here we have concentrated on the cases when $k < d_a \leq n$, $k < d_b \leq n$ and $d_a \geq d_b$. The cases when $k < d_a \leq n$, $k < d_b \leq n$ and $d_a < d_b$ are similar and hence omitted.

**20. The spectrum.** The spectrum of a self-adjoint ordinary differential operator $S$, $S_{\min}(A) \subset S = S^* \subset S_{\max}(A)$, $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geq 1$, with $A$ Lagrange symmetric matrix, constructed above in the Hilbert space $H = L^2(J, w)$, $J = (a, b)$, with $w$ any weight function, is real, consists of eigenvalues of finite multiplicity, of essential and of continuous spectrum. A number $\lambda$ is an eigenvalue of $S$ if the corresponding differential equation $My = \lambda wy$ on $J$ has a nontrivial solution in $H$ which satisfies the boundary condition of $S$. On the other hand, the essential spectrum is independent of the boundary conditions and thus depends only on the coefficients, including the weight function $w$, of the equation. This dependence is implicit and highly complicated. The coefficients and the weight function also determine the deficiency index $d$ of the minimal operator $S_{\min}$ determined by the equation. This is the number of linearly independent solutions in $H$ for nonreal values of the spectral parameter $\lambda$, and this number is independent of $\lambda$ provided $\operatorname{Im}(\lambda) \neq 0$. For real values of $\lambda$, the number of linearly independent solutions $r(\lambda)$ in $H$ varies with $\lambda$. It is this dependence on $\lambda$ which we exploit below to get information about the spectrum. One advantage of using $r(\lambda)$ to study the spectrum of these operators is that it makes available the theory of ordinary linear differential equations to use as a tool to get spectral information. A number of the results discussed here are surprisingly recent.

The contrasting behavior of $r(\lambda)$ for the two singular endpoint case from the one singular endpoint case has some interesting consequences. In the case of only one singular endpoint we have $r(\lambda) \leq d$, whereas in the two singular endpoint case $r(\lambda)$ may assume values less than $d$, equal to $d$, or greater then $d$. The case $r(\lambda) > d$ leads to the surprising result that this value of $\lambda$ is an eigenvalue of *every* self-adjoint extension, i.e., for any given self-adjoint boundary condition, there are eigenfunctions of $\lambda$ which satisfy this boundary condition. The one singular endpoint case is discussed in Section 21, the two singular endpoint case in Section 22.

But first we make a remark.

**Remark 44.** There is a vast literature studying the dependence of the spectrum and of the deficiency index of ordinary differential operators in Hilbert space $L^2(J, w)$ on the coefficients. Most of these papers are for the constant weight function $w = 1$, the interval $J = (0, \infty)$ with

0 a regular endpoint and either the classical expressions with smooth coefficients discussed in Section 2 or the quasi-derivative formulation of these discussed in Section 3. See the books or monographs by Weidmann [**116**], Kauffman, Read and Zettl [**69**], Coddington and Levinson [**21**], Dunford and Schwartz [**24**], Naimark [**84**] and the papers [**7, 8, 9, 10, 27, 29, 55, 66, 73, 82, 83**], and the references therein for an introduction to this literature. Note: This list is not intended to be comprehensive or up to date.

Next we give definitions of the parts of the spectrum discussed below. In some of the literature 'essential spectrum' and 'continuous spectrum' are used interchangeably, we use Weidmann's definitions from his well-known book [**115**] which differentiate between these terms. For the self-adjoint differential operators studied in this paper every eigenvalue has finite multiplicity.

**Definition 20.** The essential spectrum $\sigma_e(S)$ of a self-adjoint operator $S$ in a Hilbert space $H$ is the set of those points of $\sigma(S)$ that are either accumulation points of $\sigma(S)$ or isolated eigenvalues of infinite multiplicity. The set $\sigma_d(S) = \sigma(S)\backslash\sigma_e(S)$ is called the discrete spectrum of $S$ and consists of the isolated eigenvalues for the operators $S$ studied here. Below, by the multiplicity of an eigenvalue, we mean its geometric multiplicity. We say that the spectrum of $S$ is discrete if $\sigma_e(S)$ is empty.

**Definition 21.** Let $S$ be a self-adjoint operator on an abstract Hilbert space $H$. Let $H_p$ denote the closed linear hull of all eigenfunctions of $S$, we call $H_p = H_p(S)$ the discontinuous subspace of $H$ with respect to $S$. The orthogonal complement of $H_p$ is called the continuous subspace of $H$ with respect to $S$. This is denoted by $H_c = H_c(S)$. We denote by $S_p$ and $S_c$ the restrictions of $S$ to $H_p$ and $H_c$, respectively. These operators are called the (spectral) discontinuous, and continuous parts of $S$, respectively.

**Definition 22.** The continuous spectrum $\sigma_c(S)$ of $S$ is defined as the spectrum of $S_c$.

**Proposition 4.** [**115**]. *Any isolated point $\lambda$ of the spectrum of a self-adjoint operator $S$ is an eigenvalue of $S$.*

*Proof.* This is well known, see [**115**]. □

**21. One regular endpoint.** In this section, we study spectral properties of the self-adjoint realizations $S$ of the equation

$$My = \lambda wy \quad \text{on } J = (a, b), \quad -\infty \le a < b \le \infty, \tag{21.1}$$

in the Hilbert space $H = L^2(J, w)$ where $M = M_A$, $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \ge 1$, $A$ is Lagrange symmetric, $d(M) = d$, and $w$ is a weight function on $J$ but with the additional hypotheses:

The endpoint a is regular and there exist $d$ linearly independent solutions of (21.1) in $H$ for some real $\lambda$.

**Remark 45.** We comment on the additional hypotheses. Since $a$ is regular, $d$ satisfies $k \le d \le 2k$, and all values of $d$ in this range occur. If there is no real $\lambda$ for which there are $d$ linearly independent solutions of (21.1) in $H$, then by Theorem 45 below, the essential spectrum covers the whole real line. In this case any eigenvalue of any self-adjoint realization $S$ is embedded in the essential spectrum. Such eigenvalues seem to occur 'coincidentally,' and not much is known about them other than examples showing they exist. In particular, there seems to be nothing known in general about the dependence of such eigenvalues on the boundary conditions.

Recall that, in this case, we have

$$k \le d \le 2k = n. \tag{21.2}$$

Our first result is contained in the 2012 paper **[58]** by Hao, Sun, Wang and Zettl, but may have been known before.

**Theorem 44.** *For every $\lambda \in \mathbb{R}$, we have $r(\lambda) \le d$.*

*Proof.* Since the real and imaginary parts of a complex solution are real solutions, if there are $r(\lambda)$ linearly independent solutions which lie in $H = L^2(J, w)$, then there exist $r$ linearly independent real-valued solutions $u_1, \dots, u_r$ in $H$ for this $\lambda$. Suppose that $r > d$, and define

$$\begin{aligned} D_u &= D_{\min} \dotplus \operatorname{span}\{u_1, \dots, u_d\}, \\ D_r &= D_{\min} \dotplus \operatorname{span}\{u_1, \dots, u_r\}. \end{aligned}$$

We show that it follows from the GKN theorem that $D_u$ is the domain of a self-adjoint operator $S$ in $H$: To prove part (i) of GKN, assume that some linear combination

$$z = \sum_{i=1}^{d} c_i u_i$$

is in $D_{\min}$. Since the endpoint $a$ is regular, it follows that $z^{[j]}(a) = 0$, $j = 0, \dots, n-1$, which implies that $z$ is the trivial solution contradicting the linear independence of $u_1, \dots, u_d$. Part (ii) follows from the Lagrange identity and the fact that the $u_i, u_j, (i, j = 1, \dots, d)$ are all real-valued solutions for the same $\lambda$,

$$\begin{aligned}[u_i, u_j](b) - [u_i, u_j](a) &= \int_a^b \{\overline{u_j} M u_i - u_i \overline{M u_j}\} \\ &= \int_a^b \{\lambda u_j u_i - \lambda u_i u_j\} = 0.\end{aligned}$$

Part (iii) follows from the fact that the deficiency index of $S_{\min}$ is $d$ and $D_u$ is a $d$ dimensional extension of $D_{\min}$.

From the proofs of (i) and (ii), it also follows that (i) and (ii) hold for $u_j$, $j = 1, \dots, r$. Hence, $D_r$ is the domain of a proper symmetric extension $S_r$ of $S$. But a self-adjoint operator in a Hilbert space has no proper symmetric extensions:

$$S \subset S_r \subset S_r^* \subset S^*$$

implies that $S = S_r$. This contradiction completes the proof. □

**Theorem 45.** *The following results hold*:

(i) *If $r(\lambda) < d$ for some $\lambda \in \mathbb{R}$, then $\lambda$ is in the essential spectrum of every self-adjoint extension $S$ of $S_{\min}$. In particular, if $r(\lambda) < d$ for every $\lambda \in \mathbb{R}$, then $\sigma_e(S) = (-\infty, \infty)$ for every self-adjoint extension $S$.*

(ii) *If, for some $\lambda \in \mathbb{R}$, $r(\lambda) = d$, then $\lambda$ is an eigenvalue of geometric multiplicity $d$ for some self-adjoint realization $S$.*

*Proof.* Part (i) is well known [**116**]. See also the proof given in [**84**] for a special case, it extends readily to our hypotheses. For part (ii) let $u_1, \dots, u_d$ be linearly independent real solutions in $H$ for some $\lambda_1 \in \mathbb{R}$.

From the proof of Theorem 44 it follows that the operator $S_u$ with domain $D_u$ given by

$$D_u = D_{\min} \dot{+} \operatorname{span}\{u_1, \ldots, u_d\}$$

is a self-adjoint extension. Hence, each $u_j$, $j = 1, \ldots, d$, is an eigenfunction of this $\lambda_1$. □

Next we explore the relationship between $r(\lambda)$ and the continuous spectrum for arbitrary deficiency index $d$.

**Theorem 46.** *Assume there exists an open interval $I = (\mu_1, \mu_2)$, $-\infty \leq \mu_1 < \mu_2 \leq \infty$, of the real line such that the equation* (21.1) *has $d$ linearly independent solutions which lie in $H$ for every $\lambda \in I$. Then*

(i) *for any self-adjoint realization $S$ of* (21.1)*, the intersection $\sigma_c(S) \cap I$ is empty.*
(ii) *for any self-adjoint realization $S$ of* (21.1)*, the point spectrum $\sigma_p$ is nowhere dense in $I$.*

*Proof.* See [**59**] and the next remark. □

**Remark 46.** The special case when $d = k$ and $w = 1$ is due to Weidmann [**116**]. In [**102**], it is shown that, for arbitrary deficiency index $d$, there exists a self-adjoint realization $S$ with separated boundary conditions for which the conclusion holds. When $d = 2k$, it is well known that the spectrum is discrete, and so the conclusion holds automatically.

**Remark 47** (Historical comment)**.** We comment on Theorem 46. When $d = n$, the conclusions follow from the well-known fact that the spectrum is discrete. The special cases when $d = k$ and $w = 1$ are established in Weidmann [**116**]. The extension to general $w$ is routine. The extension to $k < d < n$ is not routine. There are two major obstacles: (i) When $d = k$, there is no boundary condition at the singular endpoint. When $k < d < n$, there are exactly $d$ singular boundary conditions. What are they? This answer is given in Section 16 in terms of the LC solutions constructed in Section 15. Obstacle (ii) involves an approximation method, which depends on the

LC and LP solutions constructed in Section 15. It is somewhat similar in spirit to the approximations based on 'inherited' boundary conditions used in an algorithm of the Bailey-Everitt-Zettl code SLEIGN2 for the computation of eigenvalues of singular Sturm-Liouville problems.

The continuous spectrum is contained in the essential spectrum. Can the conclusion of part (i) of Theorem 46 be strengthened to 'the intersection $\sigma_e(S) \cap I$ is empty'? This is conjectured in [**59**], but the answer is no, even in the second order case where the question dates back to Hartman and Wintner [**63**].

**Theorem 47.** *Assume that $My = -y'' + qy = \lambda y$ has an $L^2$ solution for all $\lambda$ in some interval $I = (\mu_1, \mu_2)$. Then, for every self-adjoint extension of $S_{\min}$*:

(i) *There is no continuous spectrum in $I$.*
(ii) *The point spectrum $\sigma_p$ is nowhere dense in $I$, i.e., its closure does not contain a nonempty open set.*

Hartman and Wintner [**63**] as well as others conjectured that part (ii) could be improved to: $\sigma_p$ has no accumulation points of eigenvalues in $I$. But this is false. In fact, the next theorem not only disproves this conjecture, but it also shows that Theorem 47 (i) is sharp!

**Remark 48.** Del Rio [**23**] and Remling [**92**] have clarified the complicated relationship between the essential spectrum and the real numbers $\lambda$ for which $r(\lambda) = d$. In particular, they have shown that, in general, $r(\lambda) = d$ for all $\lambda$ in some open interval $I$ does not imply that there is no essential spectrum in $I$. In 1996, Remling [**92**] proved that, in the second order case, "continuous spectrum is empty in $I$" cannot be strengthened to "essential spectrum is empty in $I$" and in fact the continuous spectrum result of Theorem 47 is 'best possible.'

**Theorem 48.** [**92**]. *Let $I$ be a finite, open interval, and let $I_1 \subset I$ be a closed, nowhere dense set. Then there exists a potential $q$ such that*:

(i) $\sigma_e = I_1$,
(ii) $My = -y'' + qy = \lambda y$ *has an $L^2$-solution for all $\lambda \in I$.*

Note that $I_1$ can be an uncountable set.

Thus, a natural question is: under what additional condition is the essential spectrum empty in an interval $I$? This question is answered by Assumption (A) stated next and the following theorem.

**Assumption (A).** There exists an open set $G$ in the complex plane containing $I = (\alpha, \beta)$ with $\alpha$ and $\beta$ on the boundary of $G$, and there exist solutions $u_1(t,\lambda), \dots, u_d(t,\lambda)$ which lie in $H$, are real valued for $\lambda \in I$, and are analytic on $G$ for each fixed $t \in I$.

**Theorem 49.** *Let the notation and hypotheses of Theorem* 46 *hold. If* $r(\lambda) = d$ *for all* $\lambda$ *in some open interval* $I = (\alpha, \beta)$, *and assumption* (***A***) *holds on* $I$. *Then eigenvalues of every self-adjoint realization have no accumulation point in* $I$.

*Proof.* Let $S$ be an arbitrary self-adjoint extension of $S_{\min}$. By Theorem 38, the domain of $S$, $D(S)$ is given by:

(21.3)

$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_m](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\},$$

where $A, B$ satisfy conditions (i) and (ii) of Theorem 38, and $u_i$ ($i = 1, 2, \dots, m,\ m = 2d - 2k$) are square-integrable real-parameter LC solutions of differential equation:

$$My = \lambda_0 w\, y \quad \text{on } J = (a,b),\ -\infty < a < b \le \infty,$$

for some fixed $\lambda_0 \in (\alpha, \beta)$. The boundary conditions (21.3) consist of the system of $d$ equations:

$$U_i(y) = \sum_{j=1}^{n} a_{ij} y^{[j-1]}(a) + \sum_{j=1}^{m} b_{ij} [y, u_j](b), \quad i = 1, \dots, d. \tag{21.4}$$

Let $\varphi_j(\cdot, \lambda)$, $\lambda \in I$, $j = 1, \dots, d$, denote $d$ linearly independent solutions of (22.1) which satisfy the assumption $(A)$ on $G$. If $y(\cdot, \lambda)$ is an eigenfunction of $S$ for some $\lambda \in I$, then $y$ is a nontrivial linear combination of the solutions $\varphi_j(\cdot,\lambda)$, $j=1,\dots,d$, i.e.,

$$y(\cdot,\lambda)=\sum_{j=1}^{d} c_j\varphi_j(\cdot,\lambda), \quad c_j\in\mathbb{C}, \tag{21.5}$$

and $y(\cdot,\lambda)$ satisfies the boundary conditions (15.12). Substituting $y(\cdot,\lambda)$ into these boundary conditions, we have

$$U_i(y)=U_i\Big(\sum_{j=1}^{d} c_j\varphi_j(\cdot,\lambda)\Big)=\sum_{j=1}^{d} c_jU_i(\varphi_j(\cdot,\lambda))=0, \quad i=1,\dots,d. \tag{21.6}$$

This is a homogeneous system of linear equations in $c_1,\dots,c_d$. Let

$$\Delta(\lambda)=\det[U_i(\varphi_j(\cdot,\lambda))], \quad i,j=1,\dots,d. \tag{21.7}$$

Note that $\Delta(\lambda)$ is the determinant of the matrix of coefficients of the system of linear equations (21.6) and that the number of linear equations in (21.6) is equal to the number of coefficients $c_1,\dots,c_d$. Therefore, the system of linear equations (21.6) has a nontrivial solution for $c_1,\dots,c_d$ if and only if $\Delta(\lambda)=0$. Therefore, $\lambda\in I$ is in $\sigma_p(S)$ if and only if $\Delta(\lambda)=0$.

By (15.12) and (21.6), we have

$$U_i(\varphi_j(\cdot,\lambda))=\sum_{j=1}^{n} a_{ij}\varphi_j^{[j-1]}(a,\lambda)+\sum_{j=1}^{m} b_{ij}[\varphi_i,u_j](b,\lambda), \quad i=1,\dots,d,$$

so

$$\begin{aligned}\Delta(\lambda)&=\det[U_i(\varphi_r(\cdot,\lambda)]\\ &=\det\left[\begin{array}{c}\begin{bmatrix}\sum_{j=1}^n a_{1j}\varphi_1^{[j-1]}(a,\lambda) & \cdots & \sum_{j=1}^n a_{1j}\varphi_d^{[j-1]}(a,\lambda)\\ \cdots & \cdots & \cdots\\ \sum_{j=1}^n a_{dj}\varphi_1^{[j-1]}(a,\lambda) & \cdots & \sum_{j=1}^n a_{dj}\varphi_d^{[j-1]}(a,\lambda)\end{bmatrix}\\ +\begin{bmatrix}\sum_{j=1}^m b_{1j}[\varphi_1,u_j](b,\lambda) & \cdots & \sum_{j=1}^m b_{1j}[\varphi_d,u_j](b,\lambda)\\ \cdots & \cdots & \cdots\\ \sum_{j=1}^m b_{dj}[\varphi_1,u_j](b,\lambda) & \cdots & \sum_{j=1}^m b_{dj}[\varphi_d,u_j](b,\lambda)\end{bmatrix}\end{array}\right]\end{aligned}$$

$$= \det \begin{bmatrix} A \begin{bmatrix} \varphi_1(a,\lambda) & \cdots & \varphi_d(a,\lambda) \\ \cdots & \cdots & \cdots \\ \varphi_1^{[n-1]}(a,\lambda) & \cdots & \varphi_d^{[n-1]}(a,\lambda) \end{bmatrix} \\ +B \begin{bmatrix} [\varphi_1, u_1](b,\lambda) & \cdots & [\varphi_d, u_1](b,\lambda) \\ \cdots & \cdots & \cdots \\ [\varphi_1, v_m](b,\lambda) & \cdots & [\varphi_d, v_m](b,\lambda) \end{bmatrix} \end{bmatrix},$$

By the assumption of analytic dependence of solutions on $\lambda \in G$, we conclude that $\Delta(\lambda)$ is an analytic function of $\lambda$ in the open set $G$ of the complex plane which contains the real interval $(\alpha, \beta)$. Note that $\Delta(\lambda)$ is not identically zero on $G$ since all eigenvalues of $S$ are real. Therefore, from the well-known distribution of zeros of analytic functions, we conclude that the eigenvalues of $S$ have no accumulation point in the interval $I = (\alpha, \beta)$. This completes the proof of Theorem 49. □

**Theorem 50.** *Let the notation and hypotheses of Theorem* 49 *hold. Then there is no essential spectrum in $I$ for any self-adjoint realization $S$.*

*Proof.* Let $S$ be a self-adjoint realization. By Theorem 46, $\sigma_c(S) \cap I$ is empty and, from Theorem 49, we have that $\overline{\sigma_p(S)} \setminus \sigma_d(S)$ is empty. By Definitions 20 and 22, we have

$$\sigma_e(S) = \sigma_c(S) \cup (\overline{\sigma_p(S)}) \setminus \sigma_d(S),$$

and the conclusion follows. □

**Remark 49.** We prove Theorem 49 by constructing, for any given self-adjoint realization $S$ of (21.1), a characteristic function $\Delta(\lambda)$ whose zeros in the interval $(\mu_1, \mu_2)$ are precisely the eigenvalues of $S$ in this interval. The construction of $\Delta(\lambda)$ uses LC solutions and other notation and definitions from Sections 15 and 16. The proof of Theorem 49 provides a very good illustration of the use of the LC and LP constructions in Section 15 to get information about the spectrum.

**22. Two singular endpoints.** In this section, we study spectral properties of the self-adjoint realizations $S$ of the equation

$$My = \lambda wy \quad \text{on } J = (a, b), \quad -\infty \leq a < b \leq \infty, \tag{22.1}$$

in the Hilbert space $H = L^2(J, w)$, where $M = M_A$, $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geq 1$, $A$ is Lagrange symmetric and $w$ is a weight function on $J$. Note that we do not assume the additional hypothesis of the previous section. Its replacement will be assumed here as needed.

In this section, both endpoints may be singular. As mentioned above, the behavior of $r(\lambda)$ in the two singular endpoint case is dramatically different from the one singular endpoint case. Each endpoint has an influence which is independent of the other endpoint. So we study equation (22.1) on the intervals

$$J = (a, b), \quad J_a = (a, c), \quad \text{and} \quad J_b = (c, b), \quad -\infty \leq a < c < b \leq \infty,$$

in the Hilbert spaces $H = L^2(J, w)$, $H_a = L^2(J_a, w)$, $H_b = L^2(J_b, w)$. Here $c$ is an arbitrarily chosen point in $J$. Note that the results of the preceding section apply to both intervals $(a, c)$ and $(c, b)$ since $c$ is a regular endpoint for both.

Recall that $A \in Z_n(J, \mathbb{R})$ implies $A \in Z_n(J_a, \mathbb{R})$, $A \in Z_n(J_b, \mathbb{R})$, and recall our notation: $d_a$ for the deficiency index of $S_{\min}(a, c)$ in $L^2(J_a, w)$, $d_b$ denotes the deficiency index of $S_{\min}(c, b)$ in $L^2(J_b, w)$, $r_a(\lambda) = d_a(\lambda)$, $r_b(\lambda) = d_b(\lambda)$ for $\lambda \in \mathbb{R}$, etc.

The next lemma summarizes some basic facts for the one regular endpoint case. It is stated here for convenience to make the comparison with the two singular endpoint case easier.

**Lemma 15.** *Let $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geq 1$, assume that $A$ is Lagrange symmetric and $w$ is a weight function on $J$.*

(i) *Then $M = M_A$ is a symmetric differential expression on $J$, $J_a$ and on $J_b$.*
(ii) *The deficiency indices $d_a$ and $d_b$ are independent of the choice of $c \in (a, b)$.*
(iii) *For any $\lambda \in \mathbb{R}$, $d_a(\lambda) \leq d_a$, $d_b(\lambda) \leq d_b$, and strict inequality does occur.*
(iv)

$$k \leq d_a, \qquad d_b \leq 2k = n,$$

*and all values of $d_a$, $d_b$ in this range are realized.*

(v)
$$d = d_a + d_b - n.$$

(vi) *We have*
$$0 \leq d \leq 2k = n,$$

*and all values in this range occur.*

(vii) *If one endpoint is regular and $r(\lambda) = d$, then $\lambda$ is an eigenvalue of multiplicity $d$ for some self-adjoint extension.*

(viii) *If one endpoint is regular and $r(\lambda) = d$ for all $\lambda$ in some open interval $I$, then there is no continuous spectrum in $I$ for any self-adjoint extension $S$. Moreover, the eigenvalues of any self-adjoint extension are nowhere dense in $I$.*

*Proof.* See Section 21 and [**59**]. □

The next theorem shows that the behavior of $r_a(\lambda)$ on $(a, c)$ and the behavior of $r_b(\lambda)$ on $(c, b)$ affect the spectrum on the whole interval $(a, b)$.

**Theorem 51.** *Let the hypothesis and notation of Lemma* 15 *hold. Then*

(i) *We have*

$$\sigma_e(a, b) = \sigma_e(a, c) \cup \sigma_e(c, b). \tag{22.2}$$

(ii) *If $\lambda \notin \sigma_e(a, b)$, then*

$$r_a(\lambda) + r_b(\lambda) = d_a + d_b. \tag{22.3}$$

(iii) *If $r_a(\lambda) < d_a$ or $r_b(\lambda) < d_b$, then $\lambda \in \sigma_e(a, b)$.*

(iv) *If $r_a(\lambda) + r_b(\lambda) < d_a + d_b$, then $\lambda \in \sigma_e(a, b)$.*

(v)

$$r_a(\lambda) + r_b(\lambda) - n \leq r(\lambda) \leq \min\{r_a(\lambda), r_b(\lambda)\}. \tag{22.4}$$

(vi) *If $\lambda \notin \sigma_e(a, b)$, then*

$$d = d_a + d_b - n \leq r(\lambda) \leq \min\{r_a(\lambda), r_b(\lambda)\}. \tag{22.5}$$

*Proof.* Parts (ii), (iii) and (iv) follow from (i) and Theorem 51, (vi) follows from (v) and (ii). For details of the proof of (v) see [**60,** Section

4]. For smooth coefficients (i) follows from Dunford and Schwartz [**24,** page 1438, Theorem 4]. This proof can readily be adapted to our hypotheses on the coefficients.

An alternative proof of (i) can be constructed from the "two-interval" theory developed by Everitt and Zettl in [**46, 47**]: Consider the "two-interval" minimal operator $S_{2\min}$ given by

$$S_{2\min} = S_{\min}(a, c) \dotplus S_{\min}(c, b) \tag{22.6}$$

in the direct sum space $L^2((a, c), w) \dotplus L^2((c, b), w)$ which can be identified with $H = L^2((a, b), w)$. Let $S_a, S_b$ be self-adjoint extensions of $S_{\min}(a, c)$ and $S_{\min}(c, b)$, respectively, and let $S = S_a \dotplus S_b$. Then $S$ is a self-adjoint extension in $H$. It is well known that the essential spectrum of the direct sum of two self-adjoint operators in Hilbert space is the union of their essential spectra, and (22.2) follows. □

**Remark 50.** For problems with only one singular endpoint, the 'decomposition method of Glazman' [**55**] shows that the essential spectrum depends only on the coefficients near the singular endpoint. In Theorem 51, both endpoints are singular. The 'two-interval' proof of part (i) of Theorem 51 essentially consists in showing that the two one interval results for the intervals $(a, c)$ and $(c, b)$ can be combined to prove (22.2). Although this is conceptually simple, the technical details involve the 'two-interval' theory of Everitt and Zettl [**46, 47**] as described above and the 'Naimark patching lemma' which 'connects' these two intervals through the interior to obtain the result (22.2) for the whole interval $(a, b)$.

**Theorem 52.** *Assume that $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geq 1$, is Lagrange symmetric, and $w$ is a weight function on $J$. Suppose $a$ and $b$ are singular. If $r(\lambda) < d$, then $\lambda \in \sigma_e(a, b)$.*

*Proof.* Suppose $\lambda \notin \sigma_e(a, b)$. Then $r_a(\lambda) = d_a$ and $r_b(\lambda) = d_b$, since otherwise $r_a(\lambda) < d_a$ or $r_b(\lambda) < d_b$ would imply that $\lambda \in \sigma_e$ by Theorem 51 (ii). But $r_a(\lambda) = d_a$ and $r_b(\lambda) = d_b$ implies, by Theorem 51 (vi), that $d = d_a + d_b - n = r_a(\lambda) + r_b(\lambda) - n \leq r(\lambda)$, which contradicts the hypothesis $r(\lambda) < d$. □

**Remark 51.** We comment on Theorem 22.4. Weidmann [**116,** Theorem 11.1] proves this result under the additional assumption that there exists a self-adjoint extension for which $\lambda$ is not an eigenvalue, and he comments [**116,** pages 162, 163] that (1) this theorem is the basis for all other results in Chapter 11 and (2) that he does not know if the additional assumption is really necessary. This additional assumption in [**116**] has also been eliminated independently by Qi and Chen [**89**] with a different proof based on functional analysis.

The contrasting behavior of $r(\lambda)$ for the two singular endpoint case from the one singular endpoint case has some interesting consequences. In the case of only one singular endpoint we have $r(\lambda) \leq d$, whereas in the two singular endpoint case $r(\lambda)$ may assume values less than $d$, equal to $d$ or greater than $d$. The case $r(\lambda) > d$ leads to a surprising and counter-intuitive result that this value of $\lambda$ is an eigenvalue of *every* self-adjoint extension, i.e., for any given self-adjoint boundary condition, there are eigenfunctions of $\lambda$ which satisfy this boundary condition (see Example 13).

**Theorem 53.** *Assume that $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geq 1$, is Lagrange symmetric and $w$ is a weight function on $J$. Suppose $a$ and $b$ are singular. If, for some $\lambda \in \mathbb{R}$, $d < r(\lambda) < \min\{r_a(\lambda), r_b(\lambda)\}$, then $\lambda$ is an eigenvalue of every self-adjoint extension.*

*Proof.* See [**60**]. □

**Corollary 15.** *Assume the hypotheses of Theorem* 53 *hold for all $\lambda \in I$ and $\lambda$ is an accumulation point of $I$. Then $\lambda$ is an accumulation point of the eigenvalues of each self-adjoint extension, and therefore $\lambda$ is in $\sigma_e$.*

*Proof.* This follows from Theorem 53 and the definition of essential spectrum. □

**Remark 52.** We comment on the hypotheses of Theorem 53. If one endpoint is regular, then $r(\lambda) \leq d$ by Theorem 51. If $r_a(\lambda) = n$, then $r(\lambda) = r_b(\lambda) \leq d$ by Theorem 51 (v), and similarly for $r_b(\lambda) = n$.

**Remark 53.** Now we comment on the conclusion of Theorem 53. The same real number $\lambda$ is an eigenvalue of *every* self-adjoint extension. In other words, given an arbitrary self-adjoint boundary condition, there exists an eigenfunction of this $\lambda$ which satisfies all $d$ conditions of any given self-adjoint boundary condition; thus, $r(\lambda) > d$ provides a sufficient supply of eigenfunctions for all the conditions of *every* self-adjoint boundary condition (see Example 13). For $n = 8$, $d_a = 6$, $d_b = 7$, we have by Theorem 51 (vi): $d = 5 \le r(\lambda) \le 6$, thus allowing $r(\lambda) = 6 > d = 5$. This case actually occurs since all intermediate values of the deficiency indices $d_a$ and $d_b$ are realized.

On the relationship between the essential spectrum and the numbers of real parameter square-integrable solutions of differential equation, Weidmann conjectured if there exist 'sufficiently many' $L^2$-solutions of $(M - \lambda)u = 0$, for $\lambda \in I = (\mu_1, \mu_2)$, then $I$ contains no points of the essential spectrum [**116**]. But, as mentioned in Remark 48, Del Rio [**23**] and Remling [**92**] have clarified the complicated relationship between the essential spectrum and the real numbers $\lambda$ for which $r(\lambda) = d$. In fact, knowing only the number of real parameter solutions is not sufficient to guarantee the discreteness of spectrum of differential operators.

For the two singular endpoints case, using the "two-interval" theory developed by Everitt and Zettl in [**46, 47**] and Theorem 50, we have the following theorem. Recall

**Assumption (A).** There exists an open set $G$ in the complex plane containing $I = (\alpha, \beta)$ with $\alpha, \beta$ on the boundary of $G$, and there exist solutions $u_1(t, \lambda), \ldots, u_{d_a}(t, \lambda)$ on $(a, c)$ and $v_1(t, \lambda), \ldots, v_{d_b}(t, \lambda)$ on $(c, b)$ which lie in $H$, are real valued for $\lambda \in I$, and are analytic on $G$ for each fixed $t \in I$.

**Theorem 54.** *Assume that $A \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \ge 1$, is Lagrange symmetric and $w$ is a weight function on $J$. Each endpoint may be regular or singular. Let $d_a$ denote the deficiency index of $S_{\min}(a, c)$ in $L^2(J_a, w)$, and let $d_b$ denote the deficiency index of $S_{\min}(c, b)$ in $L^2(J_b, w)$. If $r_a(\lambda) = d_a(\lambda)$ on $(a, c)$, $r_b(\lambda) = d_b(\lambda)$ on $(c, b)$ for all $\lambda$ in some open interval $I = (\alpha, \beta)$, and assumption $(\widetilde{A})$ holds on $I$, then there is no essential spectrum in $I$ for any self-adjoint extension*

*$S(a,c)$, $S(c,b)$, $S(a,b)$. In particular, the eigenvalues of none of these extensions $S(a,c)$, $S(c,b)$, $S(a,b)$ can have an accumulation point in $I$.*

*Proof.* By Theorem 50, we have $\sigma_e(a,c) \cap I = \emptyset$ and $\sigma_e(c,b) \cap I = \emptyset$. Therefore by Theorem 51 (i), we conclude that $\sigma_e(a,b) \cap I = \emptyset$. Since such accumulation point of eigenvalues would be in the essential spectrum, we may conclude that the eigenvalues of none of these extensions $S(a,c)$, $S(c,b)$, $S(a,b)$ can have an accumulation point in $I$. □

**23. Examples.** In this section, we give some examples. Although these examples are simple and their spectrum is well known, we believe that, nevertheless, they illustrate how the number of real-parameter square-integrable solutions $r(\lambda)$ influence the spectrum, and this influence can be used to obtain spectral information.

The first two examples consider the Fourier and Hermite differential equations.

**Example 12.** Let

$$My = -y'' = \lambda y \quad \text{on} \quad J = (-\infty, \infty). \tag{23.1}$$

Observe that $1, t$ are two linearly independent solutions for $\lambda = 0$ and both are not in $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Therefore, the deficiency index of $(M, w)$ on $(-\infty, 0]$, and on $[0, \infty)$ are both 1. Hence, the deficiency index of $(M, w)$ on $(-\infty, \infty)$ is $d = 0$. Thus, $S_{\min}$ is a self-adjoint operator realization of (23.1) in the Hilbert space $H = L^2(-\infty, \infty)$.

Firstly, we consider

$$My = -y'' = \lambda y \quad \text{on } J = (0, \infty), \tag{23.2}$$

with self-adjoint boundary conditions

$$\cos(\alpha)y(0) + \sin(\alpha)y'(0) = 0, \quad 0 \le \alpha < \pi.$$

(i) For $\lambda > 0$, $e^{i\sqrt{|\lambda|}t}$ and $e^{-i\sqrt{|\lambda|}t}$ are two linearly independent solutions of (23.2) and both are not in $L^2[0, \infty)$. So, for $\lambda \ge 0$, $r(\lambda) = 0 < 1$. By Theorem 52, we conclude that, for any self-adjoint extension $S(\alpha)$ of equation (23.2), $[0, \infty) \subset \sigma_e(S)$.

(ii) For $\lambda < 0$, $e^{\sqrt{|\lambda|}t}$ and $e^{-\sqrt{|\lambda|}t}$ are two linearly independent solutions of (23.2), and

$$e^{-\sqrt{|\lambda|}t} \in L^2[0,\infty), \quad e^{\sqrt{|\lambda|}t} \notin L^2[0,\infty),$$

so $r(\lambda) = 1$.

Let $\lambda \in G$, where $G = \{c + hi | c < 0, -\varepsilon < h < \varepsilon\}$. If $\operatorname{Im} \lambda = h \geq 0$, let $\lambda = re^{i\theta}$, $\pi/2 < \theta \leq \pi$, $e^{i\sqrt{\lambda}t}$ and $e^{-i\sqrt{\lambda}t}$ are two linearly independent solutions of (23.2), and

$$e^{i\sqrt{\lambda}t} \in L^2(0,\infty), \quad e^{-i\sqrt{\lambda}t} \notin L^2(0,\infty),$$

where $e^{i\sqrt{\lambda}t} = e^{i\sqrt{r}(\cos\theta/2 + i\sin\theta/2)t}$. The initial value of the $L^2$ solution $e^{i\sqrt{\lambda}t}$ is $(1, i\sqrt{\lambda})$. Since $\lambda \neq 0$, $(1, i\sqrt{\lambda})$ is analytically dependent on $\lambda \in G$, and if $h \to 0$, $\theta \to \pi$, $e^{i\sqrt{\lambda}t} \to e^{-\sqrt{r}t} = e^{-\sqrt{|\lambda|}t}$ is a real parameter solution lying in $L^2[0,\infty)$. Similarly, we have that if $\operatorname{Im}\lambda = h < 0$, the initial value $(1, -i\sqrt{\lambda})$ of the $L^2[0,\infty)$ solution $e^{-i\sqrt{\lambda}t}$ depends analytically on $\lambda \in G$.

By Theorem 54, we have $\sigma_e(S(\alpha)) \cap (-\infty, 0) = \emptyset$ for any self-adjoint extension $S(\alpha)$ of equation (23.2), i.e., the spectrum of $S(\alpha)$ is discrete in $(-\infty, 0)$, and thus from this and (i), we conclude that $\sigma_e(0,\infty) = [0, +\infty)$.

Similarly, we consider

$$My = -y'' = \lambda y \quad \text{on } J = (-\infty, 0), \tag{23.3}$$

with self-adjoint boundary conditions

$$\cos(\alpha)y(0) + \sin(\alpha)y'(0) = 0, \quad 0 \leq \alpha < \pi$$

and conclude that $\sigma_e(-\infty, 0) = [0, +\infty)$.

By Theorem 51, we have that

$$\sigma_e(-\infty,\infty) = \sigma_e(-\infty, 0) \cup \sigma_e(0,\infty) = [0, +\infty).$$

**Example 13.** The Hermite differential expression $M$. Let

$$My = -y'' + t^2 y = \lambda y \quad \text{on } J = (-\infty, \infty). \tag{23.4}$$

$M$ is LP singular at $-\infty$ and $+\infty$, and the deficiency index of $M$ is 0. Therefore, $S_{\min}$ is self-adjoint with no proper self-adjoint extension.

It is well known that

$$\sigma(S_{\min}) = \sigma_p(S_{\min}) = \{\lambda_n = 2n+1,\ n = 0, 1, \dots\}. \tag{23.5}$$

Thus, $r(\lambda_n) = 1 > 0 = d$.

Next, we discuss the periodic coefficient case to illustrate the close connection between $r(\lambda)$ and the spectrum of self-adjoint realizations of (21.1).

**Example 14** (Floquet theory)**.** Assume that each of $p, q, w$ is $s$-periodic with fundamental interval $[a, a+s]$. Then, see [**131,** pages 210–211], equation (21.1) on $(a,\infty)$ is LP at $\infty$ and the endpoint $a$ is regular. So each of the self-adjoint realizations of (21.1) on $(a,\infty)$ in the Hilbert space $H = L^2((a,\infty), w)$ is determined by a boundary condition at $a$ only. These have the form

$$\cos(\alpha)y(a) + \sin(\alpha)(py')(a) = 0, \quad \alpha \in [0,\pi). \tag{23.6}$$

If $S(\alpha)$ is any one of the self-adjoint realizations determined by (23.6) for any $\alpha \in [0,\pi)$, then the essential spectrum $\sigma_e(S(\alpha))$ of $S(\alpha)$ is independent of $\alpha$ and is given by

$$\sigma_e(S(\alpha)) = \cup_{j=0}^{\infty} I_j, \tag{23.7}$$

where the compact intervals $I_j$ are given by

$$\begin{aligned} I_0 &= [\lambda_0^P, \lambda_0^S], \qquad I_1 = [\lambda_1^S, \lambda_1^P], \\ I_2 &= [\lambda_2^P, \lambda_2^S], \qquad I_3 = [\lambda_3^S, \lambda_3^P], \qquad I_4 = [\lambda_4^P, \lambda_4^S], \dots, \end{aligned} \tag{23.8}$$

with $\lambda_j^P$ and $\lambda_j^S$ denoting the eigenvalues of the regular problems consisting of equation (21.1) on the interval $[a, a+s]$ with periodic (P)

$$y(a) = y(a+s) \quad \text{and} \quad (py')(a) = (py')(a+s)$$

and semi-periodic (S)

$$y(a) = -y(a+s) \quad \text{and} \quad (py')(a) = -(py')(a+s)$$

boundary conditions, respectively.

The compact intervals $I_j$ are called the spectral bands, and the complementary open intervals

$$(\lambda_0^S, \lambda_1^S), (\lambda_1^P, \lambda_2^P), (\lambda_2^S, \lambda_3^S), (\lambda_3^P, \lambda_4^P), \ldots$$

are called the spectral gaps. If $\lambda_j^S = \lambda_{j+1}^S$ or $\lambda_j^P = \lambda_{j+1}^P$ for some $j$, then the corresponding 'gap' is missing. Thus, there may be no gap, a finite number of gaps, an infinite number but not all of them, or all gaps may be 'present.' From Theorem 45, we get the following corollary.

**Corollary 16.** *If $\lambda$ is in any gap, including the 'gap' $(-\infty, \lambda_0^P)$, then $r(\lambda) = 1$ and $\lambda$ is an eigenvalue with multiplicity one of some self-adjoint realization $S(\alpha)$. If $\lambda \in \sigma_e$, then either $r(\lambda) = 0$ or $r(\lambda) = 1$. If $r(\lambda) = 1$, then $\lambda$ is an embedded eigenvalue of some self-adjoint realization $S(\alpha)$. If $r(\lambda) = 0$, then $\lambda$ is not an eigenvalue, because otherwise its eigenfunction $u$ would be in $H$ making $r(\lambda) = 1$.*

**Remark 54.** Equation (21.1) on $(a, \infty)$ has deficiency index $d = 1$. By Theorem 44, $r(\lambda) \leq 1$ but $r(\lambda) = 0$ implies, by Theorem 45 (i), that $\lambda \in \sigma_e$. Hence, $r(\lambda) = 1$ for any $\lambda$ in any gap including $(-\infty, \lambda_0^P)$ and, by Theorem 45 (ii), $\lambda$ is an eigenvalue of some self-adjoint realization $S(\alpha)$ of (21.1) on $(a, \infty)$. Also, if $\lambda \in \sigma_e$ and $r(\lambda) = 1$, then $\lambda$ is an eigenvalue of some self-adjoint realization $S(\alpha)$ by Theorem 45 (ii). Not much seems to be known about the relationship between the eigenvalues and $\alpha$.

**Remark 55.** The endpoints $\lambda_j^S$ and $\lambda_j^P$ of the spectral bands and gaps can be computed with the Bailey-Everitt-Zettl code SLEIGN2. When the coefficients are not periodic, this code can also be used to approximate the first few spectral bands and gaps. See **[131]** for more information about the code, including how to download it from the Web, together with its user friendly interface and other associated helpful files.

**24. W.N. Everitt.** William Norrie Everitt (1924–2011), known to everyone as Norrie, was one of the most influential mathematicians of his time in the field of ordinary differential equations and operators, not only through his own work consisting of over 200 papers with more than 70 coauthors from more than a dozen countries on four continents, but through his influence on others with his lectures at conferences and universities, his legendary Dundee conferences, his conversion of the journal 'Proceedings of the Royal Society of Edinburgh, A', from a general Science and mathematics journal to a mathematics journal, his uncanny ability to 'get to the bottom of things,' his tireless pursuit of mathematical truths, his determination that previous authors get appropriate credit for their work, his work ethic, etc.

Norrie was an enthusiastic supporter of the 'Hardy-Littlewood philosophy' regarding joint publications: If one starts a joint project with a colleague and succeed with it, then one publishes jointly regardless of who did what. As a consequence, he was very selective in choosing his coauthors.

His writing style was unique and his papers are influential because they extended well beyond their content. He carefully presented the context for the new results, then discussed the previous work on which the present work was built (when we started working jointly on a paper we always started with our list of references), then presented the new results often followed by comments. Each new theorem is clearly stated in detail in such a way that the reader can understand it without having to search through all the previous pages for definitions and notations–all these are clearly marked.

His lectures were works of art: he started writing on the top left corner of the first blackboard and finished with an emphatic period at the bottom right corner of the last blackboard exactly 50 minutes later. Rarely, if ever, did he have to erase anything. The first third of his lecture gave a slow and gentle overview of the general area of his topic, the middle third introduced some of the needed technical background, the last third gave the new results ending with a 'punch line.' One of these lectures was given at the University of Inner Mongolia in Hohhot at the invitation of Zhijiang Cao who founded, 36 years ago, the differential equations group now led by Sun, second author of this article.

The Dundee conferences and their published proceedings were hugely influential and spawned innumerable papers, many of them joint work by people from different countries who became acquainted at these conferences.

The next comments are personal recollections of the first author of this article.

Norrie and I had a very active collaboration for more than 25 years. During this time we and our families got to know each other well. In 1972–1973, I spent a sabbatical year in Dundee with my family; my kids went to school there. My daughter was called ‘Erika America’ by her friends. My wife and I also spent a semester in Birmingham when Norrie and I worked with Paul Bailey on the SLEIGN2 [**5**] project. This code is still the only general purpose code available for the numerical computation of eigenvalues and eigenfunctions of regular and singular Sturm-Liouville problems. The code was written by Paul Bailey; Norrie and I discovered some new algorithms which were implemented in the code for the first time.

When Norrie stayed with us or I with them, he was always still working when I went to bed and was up and working in the morning when I got up. We took two RV trips together. Norrie, Kit, Sandra and I explored eastern Canada, and our kids Erika and Karl joined the four of us for a trip west to see Colorado and Utah.

Norrie regaled me with many stories of how he persuaded the Board of the general Science journal to specialize to mathematics; this became the journal ‘Proceedings of the Royal Society of Edinburgh A’. He was as relentless in his administrative work as he was in his research. He served a couple of terms chairing the Dundee Mathematics Department and was head of the Mathematics Department of the University of Birmingham, U.K., for many years. The writers of this article first met while visiting Norrie in Birmingham in 1989.

Regarding Norrie’s relentless efforts to ‘get to the bottom of things,’ his tireless pursuit of mathematical truths, his work ethic, etc., I mention the following. When Norrie and I first met we had both written papers on the Sturm-Liouville equation and on higher order differential equations where my hypotheses were more general than his. On the blackboard in his office he grilled me mercilessly to justify my weaker assumptions. After several hours of vigorous discussions, going back to the definition of the Lebesgue integral, he convinced himself that my weaker assumptions were sufficient and used them from then on.

He was particularly concerned that Shin was credited for his work on quasi-differential expressions. He and I tried hard to get some information about Shin’s life with no success. Shin seems to have disappeared after his 1943 paper was published so we assumed that he died during WWII.

Despite Norrie's unquenchable thirst for mathematics, he had other interests, in particular, opera, railroads, trees, history. In addition to regaling me with funny anecdotes about mathematicians and their peculiarities, he seemed to know about all the peccadillos of the English kings and queens. When walking in a forest he would rattle off both the common and scientific names of the trees we encountered, when riding with him in a train he would tell me all about the railroad and its history, he knew what kinds of rails we were on from the sound, etc. All this was accomplished with a wonderful sense of humor. William N. (Norrie) Everitt was a truly remarkable man.

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## REFERENCES

**1**. N.I. Akhiezer and I.M. Glazman, *Theory of linear operators in Hilbert space*, II, Ungar, New York, 1963.

**2**. J. Ao, J. Sun and A. Zettl, *Matrix representations of fourth order boundary value problems with finite spectrum*, Linear Alg. Appl. **436** (2012), 2359–2365.

**3**. ______, *Equivalence of fourth order boundary value problems and matrix eigenvalue problems*, Res. Math. **63** (2013), 581–595.

**4**. F.V. Atkinson, *Discrete and continuous boundary value problems*, Academic Press, New York, 1964.

**5**. P.B. Bailey, W.N. Everitt and A. Zettl, *The SLEIGN*2 *Sturm-Liouville code*, ACM TOMS, Trans. Math. Software **21** (2001), 143–192.

**6**. J.H. Barrett, *Oscillation theory of ordinary linear differential equations*, Adv. Math. (1969), 415–509.

**7**. H. Behncke, *Spectral theory of higher order differential operators*, Proc. Lond. Math. Soc. **3**, (2006), 139–160.

**8**. ______, *Spectral analysis of fourth order operators* II, Math. Nach. (2007).

**9**. H. Behncke and D.B. Hinton, *Eigenfunctions, deficiency indices and spectra of odd order differential operators*, Proc. Lond. Math. Soc. (2008).

**10**. ______, *Spectral theory of higher order operators* II, pre-print.

**11**. Z.J. Cao, *On self-adjoint domains of second order differential operators in limit-circle case*, Acta Math. Sinica **1** (1985), 175–180.

**12**. ______, *On self-adjoint extensions in the limit-circle case of differential operators of order $n$*, Acta Math. Sinica **28** (1985), 205–217 (in Chinese).

**13**. ______, *Ordinary differential operators*, Shanghai Science Tech. Press, 1987 (in Chinese).

**14**. ______, *Analytic descriptions of self-adjoint ordinary differential operators*, J. Inner Mongolia Univ. **18** (1987), 393–401 (in Chinese).

**15**. Z.J. Cao and J.L. Liu, *On the deficiency index theory for singular symmetric differential operators*, Adv. Math. **12** (1983), 161–178 (in Chinese).

**16**. Z.J. Cao and J. Sun, *Self-adjoint operators generated by symmetric quasi-differential expressions*, Acta Sci. Natur., Univ. NeiMongol (1986), 7–15 (in Chinese).

**17**. ______, *Collection of differential operators*, Inner Mongolia Univ. Press, Hohhot, 1992.

**18**. ______, *On completion and prolongation of classical Sturm-Liouville theory*, Adv. Math. (China) **22** (1993), 97–117 (in Chinese).

**19**. Z.J. Cao, J. Sun and D.E. Edmunds, *On self-adjointness of the product of two* 2*-order differential operators*, Acta Math. Sinica, English series, **15** (1999), 375–386.

**20**. J. Chaudhuri and W.N. Everitt, *On the square of formally self-adjoint differential expressions*, J. Lond. Math. Soc. **1** (1969), 661–673.

**21**. E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

**22**. E.A. Coddington and A. Zettl, *Hermitian and anti-hermitian properties of Green's matrices*, Pac. J. Math. (1966), 451–454.

**23**. R. Del Rio, *On a problem of P. Hartman and A. Wintner*, Aport. Matem. **16** (1995), 119–123.

**24**. N. Dunford and J.T. Schwartz, *Linear operators*, II, Wiley, New York, 1963.

**25**. M.S.P. Eastham, *The limit-*3 *case of self-adjoint differential expressions of the fourth order with oscillating coefficients*, J. Lond. Math. Soc. **8** (1974), 427–437.

**26**. M.S.P. Eastham and A. Zettl, *Second order differential expressions whose squares are limit-*3, Proc. Roy. Soc. Edinb. **76** (1977), 233–238.

**27**. W.D. Evans, M.K. Kwong and A. Zettl, *Lower bounds for the spectrum of ordinary differential operators*, J. Diff. Equat. **48** (1983), 123–155.

**28**. W.D. Evans and E.I. Sobhy, *Boundary conditions for general ordinary differential operators and their adjoints*, Proc. Roy. Soc. Edinb. **114** (1990), 99–117.

**29**. W.D. Evans and A. Zettl, *On the deficiency indices of powers of real 2nth order symmetric differential expressions*, J. Lond. Math. Soc. **2** (1976), 543–556.

**30**. ______, *Levinson's limit-point criterion and powers*, J. Funct.l Anal. Appl. **62** (1978), 629–639.

**31**. W.N. Everitt, *A note on the self-adjoint domains of* $2n$*th-order differential equations*, Quart. J. Math. **14** (1963), 41–45.

**32**. ______, *Integrable-square solutions of ordinary differential equations* (III), Quart. J. Math. **14** (1963), 170–180.

**33**. ______, *Singular differential equations* I: **The even case**, Math. Ann. **156** (1964), 9–24.

**34**. ______, *Singular differential equations* II: *Some self-adjoint even case*, Quart. J. Math. **18** (1967), 13–32.

**35**. W.N. Everitt and M. Giertz, *On some properties of the powers of a formally self-adjoint differential expression*, Proc. Lond. Math. Soc. **74** (1972), 149–170.

**36**. ______, *On the deficiency indices of powers of formally symmetric differential expression*, Lect. Notes Math. **448**, Springer Verlag, New York, 1975.

**37**. ______, *On the deficiency indices of powers of formally symmetric differential expressions*, Lect. Notes Math. **448**, Springer Verlag, New York, 1975.

**38**. W.N. Everitt and V.K. Kumar, *On the Titchmarsh-Weyl theory of ordinary symmetric differential expressions* I: *The general theory*, Nieuw Arch. Wisk. **34** (1976), 1–48.

**39**. ______, *On the Titchmarsh-Weyl theory of ordinary symmetric differential expressions* II: *The odd order case*, Nieuw Arch. Wisk. **34** (1976), 109–145.

**40**. W.N. Everitt and L. Markus, *Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators*, Math. Surv. Mono. **61**, American Mathematical Society, 1999.

**41**. ______, *Complex symplectic geometry with applications to ordinary differential operators*, Trans. Amer. Math. Soc. **351** (1999), 4905–4945.

**42**. W.N. Everitt and F. Neuman, *A concept of adjointness and symmetry of differential expressions based on the generalized Lagrange identity and Green's formula*, Lect. Notes Math. **1032**, Springer-Verlag, Berlin, 1983.

**43**. W.N. Everitt and R. Race, *On necessary and sufficient conditions for caratheodory type solutions of ordinary differential equations*, Quaest. Math. (1978), 507–512.

**44**. W.N. Everitt and A. Zettl, *The number of integrable-square solutions of products of differential expressions*, Proc. Roy. Soc. Edinb. **76** (1977), 215–226.

**45**. ______, *Generalized symmetric ordinary differential expressions* I: *The general theory*, Nieuw Arch. Wisk. **27** (1979), 363–397.

**46**. W.N. Everitt and A. Zettl, *Sturm-Liouville differential operators in direct sum spaces*, Rocky Mountain J. Mathematics (1986), 497–516.

**47**. ______, *Differential operators generated by a countable number of quasi-differential expressions on the line*, Proc. Lond. Math. Soc. **64** (1992), 524–544.

**48**. H. Frentzen, *Equivalence, adjoints and symmetry of quasi-differential expressions with matrix-valued coefficients and polynomials in them*, Proc. Roy. Soc. Edinb. **92** (1982), 123–146.

**49**. K. Friedrichs, *Spektraltheorie halbbeschränkter operatoren und anwendungen auf die spektralzerlegung von differentialoperatoren* I, II, Math. Ann. **109** (1933), 467–487; corrections, ibid, **110** (1934), 777–779.

**50**. ______, *Über die ausgezeichnete randbedingung in der spectraltheorie gewöhnlicher differential operatoren zweiter ordnung*, Math. Ann. **112** (1935), 1–23.

**51**. S.Z. Fu, *On the self-adjoint extensions of symmetric ordinary differential operators in direct sum spaces*, J. Diff. Equat. **100** (1992), 269–291.

**52**. R.C. Gilbert, *Asymptotic formulas for solutions of a singular ordinary differential equation*, Proc. Roy. Soc. Edinb. **81** (1978), 57–70.

**53**. ______, *A class of symmetric ordinary differential operators whose deficeinecy numbers differ by an integer*, Proc. Roy. Soc. Edinb. **82** (1979), 117–134.

**54**. I.M. Glazman, *On the theory of singular differential operators*, Uspek. Math. Nauk. (1950), 102–135; Amer. Math. Soc. Trans. **4** (1962), 331–372.

**55**. ______, *Direct methods of qualitative spectral analysis of singular differential operators*, Israel Program for Scientific Translation, 1965.

**56**. A. Gorianov, V. Mikhailets and K. Pankrashkin, *Formally self-adjoint quasi-differential operators and boundary value problems*, `arXiv.12051810v3`.

**57**. I. Halperin, *Closures and adjoints of linear differential operators*, Annals Math. (1937), 880–919.

**58**. X. Hao, J. Sun, A. Wang and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators* II, Res. Math. **61** (2012), 255–281.

**59**. X. Hao, J. Sun and A. Zettl, *Real-parameter square-integrable solutions and the spectrum of differential operators*, J. Math. Anal. Appl. **376** (2011), 696–712.

**60**. ______, *The spectrum of differential operators and square-integrable solutions*, J. Funct. Anal. **262** (2012), 1630–1644.

**61**. ______, *Canonical forms of self-adjoint boundary conditions for differential operators of order four*, J. Math. Anal. Appl. **387** (2012), 1176–1187.

**62**. ______, *Fourth order canonical forms of singular self-adjoint singular boundary conditions*, Linear Alg. Appl. **437** (2012), 899–916.

**63**. P. Hartman and A. Winter, *A separation theorem for continuous spectra*, Amer. J. Math. **71** (1949), 650–662.

**64**. D.B. Hinton, *Disconjugacy properties of a system of differential equations*, J. Diff. Equat. (1966), 420–437.

**65**. ______, *Limit-point criteria for differential equations*, Canad. J. Math. **24** (1972), 293–305.

**66**. P.D. Hislop and I.M. Sigal, *Introduction to spectral theory, with applications to Schrōdinger operators*, Appl. Math. Sci. **113**, Springer-Verlag, New York, 1966.

**67**. R.M. Kauffman, *Polynomials and the limit-point condition*, Trans. Amer. Math. Soc. **201**, (1975), 347–366.

**68**. ______, *A rule relating the deficiency index of $L^j$ to that of $L^k$*, Proc. Roy. Soc. Edinb. **74** (1976), 115–118.

**69**. R.M. Kauffman, T.T. Read and A. Zettl, *The defiency index problem for powers of ordinary differential expressions*, Lect. Notes Math. **621**, Springer Verlag, New York, 1977.

**70**. K. Kodaira, *On ordinary differential equations of any even order and the corresponding eigenfunction expansions*, Amer. J. Math. (1950), 502–544.

**71**. V.I. Kogan and F.S. Rofe-Beketov, *On the question of deficiency indices of differential operators with complex coefficients*, Proc. Roy. Soc. Edinb. **74** (1973), 281–298.

**72**. Q. Kong, H. Volkmer and A. Zettl, *Matrix representations of Sturm-Liouville problems*, Res. Math. **54** (2009), 103–116.

**73**. M.K. Kwong and A. Zettl, *Discreteness conditions for the spectrum of ordinary differential operators*, J. Diff. Equat. **40** (1981), 53–70.

**74**. W.M. Li, *The high order differential operators in direct sum spaces*, J. Diff. Equat. **84** (1990), 273–289.

**75**. ______, *The domains of self-adjoint enxtensions of symmetric ordinary differential operators with two singular points*, J. Inner Mongolia University (Natural Science) **20** (1989), in Chinese.

**76**. L.L. Littlejohn and A. Zettl, *The Legendre equation and its self-adjoint operators*, Electr. J. Differ. Equat. **2011** (69), 1–33.

**77**. N. Macrae and John von Neumann, American Mathematical Society, 1992.

**78**. M. Marletta and A. Zettl, *The Friedrichs extension of singular differential operators*, J. Differ. Equat. **160** (2000), 404–421.

**79**. J.B. McLeod, *The number of integrable-square solutions of ordinary differential equations*, Quart. J. Math (1966), 285–290.

**80**. M. Möller, *On the unboundedness below of the Sturm-Liouville operator*, Proc. Roy. Soc. Edinb. **129** (1999), 1011–1015.

**81**. M. Möller and A. Zettl, *Weighted norm-inequalities for quasi-derivatives*, Res. Math. **24** (1993), 153–160.

**82**. ______, *Symmetric differential operators and their Friedrichs extension*, J. Diff. Equat. **115** (1995), 50–69.

**83**. ______, *Semi-boundedness of ordinary differential operators*, J. Diff. Equat. **115** (1995), 24–49.

**84**. M.A. Naimark, *Linear differential operators*, Ungar, New York, 1968.

**85**. J.W. Neuberger, *Concerning boundary value problems*, Pacific J. Math. **10** (1960), 1385–1392.

**86**. H.-D. Niessen and A. Zettl, *The Friedrichs extension of regular ordinary differential operators*, Proc. Roy. Soc. Edinb. **114** (1990), 229–236.

**87**. ______, *Singular Sturm-Liouville problems*: *The Friedrichs extension and comparison of eigenvalues*, Proc. Lond. Math. Soc. **64** (1992), 545–578.

**88**. S.A. Orlov, *On the deficiency indices of differential operators*, Dokl. Akad. Nauk **92** (1953), 483–486.

**89**. J. Qi and S. Chen, *On an open problem of J. Weidmann*: *Essential spectra and square-integrable solutions*, Proc. Roy. Soc. Edinb. **141** (2011), 417–430.

**90**. T.T. Read, *Sequences of deficiency indices*, Proc. Roy. Soc. Edinb. (1976), 157–164.

**91**. W.T. Reid, *Oscillation criteria for self-adjoint differential systems*, Trans. Amer. Math. Soc. (1961), 91–106.

**92**. C. Remling, *Essential spectrum and $L_2$-solutions of one dimensional Schrödinger operators*, Proc. Amer. Math. Soc. (1996), 2097–2100.

**93**. F.S. Rofe-Beketov and A. Kholkin, *Spectral analysis of differential operators, Interplay between spectral and oscillatory properties*, World Sci. Mono. Math. **7** (2005).

**94**. Z.J. Shang, *On $J$-selfadjoint extensions of $J$-symmetric ordinary differential operators*, J. Diff. Equat. **73** (1988), 153–177.

**95**. Z.J. Shang and R.Y. Zhu, *The domains of self-adjoint extensions of ordinary symmetric differential operators over* $(-\infty,\infty)$, J. Inner Mongolia Univ. **17** (1986), 17–28.

**96**. D. Shin, *On the solutions in $L^2(0,\infty)$ of the self-adjoint differential equation* $u^{(n)}=lu, I(l)=0$, Dokl. Akad. Nauk **18** (1938), 519–522.

**97**. ______, *On quasi-differential operators in Hilbert space*, Dokl. Akad. Nauk **18** (1938), 523–526.

**98**. ______, *On the solutions of linear quasi-differential equations of the nth order*, Mat. Sbor. **7** (1940), 479–532.

**99**. ______, *On quasi-differential operators in Hilbert space*, Mat. Sbor. **13** (1943), 39–70.

**100**. M.H. Stone, *Linear transformations in Hilbert space and their applications in analysis*, Amer. Math. Soc. Colloq. **15** (1932).

**101**. J. Sun, *On the self-adjoint extensions of symmetric ordinary differential operators with middle deficiency indices*, Acta Math. Sinica **2** (1986), 152–167.

**102**. J. Sun, A. Wang and A. Zettl, *Continuous spectrum and square-integrable solutions of differential operators with intermediate deficiency index*, J. Funct. Anal. **255** (2008), 3229–3248.

**103**. J. Sun and Z. Wang, *The spectrum analysis of linear oparator*, Science Press, Beijing, 2005.

**104**. E.C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations* [$J$], Part I, Second edition, Oxford, 1962.

**105**. P. Walker, *A vector-matrix formulation for formally symmetric ordinary differential equations with applications to solutions of integrable-square*, J. Lond. Math. Soc. (1974), 151–159.

**106**. A. Wang, J. Ridenhour and A. Zettl, *Construction of regular and singular Green's functions*, Proc. Roy. Soc. Edinb. **142** (2012), 171–198.

**107**. A. Wang, J. Sun and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators*, J. Diff. Equat. **246** (2009), 1600–1622.

**108**. ______, *The classification of self-adjoint boundary conditions*: *Separated, coupled, and mixed*, J. Funct. Anal. (2008), 1554–1573.

**109**. ______, *The classification of self-adjoint boundary conditions of differential operators with two singular endpoints*, J. Math. Anal. Appl. **378** (2011), 493–506.

**110**. A. Wang and A. Zettl, *The Green's function of two-interval Sturm-Liouville problems*, Electr. J. Diff. Equat. **76** (2013), 1–13.

**111**. ______, *Self-adjoint Sturm-Liouville problems with discontinuous boundary conditions*, Math. Meth. Appl. Anal. **22** (2015), 37–66.

**112**. W.Y. Wang and J. Sun, *Complex $J$-symplectic geometry characterization for $J$-symmetric extensions of $J$-symmetric differential operators*, Adv. Math. **32** (2003), 481-484.

**113**. ______, *The self-adjoint extensions of singular differential operators with a real regularity point*, J. Spectral Math. Appl. (2006).

**114**. ______, *$J$-self-adjoint extensions of $J$-symmetric operators with interior singular points*, J. Nanjing Univ. Sci. Tech. **31** (2007), 673–678.

**115**. J. Weidmann, *Linear operators in Hilbert spaces*, Grad. Texts Math., Springer-Verlag, Berlin, 1980.

**116**. ______, *Spectral theory of ordinary differential operators*, Lect. Notes Math. **1258**, Springer-Verlag, Berlin, 1987.

**117**. H. Weyl, *Über Gewönliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen*, Math. Ann. (1910), 220–269.

**118**. W. Windau, *On linear differential equations of the fourth order with singularities, and the related representations of arbitrary functions*, Math. Annal. **83** (1921), 256–279.

**119**. S. Yao, J. Sun and A. Zettl, *Self-adjoint domains, symplectic geometry and limit-circle solutions*, J. Math. Anal. Appl. **397** (2013), 644–657.

**120**. ______, *Self-adjoint domains, symplectic geometry, and limit-circle solutions*, J. Math. Anal. Appl. **386** (2013), 644–657.

**121**. ______, *Symplectic geometry and dissipative differential operators*, J. Math. Anal. Appl. **419** (2014), 218–230.

**122**. A. Zettl, *The limit-point and limit-circle cases for polynomials in a differential expression*, Proc. Roy. Soc. Edinb. **72** (1974), 219–224.

**123**. ______, *Adjoint linear differential operators*, Proc. Amer. Math. Soc. **16** (1965), 1239–1241.

**124**. A. Zettl, *Formally self-adjoint quasi-differential operators*, Rocky Mountain J. Mathematics **5** (1975), 453–474.

**125**. ______, *Adjoint and self-adjoint problems with interface conditions*, SIAM J. Appl. Math. **16** (1968), 851–859.

**126**. ______, *Adjointness in non-adjoint boundary value problems*, SIAM J. Appl. Math. **16** (1969), 1268–1279.

**127**. ______, *The lack of self-adjointness in three point boundary value problems*, Proc. Amer. Math. Soc. **17** (1966), 368–371.

**128**. ______, *Formally self-adjoint quasi-differential operators*, Rocky Mountain J. Math. **5** (1975), 453–474.

**129**. ______, *Powers of symmetric differential expressions without smoothness assumptions*, Quaest. Math. (1976), 83–94.

**130**. ______, *On the Friedrichs extension of singular differential operators*, Comm. Appl. Anal. **2** (1998), 31–36.

**131**. ______, *Sturm-Liouville theory*, AMS Surv. Mono. **121**, American Mathematical Society, Philadelphia, 2005.

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