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EXISTENCE OF SOLUTIONS FOR THE GENERALIZED p-LAPLACIAN EQUATION

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ABSTRACT. In this article, we show that, under suitable assumptions, the generalized p-Laplacian boundary value problem has at least one solution.

1. Introduction. We are interested in the existence of solutions of the equation

$$(\phi(u'))' + k(t)\phi(u') + f(t, u, u') = 0, \quad a < t < b \tag{1.1}$$

subject to boundary condition

$$u(a) = u(b) = 0. \tag{1.2}$$

Here $k : [a, b] \to \mathbb{R}$, $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homomorphism, i.e., it satisfies the following conditions:

(i) if $x \le y$, then $\phi(x) \le \phi(y)$ for all $x, y \in \mathbb{R}$;
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathbb{R}$.

Equation (1.1) with $\phi(s) = |s|^{p-2}s$, where $p > 1$, and $k(t) = (n-1)/t$ arises in the study of radial solutions for the p-Laplacian equation on the annular domain in n dimensions,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(|x|, u, |\nabla u|) = 0, \ a < |x| < b, x \in \mathbb{R}^n. \tag{1.3}$$

Problem (1.1) has been investigated a good deal in the last 20 years or so under the general heading of the p-Laplacian. The application most authors cite nowadays is to highly viscid fluid flow (cf., Ladyzhenskaya **[10]** and Lions **[12]**). This involves partial differential equations,

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but for symmetric flows, the ordinary differential operator (perhaps in radial form) is involved, see, e.g., Kusano and Swanson [**9**], del Pino and Manasevich [**4**], Rabinowitz [**14**] and Walter [**15**]. For the interesting results of this problem, we refer the reader to Dang and Oppenheimer [**2**], Bobisud [**1**], Guo [**6**], Herrero and Vazquez [**7**], Kaper, Knapp and Kwong [**8**] and Mawhin [**13**].

In this paper, we shall establish existence results for (1.1) with the boundary condition (1.2) under various growth conditions on f. In particular, our results give the existence of radial solutions to (1.3) on the annulus $\{x \in \mathbb{R} : a < |x| < b\}$ with Dirichlet boundary conditions with suitable growth conditions on f. The results obtained may be considered as extensions of results in Bobisud [**1**] and Mawhin [**13**]. Our approach is based on a direct application of the Leray-Schauder alternative theorem.

2. Main results. We shall denote the norms in C^r, L^p and $W^{1,p}$ by $|\cdot|_r$, $||\cdot||_p$ and $||\cdot||_{1,p}$, respectively. Here $L^p = L^p(a,b)$, $C^r = C^r[a,b]$ and $W^{1,p} = W^{1,p}(a,b)$. We assume throughout that ϕ is an odd, increasing homomorphism on $\mathbb{R}$ and that $k \geq 0$ is a continuous function on $[a,b]$ with the primitive $K(t) = \int_a^t k(s)ds$.

In order to discuss our results, we need the following lemmas:

Lemma 2.1. (Lasalle's inequality[11]). Let $G \in C([0,\infty];[0,\infty))$ be continuous and increasing, the functions $h \in L^1([a,b];[0,\infty))$, $y \in C([a,b];[0,\infty))$. Then the inequality

$$y(t) \leq \int_a^t h(s)G(y(s))\,ds \quad \text{on } [a,b]$$

implies

$$\int_0^{y(t)} \frac{ds}{G(s)} \leq \int_a^t h(s)\,ds \quad \text{on } [a,b].$$

Theorem 2.2. (**Leray-Schauder alternative** cf., [**5**], Theorem (5.4)). Let C be a convex subset of a normed linear space E, and assume $0 \in C$. Let $F : C \to C$ be a completely continuous operator, and let

$$\varepsilon(F) = \{x \in C | x = \lambda F(x) \quad \text{for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or F has a fixed point.

Lemma 2.3. For each $v \in L^1$, there exists a unique solution $u = Av$ of

$$\begin{cases} (\phi(u'))' + k(t)\phi(u') = v, \\ u(a) = u(b) = 0. \end{cases} \tag{2.1}$$

Proof. For each $v \in L^1$, let $u = Av$ be the solution of (2.1). Then we have

$$(e^{K(t)}\phi(u'))' = e^{K(t)}v. \tag{2.2}$$

Integrating (2.2) on $[a, t]$ gives

$$\phi(u'(t)) = e^{-K(t)}\phi(u'(a)) + e^{-K(t)} \int_a^t e^{K(s)}v\, ds,$$

and so

$$u(t) = \int_a^t \phi^{-1}\Big[e^{-K(s)}\phi(u'(a)) + e^{-K(s)} \int_a^s e^{K(\tau)}v\, d\tau\Big]\, ds.$$

Since $u(b) = 0$ and ϕ^{-1} is increasing, we have that $\phi(u'(a)) = C$, where C is the unique number such that

$$\int_a^b \phi^{-1}\Big[Ce^{-K(s)} + e^{-K(s)} \int_a^s e^{K(\tau)}v\, d\tau\Big]\, ds = 0. \tag{2.3}$$

Note that $|C| < \int_a^b |e^{K(s)}v|\, ds$. Conversely, if

$$u(t) = \int_a^t \phi^{-1}\Big[Ce^{-K(s)} + e^{-K(s)} \int_a^s e^{K(\tau)}v\, d\tau\Big]\, ds,$$

where C satisfies (2.3), then it is easy to see that u is a solution of (2.1).

Theorem 2.4. (Existence theorem). Let $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Caratheodory conditions, i.e., $f(\cdot, u, v)$ is measurable for every $u, v \in \mathbb{R}$; $f(t, \cdot, \cdot)$ is continuous for almost every $t \in (a, b)$, and for every $r > 0$, there exists $p_r \in L^1$ such that

$$|f(t, u, v)| \leq p_r(t)$$

for almost every $t \in (a,b)$, and all $u, v \in \mathbb{R}$ with $|u| \leq r$, $|v| \leq r$. Suppose that

$$0 \leq k(t) \leq k_1, \quad t \in [a,b]$$

and

$$|f(t,u,v)| \leq F(\phi(|u|), \phi(|v|)), \tag{2.4}$$

where $F(x,y)$ is increasing in x and y, respectively. Then, the boundary value problem (1.1)–(1.2) has at least one solution.

Proof. Since u is a solution of (1.1) if and only if $v(t) = u(a+b-t)$ is a solution of

$$(\phi(v'))' - k(t)\phi(v') + g(t,v,v') = 0,$$

where $g(t,u,v) = f(a+b-t, u, -v)$. Since $k(t) \equiv 0$ is obvious, we can assume that $k(t) > 0$ for some $t \in (a,b)$. For each $v \in C^1$, let $u = Bv$ be the solution of

$$\begin{cases} (\phi(u'))' + k(t)\phi(u') = Nv, \\ u(a) = u(b) = 0, \end{cases}$$

where $Nv(t) = -f(t, v(t), v'(t))$. Since $B = AN$, $B : C^1 \to C^1$ is completely continuous. We shall apply the Leray-Schauder alternative theorem to show that B has a fixed point. Let $u \in C^1$ and $\lambda \in (0,1)$ be such that $u = \lambda B u$. Then

$$\left(\phi\left(\frac{u'}{\lambda}\right)\right)' + k(t)\phi\left(\frac{u'}{\lambda}\right) = -f(t,u,u'). \tag{2.5}$$

Let $|u_0| = |u(t_0)| = \max_{t\in[a,b]} |u(t)|$ for some $t_0 \in [a,b]$. Without loss of generality, we consider $t \in [t_0, b]$. Multiplying (2.5) by $e^{K(t)}$ and integrating over $[t_0, t]$ gives

$$\phi\left(\frac{u'(t)}{\lambda}\right) = -e^{-K(t)} \int_{t_0}^{t} e^{K(s)} f(s, u(s), u'(s))\, ds.$$

Since

$$\phi(|x|) = \begin{cases} \phi(1)\phi(x), & x \geq 0, \\ \phi(-1)\phi(x), & x \leq 0. \end{cases}$$

Thus,

$$
\begin{aligned}
|\phi(|x|)| = &\begin{cases} |\phi(1)\phi(x)|, & x \geq 0 \\ |\phi(-1)\phi(x)|, & x \leq 0 \end{cases} \\
&\leq \max\{|\phi(-1)|, |\phi(1)|\}|\phi(x)|.
\end{aligned}
$$

On the other hand, it follows from $\phi(1) = \phi(1)\phi(1)$, $\phi(1) = \phi(-1)\phi(-1)$ and ϕ is a bijection that $\phi(1) = 1$ and $\phi(-1) = -1$. Thus, (2.4) implies

$$
\begin{aligned}
\phi(|u'(t)|) &\leq \phi\left(|\frac{u'(t)}{\lambda}|\right) \leq |\phi\left(\left|\frac{u'(t)}{\lambda}\right|\right)| \\
&\leq \max\{|\phi(-1)|, |\phi(1)|\}\left|\phi\left(\frac{u'(t)}{\lambda}\right)\right| \\
&\leq e^{-K(t)} \int_{t_0}^{t} |e^{K(s)} f(s, u(s), u'(s))|\, ds \\
&\leq \int_{t_0}^{t} e^{K(s)} |f(s, u(s), u'(s))|\, ds \\
&\leq \int_{t_0}^{t} e^{K(s)} F(\phi(|u(s)|), \phi(|u'(s)|))\, ds. \qquad (2.6)
\end{aligned}
$$

Since

$$
|u(t)| = \left| \int_{t}^{b} u' \right| \leq (b-a) \sup_{t_0 \leq s \leq b} |u'(s)| \equiv (b-a)\delta \quad \text{for } t \geq t_0,
$$

it follows from (2.6) that

$$
\phi(|u'(t)|) \leq \int_{t_0}^{t} e^{K(s)} F(\phi((b-a)\delta), \phi(|u'(s)|))\, ds, \tag{2.7}
$$

and Lemma 2.1 imply that

$$
\begin{aligned}
\int_{0}^{\phi(|u'(t)|)} \frac{ds}{F(\phi((b-a)\delta), s)} &\leq \int_{t_0}^{t} e^{K(s)} ds = \int_{t_0}^{t} e^{\int_a^s k(u)du} ds \\
&\leq \int_{t_0}^{t} e^{\int_a^s k_1 du} ds = \int_{t_0}^{t} e^{k_1(b-a)} ds \\
&\leq (b-a)e^{k_1(b-a)} \quad \text{for } t \in [t_0, b].
\end{aligned}
$$

Thus, we have

$$\int_0^{\phi(\delta)} \frac{ds}{G(s)} \le (b-a)e^{k_1(b-a)},$$

where $G(t) \equiv F(\phi((b-a)\delta), t)$ on $[0,\infty)$. That is,

$$\phi(\delta) \le H^{-1}((b-a)e^{k_1(b-a)}) \equiv M_1, \quad \text{(independent of } u \text{ and } u'), \tag{2.8}$$

where $H(u) = \int_0^u ds/G(s)$ is increasing on $[0,\infty)$. Consequently,

$$|u(t_0)| = \left| \int_{t_0}^b u' \right| \le (b-a)\delta \le (b-a)\phi^{-1}(M_1) \equiv M_2 \tag{2.9}$$
$$\text{(independent of } u \text{ and } u').$$

Combining (2.7) and (2.9), we obtain

$$\phi(|u'(t)|) \le \int_{t_0}^t e^{K(s)} F(\phi(M_2), \phi(|u'(s)|))\, ds,$$

which along with Lemma 2.1 implies

$$\begin{aligned}\int_0^{\phi(|u'(t)|)} \frac{ds}{F(\phi(M_2), s)} &\le \int_{t_0}^t e^{K(s)} ds = \int_{t_0}^t e^{\int_a^s k(u)du} ds \\ &\le \int_{t_0}^t e^{\int_a^s k_1 du} ds = \int_{t_0}^t e^{k_1(b-a)} ds \\ &\le (b-a)e^{k_1(b-a)} \quad \text{for } t \in [t_0, b].\end{aligned}$$

Thus, we have

$$\int_0^{\phi(|u'(t)|)} \frac{ds}{G^*(s)} \le (b-a)e^{k_1(b-a)},$$

where $G^*(t) \equiv F(\phi(M_2, t)$ on $[0,\infty)$, that is,

$$\phi(|u'(t)|) \le H^{*^{-1}}((b-a)e^{k_1(b-a)}) \equiv M_3, \quad \text{(independent of } u \text{ and } u'),$$

where $H^*(u) = \int_0^u ds/G^*(s)$ is increasing on $[0,\infty)$.

This implies

$$|u'| \le \phi^{-1}(M_3) \equiv M_4 \quad \text{(independent of } u \text{ and } u').$$

By the Leray-Schauder alternative theorem, B has a fixed point u, which is a solution of (1.1)–(1.2). Thus, we obtain the desired results.

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