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PSEUDO-HYPERBOLIC DISTANCE AND GLEASON PARTS OF THE ALGEBRA OF BOUNDED HYPER-ANALYTIC FUNCTIONS ON THE BIG DISK

DIMCHO K. STANKOV

ABSTRACT. Let G be the compact group of all characters of the additive group of rational numbers, and let H_G^∞ be the Banach algebra of so-called bounded hyper-analytic functions on the big-disk Δ_G. We characterize the pseudo-hyperbolic distance of the algebra H_G^∞ in terms of the pseudo-hyperbolic distance of the algebra H^∞ and establish relationships between Gleason parts in $M(H_G^\infty)$ and $M(H^\infty)$.

1. Introduction. Let Γ be a subgroup of the additive group of real numbers $\mathbb{R}$ with the discrete topology, and let $G = \widehat{\Gamma}$ be its dual group, i.e., the (compact) group of all continuous characters on Γ. By the celebrated Pontryagin theorem [**1**], each continuous character on G is of type $\chi_p(g)$, $p \in \Gamma$, where $\chi_p(g) = g(p)$, $g \in G$. The uniform closure A_G of finite linear combinations of 'non-negative' characters χ_p, $p \in \Gamma_+ = \Gamma \cap [0, \infty)$, with complex coefficients, i.e., of *generalized polynomials*, is the *big-disk algebra* on G [**2**]. A_G is a uniform algebra on G, and its elements are called *generalized-analytic functions* in the sense of Arens and Singer [**2**]. The maximal ideal space $M(A_G)$ of the big-disk algebra is the closed unit *big-disk* $\overline{\Delta}_G$ over G, i.e., the cone

$$\overline{\Delta}_G = [0,1] \times G/\{0\} \times G.$$

The points of $\overline{\Delta}_G$ are denoted by $r \cdot g$, $r \leq 1$, with the understanding that all the points of type $0 \cdot g$ are identified into a single point, $\{*\}$, the *origin* (or the *center*) of the closed big-disk $\overline{\Delta}_G$. Each character

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χ_p, $p \in \Gamma_+$, admits a continuous extension from the group G to the closed big disk $\overline{\Delta}_G$ of G, as follows (e.g., [**10**]):

$$\widetilde{\chi}_p(r \cdot g) = \begin{cases} r^p \chi_p(g) & \text{when } 0 < r < 1 \text{ and } p > 0, \\ 0 & \text{when } r = 0 \text{ and } p > 0, \\ 1 & \text{when } p = 0 \text{ for any } 0 \le r < 1. \end{cases}$$

Each function $\widetilde{\chi}_p$, $p \in \Gamma_+ \setminus \{0\}$, projects the closed big-disk $\overline{\Delta}_G$ onto the closed unit disk $\overline{\Delta}$ and the open big disk $\Delta_G = [0,1) \times G/\{0\} \times G$ onto the open unit disk Δ in the complex plane.

Note that, if Γ is the (additive) group of integers $\mathbb{Z}$, then its dual, $\widehat{\Gamma} = \widehat{\mathbb{Z}}$, is the unit circle $\mathbb{T}$ in the complex plane, the open big-disk $\Delta_G = \Delta_{\mathbb{T}}$ is the open unit disk Δ in the complex plane, and the corresponding big-disk algebra, $A_{\mathbb{T}} = A(\Delta)$, the classical disk algebra.

The object of this paper is, as introduced in [**9**] (see also [**10**]), the Banach algebra of *hyper-analytic functions* on the big-disk Δ_G over the dual group G of the (additive) group of rational numbers $\mathbb{Q}$.

Definition 1.1. [**9, 10**] Let Γ be the group of rational numbers $\mathbb{Q}$ and $G = \widehat{\mathbb{Q}}$. A function f on the open unit big-disk Δ_G over G is said to be *hyper-analytic* on Δ_G if f can be approximated uniformly on Δ_G by functions of type $h \circ \widetilde{\chi}_{1/n}$, where $n \in \mathbb{Z}_+ = \mathbb{Z} \cap (0, \infty)$ and h is analytic on the unit disk Δ.

The algebra of all *bounded hyper-analytic functions* on Δ_G is denoted by H^∞_G. Under the *sup*-norm $\|f\| = \sup\{|f(r \cdot g)| : r \cdot g \in \Delta_G\}$, H^∞_G is a commutative Banach algebra with unit. As is customary, we identify the functions $f \in H^\infty_G$ with their Gelfand transforms $\widehat{f} \in C(M(H^\infty_G))$, defined by $\widehat{f}(\phi) = \phi(f)$, where ϕ runs in $M(H^\infty_G)$.

Recall that, by the classical corona theorem, Δ can be identified with a dense subset of the maximal ideal space $M(H^\infty)$ (e.g., [**4**]). Namely, there exists a continuous mapping τ from $M(H^\infty)$ onto $\overline{\Delta}$ which is one-to-one and homeomorphic on $\tau^{-1}(\Delta)$. Actually, τ is the Gelfand transform of the identity mapping id: $z \mapsto z$ in Δ, i.e., $\tau(\phi) = \phi(\text{id})$, where ϕ runs in $M(H^\infty)$. For any $\alpha \in \mathbb{T}$, the set $S_\alpha = \{\phi \in M(H^\infty) : \tau(\phi) = \alpha\}$ is the *fibre* of $M(H^\infty)$ over α. Different fibres of $M(H^\infty)$ are disjoint and homeomorphic to each other (e.g., [**6**]). The union of all fibres of $M(H^\infty)$ is the complement of the open unit disk Δ in $M(H^\infty)$, i.e., $M(H^\infty) \setminus \Delta = M(H^\infty) \setminus \tau^{-1}(\Delta) = \bigcup_{\alpha\in\mathbb{T}} S_\alpha$.

In a similar way, there is a continuous map, τ_G, from the maximal ideal space, $M(H^\infty_G)$, of bounded hyper-analytic functions onto the closed unit big-disk $\overline{\Delta}_G$ with properties similar to the ones of τ. The map τ_G is defined as follows. For any $\phi \in M(H^\infty_G)$, define the function

$$g_\phi(p) = \begin{cases} \overline{\phi(\widetilde{\chi}_p)/|\phi(\widetilde{\chi}_p)|} & \text{when } p \in \Gamma_+ \\ g_\phi(-p) & \text{when } p < 0, \end{cases} \tag{1.1}$$

which is a continuous character of Γ. Therefore, g_ϕ can be interpreted as a point, written again as g_ϕ, in the dual group $G = \widehat{\Gamma}$. The mapping $\tau_G\colon M(H^\infty_G) \to \overline{\Delta}_G$ is defined by

$$\tau_G(\phi) = r_\phi \cdot g_\phi,\ \phi \in H^\infty_G, \tag{1.2}$$

where $r_\phi = |\phi(\widetilde{\chi}_1)|$.

In **[9]** (see also **[10]**) it is shown that, similarly to H^∞, the algebra H^∞_G of bounded hyper-analytic functions does not have corona. Namely,

Theorem 1.2 ([**9, 10**]).

(i) *τ_G maps $M(H^\infty_G)$ onto $\overline{\Delta}_G$.*
(ii) *The set $\tau_G^{-1}(\Delta)$ is dense in $M(H^\infty_G)$.*
(iii) *τ_G is one-to-one and homeomorphic on $\tau_G^{-1}(\Delta)$.*

If we identify the sets Δ_G and $\tau_G^{-1}(\Delta)$, then Theorem 1.2 asserts that the big-disk Δ_G is dense in $M(H^\infty_G)$, thus H^∞_G does not have corona. The *fibre* of $M(H^\infty_G)$ over a $g \in G$ is the set $S_g = \tau_G^{-1}(1\cdot g) = \{\phi \in M(H^\infty_G)\colon \tau_G = (1\cdot g)\}$. Any fibre S_g of $M(H^\infty_G)$ is a compact subset of $M(H^\infty_G)$, different fibres are disjoint and homeomorphic to each other, and $M(H^\infty_G) \setminus \tau_G(\Delta_G) = M(H^\infty_G) \setminus \Delta_G = \bigcup_{g\in G} S_g$ **[10]**.

Let A be a uniform algebra with maximal ideal space $M(A)$ and Shilov boundary ∂A. The function

$$\rho_A(\phi,\psi) = \sup\{|f(\psi)|\colon f \in A,\ \|f\| \le 1,\ f(\phi) = 0\} \tag{1.3}$$

on $M(A) \times M(A)$ is a metric in $M(A)$, called the *pseudo-hyperbolic distance* of A. Note that in (1.3) we can consider only $f \in A$ with

$\|f\| = 1$. For any $\phi, \psi \in M(A)$, the inequality $\|\phi - \psi\| < 2$ holds if and only if $\rho_A(\phi, \psi) < 1$ and defines an equivalent relation in $M(A)$, namely, $\phi \sim \psi$ if and only if $\|\phi - \psi\| < 2$ (or if $\rho_A(\phi, \psi) < 1$) (e.g., [**3**]). The equivalent classes of this relation are the *Gleason parts* of A (or, in $M(A)$). The Gleason part containing an $\phi \in M(A)$ is denoted by $P(\phi)$, i.e., $P(\phi) = \{\psi \in M(A) \colon \|\phi - \psi\| < 2\} = \{\psi \in M(A) \colon \rho_A(\phi, \psi) < 1\}$ [**3**]. If $P(\phi)$ is a singleton, then it is called a *trivial* Gleason part.

In the classical situation of H^∞ the open unit disk Δ is a Gleason part, the pseudo-hyperbolic distance ρ_{H^∞} is lower semi-continuous on $M(H^\infty) \times M(H^\infty)$ and its restriction on $\Delta \times \Delta$ is invariant under Möbius transformation (e.g., [**4, 6**]). In addition, $\rho_{H^\infty}(z, w) = \sup\{\rho_{H^\infty}(f(\phi), f(\psi)) \colon f \in H^\infty,\ \|f\| \leq 1\}$. Moreover, by the Schwarz-Pick's lemma (cf., [**4**])

$$\rho_{H^\infty}(z, w) = \frac{|z - w|}{|1 - \bar{z}w|} \tag{1.4}$$

for any z and w in Δ. If $\Gamma = \mathbb{Q}$ (or, more generally, if Γ is dense in $\mathbb{R}$ in the usual topology) the only trivial Gleason parts of A_G are the points in $G = \partial A_G$ and the origin $\{*\}$ of the big-disk $\overline{\Delta}_G$ (e.g. [**3**]).

In this paper, we study the maximal ideal space $M(H_G^\infty)$ of the algebra of bounded hyper-analytic functions on the big-disk Δ_G, where $G = \widehat{\mathbb{Q}}$. In Section 2, we consider a natural extensions of the "positive" characters χ_p, $p \in \mathbb{Q}_+$, and establish a formula for the pseudo-hyperbolic distance in $M(H_G^\infty)$, based on the pseudo-hyperbolic distance in $M(H^\infty)$. In Section 3, we investigate the restriction of the pseudo-hyperbolic distance $\rho_{H_G^\infty}$ on the big disk Δ_G. In Section 4, we study the relationships between Gleason parts of $M(H_G^\infty)$ and $M(H^\infty)$.

2. The pseudo-hyperbolic distance in $M(H_G^\infty)$. In [**8**] (see also [**10**]) it is shown that every character $\chi_{1/m}$, $m \in \mathbb{Z}_+$ extends continuously to a projection, π_m, from $M(H_G^\infty)$ onto $M(H^\infty)$. Namely, given a $\phi \in M(H_G^\infty)$, π_m is defined as

$$(\pi_m(\phi))(h) = \phi(h \circ \widetilde{\chi}_{1/m}), \tag{2.1}$$

where h runs in H^∞.

Proposition 2.1. *Let $m \in \mathbb{Z}_+$, and let $\pi_m \colon M(H_G^\infty) \to M(H^\infty)$ be the mapping defined in* (2.1). *Then*

(i) *π_m is a continuous extension of the character $\chi_{1/m}$ from G to $M(H^\infty_G)$;*
(ii) *π_m is surjective, i.e., $\pi_m(M(H^\infty_G)) = M(H^\infty)$;*
(iii) *$\pi_m(\partial H^\infty_G) = \partial H^\infty$;*
(iv) *The maps $\{\pi_m\}_{m=1}^\infty$ separate the points of $M(H^\infty_G)$;*
(v) *If $f \in H^\infty_G$ and $h_{n_k} \in H^\infty$ be such that*

$$\lim_{k\to\infty} h_{n_k} \circ \widetilde{\chi}_{1/n_k} = f \quad \text{in } H^\infty_G,$$

then

$$\lim_{k\to\infty} \widehat{h}_{n_k} \circ \pi_{n_k} = \widehat{f} \quad \text{in } C(M(H^\infty_G));$$

(vi) *If $\chi_{1/m}(g) = \alpha \in \mathbb{T}$, then $\pi_m(S_g) = S_\alpha$; hence, $\pi_m^{-1}(S_\alpha) = \bigcup_{g\in G}\{S_g : g \in \chi^{-1}_{1/m}(\alpha)\}$.*

Proof.

(i) If $\phi_\alpha \to \phi_0$ in $M(H^\infty_G)$, then, according to (2.1),

$$(\pi_m(\phi_\alpha))(h) = \phi_\alpha(h \circ \widetilde{\chi}_{1/m}) \to \phi_0(h \circ \widetilde{\chi}_{1/m}) = (\pi_m(\phi_0))(h)$$

for every $h \in H^\infty$. Hence, $\pi_m(\phi_\alpha) \to \pi_m(\phi_0)$ in $M(H^\infty)$ and therefore π_m is continuous. Let $r \cdot g \in \Delta_G$ and $h \in H^\infty$. For the point evaluation $\phi_{r\cdot g}$, we have $(\pi_m(\phi_{r\cdot g}))(h) = \phi_{r\cdot g}(h \circ \widetilde{\chi}_{1/m}) = h(\widetilde{\chi}_{1/m}(r \cdot g))$. Hence, $\pi_m(\phi_{r\cdot g})$ is the evaluation at the point $\widetilde{\chi}_{1/m}(r \cdot g) \in \Delta$. Consequently, $\pi_m|_{\Delta_G} = \widetilde{\chi}_{1/m}$, i.e., π_m is a continuous extension of $\widetilde{\chi}_{1/m}$ to $M(H^\infty_G)$, and therefore it extends also $\chi_{1/m}$ from G to $M(H^\infty_G)$.

(ii) As shown in the proof of (i), $\pi_m(\Delta_G) = \widetilde{\chi}_{1/m}(\Delta_G) = \Delta$. Therefore, $\pi_m(M(H^\infty_G)) = M(H^\infty)$ since π_m is continuous and Δ_G, Δ are dense in the compact sets $M(H^\infty_G)$ and $M(H^\infty)$ correspondingly. Hence, π_m is surjective. In addition, $\pi_m(M(H^\infty_G) \setminus \Delta_G) = M(H^\infty) \setminus \Delta$.

(iii) This is shown in [**8**] (see also [**10**]).

(iv) Let $\phi_1 \neq \phi_2$ be two points in $M(H^\infty_G)$ with $\pi_m(\phi_1) = \pi_m(\phi_2)$ for every $m \in \mathbb{Z}_+$. Then $\phi_1(h\circ\widetilde{\chi}_{1/m}) = (\pi_m(\phi_1))(h) = (\pi_m(\phi_2)(h) = \phi_2(h \circ \widetilde{\chi}_{1/m})$ for every $m \in \mathbb{Z}_+$ and all $h \in H^\infty$. Since functions of type $h \circ \widetilde{\chi}_{1/m}$, where $m \in \mathbb{Z}_+$ and $h \in H^\infty$, are dense in H^∞_G it follows that $\phi_1 = \phi_2$.

(v) Assume that $h_{n_k} \in H^\infty$, and let $f \in H^\infty_G$ be such that $\lim_{k\to\infty} h_{n_k} \circ \widetilde{\chi}_{1/n_k} = f$ in H^∞_G. Fix an $\varepsilon > 0$. Since $\pi_m|_{\Delta_G} =$

$\widetilde{\chi}_{1/m}$, we can find a $k_0 \in \mathbb{Z}_+$ such that, for every $r \cdot g \in \Delta_G$,

$$|(\widehat{h}_{n_k} \circ \pi_{n_k})(\phi_{r\cdot g}) - \widehat{f}(\phi_{r\cdot g})| = |(h_{n_k} \circ \widetilde{\chi}_{1/n_k})(r \cdot g) - f(r \cdot g)| < \varepsilon$$

for all $k > k_0$. Therefore, $|(\widehat{h}_{n_k} \circ \pi_{n_k})(\phi) - \widehat{f}(\phi)| \leq \varepsilon$ for all $\phi \in M(H^\infty_G)$ and each $k > k_0$, since Δ_G is dense in $M(H^\infty_G)$. Consequently, $\lim_{k\to\infty} \widehat{h}_{n_k} \circ \pi_{n_k} = \widehat{f}$ in $C(M(H^\infty_G))$, as claimed.

(vi) We claim that the following diagram is commutative:

$$\begin{array}{ccc} M(H^\infty_G) & \xrightarrow{\pi_m} & M(H^\infty) \\ \tau_G \downarrow & & \tau \downarrow \\ \overline{\Delta}_G & \xrightarrow{\widetilde{\chi}_{1/m}} & \overline{\Delta} \end{array}$$

Indeed, let $\phi \in M(H^\infty_G)$. By (1.1), (1.2) and Theorem 1.2, we have:

$$\begin{aligned} \widetilde{\chi}_{1/m}(\tau_G(\phi)) &= \widetilde{\chi}_{1/m}(r_\phi \cdot g_\phi) = r_\phi^{1/m} \cdot g_\phi(1/m) \\ &= |\phi(\widetilde{\chi}_1)|^{1/m} \cdot \frac{\phi(\widetilde{\chi}_{1/m})}{|\phi(\widetilde{\chi}_{1/m})|} \\ &= |\phi(\widetilde{\chi}_{1/m})| \cdot \frac{\phi(\widetilde{\chi}_{1/m})}{|\phi(\widetilde{\chi}_{1/m})|} \\ &= \phi(\widetilde{\chi}_{1/m}) = \phi(\mathrm{id} \circ \widetilde{\chi}_{1/m}) \\ &= (\pi_m(\phi))(\mathrm{id}) = \tau(\pi_m(\phi)), \end{aligned}$$

i.e., the diagram is commutative, as claimed. Assume now that $g \in G$, $\phi \in S_g$, $\chi_{1/m}(g) = \alpha \in \mathbb{T}$ and $\tau_G(\phi) = 1 \cdot g \in \overline{\Delta}_G$. The commutativity of the diagram from the above implies that $\tau(\pi_m(\phi)) = \widetilde{\chi}_{1/m}(\tau_G(\phi)) = \widetilde{\chi}_{1/m}(1 \cdot g) = \chi_{1/m}(g) = \alpha$; thus, $\pi_m(\phi) \in S_\alpha$. Conversely, let ψ belong to $S_{\alpha_0} \subset M(H^\infty)\setminus\Delta$, where $\alpha_0 \in \mathbb{T}$. Since, as we saw in the proof of (ii), $\pi_m(\bigcup_{g\in G} S_g) = \pi_m(M(H^\infty_G) \setminus \Delta_G) = M(H^\infty) \setminus \Delta = \bigcup_{\beta\in\mathbb{T}} S_\beta$, there are $g_0 \in G$ and $\phi \in S_{g_0}$ such that $\pi_m(\phi) = \psi$ and $\chi_{1/m}(g_0) = \widetilde{\chi}_{1/m}(1 \cdot g_0) = \widetilde{\chi}_{1/m}(\tau_G(\phi)) = \tau(\pi_m(\phi)) = \alpha_0$. Therefore, $\pi_m(S_g) = S_\alpha$, and consequently, $\pi_m^{-1}(S_\alpha) = \bigcup_{g\in G}\{S_g \colon \chi_{1/m}(g) = \alpha\}$, as desired.

□

In general, the mapping π_m is not injective, and, for every functional $\phi \in M(H^\infty_G) \setminus \{*\}$, there are $m, n \in \mathbb{Z}_+$ with $\pi_m(\phi) \neq \pi_n(\phi)$.

Given an $m \in \mathbb{Z}_+$, the set $H^\infty_{1/m} = \{h \circ \widetilde{\chi}_{1/m} \colon h \in H^\infty\}$ is a subalgebra of H^∞_G. It is easy to see that $H^\infty_{1/n} \subset H^\infty_{1/m}$ whenever $m = kn$ for some $k \in \mathbb{Z}_+$. The map $h \mapsto h \circ \widetilde{\chi}_{1/m}$ is an isometric algebra isomorphism between H^∞ and $H^\infty_{1/m}$. Its conjugate, $\phi \to \psi \colon M(H^\infty_{1/m}) \to M(H^\infty)$, defined by $\psi(h \circ \widetilde{\chi}_{1/m}) = \phi(h)$, where h runs in H^∞, is a homeomorphism between the corresponding maximal ideal spaces. Therefore, any property of the algebra H^∞ has an identical property for $H^\infty_{1/m}$. By identifying $M(H^\infty_{1/m})$ with $M(H^\infty)$, we may assume that π_m maps $M(H^\infty_G)$ onto $M(H^\infty_{1/m})$ and that $\pi_m(\phi)$ is the restriction of ϕ on the algebra $H^\infty_{1/m} \subset H^\infty_G$.

The equality (1.3) implies that the pseudo-hyperbolic distance in $M(H^\infty_G)$ is given by

$$\rho_{H^\infty_G}(\phi_1, \phi_2) = \sup\{|f(\phi_2)| \colon f \in H^\infty_G,\ \|f\| = 1,\ f(\phi_1) = 0\}, \tag{2.2}$$

where $\phi_1, \phi_2 \in M(H^\infty_G)$.

In the sequel we will need the following

Lemma 2.2. [**7**] *If f is a hyper-analytic function in the open big-disk Δ_G, then there exists a sequence of functions of type $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^\infty$ that converges uniformly to f on Δ_G and such that*

(i) *h_{n_k} is analytic in Δ for every $k \in \mathbb{Z}_+$, and*
(ii) *For every $m > s$ there is a $k_{m,s} \in \mathbb{Z}_+$ so that $n_m = n_s k_{m,s}$.*

The next theorem describes the pseudo-hyperbolic distance in $M(H^\infty_G)$ in terms of the pseudo-hyperbolic distance in $M(H^\infty)$.

Theorem 2.3. *If $\phi_1, \phi_2 \in M(H^\infty_G)$, then*

$$\begin{gathered} \rho_{H^\infty_G}(\phi_1, \phi_2) = \sup\nolimits_{m \in \mathbb{Z}_+} \rho_{H^\infty}(\pi_m(\phi_1), \pi_m(\phi_2)) = \\ \sup\nolimits_{m \in \mathbb{Z}_+} \sup\{|(\widehat{h} \circ \pi_m)(\phi_2)| \colon h \in H^\infty,\ \|h\| = 1,\ (\widehat{h} \circ \pi_m)(\phi_1) = 0\}. \end{gathered} \tag{2.3}$$

Proof. Let $\phi_1, \phi_2 \in M(H^\infty_G)$, $\phi_1 \neq \phi_2$ and denote $\gamma = \rho_{H^\infty_G}(\phi_1, \phi_2) > 0$. Choose a λ with $0 < \lambda < \gamma$. By (2.2), there is an $f \in H^\infty_G$ such that $\|f\| = 1$, $f(\phi_1) = 0$ and $\gamma - \lambda < |f(\phi_2)| \leq \gamma$. According to

Lemma 2.2, there is a sequence of type $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^{\infty}$ converging uniformly on Δ_G to f, where h_{n_k} are analytic in $\Delta = \widetilde{\chi}_{1/n_k}(\Delta_G)$ and such that, if $m > s$, then $n_m = n_s k_{m,s}$ for some $k_{m,s} \in \mathbb{Z}_+$. Without loss of generality, we may assume that $\widehat{h}_{n_k} \circ \pi_{n_k}(\phi_1) = 0$ for every k. Indeed, let $h'_{n_k} = h_{n_k} - h_{n_k}(\pi_{n_k}(\phi_1))$. By Proposition 2.1 (v), for any $\varepsilon > 0$, there is a $k_0 > 0$ such that, for all $k > k_0$, we have $\|h_{n_k} \circ \pi_{n_k} - f\| < \varepsilon$ and $|\widehat{h}_{n_k} \circ \pi_{n_k}(\varphi_1)| < \varepsilon$. Therefore, $\widehat{h}'_{n_k} \circ \pi_{n_k}(\phi_1) = 0$, and $\|h'_{n_k} \circ \widetilde{\chi}_{1/n_k} - f\| = \|h'_{n_k} \circ \pi_{n_k} - f\| \leq \|h'_{n_k} \circ \pi_{n_k} - h_{n_k} \circ \pi_{n_k}\| + \|h_{n_k} \circ \pi_{n_k} - f\| < 2\varepsilon$ for all $k > k_0$. Also, we can assume that $\|h_{n_k}\| = 1$ for each $k \in \mathbb{Z}_+$. Indeed, if $g_{n_k} = h_{n_k}/\|h_{n_k}\|$, then

$$\begin{aligned}\|g_{n_k} \circ \widetilde{\chi}_{1/n_k} - f\| &= (1/\|h_{n_k}\|)(\|(h_{n_k} \circ \widetilde{\chi}_{1/n_k}) - f + (1 - \|h_{n_k}\|)\, f\|) \\ &\leq (1/\|h_{n_k}\|)(\|(h_{n_k} \circ \widetilde{\chi}_{1/n_k}) - f\| + (1 - \|h_{n_k}\|)\|f\|)\end{aligned}$$

which converges to 0 as $k \to \infty$, since $\|h_{n_k}\| = \|h_{n_k} \circ \widetilde{\chi}_{1/n_k}\| \to \|f\| = 1$. Let $\lambda' > 0$ be such that $\lambda' < |\widehat{f}(\phi_2)| - \gamma + \lambda$. By Proposition 2.1 (v), there is a $k_1 \in \mathbb{Z}_+$ such that $|(\widehat{h}_{n_k} \circ \pi_{n_k})(\phi) - f(\phi)| < \lambda'$ for any $k > k_1$ and every $\phi \in M(H_G^{\infty})$. Then $|(\widehat{h}_{n_k} \circ \pi_{n_k})(\phi_2)| > |f(\phi_2)| - \lambda' > \gamma - \lambda$ for all $k > k_1$. On the other hand, $h_{n_k} \circ \widetilde{\chi}_{1/n_k} \in H_G^{\infty}$, $\|\widehat{h}_{n_k} \circ \widetilde{\chi}_{1/n_k}\| = 1$ and $\widehat{h}_{n_k} \circ \pi_{n_k}(\phi_1) = 0$. Therefore, $|(\widehat{h}_{n_k} \circ \pi_{n_k})(\phi_2)| \leq \gamma$. Consequently, for any $m > k_1$ we have

$$\begin{aligned}\gamma - \lambda < \sup\{|(\widehat{h} \circ \pi_m)(\phi_2)| \colon h \in H^{\infty},& \\ \|h\| = 1,\ (\widehat{h} \circ \pi_m)(\phi_1)\} = 0\} \leq \gamma.&\end{aligned}$$

Since λ can be chosen arbitrarily close to γ it follows that $\sup\{|(\widehat{h} \circ \pi_m)(\phi_2)| \colon h \in H^{\infty},\ \|h\| = 1,\ (\widehat{h} \circ \pi_m)(\phi_1)\} = 0\} = \gamma = \rho_{H_G^{\infty}}(\phi_1, \phi_2)$, as claimed. □

Grigorian and Tonev **[5]** have generalized the construction of the algebra H_G^{∞} and have considered inductive limits $H^{\infty}(I)$ of algebras H^{∞} linked by a sequence $I = \{I_k\}_1^{\infty}$ of general inner functions and prove a version of the corona theorem with estimates for them. Whether Theorem 2.3 holds in general for algebras of type $H^{\infty}(I)$ is not known.

3. The pseudo-hyperbolic distance $\rho_{H^\infty_G}$ on the big-disk Δ_G. In the case when $\phi_i = \phi_{r_i \cdot g_i}$, $i = 1, 2$, we will write for short $\rho_{H^\infty_G}(r_1 \cdot g_1, r_2 \cdot g_2)$ instead of $\rho_{H^\infty_G}(\phi_{r_1\cdot g_1}, \phi_{r_2 \cdot g_2})$. Since $\pi_m(\phi_{r\cdot g}) = \widetilde{\chi}_{1/m}(r \cdot g)$, the following corollary follows directly from Theorem 2.3.

Corollary 3.1. *If $r_1 \cdot g_1$ and $r_2 \cdot g_2$ are points in the big-disk Δ_G, then*

$$\begin{aligned}\rho_{H^\infty_G}(r_1 \cdot g_1, r_2 \cdot g_2) &= \sup_{m\in\mathbb{Z}_+} \rho_{H^\infty}(\widetilde{\chi}_{1/m}(r_1 \cdot g_1), \widetilde{\chi}_{1/m}(r_2 \cdot g_2)) \\ &= \sup_{m\in\mathbb{Z}_+} \frac{|\widetilde{\chi}_{1/m}(r_1\cdot g_1) - \widetilde{\chi}_{1/m}(r_2 \cdot g_2)|}{|1 - \overline{\widetilde{\chi}_{1/m}(r_1 \cdot g_1)}\widetilde{\chi}_{1/m}(r_2 \cdot g_2)|} \\ &= \sup_{m\in\mathbb{Z}_+} \sup\{|(h \circ \widetilde{\chi}_{1/m})(r_2 \cdot g_2)| \colon h \in H^\infty, \\ &\qquad \|h\| = 1,\ (h \circ \widetilde{\chi}_{1/m})(r_1 \cdot g_1) = 0\}.\end{aligned}$$

Corollary 3.2. *The pseudo-hyperbolic distance $\rho_{H^\infty_G}$ on Δ_G is lower semi-continuous on $\Delta_G \times \Delta_G$.*

Proof. Denote $B_\delta = \{(r_1 \cdot g_1, r_2 \cdot g_2) \in \Delta_G \times \Delta_G \colon \rho_{H^\infty_G}(r_1 \cdot g_1, r_2 \cdot g_2) > \delta\}$, and $C_\delta = \{(z_1, z_2) \in \Delta \times \Delta \colon \rho_{H^\infty}(z_1, z_2) > \delta\}$. Let $(r_1^0 \cdot g_1^0, r_2^0 \cdot g_2^0) \in B_\delta$. Corollary 3.1 implies that $\sup_{m\in\mathbb{Z}_+} \rho_{H^\infty}(\widetilde{\chi}_{1/m}(r_1^0 \cdot g_1^0), \widetilde{\chi}_{1/m}(r_2^0 \cdot g_2^0)) = \rho_{H^\infty_G}(r_1^0 \cdot g_1^0, r_2^0 \cdot g_2^0) > \delta$, and there is an $m_0 \in \mathbb{Z}_+$ such that $(\widetilde{\chi}_{1/m_0}(r_1^0 \cdot g_1^0), \widetilde{\chi}_{1/m_0}(r_2^0 \cdot g_2^0)) \in C_\delta$. Since the pseudo-hyperbolic distance ρ_{H^∞} is lower semi-continuous on $\Delta \times \Delta$ we can find neighborhoods U_1 and U_2 of $\widetilde{\chi}_{1/m_0}(r_1^0 \cdot g_1^0)$ and $\widetilde{\chi}_{1/m_0}(r_2^0 \cdot g_2^0)$ correspondingly, such that $U_1 \times U_2 \subset C_\delta$. Clearly, $(r_1^0 \cdot g_1^0, r_2^0 \cdot g_2^0) \in \widetilde{\chi}^{-1}_{1/m_0}(U_1) \times \chi^{-1}_{1/m_0}(U_2)$. Since $\widetilde{\chi}_{1/m_0}$ is continuous on Δ_G, then the set $\widetilde{\chi}^{-1}_{1/m_0}(U_1) \times \widetilde{\chi}^{-1}_{1/m_0}(U_2)$ is open. Corollary 3.1 implies that $\widetilde{\chi}^{-1}_{1/m_0}(U_1) \times \widetilde{\chi}^{-1}_{1/m_0}(U_2) \subset B_\delta$. □

Corollary 3.3. *$\rho_{H^\infty}(f(r_1 \cdot g_1), f(r_2 \cdot g_2)) \le \rho_{H^\infty_G}(r_1 \cdot g_1, r_2 \cdot g_2)$ for every $f \in H^\infty_G$ with $\|f\| \le 1$.*

Proof. Suppose, on the contrary, that there is a function $f \in H^\infty_G$ with $\|f\| \le 1$ such that $\rho_{H^\infty}(f(r_1 \cdot g_1), f(r_2 \cdot g_2)) > \rho_{H^\infty_G}(r_1 \cdot g_1, r_2 \cdot g_2)$. Let $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^\infty$, $h_{n_k} \in H^\infty$, be a sequence as in Lemma 2.2 that approximates uniformly f on Δ_G. The lower semi-continuity of ρ_{H^∞}

on $\Delta \times \Delta$ implies

$$\rho_{H^\infty}(h_{n_k} \circ \widetilde{\chi}_{1/n_k}(r_1 \cdot g_1), h_{n_k} \circ \widetilde{\chi}_{1/n_k}(r_2 \cdot g_2)) > \rho_{H_G^\infty}(r_1 \cdot g_1, r_2 \cdot g_2) \tag{3.1}$$

for sufficiently large k. We may assume that $\|h_{n_k}\| \le 1$ for every $k \in \mathbb{Z}_+$. If $z_1^k = \widetilde{\chi}_{1/n_k}(r_1 \cdot g_1)$, $z_2^k = \widetilde{\chi}_{1/n_k}(r_2 \cdot g_2)$, then from (3.1) it follows that, for sufficiently large k,

$$\begin{aligned}
\rho_{H^\infty}((h_{n_k} \circ \widetilde{\chi}_{n_k})(r_1 \cdot g_1), (h_{n_k} \circ \widetilde{\chi}_{n_k})(r_2 \cdot g_2))& \\
= \rho_{H^\infty}(h_{n_k}(z_1^k), h_{n_k}(z_2^k)) \le \rho_{H^\infty}(z_1^k, z_2^k)& \\
= \rho_{H^\infty}(\widetilde{\chi}_{1/n_k}(r_1 \cdot g_1), \widetilde{\chi}_{1/n_k}(r_2 \cdot g_2)) \le \rho_{H_G^\infty}(r_1 \cdot g_1, r_2 \cdot g_2),&
\end{aligned}$$

by Corollary 3.1 and the corresponding classical results for ρ_{H^∞}. This contradicts (3.1). □

Let g_0 be a fixed point of G, and let $R_{g_0} : \Delta_G \to \Delta_G$ be the rotation $R_{g_0}(r \cdot g) = r \cdot g g_0$ in the big-disk Δ_G by g_0.

Corollary 3.4. *The restriction of the pseudo-hyperbolic distance $\rho_{H_G^\infty}$ on the big-disk Δ_G is invariant under any rotation R_{g_0}, i.e.,*

$$\rho_{H_G^\infty}(R_{g_0}(r_1 \cdot g_1), R_{g_0}(r_2 \cdot g_2)) = \rho_{H_G^\infty}(r_1 \cdot g_1, r_2 \cdot g_2)$$

for any $r_1 \cdot g_1$ and $r_2 \cdot g_2$ in Δ_G.

Proof. Let $r_1 \cdot g_1$ and $r_2 \cdot g_2$ be points in the big-disk Δ_G. According to (1.4) for every $m \in \mathbb{Z}_+$ we have:

$$\begin{aligned}
&\rho_{H^\infty}(\widetilde{\chi}_{1/m}(R_{g_0}(r_1 \cdot g_1)), \widetilde{\chi}_{1/m}(R_{g_0}(r_2 \cdot g_2))) \\
&\quad = \left| \frac{\widetilde{\chi}_{1/m}(r_1 \cdot g_1 g_0) - \widetilde{\chi}_{1/m}(r_2 \cdot g_2 g_0)}{1 - \overline{\widetilde{\chi}_{1/m}(r_2 \cdot g_2 g_0)}\widetilde{\chi}_{1/m}(r_1 \cdot g_1 g_0)} \right| \\
&\quad = \left| \frac{r_1^{1/m} \cdot g_1(1/m)\, g_0(1/m) - r_2^{1/m} \cdot g_2(1/m)\, g_0(1/m)}{1 - r_2^{1/m}\overline{g_2(1/m)\, g_0(1/m)} r_1^{1/m} \cdot g_1(1/m)\, g_0(1/m)} \right| \\
&\quad = \left| \frac{r_1^{1/m} \cdot g_1(1/m) - r_2^{1/m} \cdot g_2(1/m)}{1 - r_2^{1/m}\overline{g_2(1/m)} r_1^{1/m} \cdot g_1(1/m)} \right| \\
&\quad = \left| \frac{\widetilde{\chi}_{1/m}(r_1 \cdot g_1) - \widetilde{\chi}_{1/m}(r_2 \cdot g_2)}{1 - \overline{\widetilde{\chi}_{1/m}(r_2 \cdot g_2)}\widetilde{\chi}_{1/m}(r_1 \cdot g_1)} \right| \\
&\quad = \rho_{H^\infty}(\widetilde{\chi}_{1/m}(r_1 \cdot g_1), \widetilde{\chi}_{1/m}(r_2 \cdot g_2)).
\end{aligned}$$

Corollary 3.1 implies that $\rho_{H_G^\infty}(R_{g_0}(r_1 \cdot g_1), R_{g_0}(r_2 \cdot g_2)) = \rho_{H_G^\infty}(r_1 \cdot g_1, r_2 \cdot g_2)$, as claimed. $\square$

4. Gleason parts in $M(H_G^\infty)$. The next proposition follows directly from Theorem 2.3.

Proposition 4.1. *If $\phi \in M(H_G^\infty)$, then $\pi_m(P(\phi)) \subset P(\pi_m(\phi))$ for every $m \in \mathbb{Z}_+$.*

Indeed, let $\psi \in P(\varphi)$, $\varphi \in M(H_G^\infty)$ and $m \in \mathbb{Z}_+$. Theorem 2.3 implies

$$\rho_{H^\infty}\left(\pi_m\left(\varphi\right), \pi_m\left(\psi\right)\right) \leqslant \sup_{m \in \mathbb{Z}_+} \rho_{H^\infty}\left(\pi_m\left(\varphi\right), \pi_m\left(\psi\right)\right) = \rho_{H_G^\infty}\left(\varphi, \psi\right) < 1,$$

and therefore $\pi_m(\psi) \in P(\pi_m(\varphi))$.

One can prove Proposition 4.1 also directly. Note that, for every $m \in \mathbb{Z}_+$, the map $\pi_m \colon M(H_G^\infty) \to M(H^\infty)$ is the restriction on $M(H_G^\infty)$ of the linear map $\widetilde{\pi}_m \colon (H_G^\infty)^* \to (H^\infty)^*$, $(\widetilde{\pi}_m(\varphi))(h) = \varphi(h \circ \widetilde{\chi}_{1/m})$ for $\varphi \in (H_G^\infty)^*$ and $h \in H^\infty$. Note that $\widetilde{\pi}_m$ is a contraction. Indeed, $\|\widetilde{\pi}_m(\phi)\| = \sup_{h \neq 0} \frac{|(\widetilde{\pi}_m)(\phi)(h)|}{\|h\|} \leq \sup_{h \neq 0} \frac{\|\phi\| \|h \circ \widetilde{\chi}_{1/m}\|}{\|h\|} = \|\phi\|$. Now, if $\varphi \in M(H_G^\infty)$, $\psi \in P(\varphi)$, then

$$\left\|\pi_m\left(\psi\right) - \pi_m\left(\varphi\right)\right\|_{(H^\infty)^*} = \left\|\pi_m\left(\psi - \varphi\right)\right\|_{(H^\infty)^*} \leqslant \left\|\psi - \varphi\right\|_{\left(H_G^\infty\right)^*} < 2.$$

Therefore, $\pi_m(\psi) \in P(\pi_m(\varphi))$ and, consequently, $\pi_m(P(\varphi)) \subset P(\pi_m(\varphi))$ for every $m \in \mathbb{Z}_+$.

As mentioned in the introduction, the big-disk Δ_G can be interpreted as a subset of the maximal ideal spaces of both algebras A_G and H_G^∞. Since, as it is easy to see, $\rho_{A_G}(r_1 \cdot g_1, r_2 \cdot g_2) = \rho_{H_G^\infty}(r_1 \cdot g_1, r_2 \cdot g_2)$, the Gleason parts of $M(A_G)$ and of $M(H_G^\infty)$ inside the big-disk Δ_G coincide. In particular, the center $\{*\}$ of the big-disk is a singleton Gleason part for both algebras A_G and H_G^∞. Other trivial Gleason parts of both algebras outside $\{*\}$ are the points of their corresponding Shilov boundaries.

Proposition 4.2. *If the Gleason part of $\phi \in M(H_G^\infty)$ is non-trivial, then there exists an $m_0 \in \mathbb{Z}_+$ such that the Gleason part of $\pi_{m_0}(\phi)$ with respect to the algebra H^∞ is also non-trivial.*

Proof. Let the Gleason part $P(\phi)$ of a $\phi \in M(H_G^\infty)$ be non-trivial, and let $\phi_1 \in P(\phi) \setminus \{\phi\}$. By Proposition 2.1 (iv), there is an $m_0 \in \mathbb{Z}_+$ such that $\pi_{m_0}(\phi_1) \neq \pi_{m_0}(\phi)$. From Proposition 4.1, it follows that $P(\pi_{m_0}(\phi))$ is non-trivial. $\square$

Observe that the statement of Proposition 4.2 cannot be reversed. Indeed, while the Gleason part of $\pi_m(*) = 0$ in H^∞ and the open unit disk Δ, is not trivial, the center $\{*\}$ of the big-disk Δ_G itself is a trivial part of H_G^∞.

Denote by Ψ the set of all trivial Gleason parts of the algebra H^∞. Since $\partial H^\infty \subset \Psi$, Proposition 2.1 (iii) implies that $\partial H_G^\infty \subset \bigcap_{m\in\mathbb{Z}_+} \pi_m^{-1}(\partial H_{1/m}^\infty) \subset \bigcap_{m\in\mathbb{Z}_+} \pi_m^{-1}(\Psi)$. Note that the center, $\{*\}$, of the big-disk Δ_G is a trivial Gleason part of H_G^∞ that is outside $\bigcap_{m\in\mathbb{Z}_+} \pi_m^{-1}(\Psi)$.

Proposition 4.3. *The points of the set $\bigcap_{m\in\mathbb{Z}_+} \pi_m^{-1}(\Psi)$ are trivial Gleason parts of H_G^∞.*

Proof. Let $\phi_0 \in \bigcap_{m\in\mathbb{Z}_+} \pi_m^{-1}(\Psi)$, and assume that $\phi \in M(H_G^\infty)$, $\phi \neq \phi_0$. According to Proposition 2.1 (iv), there is an $m_0 \in \mathbb{Z}_+$ such that $\pi_{m_0}(\phi) \neq \pi_{m_0}(\phi_0)$. Since $\pi_{m_0}(\phi_0)$ is a trivial Gleason part of H^∞, $\pi_{m_0}(\phi) \notin P(\pi_{m_0}(\phi_0))$. Therefore, $\rho_{H^\infty}(\pi_{m_0}(\phi), \pi_{m_0}(\phi_0)) = 1$. By Theorem 2.3,

$$\rho_{H_G^\infty}(\phi, \phi_0) = \sup_{m\in\mathbb{Z}_+} \rho_{H^\infty}(\pi_m(\phi), \pi_m(\phi_0)) = 1,$$

and hence ϕ and ϕ_0 belong to different Gleason parts of H_G^∞. $\square$

Proposition 4.4. *For every $\phi \in M(H_G^\infty) \setminus \Delta_G$ and each $m \in \mathbb{Z}_+$, there is a $g_0 \in G$ such that $P(\phi) \subset S_{g_0} \cap \pi_m^{-1}(P(\pi_m(\phi)))$.*

Proof. First we will show that points of different fibres S_g belong to different Gleason parts of H_G^∞. Let $g_1 \neq g_2$, and let $\phi_i \in S_{g_i}$, $i = 1, 2$, where $S_{g_i} = \tau_G^{-1}(1 \cdot g_i)$ are the fibres over g_i. We will show that $\rho_{H_G^\infty}(\phi_1, \phi_2) = 1$. Since the family of functions $\{\chi_{1/m}\}_{m=1}^\infty$ separates the points of G, there is an $m_0 \in \mathbb{Z}_+$ such that $\alpha_1 = \chi_{1/m_0}(g_1) \neq \chi_{1/m_0}(g_2) = \alpha_2$. Hence, $S_{\alpha_1} \cap S_{\alpha_2} = \emptyset$. Proposition 2.1 (vi) implies that $\pi_{m_0}(\phi_i) \in S_{\alpha_i}$, and hence $P(\pi_{m_0}(\phi_i)) \subset S_{\alpha_i}$, $i = 1, 2$ (cf., **[6]**).

By Theorem 2.3, $\rho_{H^\infty_G}(\phi_1, \phi_2) \geq \rho_{H^\infty}(\pi_{m_0}(\phi_1), \pi_{m_0}(\phi_2)) = 1$, and therefore, $\phi_2 \notin P(\phi_1)$. If $\phi \in M(H^\infty_G) \setminus \Delta_G$, then, by Proposition 4.1, $\pi_m(P(\phi)) \subset P(\pi_m(\phi))$ for any $m \in \mathbb{Z}_+$ and, therefore, $P(\phi) \in \pi_m^{-1}(P(\pi_m(\phi)))$. $\square$

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Faculty of Mathematics and Informatics, Shumen University Konstantin Preslavsky, 115 Universitetska Str., 9712, Shumen, Bulgaria
Email address: **stankov.d@abv.bg**