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FULL-DERIVABLE POINTS OF $\mathcal{J}$-SUBSPACE LATTICE ALGEBRAS

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ABSTRACT. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a complex Banach space X and $\operatorname{Alg}\mathcal{L}$ the associated $\mathcal{J}$-subspace lattice algebra. We say that an operator $Z \in \operatorname{Alg}\mathcal{L}$ is a full-derivable point of $\operatorname{Alg}\mathcal{L}$ if every linear map δ from $\operatorname{Alg}\mathcal{L}$ into itself derivable at Z (i.e., $\delta(A)B + A\delta(B) = \delta(Z)$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB = Z$) is a derivation and is a full-generalized-derivable point of $\operatorname{Alg}\mathcal{L}$ if every linear map δ from $\operatorname{Alg}\mathcal{L}$ into itself generalized derivable at Z (i.e., $\delta(A)B + A\delta(B) - A\delta(I)B = \delta(Z)$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB = Z$) is a generalized derivation. In this paper, we prove that if $Z \in \operatorname{Alg}\mathcal{L}$ is an injective operator or an operator with dense range, then Z is a full-derivable point as well as a full-generalized-derivable point of $\operatorname{Alg}\mathcal{L}$.

1. Introduction. Let $\mathcal{A}$ be an algebra with unit I and $\delta : \mathcal{A} \to \mathcal{A}$ a linear map. Recall that a linear map δ from $\mathcal{A}$ into itself is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$ and is called a generalized derivation if $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for all $A, B \in \mathcal{A}$. As is well known, derivations and generalized derivations are very important linear maps both in theory and applications, and are studied intensively. The question under what conditions a linear (even additive) map becomes a derivation or a generalized derivation attracts much attention from mathematicians and has been studied (for instance, see **[1]**). We say that δ is a map derivable at Z if $\delta(A)B + A\delta(B) = \delta(Z)$ for any $A, B \in \mathcal{A}$ with $AB = Z$ and δ is a map generalized derivable at Z if $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for any $A, B \in \mathcal{A}$ with $AB = Z$. In addition, we say that an element

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$Z \in \mathcal{A}$ is a full-derivable point (a full-generalized-derivable point) of $\mathcal{A}$ if every linear map from $\mathcal{A}$ into itself which is derivable at Z (generalized derivable at Z) is in fact a derivation (a generalized derivation).

It is natural and interesting to discuss what kind of elements are full-derivable points (full-generalized-derivable points) of an algebra, especially, of an operator algebra, and thus to be able to get some new characterizations and criteria of derivations (generalized derivations).

In this direction, some work has been done.

Jing et al. in **[4]** showed that, for the case of nest algebras on a Hilbert space, every linear map derivable at zero point with $\delta(I) = 0$ is an inner derivation. Zhu and Xiong showed in **[13]** that every norm continuous linear map generalized derivable at zero point between finite nest algebras on Hilbert spaces is a generalized inner derivation (i.e., has the form $A \mapsto TA + AS$) and it was shown in **[14]** that every norm continuous linear map on a finite CSL algebra that generalized derivable at a zero point is a generalized derivation. Recently, the above results for the nest algebra case have been improved. Let $\mathrm{Alg}\,\mathcal{N}$ and $\mathrm{Alg}\,\mathcal{M}$ be a nest algebra of a real or complex Banach space X with $N \in \mathcal{N}$ complemented whenever $N_- = N$ (this assumption is superfluous when X is a Hilbert space). Hou and Jiao in **[2]** proved that, if $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{M}$ is an additive map derivable at zero point, then $\delta(A) = \tau(A) + cA$ for some (linear) derivation τ and some scalar c; if δ is an additive map generalized derivable at zero point, then δ is a generalized derivation. Thus, zero is not a full-derivable point of nest algebras but a full-generalized-derivable point of nest algebras.

For the nonzero elements case, let $\mathcal{N}$ be a nest of a complex separable Hilbert space H. Zhu and Xiong proved in **[15]** that the unit operator I and, further, Zhu proved in **[11]** that every invertible operator in nest algebra $\mathrm{Alg}\,\mathcal{N}$ is a full-derivable point (there "all-derivable point" is used) of $\mathrm{Alg}\,\mathcal{N}$ *for the strongly operator topology*, that is, if $Z \in \mathrm{Alg}\,\mathcal{N}$ is invertible, then every *strongly continuous* linear map that is derivable at Z is a derivation. Zhu also proved in **[16]** that if $\mathcal{N}$ is a continuous nest, then every orthogonal projection $P \neq 0$ with $P(H) \in \mathcal{N}$ is a full-derivable point of $\mathrm{Alg}\,\mathcal{N}$ *for the strongly operator topology.* Note that Hilbert space and some kind of continuity of maps are assumed in all the above results.

The class of JSL algebras is another important kind of lattice algebra, and it is interesting to ask the question what kind of elements are full-derivable points or full-generalized-derivable points of JSL algebras. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over real or complex field $\mathbb{F}$ with $\dim X > 2$ and $\operatorname{Alg}\mathcal{L}$ the associated $\mathcal{J}$-subspace lattice algebra. Hou and Qi in [**3**] proved, with no continuity assumption on the maps, that every additive map derivable at a zero point on $\operatorname{Alg}\mathcal{L}$ is of the form $\delta(A) = \tau(A) + cA$ for all A, where τ is an additive derivation and c is a scalar and every additive map generalized derivable at zero point on $\operatorname{Alg}\mathcal{L}$ is a generalized derivation. Tt is also shown that, if X is complex, then every linear map on JSL algebra $\operatorname{Alg}\mathcal{L}$ derivable at unit operator I is a derivation. Hence, zero is a full-generalized-derivable point of JSL algebras on Banach spaces, and, as the first known example of full derivable points, the unit I is a full-derivable point of JSL algebras on complex Banach spaces.

The main purpose of the present paper is to find more full-derivable points and full-generalized-derivable points of JSL algebras. In fact, we prove that injective operators and operators with dense range are full-derivable points as well as full-generalized-derivable points of JSL algebras (Theorem 2.1). Note that, by [**10**], derivations on JSL algebras are quasi-spacial.

Let X be a Banach space over the real or complex field $\mathbb{F}$. A family $\mathcal{L}$ of subspaces of X is a subspace lattice of X which contains $\{0\}$ and X and is closed under the operations closed linear span $\vee$ and intersection $\wedge$ in the sense that $\vee_{\gamma\in\Gamma}L_\gamma \in \mathcal{L}$ and $\wedge_{\gamma\in\Gamma}L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma : \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ of X, the associated subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ is the set of operators in $\mathcal{B}(X)$ leaving every subspace in $\mathcal{L}$ invariant. Given a subspace lattice $\mathcal{L}$ of X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq \{0\} \quad \text{and} \quad K_- \neq X\},$$

where $K_- = \vee\{L \in \mathcal{L} : K \nsubseteq L\}$. Call $\mathcal{L}$ a $\mathcal{J}$-subspace lattice (simply, JSL) on X if it satisfies the following conditions:

(1) $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$;
(2) $\wedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = \{0\}$;
(3) $K \vee K_- = X$, for all $K \in \mathcal{J}(\mathcal{L})$;
(4) $K \wedge K_- = \{0\}$, for all $K \in \mathcal{J}(\mathcal{L})$.

If $\mathcal{L}$ is a JSL, the associated subspace lattice algebra $\mathrm{Alg}\,\mathcal{L}$ is called a $\mathcal{J}$-subspace lattice algebra, briefly, JSL algebra. It should be mentioned that both atomic Boolean subspace lattices and pentagon subspace lattices are members of the class of $\mathcal{J}$-subspace lattices [**9**]. We will denote by $\mathcal{F}(\mathcal{L})$ the ideal of all finite rank operators in $\mathrm{Alg}\,\mathcal{L}$. For $L \in \mathcal{L}$, denote $L_-^{\perp} = (L_-)^{\perp}$, where $L^{\perp}$ denotes the annihilator of L. Denote by $\langle \mathcal{J}(\mathcal{L}) \rangle$ and $\langle \mathcal{J}(\mathcal{L})_-^{\perp} \rangle$ the (not necessarily closed) linear span of $\cup\{K : K \in \mathcal{J}(\mathcal{L})\}$ and the linear span of $\cup\{K_-^{\perp} : K \in \mathcal{J}(\mathcal{L})\}$, respectively. For $x \in X$ and $f \in X^*$, $x \otimes f$ stands for the operator on X with rank not greater than one defined by $(x \otimes f)y = f(y)x$. Sometimes we use $\langle x, f \rangle$ to present the value $f(x)$ of f at x. We refer readers to [**7, 8, 9**] for more properties of JSL algebras.

2. Results and proofs. In this section, we state the results promised in the introduction and give their proofs. Let $\mathrm{Alg}\,\mathcal{L}$ be a JSL algebra. We show that every linear map on $\mathrm{Alg}\,\mathcal{L}$ is derivable (generalized derivable) at an injective operator or an operator with dense range is a derivation (generalized derivation). Hence, injective operators as well as operators with dense range are both full-derivable points and full-generalized-derivable points of JSL algebras.

The following is our main result.

Theorem 2.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a complex Banach space X and $\mathrm{Alg}\,\mathcal{L}$ the associated $\mathcal{J}$-subspace lattice algebra. Let $\delta : \mathrm{Alg}\,\mathcal{L} \to \mathrm{Alg}\,\mathcal{L}$ be a linear map and $Z \in \mathrm{Alg}\,\mathcal{N}$ an injective operator or an operator with dense range. If δ is derivable at Z, then δ is a derivation.*

It follows from Theorem 2.1 that a linear map that is generalized derivable at Z, an injective operator or one with dense range, must be a generalized derivation, and therefore the injective operators and the operators with dense range are full-generalized-derivable points for JSL algebras. This is a short calculation with the map $\tau(T) = \delta(T) - \delta(I)T$, which is derivable at Z. For details, see the proof of [**3,** Theorem 2.2].

To prove Theorem 2.1, we need several lemmas which are valid for both real and complex cases.

Lemma 2.2 ([**6**]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists a subspace $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$.*

Lemma 2.3 ([**9**]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then the following statements hold true.*

(1) *For any $K, L \in \mathcal{J}(\mathcal{L})$, $K \neq L$ implies that $K \cap L = (0)$.*
(2) *If $K \in \mathcal{J}(\mathcal{L})$, then, for any nonzero vector $x \in K$, there exists $f \in K_-^\perp$ such that $f(x) = 1$; dually, for any nonzero functional $f \in K_-^\perp$, there exists $x \in K$ such that $f(x) = 1$.*

Lemma 2.4. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Suppose that $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map. If there exists an injective operator or an operator with dense range $Z \in \operatorname{Alg}\mathcal{L}$ such that δ is derivable at Z, then $\delta(I) = 0$.*

Proof. Since δ is derivable at Z and $Z = IZ = ZI$, we have $\delta(Z) = \delta(I)Z + I\delta(Z) = \delta(Z) + Z\delta(I)$. So $\delta(I)Z = Z\delta(I) = 0$. If Z is injective, by $Z\delta(I) = 0$, we get $\delta(I) = 0$; if Z is an operator with dense range, then, by $\delta(I)Z = 0$, we get again $\delta(I) = 0$. $\square$

Lemma 2.5. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ a linear map derivable at $Z \in \operatorname{Alg}\mathcal{L}$.*

If Z is an operator with dense range, then

(1) *for every idempotent operator $P \in \operatorname{Alg}\mathcal{L}$, we have $\delta(PZ) = \delta(P)Z + P\delta(Z)$ and $\delta(P) = \delta(P)P + P\delta(P)$;*
(2) *for every operator $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$, we have $\delta(NZ) = \delta(N)Z + N\delta(Z)$ and $\delta(N)N + N\delta(N) = 0$.*

If Z is an injective operator, then

(1′) *for every idempotent operator $P \in \operatorname{Alg}\mathcal{L}$, we have $\delta(ZP) = \delta(Z)P + Z\delta(P)$ and $\delta(P) = \delta(P)P + P\delta(P)$;*
(2′) *for every operator $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$, we have $\delta(ZN) = \delta(Z)N + Z\delta(N)$ and $\delta(N)N + N\delta(N) = 0$.*

Proof. Let $P \in \operatorname{Alg}\mathcal{L}$ be any idempotent operator. If Z is an operator with dense range, then by Lemma 2.4, we have

$$\begin{aligned}\delta(Z) = &\ \delta(4P-I)(\tfrac{4}{3}P-I)Z+(4P-I)\delta(\tfrac{4}{3}PZ-Z)\\ = &\ \tfrac{16}{3}\delta(P)PZ-4\delta(P)Z+\tfrac{16}{3}P\delta(PZ)-4P\delta(Z)\\ - &\ \tfrac{4}{3}\delta(PZ)+\delta(Z)\end{aligned}$$

since $Z=(4P-I)(\frac{4}{3}P-I)Z$. That is,

$$\tag{1} \tfrac{16}{3}\delta(P)PZ-4\delta(P)Z+\tfrac{16}{3}P\delta(PZ)-4P\delta(Z)-\tfrac{4}{3}\delta(PZ)=0.$$

On the other hand, since $Z=(\frac{4}{3}P-I)(4P-I)Z$, we have

$$\tag{2} \tfrac{16}{3}\delta(P)PZ-\tfrac{4}{3}\delta(P)Z+\tfrac{16}{3}P\delta(PZ)-\tfrac{4}{3}P\delta(Z)-4\delta(PZ)=0.$$

Combining equation (1) with (2) yields that

$$\tag{3} \delta(PZ)=\delta(P)Z+P\delta(Z).$$

Then, replacing $\delta(PZ)$ by $\delta(P)Z+P\delta(Z)$ in equation (2), we see that $\delta(P)Z=\delta(P)PZ+P\delta(P)Z$. Since Z is an operator with dense range, it follows that $\delta(P)=\delta(P)P+P\delta(P)$. This completes the proof of assertion (1).

If Z is an injective operator, one can discuss dually. In fact, by the equation

$$Z=Z(4P-I)(\tfrac{4}{3}P-I)=Z(\tfrac{4}{3}P-I)(4P-I),$$

and using a similar argument as above, one can get that $\delta(ZP)=\delta(Z)P+Z\delta(P)$ and $\delta(P)=\delta(P)P+P\delta(P)$. Hence, ($1'$) holds true.

For every operator $N\in\operatorname{Alg}\mathcal{L}$ with $N^2=0$, if Z is an operator with dense range, then, noting that $Z=(I-N)(I+N)Z=(I+N)(I-N)Z$, we have

$$\tag{4} \delta(N)Z-\delta(N)NZ-\delta(NZ)+N\delta(Z)-N\delta(NZ)=0$$

and

$$\tag{5} -\delta(N)Z-\delta(N)NZ+\delta(NZ)-N\delta(Z)-N\delta(NZ)=0$$

since δ is derivable at Z. Comparing the above two equations, we get

$$\tag{6} \delta(NZ)=\delta(N)Z+N\delta(Z).$$

Replacing $\delta(NZ)$ by $\delta(N)Z+N\delta(Z)$ in equation (4) and noting that the range of Z is dense, it follows that $\delta(N)N+N\delta(N)=0$.

If Z is an injective operator, consider the equation $Z = Z(I-N)(I+N) = Z(I+N)(I-N)$. Hence (2) and (2′) hold true, completing the proof. $\square$

Lemma 2.6. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ be a linear map derivable at $Z \in \operatorname{Alg}\mathcal{L}$. For any rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$, we have*

$$\delta(x \otimes f)\ker(x \otimes f) \subseteq \operatorname{span}\{x\}.$$

Proof. By Lemma 2.5, we have proved that $\delta(P) = \delta(P)P + P\delta(P)$ holds for every idempotent operator $P \in \operatorname{Alg}\mathcal{L}$ and $\delta(N)N + N\delta(N) = 0$ holds for every operator $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$. The following argument is similar to the proof of Claim 1 in the proof of [**3,** Theorem 3.1]. We give the details here for reader's convenience.

We shall prove the lemma by considering two cases.

Case 1. $\langle x, f\rangle = \lambda \neq 0$. By the linearity of δ, we have

$$\delta(\lambda^{-1}x \otimes f) = \delta(\lambda^{-1}x \otimes f)(\lambda^{-1}x \otimes f) + (\lambda^{-1}x \otimes f)\delta(\lambda^{-1}x \otimes f),$$

that is, $\delta(x \otimes f) = \lambda^{-1}\delta(x \otimes f)(x \otimes f) + \lambda^{-1}(x \otimes f)\delta(x \otimes f)$, which implies that the lemma holds true.

Case 2. $\langle x, f\rangle = 0$. By Lemma 2.2, there exists $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$. Then, by Lemma 2.3 (2), there exists $z \in K$ such that $\langle z, f\rangle = 1$. Thus, $(x+z) \otimes f$, $z \otimes f \in \operatorname{Alg}\mathcal{L}$ are both idempotents. So we have $\delta((x+z)\otimes f)\ker(f) \subseteq \operatorname{span}\{x+z\}$ and $\delta(z\otimes f)\ker(f) \subseteq \operatorname{span}\{z\}$. Note that $\delta(x\otimes f) = \delta((x+z)\otimes f) - \delta(z\otimes f)$. Hence, $\delta(x \otimes f)\ker(f) \subseteq (\operatorname{span}\{x+z\} - \operatorname{span}\{z\})$. Taking any $y \in \ker(f)$, and then there exist $\alpha(y), \beta(y) \in \mathbb{C}$ such that

$$\delta(x \otimes f)y = \alpha(y)(x+z) - \beta(y)z = \alpha(y)x + (\alpha(y) - \beta(y))z. \tag{7}$$

Since $(x \otimes f)^2 = 0$, we get

$$\delta(x \otimes f)(x \otimes f) + (x \otimes f)\delta(x \otimes f) = 0.$$

It follows that $0 = (\delta(x\otimes f)(x\otimes f) + (x\otimes f)\delta(x\otimes f))y = \langle\delta(x\otimes f)y, f\rangle x$ for every $y \in \ker(f)$, that is, $\langle\delta(x \otimes f)y, f\rangle = 0$. Thus, we get from equation (7) that

$$\begin{aligned} 0 = {} & \langle\delta(x \otimes f)y, f\rangle = \langle\alpha(y)x + (\alpha(y) - \beta(y))z, f\rangle \\ = {} & (\alpha(y) - \beta(y))\langle z, f\rangle = \alpha(y) - \beta(y), \end{aligned}$$

that is, $\delta(x\otimes f)y=\alpha(y)x$ for every $y\in\ker(f)$, completing the proof of the lemma. $\square$

Now we are at a position to give the proof of our main result in this section.

Proof of Theorem 2.1. We only give the proof for the case that Z is an operator with dense range. For the case that Z is an injective operator, one can deal with dually. Hence, in the sequel, we always assume that Z is an operator with dense range. We'll complete the proof of the theorem by checking several claims.

Claim 2.7. *For each $K\in\mathcal{J}(\mathcal{L})$, there exist linear maps $C_K:K_-^\perp\to K_-^\perp$ and $B_K:K\to K$ such that*

$$\delta(x\otimes f)=x\otimes C_Kf+B_Kx\otimes f\quad \textit{for all } x\in K \textit{ and } f\in K_-^\perp.$$

Since, by Lemma 2.6, $\delta(x\otimes f)\ker(x\otimes f)\subseteq\operatorname{span}\{x\}$ holds for all rank one operator $x\otimes f\in\operatorname{Alg}\mathcal{L}$, by similar arguments as those in the proofs of [**12,** Theorem 3], one can show that the claim holds. We give an outline of its proof here.

For any $K\in\mathcal{J}(\mathcal{L})$ and any nonzero vector $f\in K_-^\perp$, by Lemma 2.6, there exists a continuous linear functional $\lambda_{f,x}^K$ on $\ker(f)$ such that $\delta(x\otimes f)(u)=\langle u,\lambda_{f,x}^K\rangle x$ for all $u\in\ker(f)$. So, for any $y\in K$, on the one hand, we have

$$\delta((x+y)\otimes f)(u)=\langle u,\lambda_{f,x+y}^K\rangle(x+y). \tag{8}$$

On the other hand,

$$\delta((x+y)\otimes f)(u)=\delta(x\otimes f)(u)+\delta(y\otimes f)(u)=\langle u,\lambda_{f,x}^K\rangle x+\langle u,\lambda_{f,y}^K\rangle y. \tag{9}$$

Comparing equation (8) with (9), we see that $\langle u,\lambda_{f,x+y}^K-\lambda_{f,x}^K\rangle x+\langle u,\lambda_{f,x+y}^K-\lambda_{f,y}^K\rangle y=0$, which implies that $\lambda_{f,x+y}^K=\lambda_{f,x}^K=\lambda_{f,y}^K$. So $\lambda_{f,x}^K$ is independent of x. Write $\lambda_f^K=\lambda_{f,x}^K$. Let g_f^K be a continuous linear extension of λ_f^K to K. Then we have

$$\delta(x\otimes f)(u)=\langle u,g_f^K\rangle x\quad\text{for all } u\in\ker(f). \tag{10}$$

For any $u \in K \setminus \ker(f)$, we define an operator $B^K_{u,f} : K \to K$ as follows:

$$B^K_{u,f}x = \langle u, f\rangle^{-1}(\delta(x \otimes f)(u) - \langle u, g^K_f\rangle x).$$

It is obvious that $B^K_{u,f}$ is a linear operator on K, and that

$$\delta(x \otimes f)(u) = \langle u, f\rangle B^K_{u,f}x + \langle u, g^K_f\rangle x \quad \text{for all } x \in K. \tag{11}$$

For any $v \in K \setminus \ker(f)$ with $v \neq -u$, $v \neq 0$, we have

$$\delta(x \otimes f)(v) = \langle v, f\rangle B^K_{v,f}x + \langle v, g^K_f\rangle x, \tag{12}$$

$$\delta(x \otimes f)(u+v) = \langle u+v, f\rangle B^K_{u+v,f}x + \langle u+v, g^K_f\rangle x. \tag{13}$$

Comparing equations (11)–(13), we get that $B^K_{u,f} = B^K_{v,f}$. Thus, $B^K_{u,f}$ is independent to u, and we may write $B^K_f = B^K_{u,f}$. It follows from equation (11) that

$$\delta(x \otimes f)(u) = \langle u, f\rangle B^K_f x + \langle u, g^K_f\rangle x \quad \text{for every } u \in K \setminus \ker(f). \tag{14}$$

Combining equation (14) with (10), we obtain

$$\delta(x \otimes f)(u) = \langle u, f\rangle B^K_f x + \langle u, g^K_f\rangle x = (x \otimes g^K_f + B^K_f x \otimes f)(u), \quad \text{for all } u \in K.$$

Hence,

$$\delta(x \otimes f) = x \otimes g^K_f + B^K_f x \otimes f \quad \text{for all} \quad x \in K. \tag{15}$$

Next, we'll prove that there exists a linear map $B_K : K \to K$ such that

$$\delta(x \otimes f) = x \otimes g^K_f + B_K x \otimes f \quad \text{holds for all } x \in K. \tag{16}$$

In fact, fix a nonzero vector $f_0 \in K^{\perp}_{-}$. By equation (15), we may take $g^K_{f_0} \in K^*$ and $B^K_{f_0} : K \to K$ such that

$$\delta(x \otimes f_0) = x \otimes g^K_{f_0} + B^K_{f_0}x \otimes f_0 \quad \text{for all } x \in K.$$

Let $B_K = B^K_{f_0}$. If $\dim K^{\perp}_{-} = 1$, equation (16) holds. So we assume that $\dim K^{\perp}_{-} \geq 2$. For any $f \in K^{\perp}_{-}$, if f and f_0 are linearly dependent, equation (16) is true; if f and f_0 are linearly independent, we will show that $B^K_f - B_K = \lambda I$ for some $\lambda \in \mathbb{C}$. If this is the case, then for every $x \in K$, we have $\delta(x \otimes f) = x \otimes (g^K_f + \lambda f) + B_K x \otimes f$, and therefore equation (16) also holds. For $f+f_0$, by equation (15), there exists a continuous linear functional $g^K_{f+f_0}$ on K and a linear map $B^K_{f+f_0}: K \to K$ such that

$$\delta(x\otimes(f+f_0)) = x\otimes g^K_{f+f_0} + B^K_{f+f_0}x\otimes(f+f_0) \quad \text{for all } x\in K.$$

On the other hand, we have

$$\begin{aligned}\delta(x\otimes(f+f_0)) = &\ \delta(x\otimes f)+\delta(x\otimes f_0)\\ = &\ x\otimes g^K_f + B^K_f x\otimes f + x\otimes g^K_{f_0} + B_K x\otimes f_0.\end{aligned}$$

Combining the above two equations, we get that

$$\begin{aligned}(17)\quad & x\otimes g^K_{f+f_0} + B^K_{f+f_0}x\otimes(f+f_0)\\ &\qquad = x\otimes g^K_f + B^K_f x\otimes f + x\otimes g^K_{f_0} + B_K x\otimes f_0.\end{aligned}$$

Since f and f_0 are linearly independent, there exist $x_1, x_2\in K$ such that $\langle x_1, f\rangle = \langle x_2, f_0\rangle = 1$ and $\langle x_1, f_0\rangle = \langle x_2, f\rangle = 0$. Applying equation (17) to x_1 and x_2, respectively, we get that $B^K_{f+f_0} - B_K = \lambda_1 I$ for some $\lambda_1\in\mathbb{C}$ and $B^K_{f+f_0} - B^K_f = \lambda_2 I$ for some $\lambda_2\in\mathbb{C}$. So $B^K_f - B_K = (\lambda_1-\lambda_2)I$.

Now fix a nonzero vector $x\in K$, and fix $h\in K^\perp_-$ such that $\langle x, h\rangle = 1$. For each $f\in K^\perp_-$, by equation (16), we have $g^K_f = (\delta(x\otimes f) - B_K x\otimes f)^*(h)$. Define an operator C_K by $C_K(f) = g^K_f$ for every $f\in K^\perp_-$. It is obvious that $C_K: K^\perp_-\to K^\perp_-$ is linear. Hence, we have

$$\delta(x\otimes f) = x\otimes C_K f + B_K x\otimes f \quad \text{for all } x\in K.$$

Claim 2.8. *There exists a linear operator $B:\langle\mathcal{J}(\mathcal{L})\rangle\to\langle\mathcal{J}(\mathcal{L})\rangle$ such that $\delta(x\otimes f) = B(x\otimes f) - (x\otimes f)B$ for every rank one operator $x\otimes f\in\operatorname{Alg}\mathcal{L}$.*

We first prove that $B_K: K\to K$ is bounded. In fact, for any $x\otimes f\in\operatorname{Alg}\mathcal{L}$ with $x\in K$, $f\in K^\perp_-$ and $\langle x, f\rangle = 1$, by Lemma 2.5 (1) and (1′), we can easily get that $(x\otimes f)\delta(x\otimes f)(x\otimes f) = 0$. So $(\langle x, C_K f\rangle + \langle B_K x, f\rangle)x\otimes f = 0$. It follows that

$$(18)\qquad \langle x, C_K f\rangle + \langle B_K x, f\rangle = 0 \quad \text{holds for all } x\in K,\ f\in K^\perp_- \text{ with } \langle x, f\rangle = 1.$$

Now let $x \in K$ and $f \in K_-^\perp$ be arbitrary. If $\langle x, f\rangle \neq 0$, it is obvious that equation (18) holds. If $\langle x, f\rangle = 0$, there exists $f_1 \in K_-^\perp$ such that $\langle x, f_1\rangle = 1$. Let $f_2 = f_1 - f$. So $\langle x, f_2\rangle = 1$. Thus, we have

$$\begin{aligned}\langle x, C_K f\rangle + \langle B_K x, f\rangle = &\ \langle x, C_K(f_1 - f_2)\rangle + \langle B_K x, (f_1 - f_2)\rangle \\ = &\ \langle x, C_K f_1\rangle + \langle B_K x, f_1\rangle \\ - &\ \langle x, C_K f_2\rangle - \langle B_K x, f_2\rangle = 0.\end{aligned}$$

Hence, we obtain that

$$\langle x, C_K f\rangle + \langle B_K x, f\rangle = 0 \quad \text{holds for all } x \in K, \quad f \in K_-^\perp. \tag{19}$$

If $\{x_n\} \subseteq K$ so that $x_n \to x_0$ and $B_K x_n \to y_0$ as $n \to \infty$, then

$$0 = \langle x_n, C_K f\rangle + \langle B_K x_n, f\rangle \mapsto \langle x_0, C_K f\rangle + \langle y_0, f\rangle = 0.$$

Combining equation (19) with the above equation, we have $\langle B_K x_0, f\rangle = \langle y_0, f\rangle$ holds for all $f \in K_-^\perp$. This entails that $B_K x_0 = y_0$, since, otherwise, there would be some $f \in K_-^\perp$ such that $\langle B_K x_0, f\rangle \neq \langle y_0, f\rangle$. It follows from the closed graph theorem that $B_K \in \mathcal{B}(K)$. Similarly, we can check that $C_K \in \mathcal{B}(K_-^\perp)$.

Now, define a linear map $B : \langle \mathcal{J}(\mathcal{L})\rangle \to \langle \mathcal{J}(\mathcal{L})\rangle$ such that $B|_K = B_K$ for any $K \in \mathcal{J}(\mathcal{L})$ and a linear map $C : \langle \mathcal{J}(\mathcal{L})_-^\perp\rangle \to \langle \mathcal{J}(\mathcal{L})_-^\perp\rangle$ such that $C|_{K_-^\perp} = C_K$ for any $K_-^\perp \in \mathcal{J}(\mathcal{L})_-^\perp$. B and C are well defined as, by Lemma 2.3 (1), $\mathcal{J}(\mathcal{L})$ is a collection of linearly independent subspaces K and $\langle \mathcal{J}(\mathcal{L})\rangle = \operatorname{span}\{K \mid K \in \mathcal{J}(\mathcal{L})\}$.

Since $K \wedge K_- = \{0\}$ and $K \vee K_- = X$, we may regard $K_-^\perp$ as the dual space K^* of K. Thus, by equation (19), we get $C_K = -(B_K)^*$. Hence, $C = -B^*$.

Thus, there exists a linear map $B : \langle \mathcal{J}(\mathcal{L})\rangle \to \langle \mathcal{J}(\mathcal{L})\rangle$ such that

$$\delta(x \otimes f) = Bx \otimes f - x \otimes fB \quad \text{holds for all } x \in K, \quad f \in K_-^\perp.$$

Claim 2.9. *$\delta(Z) = BZ - ZB$.*

By Lemma 2.5 (1), we have $\delta(PZ) = \delta(P)Z + P\delta(Z)$ for all idempotents in $\operatorname{Alg}\mathcal{L}$. Since we know that every rank one operator in $\operatorname{Alg}\mathcal{L}$ is a linear combination of idempotents in $\operatorname{Alg}\mathcal{L}$ (see [**3,** Lemma 1.4]), it follows that, for every rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$, we have

$$\delta((x\otimes f)Z)=\delta(x\otimes f)Z+(x\otimes f)\delta(Z). \tag{20}$$

Combining Claim 2.8 with equation (20), we have

$$B(x\otimes f)Z-(x\otimes f)ZB=B(x\otimes f)Z-(x\otimes f)BZ+(x\otimes f)\delta(Z),$$

that is, $(x\otimes f)(\delta(Z)-BZ+ZB)=0$ for all $x\otimes f\in \operatorname{Alg}\mathcal{L}$, which implies $\delta(Z)=BZ-ZB$ (see [**5**]).

Now, for every invertible operator $A\in \operatorname{Alg}\mathcal{L}$, since $Z=AA^{-1}Z$ and δ is derivable at Z, we have $\delta(Z)=\delta(A)A^{-1}Z+A\delta(A^{-1}Z)$. Hence, by Claim 2.9, we get

$$\begin{aligned}\delta(A^{-1}Z)= & \ A^{-1}\delta(Z)-A^{-1}\delta(A)A^{-1}Z\\ = & \ A^{-1}BZ-A^{-1}ZB-A^{-1}\delta(A)A^{-1}Z.\end{aligned} \tag{21}$$

Claim 2.10. *For every $T\in \operatorname{Alg}\mathcal{L}$, we have $\delta(T)|_{\langle\mathcal{J}(\mathcal{L})\rangle}=BT|_{\langle\mathcal{J}(\mathcal{L})\rangle}-TB$.*

So far, the complexity of X has not be used. Since X is complex, for any T and any $x\otimes f\in \operatorname{Alg}\mathcal{L}$, we can take $\lambda\in\mathbb{C}$ such that $|\lambda|>\|T\|$ and $\|(\lambda I-T)^{-1}x\|\|f\|<1$. Then both $\lambda I-T$ and $\lambda I-T-x\otimes f=(\lambda I-T)(I-(\lambda I-T)^{-1}x\otimes f)$ are invertible with their inverses are still in $\operatorname{Alg}\mathcal{L}$. It is obvious that $(I-(\lambda I-T)^{-1}x\otimes f)^{-1}=I+(1-\alpha)^{-1}(\lambda I-T)^{-1}x\otimes f$, where $\alpha=\langle(\lambda I-T)^{-1}x,f\rangle$.

In the following, for simpleness, let $W=\lambda I-T$ and $R=x\otimes f$.

Then, by Claim 2.9, Lemma 2.4 and equation (21), we have

$$\begin{aligned}& BZ-ZB=\delta(Z)\\ = \ & \delta(W-R)(I+(1-\alpha)^{-1}W^{-1}R)W^{-1}Z\\ & +(W-R)\delta((I+(1-\alpha)^{-1}W^{-1}R)W^{-1}Z)\\ = \ & (-\delta(T)-BR+RB)[W^{-1}Z+(1-\alpha)^{-1}W^{-1}RW^{-1}Z]\\ & +(W-R)[W^{-1}BZ-W^{-1}ZB+W^{-1}\delta(T)W^{-1}Z\\ & +(1-\alpha)^{-1}BW^{-1}RW^{-1}Z-(1-\alpha)^{-1}W^{-1}RW^{-1}ZB].\end{aligned}$$

Expanding this equation, and using $RW^{-1}R=\alpha R$, it follows that

$$\begin{aligned}0= \ & (1-\alpha)^{-1}\delta(T)W^{-1}RW^{-1}Z+RW^{-1}BZ+RW^{-1}\delta(T)W^{-1}Z\\ & -(1-\alpha)^{-1}WBW^{-1}RW^{-1}Z-RBW^{-1}Z\\ & +(1-\alpha)^{-1}BRW^{-1}Z.\end{aligned}$$

Since the range of Z is dense, the above equation implies that

$$\begin{aligned} 0 = \;& (1-\alpha)^{-1}\delta(T)W^{-1}RW^{-1} + RW^{-1}B + RW^{-1}\delta(T)W^{-1} \\ & -(1-\alpha)^{-1}WBW^{-1}RW^{-1} - RBW^{-1} + (1-\alpha)^{-1}BRW^{-1}. \end{aligned}$$

Multiplying the above equation by W from the right, we get

$$\begin{aligned} 0 = \;& (1-\alpha)^{-1}\delta(T)W^{-1}R + RW^{-1}BW + RW^{-1}\delta(T) \\ & -(1-\alpha)^{-1}WBW^{-1}R - RB + (1-\alpha)^{-1}BR. \end{aligned}$$

So we have $(\delta(T)W^{-1} - WBW^{-1} + B)R = R(1-\alpha)(B - W^{-1}BW - W^{-1}\delta(T))$, that is,

$$\begin{aligned} & [\delta(T)(\lambda I - T)^{-1} - (\lambda I - T)B(\lambda I - T)^{-1} + B]x \otimes f \\ = \;& (x \otimes f)(1-\alpha)[B - (\lambda I - T)^{-1}B(\lambda I - T) - (\lambda I - T)^{-1}\delta(T)]. \end{aligned}$$

Hence, $[\delta(T)(\lambda I - T)^{-1} - (\lambda I - T)B(\lambda I - T)^{-1} + B]x$ is linearly dependent to x for every $x \in \langle \mathcal{J}(\mathcal{L})\rangle$. This entails that there is a scalar β_λ such that

$$\delta(T)(\lambda I - T)^{-1} - (\lambda I - T)B(\lambda I - T)^{-1} + B = \beta_\lambda I$$

on $\langle \mathcal{J}(\mathcal{L})\rangle$. It follows that

$$\delta(T) = BT - TB + \beta_\lambda(\lambda I - T) \tag{22}$$

holds on $\langle \mathcal{J}(\mathcal{L})\rangle$. By taking different λ in equation (22), we see that $\beta_\lambda = 0$ and consequently $\delta(T)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = BT|_{\langle \mathcal{J}(\mathcal{L})\rangle} - TB$ holds for all $T \in \operatorname{Alg}\mathcal{L}$, as desired.

Claim 2.11. *δ is a derivation.*

For any $T, S \in \operatorname{Alg}\mathcal{L}$, by Claim 2.10, we have

$$\delta(TS)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = BTS|_{\langle \mathcal{J}(\mathcal{L})\rangle} - TSB = BT|_{\langle \mathcal{J}(\mathcal{L})\rangle}S|_{\langle \mathcal{J}(\mathcal{L})\rangle} - TSB$$

and

$$\begin{aligned} (\delta(T)S + T\delta(S)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = \;& (BT|_{\langle \mathcal{J}(\mathcal{L})\rangle} - TB)S|_{\langle \mathcal{J}(\mathcal{L})\rangle} \\ & +T(BS|_{\langle \mathcal{J}(\mathcal{L})\rangle} - SB) \\ = \;& BT|_{\langle \mathcal{J}(\mathcal{L})\rangle}S|_{\langle \mathcal{J}(\mathcal{L})\rangle} - TSB. \end{aligned}$$

Comparing the above two equations, we get $\delta(TS)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = (\delta(T)S + T\delta(S)|_{\langle \mathcal{J}(\mathcal{L})\rangle}$. Thus,

$$\delta(TS) = \delta(T)S + T\delta(S)$$

holds for all $T, S \in \operatorname{Alg}\mathcal{L}$ since $\langle \mathcal{J}(\mathcal{L}) \rangle$ is dense in X. Hence, δ is a derivation. $\square$

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