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ON THE DIOPHANTINE EQUATION $F_n^x + F_{n+1}^x = F_m^y$

NORIKO HIRATA-KOHNO AND FLORIAN LUCA

ABSTRACT. Here, we find all the solutions of the title Diophantine equation in positive integer variables (m, n, x, y), where F_k is the k-th term of the Fibonacci sequence.

1. Introduction. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. The Diophantine equation

$$F_n^x + F_{n+1}^x = F_m \tag{1}$$

in positive integers (m, n, x) was studied in [**5**]. There, it was shown that no solution other than $(m, n) = (3, 1)$ exists for which $1^x + 1^x = 2$ (valid for all positive integers x), and the solutions for $x = 1$ and $x = 2$ arising via the formulas $F_n + F_{n+1} = F_{n+2}$ and $F_n^2 + F_{n+1}^2 = F_{2n+1}$. Here, we revisit equation (1) under the more general form

$$F_n^x + F_{n+1}^x = F_m^y \tag{2}$$

in positive integers (m, n, x, y). The solution with $n = 1$ arising from $1^x + 1^x = 2$ for any positive integer x with $m = 3$ and $y = 1$ will be called *trivial*. So, we shall assume that $n \geq 2$. The solutions with $(x, y) = (1, 1)$, $(2, 1)$ given by $F_n+F_{n+1} = F_{n+2}$ and $F_n^2+F_{n+1}^2 = F_{2n+1}$ will also be called trivial. In the case $x = 1$, there is a nontrivial solution arising from $F_4 + F_5 = F_6 = F_3^3$; therefore, $(m, n, x, y) = (3, 4, 1, 3)$. It is the only solution with $y > 1$ when $x = 1$ or $x = 2$ because 8 is the only Fibonacci number larger than 1 which is a perfect power of another Fibonacci number (see [**2**]). In the case $n = 2$, we get the equation $1 + 2^x = F_m^y$. When $y = 1$, there is no solution (see [**1**]), while

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for $y \geq 2$, this is Catalan's equation whose only solution $1 + 2^3 = 3^2$ yields $(m, n, x, y) = (4, 2, 3, 2)$ as a solution to our original equation.

Our main result shows that no other solution exists.

Theorem 1.1. *All positive integer solutions* (m, n, x, y) *of equation* (2) *are* $(3, 1, x, 1)$, $(n + 2, n, 1, 1)$, $(3, 4, 1, 3)$, $(2n + 1, n, 1, 1)$, $(4, 2, 3, 2)$.

Before getting to the proof, we mention that similar looking equations have already been studied. For example, in **[3]**, it was shown that the only solution in positive integers (k, ℓ, n, r) of the equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + \cdots + F_{n+r}^\ell$$

is $(k, \ell, n, r) = (8, 2, 4, 3)$, while in **[7]**, Miyazaki showed that the only positive integer solutions (x, y, z, n) of the equation

$$F_n^x + F_{n+1}^y = F_{2n+1}^z$$

are for $(x, y, z) = (2, 2, 1)$ (and for all positive integers n).

2. The proof of Theorem 1.1.

2.1. An inequality among the variables m, n, x, y**.** We write $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ and use the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{valid for all } n \geq 0. \tag{3}$$

We also use the inequality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{valid for all } n \geq 1. \tag{4}$$

We may assume that $n \geq 3$, $x \geq 3$ and $y \geq 2$ in (2) because the case $y = 1$ was treated in **[5]**. Further, by Fermat's last theorem, it follows that $d = \gcd(x, y) \in \{1, 2\}$, for if $d \geq 3$ divides both x and y, then the triple $(X, Y, Z) = (F_n^{x/d}, F_{n+1}^{x/d}, F_m^{y/d})$ is a positive integer solution to the Fermat equation $X^d + Y^d = Z^d$ with integer exponent $d \geq 3$ and coprime positive integers X and Y, which we know does not exist. In particular, since $x \geq 3$, it follows that $x \neq y$. It is clear that $m \geq 3$, but observe that in fact the inequality $m \geq 4$ holds, for if $m = 3$, then with $(a,b) = (F_n, F_{n+1})$ we are led to a solution of the equation

$$a^x + b^x = 2^y$$

in coprime integers $1 < a < b$, and integers $x \geq 3$ and $y \geq 2$. Since a and b are coprime, they are both odd, so when x is even, the left-hand side above is congruent to 2 modulo 8, which is impossible for $y > 1$, while if x is odd, then the number $(a^x + b^x)/(a+b)$ is odd, larger than 1, and divides the left-hand side of the above equation but not the right-hand side of it, which is again impossible.

Equation (2) and inequalities (4) imply the following inequalities:

$$(\alpha^{m-2})^y < F_m^y = F_n^x + F_{n+1}^x < (F_n + F_{n+1})^x = F_{n+2}^x < (\alpha^{n+1})^x,$$
$$(\alpha^{m-1})^y > F_m^y = F_n^x + F_{n+1}^x > F_{n+1}^x > (\alpha^{n-1})^x,$$

leading to

$$-2y < (n+1)x - my \quad \text{and} \quad my - (n+1)x > -2x + y,$$

so

$$|(n+1)x - my| < 2\max\{x, y\}. \tag{5}$$

We record this as a lemma.

Lemma 2.1. *If (m,n,x,y) is a solution of (2) with $n \geq 3$, $x \geq 3$ and $y \geq 2$, then inequality (5) holds.*

From now on, we put

$$M = \min\{m, n+1\} \quad \text{and} \quad N = \max\{m, n+1\}. \tag{6}$$

2.2. Bounds on x and y in terms of N. Since $n \geq 3$, we have that $F_n/F_{n+1} \leq 2/3$. Equation (2) implies that

$$F_m^y - F_{n+1}^x = F_n^x;$$

hence,

$$F_m^y F_{n+1}^{-x} - 1 = \left(\frac{F_n}{F_{n+1}}\right)^x \leq \frac{1}{1.5^x}. \tag{7}$$

We shall use several times a result of Matveev (see **[6]**, or **[2,** Theorem 9.4]), which asserts that if $\alpha_1, \alpha_2, \ldots, \alpha_K$ are positive real algebraic numbers in an algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_K$ are rational integers, and

$$\Lambda = \alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_K^{b_K} - 1$$

is not zero, then

(8)

$$|\Lambda| > \exp\left(-1.4\times 30^{K+3}K^{4.5}D^2(1+\log D)(1+\log B)A_1A_2\cdots A_K\right),$$

where

$$B \geq \max\{|b_1|, |b_2|, \ldots, |b_K|\},$$

and

$$A_i \geq \max\{Dh(\alpha_i), |\log\alpha_i|, 0.16\}, \quad \text{for all } i = 1, 2, \ldots, K. \tag{9}$$

Here, for an algebraic number η, we write $h(\eta)$ for its logarithmic absolute height whose formula is

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\max\{|\eta^{(i)}|, 1\}\right), \tag{10}$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) = a_0\prod_{i=1}^{d}(X - \eta^{(i)}) \in \mathbb{Z}[X] \tag{11}$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root. In particular, for a positive integer η, we have $h(\eta) = \log\eta$.

In a first application of Matveev's theorem, we take $K = 2$, $\alpha_1 = F_m$, $\alpha_2 = F_{n+1}$. We also take $b_1 = y$, and $b_2 = -x$. Thus,

$$\Lambda_1 = F_m^y F_{n+1}^{-x} - 1 \tag{12}$$

is the expression appearing on the left-hand side of inequality (7). Clearly, $\Lambda_1 = (F_n/F_{n+1})^x > 0$, so, in particular, it is nonzero.

We take $B = \max\{x, y\}$. Since α_1 and α_2 are integers, it follows that we can take $D = 1$. We can take $A_1 = m\log\alpha$ and $A_2 = n\log\alpha$, then by (4), inequalities (9) hold for both $i = 1, 2$. Now Matveev's theorem tells us that

$$|\Lambda_1| > \exp\left(-C_1 \times m\log\alpha \times n\log\alpha \times (1+\log B)\right), \tag{13}$$

where

$$C_1 = 1.4 \times 30^5 \times 2^{4.5} < 8 \times 10^8. \tag{14}$$

Taking logarithms in inequality (7) and comparing the resulting inequality with (13), we get

$$-C_1(\log\alpha)^2 mn(1+\log B) < \log|\Lambda_1| < -x\log(1.5),$$

so

$$x < \frac{C_1(\log\alpha)^2}{\log(1.5)} mn(1+\log B), \tag{15}$$

which leads to

$$x < 5 \times 10^8 mn(1+\log B) < 10^9 mn\log B, \tag{16}$$

because $\log B \geq \log 3 > 1$.

If $x > y$, then $B = x$ and the above inequality gives

$$x < 10^9 mn\log x. \tag{17}$$

If $y > x$, then $B = y$. Further, by Lemma 2.1, we have that

$$|my - (n+1)x| < 2y;$$

therefore,

$$y < (m-2)y < (n+1)x \leq Nx \tag{18}$$

(because $m \geq 4$), so inequality (16) shows that

$$x < 10^9 mn\log(Nx). \tag{19}$$

If

$$x \leq N, \tag{20}$$

we already have a sharp bound on x by definition of N. Otherwise, $x > N$ and inequality (19) shows that

$$x < 10^9 mn\log(Nx) < 2 \times 10^9 mn\log x. \tag{21}$$

Comparing (17), (20) and (21), we conclude that inequality (21) holds in all cases.

It is well-known and easy to prove that, if $A \geq 3$ and $x/\log x < A$, then $x < 2A \log A$. Thus, taking $A = 2 \times 10^9 mn$, inequality (21) gives us

$$\begin{aligned} x &< 4 \times 10^9 mn \log(2 \times 10^9 N^2) \\ &< 4 \times 10^9 mn(\log(2 \times 10^9) + 2 \log N) \\ &< 4 \times 10^9 mn(22 + 2 \log N) \\ &< 10^{11} mn \log N. \end{aligned} \tag{22}$$

In the above chain of inequalities, we used that fact that $N \geq 4$, which implies that $22 + 2 \log N < 24 \log N$. From estimate (18), we also deduce that

$$y < Nx < 10^{11} MN^2 \log N. \tag{23}$$

We record what we have just proved.

Lemma 2.2. *If (n, m, x, y) is a solution in positive integers of equation (2) with $n \geq 3$, $x \geq 3$ and $y \geq 2$, then both inequalities*

$$x < 10^{11} MN \log N, \qquad y < 10^{11} MN^2 \log N$$

hold.

2.3. Solutions with $N \leq 1000$. Assume that $N \leq 1000$. By Lemma 2.2, we have

$$\begin{aligned} x &< 10^{11} \times (10^3)^2 \log(10^3) < 10^{18}, \\ y &< 10^{11} \times (10^3)^3 \log(10^3) < 10^{21}. \end{aligned}$$

Put $\Gamma_1 = y \log F_m - x \log F_{n+1}$, and observe $\Gamma_1 > 0$ and $\Lambda_1 + 1 = e^{\Gamma_1}$. Hence, from (7), we get

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{1}{1.5^x}.$$

Dividing the last inequality above by $x \log F_m$, we get

$$0 < \frac{y}{x} - \frac{\log F_{n+1}}{\log F_m} < \frac{1}{x \log F_m (1.5)^x}. \tag{24}$$

Observe that

$$(\log F_m)(1.5)^x \geq (\log 3)(1.5)^x > 2x \quad \text{for all } x \geq 6.$$

In fact, the inequality $\log F_m(1.5)^x > 2x$ fails only when $x \in \{3, 4, 5\}$ and $m \in \{4, 5\}$. For such values of x and m, with $(a, b) = (F_n, F_{n+1})$, we are led to solutions of one the equations

$$a^x + b^x = 3^y \quad \text{or} \quad a^x + b^x = 5^y,$$

but none of these equations has any solutions in positive coprime integers $1 < a < b$, $x \in \{3, 4, 5\}$ and $y \geq 2$. Hence, $(\log F_m)(1.5)^x > 2x$; therefore, inequality (24) becomes

$$0 < \frac{y}{x} - \frac{\log F_{n+1}}{\log F_m} < \frac{1}{2x^2}, \tag{25}$$

which, by a known criterion of Legendre, implies that y/x is a convergent to the continued fraction of $\log F_{n+1}/\log F_m$, and it is in fact a convergent with an odd index. Recall also that $d = \gcd(x, y) \in \{1, 2\}$.

We ran a computer code that tested all possibilities (m, n) with $N \leq 1000$. Since the convergents p_k/q_k of any irrational number γ satisfy $p_k \geq F_k$, and since $F_{105} > 10^{21}$, we generated, for each pair (m, n) with $m \geq 4$, $n \geq 3$, and $m \notin \{n, n+1\}$, the first 105 convergents p_k/q_k of $\log F_{n+1}/\log F_m$ to see whether one of the pairs $(y, x) = (p_k, q_k)$, $(2p_k, 2q_k)$ for which $x \geq 3$, the congruence $F_n^x + F_{n+1}^x \equiv F_m^y \pmod{10^{10}}$ holds. That is, we only tested equation (2) modulo 10^{10}. This computation took about six hours with Mathematica, and no solution to the above congruence was found. We record our conclusion as follows.

Lemma 2.3. *If (m, n, x, y) is a solution of equation (2) with $n \geq 3$, $x \geq 3$, and $y \geq 2$, then $N > 1000$.*

2.4. Bounds for x, y and N in terms of M. By Lemmas 2.2 and 2.3, we have

$$\max\{x, y\} < 10^{11} N^3 \log N < \alpha^N. \tag{26}$$

The right-most inequality above holds in fact for all $N \geq 84$. Say $z \in \{x, y\}$ is such that (z, N) is one of the two pairs $(x, n+1)$, (y, m).

Then inequality (26) implies that

$$\frac{z}{\alpha^{2N}} < \frac{1}{\alpha^N}. \tag{27}$$

By the Binet formula (3) and the fact that $\beta = -\alpha^{-1}$, we have

$$F_N^z = \frac{\alpha^{Nz}}{5^{z/2}}\left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z = \frac{\alpha^{Nz}}{5^{z/2}} \exp\left(z \log\left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)\right).$$

We use the fact that the inequalities

$$1 + t < e^t < 1 + 2t \tag{28}$$

and

$$1 - t < e^{-t} < 1 - t/2$$

hold for all $t \in (0, 1/2)$, as well as their logarithmic versions

$$t/2 < \log(1 + t) < t \tag{29}$$

and

$$-2t < \log(1 - t) < -t \quad \text{for all } t \in (0, 1/2),$$

and (27), to deduce that if N is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z &= \left(1 + \frac{1}{\alpha^{2N}}\right)^z \\ &= \exp\left(z \log\left(1 + \frac{1}{\alpha^{2N}}\right)\right) \\ &< \exp\left(\frac{z}{\alpha^{2N}}\right) < \exp\left(\frac{1}{\alpha^N}\right) < 1 + \frac{2}{\alpha^N}, \end{aligned} \tag{30}$$

while, if N is even, then

$$\begin{aligned} 1 > \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z &= \left(1 - \frac{1}{\alpha^{2N}}\right)^z \\ &= \exp\left(z \log\left(1 - \frac{1}{\alpha^{2N}}\right)\right) \\ &> \exp\left(-\frac{2z}{\alpha^{2N}}\right) > \exp\left(-\frac{2}{\alpha^N}\right) > 1 - \frac{2}{\alpha^N}. \end{aligned} \tag{31}$$

Thus, from the two inequalities (30) and (31) above, we deduce that if we put

$$\varepsilon_{N,z} = \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z - 1,$$

then

$$F_N^z = \frac{\alpha^{Nz}}{5^{z/2}}\left(1 + \varepsilon_{N,z}\right), \quad \text{and} \quad |\varepsilon_{N,z}| < \frac{2}{\alpha^N}. \tag{32}$$

Since $x \geq 3$ and $N > 1000$, we deduce easily from (7) and (32) that

$$\frac{F_m^y}{F_{n+1}^x}, \ \frac{F_N^z}{\alpha^{Nz}/5^{z/2}} \in \left(\frac{1}{2}, 2\right). \tag{33}$$

Suppose now that $N = n+1$. Then $z = x$ and

$$F_m^y = F_{n+1}^x + F_n^x = \frac{\alpha^{(n+1)x}}{5^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right)\varepsilon_{n+1,x} + F_n^x,$$

so

$$\begin{aligned} |F_m^y \alpha^{-(n+1)x} 5^{x/2} - 1| &= \left|\varepsilon_{n+1,x} + \frac{F_n^x}{\alpha^{(n+1)x}/5^{x/2}}\right| \\ &< |\varepsilon_{n+1,x}| + \left(\frac{F_n}{F_{n+1}}\right)^x \left(\frac{F_{n+1}^x}{\alpha^{(n+1)x}/5^{x/2}}\right) \\ &< \frac{2}{\alpha^{n+1}} + \frac{2}{1.5^x} \leq \frac{4}{1.5^\lambda}, \end{aligned} \tag{34}$$

where

$$\lambda = \min\{x, N\}. \tag{35}$$

Here we used, in addition to (33), the fact that $\alpha > 1.5$. The same inequality is obtained when $N = m$, because in this case $z = y$ and

$$F_{n+1}^x = F_m^y - F_n^x = \left(\frac{\alpha^{my}}{5^{y/2}}\right) + \left(\frac{\alpha^{my}}{5^{y/2}}\right)\varepsilon_{m,y} - F_n^x,$$

so

$$
\begin{aligned}
(36)\quad |F_{n+1}^{x}\alpha^{-my}5^{y/2}-1| &= \left|\varepsilon_{m,y}-\left(\frac{F_n^x}{\alpha^{my}/5^{y/2}}\right)\right| \\
&< |\varepsilon_{m,y}| + \left(\frac{F_n}{F_{n+1}}\right)^x \left(\frac{F_{n+1}^x}{F_m^y}\right)\left(\frac{F_m^y}{\alpha^{my}/5^{y/2}}\right) \\
&< \frac{2}{\alpha^N}+\frac{2}{1.5^x}<\frac{4}{1.5^\lambda}.
\end{aligned}
$$

To summarize, from (34) and (36), we get that if we put $\{w,z\}=\{x,y\}$ such that (w,M), (z,N) are the two pairs $(x,n+1)$, (y,m), then the inequality

$$
|F_M^w\alpha^{-Nz}5^{z/2}-1|<\frac{10}{1.5^\lambda} \tag{37}
$$

holds, where λ is given by formula (35). We shall use (37) and Matveev's theorem to get an upper bound on x and N in terms of M.

We continue by getting a lower bound on the left-hand side of inequality (37). For this, we take $K=3$, $\alpha_1=F_M$, $\alpha_2=\alpha$, $\alpha_3=\sqrt{5}$. We also take $b_1=w$, $b_2=-Nz$, $b_3=z$. Hence,

$$
\Lambda_2=\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3}-1=F_M^w\alpha^{-Nz}5^{z/2}-1
$$

is the expression which appears under the absolute value in the left-hand side of inequality (37). It is easy to see that $\Lambda_2\neq 0$, for if $\Lambda_2=0$, we then get that $\alpha^{2Nz}=F_M^{2w}5^z\in\mathbb{Z}$, which is impossible since no power of α of positive integer exponent can be an integer. Observe next that $\alpha_1,\alpha_2,\alpha_3$ are all real and belong to the field $\mathbb{K}=\mathbb{Q}(\sqrt{5})$, so we can take $D=2$. Next, since $F_M<\alpha^M$ (see (4)), it follows that we can take

$$
A_1=2M\log\alpha>D\log F_M=Dh(\alpha_1).
$$

Next, since $h(\alpha_2)=(\log\alpha)/2=0.240606\ldots$, it follows that we can take $A_2=0.5>Dh(\alpha_2)$. Since $h(\alpha_3)=(\log 5)/2=0.804719\ldots$, it follows that we can take $A_3=1.61>Dh(\alpha_3)$. Finally, Lemma 2.2, and the fact that $N > 1000$, tell us that we can take

$$\begin{aligned} B = N^9 = N^4 \times N \times N^4 &> (10^3)^4 \times \log N \times MN^2 \times N \\ &> (10^{11} MN^2 \log N) \times N > \max\{Nz, z, w\} \\ &= \max\{|b_1|, |b_2|, |b_3|\}. \end{aligned}$$

Matveev's theorem tells us that

$$|\Lambda_2| > \exp(-C_2(1 + \log B) A_1 A_2 A_3), \tag{38}$$

where

$$C_2 = 1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2) < 10^{12}. \tag{39}$$

Thus,

$$\begin{aligned} & C_2(1 + \log B) A_1 A_2 A_3 \\ & < 10^{12} \times (2 \log \alpha) \times 0.5 \times 1.61 \times (1 + \log(N^9)) M \\ & < 8 \times 10^{11} M (1 + 9 \log N) \\ & < 8 \times 10^{12} M \log N. \end{aligned} \tag{40}$$

Comparing (40) with (37), we get that

$$\lambda < \frac{\log 10}{\log 1.5} + \left(\frac{1}{\log 1.5}\right) 8 \times 10^{12} M \log N < 2 \times 10^{13} M \log N. \tag{41}$$

Before proceeding further, for reasons that will become clear later, we make one comment about Mw and Nz. If $z > w$, we then have by inequality (5), that

$$Mw \leq (N + 2) z \leq 2Nz,$$

while if $z < w$, then, again by (5) and the fact that $M \geq 3$, we have

$$\frac{Mw}{3} \leq (M - 2) w \leq Nz, \quad \text{therefore } Mw \leq 3Nz.$$

So, it is always the case that $Mw \leq 3Nz$. A similar argument shows that $Nz \leq 2Mw$; therefore,

$$\frac{Nz}{Mw} \in \left(\frac{1}{3}, 2\right). \tag{42}$$

Next, we distinguish several cases.

Case 1. $\lambda = N$. Then, by (41), we get

$$N < 2 \times 10^{13} M \log N. \tag{43}$$

Hence,

$$\begin{aligned} N &< 2 \times 2 \times 10^{13} M \log(2 \times 10^{13} M) \\ &= 4 \times 10^{13} M(\log(2 \times 10^{13}) + \log M) \\ &< 4 \times 10^{13} M(31 + \log M) \\ &< 4 \times 32 \times 10^{13} M \log M \\ &< 1.5 \times 10^{15} M \log M. \end{aligned} \tag{44}$$

From Lemma 2.2, we get

$$\begin{aligned} x &< 10^{11} MN \log N \\ &< 10^{11}(1.5 \times 10^{15} M \log M) M \log(1.5 \times 10^{15} M \log M) \\ &< 1.5 \times 10^{26} M^2 (\log M)(35 + 2 \log M) \\ &< 1.5 \times 37 \times 10^{26} M^2 (\log M)^2 \\ &< 10^{28} M^2 (\log M)^2. \end{aligned} \tag{45}$$

Thus, if $w = x$, then

$$Mw = Mx < 10^{28} M^3 (\log M)^2, \tag{46}$$

while if $w = y$, then $z = x$ and

$$\begin{aligned} Nz = Nx &< (1.5 \times 10^{15} M \log M)(10^{28} M^2 (\log M)^2) \\ &< 1.5 \times 10^{43} M^3 (\log M)^3. \end{aligned} \tag{47}$$

Using containment (42), we deduce from estimates (46) and (47)

$$\max\{Nz, Mw\} < 5 \times 10^{43} M^3 (\log M)^3. \tag{48}$$

Since $My \le \max\{Mw, Nz\}$, we get from (48),

$$y < 5 \times 10^{43} M^2 (\log M)^3. \tag{49}$$

Case 2. $\lambda = x$. In this case, from inequality (41), we have

$$x = \lambda < 2 \times 10^{13} M \log N. \tag{50}$$

We now distinguish two subcases.

Case 2.1. $m = N$. Then $n+1 = M$. Further, if $x > y$, then by inequality (5), the fact that $y \geq 2$ and (50), we have

$$N = m \leq \frac{my}{2} < \frac{(n+3)x}{2} < (n+1)x \tag{51}$$
$$= Mx < 2 \times 10^{13} M^2 \log N,$$

while if $x < y$, then by inequality (5), the fact that $y \geq 4$ in this case and (50), we have

$$N = m < m(y-2) < (n+1)x < Mx < 2 \times 10^{13} M^2 \log N. \tag{52}$$

So, comparing (51) and (52), we conclude that in this case we always have

$$N < 2 \times 10^{13} M^2 \log N. \tag{53}$$

Case 2.2. $n+1 = N$. First, note that if $y > x$, then, by (5), we have

$$y < (m-2)y < (n+1)x = Nx,$$

while if $x > y$, then

$$y \leq \frac{my}{2} < \frac{(n+3)x}{2} < (n+1)x = Nx.$$

Hence, the inequality

$$y < Nx \tag{54}$$

holds in this case.

Further, observe that $x = z$. Thus, we also have

$$F_n^x = \frac{\alpha^{nx}}{5^{x/2}} \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^x.$$

Further, by (27), we have

$$\frac{x}{\alpha^{2n}} = \frac{x}{\alpha^{2N-2}} < \frac{\alpha^2}{\alpha^N}.$$

The argument from inequalities (30) and (31) now shows that

$$F_n^x = \frac{\alpha^{nx}}{5^{x/2}} \left(1 + \zeta_{n,x}\right), \quad \text{where} \quad |\zeta_{n,x}| < \frac{2\alpha^2}{\alpha^N}.$$

We thus get

$$F_m^y = F_{n+1}^x + F_n^x = \frac{\alpha^{nx}(\alpha^x+1)}{5^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right)\varepsilon_{n+1,x} + \left(\frac{\alpha^{nx}}{5^{x/2}}\right)\varepsilon_{n,x},$$

so

$$\left|F_m^y\alpha^{-nx}\left(\frac{5^{x/2}}{\alpha^x+1}\right)-1\right| < |\varepsilon_{n+1,x}|\left(\frac{\alpha^x}{\alpha^x+1}\right) + |\varepsilon_{n,x}|\left(\frac{1}{\alpha^x+1}\right) < \frac{4}{\alpha^N}, \tag{55}$$

where we used the facts that

$$|\varepsilon_{n+1,x}| < \frac{2}{\alpha^N}, \qquad |\varepsilon_{n,x}| < \frac{2\alpha^2}{\alpha^N}, \qquad \frac{\alpha^x}{\alpha^x+1} < 1,$$

and

$$\frac{1}{\alpha^x+1} < \frac{1}{\alpha^2},$$

and the right-most inequality above holds because $x \geq 3$.

We continue by getting a lower bound on the left-hand side of inequality (55) using again Matveev's theorem. For this, we take $K = 3$, $\alpha_1 = F_m$, $\alpha_2 = \alpha$, $\alpha_3 = (\alpha^x+1)/5^{x/2}$. We also take $b_1 = y$, $b_2 = -nx$, $b_3 = -1$. Hence,

$$\Lambda_3 = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1 = F_m^y\alpha^{-nx}\left(\frac{5^{x/2}}{\alpha^x+1}\right) - 1$$

is the expression which appears under the absolute value in the left-hand side of inequality (55). We first check that $\Lambda_3 \neq 0$. If $\Lambda_3 = 0$, then

$$\alpha^{2nx}(\alpha^x+1)^2 = F_m^{2y}5^x \in \mathbb{Z}.$$

Conjugating the above expression in $\mathbb{Q}(\sqrt{5})$, we get that

$$\alpha^{2nx}(\alpha^x+1)^2 = \beta^{2nx}(\beta^x+1)^2,$$

which is impossible because the left-hand side of it is very large (at least α^{2000}), while the right-hand side of it is smaller than 2 for $x \geq 3$. Observe next that $\alpha_1, \alpha_2, \alpha_3$ are all real and belong to the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, so we can take $D = 2$. Next, since $F_m = F_M < \alpha^M$, it follows that we can take

$$A_1 = 2M \log \alpha > D \log F_M = Dh(\alpha_1).$$

Next, since $h(\alpha_2) = (\log \alpha)/2 = 0.240606\ldots$, it follows that we can take $A_2 = 0.5 > Dh(\alpha_2)$. For α_3, its conjugate in $\mathbb{K}$ is $(-1)^x(\beta^x + 1)/5^{x/2}$, so its minimal polynomial over the integers is a divisor of

$$\begin{aligned} 5^x \left(X - \frac{\alpha^x + 1}{5^{x/2}}\right)\left(X - (-1)^x \frac{\beta^x + 1}{5^{x/2}}\right) & \\ = 5^x X^2 - 5^{x/2}(\alpha^x + (-1)^x \beta^x + 1 + (-1)^x)X & \\ + (-1)^x(\alpha^x + \beta^x + 1 + (-1)^x) \in \mathbb{Z}[X]. & \end{aligned}$$

Thus, with the notations from (11) for $\eta = \alpha_3$, we have $a_0 \le 5^x$,

$$|\alpha_3^{(1)}| = |\alpha_3| = \frac{\alpha^x + 1}{5^{x/2}} < 2\left(\frac{\alpha}{\sqrt{5}}\right)^x < 1$$

(because $x \ge 3$), and

$$|\alpha_3^{(2)}| = \frac{|\beta^x + 1|}{5^{x/2}} < \frac{2}{5^{x/2}} < 1.$$

Hence, $h(\alpha_3) = (\log a_0)/2$, so we can take

$$A_3 = 1.61x > x \log 5 = \frac{D \log 5^x}{2} \ge \frac{D \log a_0}{2} = Dh(\alpha_3).$$

Finally, inequality (50), the fact that $N > 1000$ as well as inequality (54), tell us that we can take

$$\begin{aligned} B = N^7 = N^4 \times N \times N^2 &> (10^3)^4 \\ &\times 20 \log N \times MN \\ &> (2 \times 10^{13} M \log N) \\ &\times N > Nx = \max\{Nx, y, 1\} \\ &= \max\{|b_1|, |b_2|, |b_3|\}. \end{aligned}$$

In the above, we used the fact that the inequality $N > 20 \log N$ holds for all $N > 1000$. We thus get that

$$|\Lambda_3| > \exp(-C_2(1 + \log B)A_1 A_2 A_3), \tag{56}$$

where $C_2 < 10^{12}$ (see inequality (39)). Thus,

$$\begin{aligned}
(57) \qquad & C_2(1+\log B)A_1A_2A_3 \\
& < 10^{12} \times (2\log\alpha) \times 0.5 \times 1.61 \times (1+\log N^7)Mx \\
& < 8 \times 10^{11} Mx(1+7\log N) \\
& < 7 \times 10^{12} Mx \log N.
\end{aligned}$$

Inserting (50) into (57), we get that

$$\begin{aligned}
(58) \qquad & C_2(1+\log B)A_1A_2A_3 \\
& < 7 \times 10^{12} M(2 \times 10^{13} M \log N)\log N \\
& < 1.5 \times 10^{26} M^2 (\log N)^2.
\end{aligned}$$

From inequalities (55), (56) and (58), we get that

$$\begin{aligned}
(59) \qquad N & < \frac{\log 4}{\log\alpha} + \left(\frac{1}{\log\alpha}\right) 1.5 \times 10^{26} M^2 (\log N)^2 \\
& < 4 \times 10^{26} M^2 (\log N)^2.
\end{aligned}$$

Taking the worst possibility between (53) and (59), we get that

$$N < 4 \times 10^{26} M^2 (\log N)^2.$$

We now use the fact that, if $A > 100$, then the inequality

$$\frac{t}{\log t} < A \quad \text{implies} \quad t < 4A(\log A)^2$$

(see [**3**]) with $A = 4 \times 10^{26} M^2$, to get that

$$\begin{aligned}
(60) \qquad N & < 4 \times 10^{26} M^2 (\log(4 \times 10^{26} M^2)^2 \\
& = 4 \times 10^{26} M^2 (\log(4 \times 10^{26}) + 2\log M)^2 \\
& < 4 \times 10^{26} M^2 (62 + 2\log M)^2 \\
& < 4 \times 10^{26} M^2 \times 64^2 (\log M)^2 \\
& < 2 \times 10^{30} M^2 (\log M)^2.
\end{aligned}$$

Using also inequality (50), we get

$$
\begin{aligned}
x &< 2 \times 10^{13} M \log N \\
&< 2 \times 10^{13} M \log(2 \times 10^{30} M^2 (\log M)^2) \\
&< 2 \times 10^{13} M (\log(2 \times 10^{30}) + 4 \log M) \\
&< 2 \times 10^{13} M (70 + 4 \log M) \\
&< 2 \times 74 \times 10^{13} M \log M \\
&< 1.5 \times 10^{15} M \log M.
\end{aligned} \tag{61}
$$

So, as in Case 1, we deduce that if $w = x$, then

$$
Mw = Mx < 2 \times 10^{15} M^2 \log M, \tag{62}
$$

while if $w = y$, then $z = x$ and

$$
\begin{aligned}
Nz &= Nx \\
&< (2 \times 10^{30} M^2 (\log M)^2)(1.5 \times 10^{15} M (\log M)) \\
&< 3 \times 10^{45} M^3 (\log M)^3.
\end{aligned} \tag{63}
$$

Using containment (42), we deduce from estimates (62) and (63)

$$
\max\{Nz, Mw\} < 10^{46} M^3 (\log M)^3. \tag{64}
$$

In particular, since $My \le \max\{Nz, Mw\}$, we also get from (64) that

$$
y < 10^{46} M^2 (\log M)^3. \tag{65}
$$

From (44), (45), (48), (49), (60), (61), (64) and (65), we record what we have just proved in the following way.

Lemma 2.4. *If (m, n, x, y) is a solution to equation (2) with $n \ge 3$, $x \ge 3$ and $y \ge 2$, then*

$$
\begin{aligned}
N &< 2 \times 10^{30} M^2 (\log M)^2, \\
x &< 10^{28} M^2 (\log M)^2, \\
y &< 10^{46} M^2 (\log M)^3, \\
\max\{Mw, Nz\} &< 10^{46} M^3 (\log M)^3.
\end{aligned}
$$

2.5. The case when $M \leq 1000$. Here, we go back to the inequality (37), which is

$$|F_M^w \alpha^{-Nz} 5^{z/2} - 1| < \frac{10}{1.5^\lambda}. \tag{66}$$

Since $M \leq 1000$, we have, by Lemma 2.4, that

$$\max\{w, Nz, z\} < (10^{46})(10^3)^3(\log(10^3))^3 < 10^{56}.$$

Assume that $\lambda \geq 8$. Then the right-hand side of (66) is at most $1/2$, so by a classical argument it follows that

$$|w \log F_M - Nz \log \alpha + z \log \sqrt{5}| < \frac{20}{1.5^\lambda}. \tag{67}$$

However, the minimum of the expression appearing on the left-hand side of inequality (67) over all possible indices $M < 3000$ and integer exponents w, Nz, z of maximal absolute values at most 5×10^{65} (hence, which includes our current range) was bounded from below using LLL in [**3,** Section 6]. The lower bound there is $100/1.5^{750}$. This immediately implies that $\lambda < 750$.

If $x > N$, we then get $N = \lambda < 750$, a contradiction with the results obtained at subsection 2.3.

Thus, $x < N$ and $x = \lambda < 750$. Hence, we are in Case 2 of the analysis from subsection 2.4. We treat the two cases from subsection 2.4.

Case 2.1. $m = N$. In this case, by inequality (5), we have

$$(M-1)x = nx > (m-2)y = (N-2)y \geq (M-1)y,$$

where the last inequality holds because otherwise $m = N = M = n+1$, which is false since F_n and F_{n+1} are coprime. Hence, $y \leq x$; therefore, $y < x$. Let

$$\lambda_1 = \min\{M, x\}. \tag{68}$$

Assume that $\lambda_1 > 20$. Then $\lambda > 20$, and inequality (67) becomes

$$|x \log F_{n+1} - my \log \alpha + y \log \sqrt{5}| < \frac{20}{1.5^\lambda}. \tag{69}$$

With the Binet formula (3) and the fact that $n+1 = M > 20$, we have

$$\begin{aligned} \log F_{n+1} &= \log\left(\frac{\alpha^{n+1}}{5^{1/2}}\right) + \log\left(1 - \frac{(-1)^{n+1}}{\alpha^{2n+2}}\right) \\ &= (n+1)\log\alpha - \log\sqrt{5} + \zeta_n, \end{aligned} \tag{70}$$

where

$$|\zeta_n| < \frac{2}{\alpha^{2n+2}}. \tag{71}$$

Inserting formula (70) with the bound (71) into (69), we get

$$|(x(n+1) - my)\log\alpha - (x-y)\log\sqrt{5}| < \frac{20}{1.5^\lambda} + \frac{2x}{\alpha^{2M}}. \tag{72}$$

Since $\alpha^M \ge \alpha^{\lambda_1} > \alpha^{20} > 1500 > 2x$, it follows that

$$\frac{2x}{\alpha^{2M}} < \frac{1}{\alpha^M} < \frac{1}{(1.5)^M} \le \frac{1}{(1.5)^{\lambda_1}}.$$

Thus, estimate (72) implies

$$\left|\frac{x(n+1) - my}{x-y} - \frac{\log\sqrt{5}}{\log\alpha}\right| < \left(\frac{21}{\log\alpha}\right)\frac{1}{(1.5)^{\lambda_1}}. \tag{73}$$

The first few convergents $(p_k/q_k)_{k\ge 0}$ of $\log\sqrt{5}/\log\alpha$ are

$$1,\ 2,\ \frac{5}{3},\ \frac{97}{58},\ \frac{199}{119},\ \frac{1888}{1129},\ \frac{2087}{1248},\ldots.$$

Since $0 < x - y < 750 < 1129$, it follows that a lower bound on the expression appearing in the left-hand side of (73) is

$$\left|\frac{\log\sqrt{5}}{\log\alpha} - \frac{p_5}{q_5}\right| > \frac{4}{10^7},$$

which together with (73) gives $\lambda_1 \le 45$. So, $y < x \le 45$ if $\lambda_1 = x$, whereas if $\lambda_1 = n+1$, then

$$y < \frac{xn}{N-2} < \frac{750 \times 45}{999} < 45.$$

Thus, $y \in [2, 44]$. We covered the rest by brute force. That is, we checked whether for some triple (n, x, y) with $n \in [3, 1000]$, $x \in [3, 750]$ satisfying $\min\{n+1, x\} \le 45$ and for $2 \le y < \min\{x, 45\}$, the number

$F_n^x + F_{n+1}^x$ is the yth power of some integer. This computation took several hours, and no new solutions were found.

Case 2.2. $n + 1 = N$. In this case, $m \le 1000$. We next use some elementary arguments to restrict the ranges of the variables x and m. We first note that x is even, for if x is odd, then

$$F_{N+1} = F_{n+2} = F_n + F_{n+1} \mid F_n^x + F_{n+1}^x = F_m^y.$$

However, by the primitive divisor theorem, we know that F_{N+1} has a primitive prime factor p, which is a prime factor that does not divide F_k for any $1 \le k \le N$. In particular, the primitive prime factor p of F_{N+1} cannot divide F_m. By the same argument, we get that in fact $4 \mid x$, since if $2 \| x$, then

$$F_{2N-1} = F_{2n+1} = F_n^2 + F_{n+1}^2 \mid (F_n^2)^{x/2} + (F_{n+1}^2)^{x/2} = F_m^y,$$

and the contradiction is again obtained by invoking the fact that F_{2N-1} possesses a primitive prime factor which cannot divide F_m. Thus, $4 \mid x$. If both F_n and F_{n+1} are odd, the left-hand side of equation (2) is congruent to 2 modulo 4, implying that $y = 1$, which is false. Thus, one of F_n and F_{n+1} is even and the other is odd, so the left-hand side of equation (2) is congruent to 1 modulo 16. Since $4 \mid x$, we conclude that y is odd, for if not, then with $X = F_n^{x/4}$, $Y = F_{n+1}^{x/4}$ and $Z = F_m^{y/2}$, we would get a solution to the equation $X^4 + Y^4 = Z^2$ in positive integers X, Y, Z, which we know does not exist. Since the left-hand side of the expression

$$F_n^x + F_{n+1}^x - 1 = F_m^y - 1 = (F_m - 1)\left(\frac{F_m^y - 1}{F_m - 1}\right),$$

is a multiple of 16 and the second factor on the right-hand side above is odd (because y is odd), we get that $F_m \equiv 1 \pmod{16}$. There are 123 values for $m \le 1000$ such that $F_m \equiv 1 \pmod{16}$. Further, observe that, since $4 \mid x$, it follows that every prime factor p of F_m must be congruent to 1 modulo 8 (this is because the multiplicative order of F_{n+1}/F_n modulo each such prime p is a multiple of 8). The factorizations of all Fibonacci numbers F_m with $m \le 1000$ are known. Testing by hand each of the 123 candidates above against this last condition leaves only

21 candidates, namely,

$$
m \in \{23, 26, 47, 71, 121, 122, 167, 191, 193, 337, 359, 383, 431, 433, 601, 647, 649, 794, 866, 911, 913\}\,. \tag{74}
$$

By a standard argument, inequality (55) together with the fact that N is very large implies that

$$
\left| y \log F_m - nx \log \alpha + \log(5^{x/2}/(\alpha^x + 1)) \right| < \frac{8}{\alpha^N}. \tag{75}
$$

Assume, for example, that the expression under the absolute value in above is positive. We then get that

$$
\begin{aligned}
0 < y\left(\frac{\log F_m}{\log \alpha}\right) - nx + \left(\frac{\log(5^{x/2}/(\alpha^x + 1)}{\log \alpha}\right) \\
< \frac{8}{(\log \alpha)\alpha^N} < \frac{20}{\alpha^N}.
\end{aligned} \tag{76}
$$

We now apply the Baker-Davenport reduction as presented in **[4]**. Namely, let m be in the list (74), and let $x < 750$ be multiple of 4. Put

$$
\gamma = \frac{\log F_m}{\log \alpha}, \qquad \mu = \frac{\log(5^{x/2}/(\alpha^x + 1))}{\log \alpha}, \quad A = 20, \ B = \alpha.
$$

Then inequality (76) is

$$
0 < u\gamma - v + \mu < \frac{A}{B^N} \tag{77}
$$

in positive integers u and v. For the purposes here, $(u, v) = (y, nx)$. We first need a bound T on y. Since

$$
y < \frac{Nx}{m-2} < \frac{750N}{20} < 4N < 4(2 \times 10^{30} \times (10^3)^2(\log(10^3))^2 < 10^{42}
$$

(where we used Lemma 2.4 for the bound on N), it follows that we can take $T = 10^{43}$. Now we need to take the denominator q of a convergent to γ such that $q > 10T$ and put $\varepsilon = \|\mu q\| - 10T\|\gamma q\|$ in such a way that $\varepsilon > 0$. It turns out that by choosing q to be the denominator of the 250th convergent of γ leads to the conclusion that $\varepsilon > 3 \times 10^{-36}$ for all choices of m and x. Further, the maximal denominator of such a convergent satisfies $q < 2 \times 10^{133}$, while the minimal one satisfies

$q > 7 \times 10^{116}$. Then the theory says that

$$N < \frac{\log(Aq/\varepsilon)}{\log B} < \frac{\log\left(20 \times 2 \times 10^{133} \times 10^{36}/3\right)}{\log \alpha} < 815,$$

a contradiction to the fact that $N > 1000$. A similar contradiction is obtained if one assumes that the expression appearing under the absolute value in (75) is negative, namely, we just change

$$(\gamma,\ \mu,\ A) \quad \text{to} \quad \left(\frac{1}{\gamma},\ \frac{\log(5^{x/2}/(\alpha^x+1))}{\log F_m},\ \frac{8}{\log F_m}\right).$$

respectively. We give no further details. This completes the analysis when $M \leq 1000$. We record what we have proved as follows.

Lemma 2.5. *If (m,n,x,y) is a solution of equation (2) with $x \geq 3$, $n \geq 3$ and $y \geq 2$, then $N > M > 1000$.*

2.6. An absolute bound on all the variables m,n,x,y. Since $N > M > 1000$, it follows, from Lemma 2.4, that

$$\max\{x,y\} < 10^{46} M^2 (\log M)^3 < \alpha^{M-2} \leq \min\{\alpha^{n-1}, \alpha^m\}.$$

The middle inequality above holds for all $M \geq 256$. Hence, all three inequalities

$$\frac{x}{\alpha^{2n}} \leq \frac{1}{\alpha^{n+1}}, \qquad \frac{x}{\alpha^{2n+2}} \leq \frac{1}{\alpha^{n+1}}, \qquad \frac{y}{\alpha^{2m}} \leq \frac{1}{\alpha^m}$$

hold; therefore, as in subsection 2.4, we may write

$$\begin{aligned} F_n^x &= \frac{\alpha^{nx}}{5^{x/2}}\left(1+\zeta_{n,x}\right), \\ F_{n+1} &= \frac{\alpha^{(n+1)x}}{5^{x/2}}\left(1+\zeta_{n+1,x}\right), \\ F_m^y &= \frac{\alpha^{my}}{5^{y/2}}\left(1+\zeta_{m,y}\right), \end{aligned} \tag{78}$$

where

$$\max\{|\zeta_{n,x}|, |\zeta_{n+1,x}|\} \leq \frac{2}{\alpha^{n+1}}, \qquad |\zeta_{m,y}| \leq \frac{2}{\alpha^m}. \tag{79}$$

We also have the analog of containment (33), namely,

$$\frac{F_n^x}{\alpha^{nx}/5^{x/2}},\ \frac{F_{n+1}^x}{\alpha^{(n+1)x}/5^{x/2}},\ \frac{F_m^y}{\alpha^{my}/5^{y/2}} \in \left(\frac{1}{2}, 2\right). \tag{80}$$

Inserting approximations (78) into equation (2) and shuffling some terms, we get

$$\frac{\alpha^{my}}{5^{y/2}} - \frac{\alpha^{(n+1)x}}{5^{x/2}} - \frac{\alpha^{nx}}{5^{x/2}} = \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right)\zeta_{n+1,x} + \left(\frac{\alpha^{nx}}{5^{x/2}}\right)\zeta_{n,x} - \left(\frac{\alpha^{my}}{5^{y/2}}\right)\zeta_{m,y},$$

which, together with (80), implies the following inequalities:

$$\begin{aligned}
&\left|\alpha^{my-(n+1)x}5^{(x-y)/2} - 1\right| \\
&< \frac{1}{\alpha^x} + |\zeta_{n+1,x}| + \left(\frac{1}{\alpha^x}\right)|\zeta_{n,x}| + \left(\frac{\alpha^{my}/5^{y/2}}{\alpha^{(n+1)x}/5^{x/2}}\right)|\zeta_{m,y}| \\
&< \frac{1}{\alpha^x} + \frac{3}{\alpha^{n+1}} + \left(\frac{\alpha^{my}/5^{y/2}}{F_m^y}\right)\left(\frac{F_m^y}{F_{n+1}^x}\right)\left(\frac{F_{n+1}^x}{\alpha^{(n+1)x}/5^{x/2}}\right)|\zeta_{m,y}| \\
&< \frac{1}{\alpha^x} + \frac{3}{\alpha^{n+1}} + \frac{16}{\alpha^{my}} < \frac{20}{\alpha^{\lambda_1}},
\end{aligned} \tag{81}$$

where $\lambda_1 = \min\{x, M\}$ has the same meaning as in (68), and

$$\begin{aligned}
&\left|\alpha^{my-nx}5^{(x-y)/2}(\alpha^x+1)^{-1} - 1\right| \\
&< \left(\frac{\alpha^x}{\alpha^x+1}\right)|\zeta_{n+1,x}| + \frac{|\zeta_{n,x}|}{\alpha^x+1} + \left(\frac{\alpha^x}{\alpha^x+1}\right)\left(\frac{\alpha^{my}/5^{y/2}}{\alpha^{(n+1)x}/5^{x/2}}\right)|\zeta_{m,y}| \\
&< \frac{3}{\alpha^{n+1}} + \frac{16}{\alpha^m} < \frac{20}{\alpha^M}.
\end{aligned} \tag{82}$$

We apply Matveev's theorem to the left-hand side of inequality (81) with $K = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \sqrt{5}$, $b_1 = my - (n+1)x$, $b_2 = x - y$ and $D = 2$. Thus,

$$\Lambda_4 = \alpha_1^{b_1}\alpha_2^{b_2} - 1 = \alpha^{my-(n+1)x}5^{(x-1)/2} - 1.$$

Observe that $\Lambda_4 \neq 0$ since otherwise we would get that $\alpha^{2my-2(n+1)x} = 5^{y-x} \in \mathbb{Z}$, and this is possible only if $my = (n+1)x$ and $y = x$, but this last equality is not allowed. We take as in prior applications of this theorem $A_1 = 0.5 > \log\alpha = Dh(\alpha_1)$ and $A_2 = 1.61 > \log 5 =$

$2Dh(\alpha_2)$. Further, since $x \geq 3$, it follows that the right-hand side in (81) is at most $20/\alpha^3 < 5$; therefore,

$$\frac{\alpha^{|my-(n+1)x|}}{5^{|y-x|/2}} < 6,$$

so

$$(83) \qquad \begin{aligned} |b_1| = |my-(n+1)x| &< \frac{\log\left(6 \times 5^{|y-x|/2}\right)}{\log \alpha} \\ &= \left(\frac{\log 5}{2\log\alpha}\right)|y-x| + \frac{\log 6}{\log\alpha} < 2|y-x| + 18 \\ &< 20\max\{x,y\}. \end{aligned}$$

Thus, using Lemmas 2.4 and 2.5, we can take

$$(84) \qquad \begin{aligned} B &= M^{20} \\ &= M^{17} \times M^2 \times M > (10^3)^{17} \times M^2 \times (\log M)^3 \\ &> 20 \times 10^{46} M^2 (\log M)^3 \\ &> 20\max\{x,y\} > \max\{|b_1|,|b_2|\}. \end{aligned}$$

Matveev's theorem tells us that

$$(85) \qquad |\Lambda_4| > \exp(-C_1(1+\log B)A_1A_2),$$

where $C_1 < 8 \times 10^8$ is given by (14). Thus,

$$(86) \qquad \begin{aligned} C_1(1+\log B)A_1A_2 &< 8 \times 10^8 \times 0.5 \times 1.61(1+\log(M^{20})) \\ &< 8 \times 0.5 \times 1.61 \times 10^8 \times 21 \log M \\ &< 2 \times 10^{10} \log M. \end{aligned}$$

Comparing estimates (81), (85) and (86), we get that

$$(87) \qquad \lambda_1 < \frac{\log 20}{\log\alpha} + \left(\frac{2 \times 10^{10}}{\log \alpha}\right)\log M < 5 \times 10^{10} \log M.$$

We now distinguish two cases.

Case 1. $\lambda_1 = M$. In this case, from (86), we get

$$M < 5 \times 10^{10} \log M;$$

therefore,

$$(88) \qquad M < 2 \times 5 \times 10^{10} \log(5 \times 10^{10}) < 3 \times 10^{12}.$$

Case 2. $\lambda_1 = x$. In this case, from (86), we get

$$x < 5 \times 10^{10} \log M. \tag{89}$$

We apply Matveev's theorem to the left-hand side of the inequality (82) with $K = 3$, $\alpha_1 = \alpha$, $\alpha_2 = \sqrt{5}$, $\alpha_3 = \alpha^x + 1$, $b_1 = my - nx$, $b_2 = x - y$, $b_3 = -1$ and $D = 2$. Thus,

$$\Lambda_5 = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1 = \alpha^{my-nx}5^{(x-y)/2}(\alpha^x + 1)^{-1} - 1.$$

Let us check that $\Lambda_5 \neq 0$. If $\Lambda_5 = 0$, we then get that

$$\alpha^x + 1 = 5^{(x-y)/2}\alpha^{mx-ny}. \tag{90}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^x + 1 = 5^{(x-y)/2}\beta^{mx-ny}. \tag{91}$$

Multiplying relations (90) and (91), we get

$$\begin{aligned} \alpha^x + \beta^x + (-1)^x + 1 &= (\alpha^x + 1)(\beta^x + 1) \\ &= (\alpha\beta)^{my-nx}5^{x-y} \\ &= (-1)^{my-nx}5^{x-y}. \end{aligned} \tag{92}$$

Since the left-hand side of equation (92) above is larger than 1 for $x \geq 3$, it follows that $my - nx$ is even and $x > y$. If x is odd, the above relation implies that $L_x = 5^{x-y}$, where $(L_k)_{k\geq 0}$ is the Lucas companion of the Fibonacci sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{k+2} = L_{k+1} + L_k$ for all $k \geq 0$. However, it is easy to check (by invoking the identity $L_k^2 - 5F_k^2 = 4(-1)^k$, for example), that $5 \nmid L_k$ for any positive integer k. Thus, x is even and the equation becomes

$$\alpha^x + \beta^x + 2 = 5^{x-y}.$$

If $4 \mid x$, the above equation gives $L_{x/2}^2 = 5^{x-y}$, which is again impossible because L_k is never a multiple of 5. Finally, when $2\|x$, we get $5F_{x/2}^2 = 5^{x-y}$; therefore, $F_{x/2} = 5^{(x-y-1)/2}$. It is well-known that the only Fibonacci number larger than 1 which is a power of 5 is $F_5 = 5$. Thus, $x = 10$ and $x - y - 1 = 2$; therefore, $y = 7$. Thus, equation (2) becomes

$$F_n^{10} + F_{n+1}^{10} = F_m^7.$$

Hence, $F_{2n+1} = F_n^2 + F_{n+1}^2 \mid F_n^{10} + F_{n+1}^{10} = F_m^7$, and, by the primitive divisor theorem, we conclude that $2n + 1 \mid m$. However, this is impossible since one can easily check that

$$F_{2n+1}^7 > F_n^{10} + F_{n+1}^{10}$$

holds for all $n \geq 1$. Indeed, one checks that the above inequality holds for $n = 1$, whereas for $n \geq 2$, we have

$$F_n^5 + F_{n+1}^5 < (F_n + F_{n+1})^5 = F_{n+2}^5 < \alpha^{5n+5} < \alpha^{14n-7} < F_{2n+1}^7.$$

Thus, indeed $\Lambda_5 \neq 0$.

We take, as in prior applications of Matveev's theorem, $A_1 = 0.5 > \log \alpha = Dh(\alpha_1)$ and $A_2 = 1.61 > \log 5 = 2Dh(\alpha_2)$. As for $\alpha_3 = \alpha^x + 1$, this is an algebraic integer whose conjugate is $\beta^x + 1$ whose absolute value is smaller than 2. Thus,

$$\begin{aligned} Dh(\alpha_3) &\leq \log(\alpha^x + 1) + \log 2 < \log(2\alpha^x) + \log 2 \\ &= x \log \alpha + 2 \log 2 \leq x\left(\log \alpha + \frac{2 \log 2}{3}\right) \\ &< x, \end{aligned}$$

so we can take $A_3 = x$. Finally, observe that by the calculation (83), we have

$$\begin{aligned} |b_1| = |my - nx| &\leq |my - (n+1)x| + x < 20|y - x| + 18 + x \\ &< 21 \max\{x, y\}. \end{aligned}$$

Hence, using Lemmas 2.4 and 2.5, we conclude, as at estimate (84), that we can take

$$\begin{aligned} B = M^{20} &= M^{17} \times M^2 \times M \\ &> (10^3)^{17} \times M^2 \times (\log M)^3 \\ &> 21 \times 10^{46} M^2 (\log M)^3 \\ &> 21 \max\{x, y\} > \max\{|b_1|, |b_2|\}. \end{aligned} \tag{93}$$

Matveev's theorem now implies that

$$|\Lambda_5| > \exp(-C_2(1 + \log B) A_1 A_2 A_3), \tag{94}$$

where $C_2 < 10^{12}$ is given by (39). Note that

$$C_2(1+\log B)A_1A_2A_3 < 10^{12} \times 0.5 \times 1.61 \times x(1+\log N^{20}) \tag{95}$$
$$< 10^{12} \times 0.5 \times 1.61 \times 21x \log M$$
$$< 2 \times 10^{13} x \log M.$$

From estimates (82), (94) and (95), we get that

$$M < \left(\frac{\log 20}{\log \alpha}\right) + \left(\frac{2 \times 10^{13}}{\log \alpha}\right) x \log M < 5 \times 10^{13} x \log M.$$

Thus,

$$M < 5 \times 10^{13} x \log M. \tag{96}$$

Inserting estimate (89) into (96), we get

$$M < 5 \times 10^{13}(5 \times 10^{10} \log M) \log M < 3 \times 10^{24} (\log M)^2.$$

Thus,

$$M < 4 \times 3 \times 10^{24} (\log(3 \times 10^{24}))^2 < 4 \times 10^{28}. \tag{97}$$

Comparing the bounds (88) with (97) on M obtained in the two cases, we conclude that the inequality (97) always holds. Inserting the above bound for M into the inequalities of Lemma 2.4, we get

$$\begin{aligned} N &< 2 \times 10^{30} (4 \times 10^{28})^2 (\log(4 \times 10^{28}))^2 < 10^{92}, \\ x &< 10^{28} M^2 (\log M)^2 \\ &< 10^{28} (4 \times 10^{28})^2 (\log(4 \times 10^{28}))^2 < 10^{89}, \\ y &< 10^{46} M^2 (\log M)^2 \\ &< 10^{46} (4 \times 10^{28})^2 (\log(4 \times 10^{28}))^2 < 10^{107}. \end{aligned}$$

We record the following conclusions.

Lemma 2.6. *If (m, n, x, y) is a solution of equation (2) with $n \geq 3$, $x \geq 3$ and $y \geq 2$, then*

$$\max\{x, y\} < 10^{107}.$$

2.7. Reducing the bound. We work some more on inequality (81). Assume that $\lambda_1 > 600$. Then $20/\alpha^{\lambda_1} < 1/2$, so by a classical argument we get that

$$\left|(my-(n+1)x)\log\alpha-(y-x)\log\sqrt{5}\right|<\frac{40}{\alpha^{\lambda_1}}.$$

Thus,

$$\left|\frac{(my-(n+1)x}{x-y}-\frac{\log\sqrt{5}}{\log\alpha}\right|<\frac{40}{(\log\alpha)|x-y|\alpha^{\lambda_1}} \tag{98}$$
$$<\frac{100}{|x-y|\alpha^{\lambda_1}}.$$

Since $\lambda_1>600$, we have, by Lemma 2.6, that

$$\alpha^{\lambda_1}>\alpha^{600}>10^{125}>200\max\{x,y\}>200|x-y|,$$

showing that the expression appearing on the right-hand side of (98) is smaller than $1/(2|x-y|^2)$, so by Legendre's result, $(my-(n+1)x)/(x-y)$ equals some convergent p_k/q_k of $\gamma=\log\sqrt{5}/\log\alpha$ for some nonnegative integer k. If $k<100$, then

$$\frac{1}{10^{100}}<\left|\gamma-\frac{p_{99}}{q_{99}}\right|\le\left|\gamma-\frac{(my-(n+1)x}{x-y}\right|<\frac{100}{\alpha^{\lambda_1}};$$

therefore,

$$\lambda_1<\frac{\log(10^{102})}{\log\alpha}<489,$$

which is false since we are assuming that $\lambda_1>600$. Thus, $k\ge 100$, and since the 214th convergent p_{214}/q_{214} of γ has $q>10^{110}>|x-y|$, we conclude that $k\in[100,213]$. Since

$$\left|\gamma-\frac{p_{214}}{q_{214}}\right|>\frac{1}{10^{222}},$$

we get that

$$\frac{1}{10^{222}}<\left|\gamma-\frac{p_k}{q_k}\right|<\frac{100}{|x-y|\alpha^{\lambda_1}}\le\frac{100}{q_{100}\alpha^{\lambda_1}}\le\frac{1}{10^{46}\alpha^{\lambda_1}},$$

where we used the fact that $|x-y|\ge q_{100}>10^{48}$, giving

$$\lambda_1<\frac{\log(10^{176})}{\log\alpha}<843.$$

Hence, $\lambda_1<843$. If $M\le x$, we then have $M=\lambda_1<843$, a contradiction. Thus, $x=\lambda_1$; therefore, $x<843$. We now get a better bound for M. That is, using estimate (96) and comparing it also with estimate (88) according to the two cases distinguished in subsection 2.6, we conclude that

$$M < 5 \times 10^{13} x \log M < 5 \times 843 \times 10^{13} \log M < 5 \times 10^{16} \log M,$$

giving

$$M < 2 \times 5 \times 10^{16} \log(5 \times 10^{16}) < 4 \times 10^{18},$$

which, via Lemma 2.4, yields

$$\begin{aligned} x &< 10^{28}(4 \times 10^{18})^2 \log(4 \times 10^{18}) < 2 \times 10^{48}, \\ y &< 10^{46}(4 \times 10^{18})^2 \log(4 \times 10^{18}) < 2 \times 10^{66}. \end{aligned} \tag{99}$$

Now the convergent p_{134}/q_{134} of γ has $q_{134} > 4 \times 10^{66} > |x - y|$ and

$$\left|\gamma - \frac{p_{134}}{q_{134}}\right| > \frac{1}{10^{134}};$$

therefore, by an argument previously used, we have

$$\lambda_1 < \frac{\log(10^{134})}{\log \alpha} < 642.$$

Thus, $x \in [3, 641]$. We now move on to inequality (82). Since $M > 1000$, we get that

$$|(x - y) \log \sqrt{5} - (nx - my) \log \alpha - \log(\alpha^x + 1)| < \frac{80}{\alpha^M}. \tag{100}$$

Here, we fix x and note that we are in a suitable position to apply the Baker-Davenport reduction method as we did in Case 2.2 of subsection 2.5. Suppose that the expression appearing inside that logarithm at (100) is positive. We then have

$$0 < u\gamma - v + \mu < \frac{A}{B^M},$$

where we take

$$\gamma = \frac{\log \sqrt{5}}{\log \alpha}, \qquad \mu = -\frac{\log(\alpha^x + 1)}{\log \alpha}, \quad A = \frac{80}{\log \alpha}, \ B = \alpha,$$

and $(u, v) = (|x - y|, |nx - my|)$. By estimates (99), we can take $T = 10^{67}$ as a bound on u. We choose the denominator q_{250} of the 250th convergent for γ. We have $q \in [10^{131}, 10^{132}]$. We compute

$\varepsilon = \|\mu q\| - 10T\|\gamma q\|$ for all possible choices of x. The minimum value satisfies

$$M < \frac{\log(Aq/\varepsilon)}{\log B} < \frac{\log(170 \times 10^{132} \times 10^{35})}{\log B} < 810,$$

a contradiction.

A similar contradiction is obtained in the case when the expression under the absolute value in (100) is negative.

The theorem is therefore proved. □

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Department of Mathematics, College of Science and Technology, Nihon University, Suruga-Dai Kanda, Chiyoda, Tokyo 101-8308, Japan
Email address: **hirata@math.cst.nihon-u.ac.jp**

School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
Email address: **florian.luca@wits.ac.za**